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Fixed points and steady solitons for the two-loop renormalization group flow

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Abstract. Solitons for the two-loop renormalization group flow are studied in the four-dimensional homogeneous setting, providing a classification of algebraic steady four-dimensional solitons.

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1. Introduction

The Ricci flow (introduced by Hamilton [16] and Friedan [11]) is of fundamental importance in mathematics. It has been shown to be an effective tool for attacking important problems in geometry like Thurston's geometrization conjecture and the Poincaré conjecture. The Ricci flow also has a similar importance in physics, arising as the first-order approximation of the renormalization group flow for the non-linear sigma model of quantum field theory.

Recently, there has been interest in the second-order approximation of this renormalization group flow, or the 2-loop renormalization group flow (*RG2 flow* for short), which mathematically is described by

$$\frac{\partial}{\partial t} g_t = RG[g], \tag{1}$$

where $RG[g] = -2\rho - \frac{\alpha}{2}\check{R}$ and α denotes a positive coupling constant. Here, ρ denotes the Ricci tensor and $\check{R}$ is the symmetric $(0,2)$-tensor field $\check{R}_{ij} = R_{iabc}R_j{}^{abc}$ of the evolving metric g_t. Independently of the physical significance of the flow (1), it is mathematically interesting as a perturbation of the Ricci flow. We refer to [7, 12–14] and references therein for more information on the RG2 flow.

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The purpose of this paper is to study fixed points of the RG2 flow in the four-dimensional case. Genuine fixed points of the flow are provided by those manifolds where the right-hand side of (1) vanishes, i.e., $\rho + \frac{\alpha}{4}\check{R} = 0$. In dimension two, this condition reduces to constant negative curvature. The analysis in dimension three has been carried out in [13], where it is shown that solutions of non-constant curvature must have Ricci curvatures $Q = \mathrm{diag}[-2/\alpha, -2\alpha, 0]$ or $Q = \mathrm{diag}[-4/\alpha, -2\alpha, -2/\alpha]$, where Q denotes the metrically equivalent $(1,1)$-tensor field associated with ρ. In the homogeneous situation, these geometries correspond to product manifolds $\mathbb{R}\times N(c)$, where $N(c)$ is a surface of constant curvature c, or a left-invariant metric on $SU(2)$ with $\alpha < 0$.

Tracing the tensor field $RG[g]$, one has that if $RG[g] = 0$, then the coupling constant α satisfies $\tau + \frac{\alpha}{4}\|R\|^2 = 0$, where τ is the scalar curvature and $\|R\|^2 = R_{ijk\ell}R^{ijk\ell}$. The previous expression does not necessarily mean that α must be constant, but it is expressible in terms of the scalar curvature and the norm of the curvature tensor, which are constant in the homogeneous case. On the other hand, the functional defined by the four-dimensional Gauss–Bonnet integrand $g \mapsto \int_M \{\|R\|^2 - 4\|\rho\|^2 + \tau^2\}\, vol_g$ is constant in dimension 4 and thus any compact four-dimensional manifold satisfies the curvature identity (see [4])

$$\left(\check{R} - \frac{\|R\|^2}{4}g\right) + \tau\left(\rho - \frac{\tau}{4}g\right) - 2\left(\check{\rho} - \frac{\|\rho\|^2}{4}g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4}g\right) = 0, \tag{2}$$

where $\check{\rho}$ and $R[\rho]$ are the symmetric $(0,2)$-tensor fields given by $\check{\rho}_{ij} = \rho_{ia}\rho^a{}_j$ and $R[\rho]_{ij} = R_{iabj}\rho^{ab}$. (See [10] for an extension of the previous identity to the non-compact case). If (M,g) is Einstein, then all terms in (2) vanish and it immediately follows that any Einstein four-dimensional manifold satisfies $\rho + \frac{\alpha}{4}\check{R} = 0$ for $\alpha = -4\tau\|R\|^{-2}$, which shows that Einstein four-manifolds are genuine fixed points of the flow (1). In the homogeneous setting the situation is rather restrictive, and we show in Sect. 7 that any other example is a product as follows:

Theorem 1.1. *A simply connected four-dimensional homogeneous manifold is a genuine fixed point of the RG2 flow if and only if it is Einstein, a product* $\mathbb{R}\times N^3(c)$, *a product* $\mathbb{R}^2 \times N^2(c)$ *or homothetic to the Lie group* $SU(2)\times\mathbb{R}$ *with left-invariant metric*

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = \tfrac{4}{3}e_2,$$

where $\{e_1, \dots, e_4\}$ *is an orthonormal basis of* $\mathfrak{su}(2)\times\mathbb{R}$.

The above result is in sharp contrast with the geometry of the Ricci flow, since genuine fixed points of the Ricci flow are Ricci-flat manifolds, which are necessarily flat in the homogeneous setting [1].

In addition to the cases above, one may consider geometrical fixed points of the RG2 flow, i.e., solutions $g(t)$ which are fixed modulo scalings and diffeomorphisms. Given a one-parameter family ψ_t of diffeomorphisms of M (with $\psi_0 = \mathrm{Id}$), a solution of the form $g(t) = \sigma(t)\psi_t^* g$ (where σ is a real-valued function with $\sigma(0) = 1$) is said to be a *self-similar solution*. A triple (M, g, X), where X is a vector field on M, is called an RG2 *soliton* if $\mathcal{L}_X g + RG[g] = \lambda g$ for some $\lambda \in \mathbb{R}$. Further, the soliton is said to be *expanding*, *steady* or *shrinking* if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$, respectively.

Any self-similar solution of the RG2 flow is an RG2 soliton just considering the vector field X generated by the one-parameter group of diffeomorphisms ψ_t (see, for example, [6] and [22]). Since the two terms comprising $RG[g]$ behave differently under homotheties ($\rho[\kappa g] = \rho[g]$ and $\check{R}[\kappa g] = \frac{1}{\kappa}\check{R}[g]$), one has that the converse holds only for steady solitons, in which case ψ_t is the one-parameter group of diffeomorphisms associated with the vector field X determined by the soliton equation $\mathcal{L}_X g + RG[g] = 0$ and $g(t) = \psi_t^* g$ is a self-similar solution (see [22]).

Let G be a Lie group with left-invariant metric $\langle \cdot, \cdot \rangle$ and let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ denote the corresponding Lie algebra. An RG2 *algebraic soliton* is a derivation of the Lie algebra $\mathfrak{g}$ given by $\mathfrak{D} = \widehat{RG}[g] - \beta\,\mathrm{Id}$, where $\widehat{RG}[g]$ is the $(1,1)$-tensor field metrically equivalent to $RG[g]$ and $\beta \in \mathbb{R}$. RG2 algebraic solitons give rise to RG2 solitons as well as in the Ricci flow case (see [19,22]).

Let Q and $\check{Q}$ denote the metrically equivalent $(1,1)$-tensor fields associated with ρ and $\check{R}$, respectively. Let $\langle \cdot, \cdot \rangle^* = \kappa \langle \cdot, \cdot \rangle$ be a homothetic deformation of a left-invariant metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Then,

$$Q^* + \frac{\kappa\alpha}{4}\check{Q}^* = \frac{1}{\kappa}Q + \frac{\kappa\alpha}{4\kappa^2}\check{Q} = \frac{1}{\kappa}\left(Q + \frac{\alpha}{4}\check{Q}\right)$$

and thus $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ with coupling constant α if and only if $\mathfrak{D}^* = Q^* + \frac{\kappa\alpha}{4}\check{Q}^*$ is a derivation of the Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle^*)$ with coupling constant $\kappa\alpha$. Aimed to describe four-dimensional RG2 algebraic steady solitons, we therefore work modulo homotheties in what follows to simplify the exposition.

Let H be a Lie group with a left-invariant metric determined by an inner product on the Lie algebra $(\mathfrak{h}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$ and let $G = \mathbb{R} \times H$ be the product Lie group with product left-invariant metric $\langle \cdot, \cdot \rangle_{\mathfrak{g}} = dt \otimes dt \oplus \langle \cdot, \cdot \rangle_{\mathfrak{h}}$. Since $\widehat{RG}_{\mathfrak{g}} = 0 \oplus \widehat{RG}_{\mathfrak{h}}$, one has that if $(\mathfrak{h}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$ is an RG2 algebraic steady soliton, then so is $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$. Conversely, assume that a (complete and simply connected) Lie group G with left-invariant metric is an RG2 algebraic steady soliton. Further, assume that there exists a parallel left-invariant vector field on G. Then it splits a one-dimensional factor so that the Lie group splits isometrically (as a Riemannian manifold) $G = \mathbb{R} \times N$, where N is a complete and simply connected three-dimensional homogeneous manifold. Hence, N is either symmetric (in which case G is also a symmetric space) or N is isometric to a Lie group H. Correspondingly, the tensor field RG also splits as $RG_{\mathfrak{g}} = 0 \oplus RG_{\mathfrak{h}}$ and so does the corresponding $(1,1)$-tensor field $\widehat{RG}_{\mathfrak{g}}$. Hence, if G is an RG2 algebraic steady soliton, then so is H just considering the derivation determined by $\widehat{RG}_{\mathfrak{h}}$.

Four-dimensional Lie groups are given by the product Lie groups $SU(2) \times \mathbb{R}$ and $SL(2, \mathbb{R}) \times \mathbb{R}$ (where we use the non-standard notation to represent the universal covering) and the semi-direct products $\mathbb{R} \ltimes E(1,1)$, $\mathbb{R} \ltimes E(2)$, $\mathbb{R} \ltimes H^3$ and $\mathbb{R} \ltimes \mathbb{R}^3$, where $E(1,1)$, $E(2)$, H^3 and $\mathbb{R}^3$ are, respectively, the three-dimensional Poincaré group, the Euclidean group, the Heisenberg group and the Abelian group (see the discussion in [3]). Let $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}(2)$, $\mathfrak{e}(1,1)$, $\mathfrak{e}(2)$, $\mathfrak{h}^3$ and $\mathfrak{r}^3$ be the Lie algebras corresponding to the three-dimensional Lie groups above. We analyze in Sect. 7 the existence of RG2 algebraic steady solitons on four-dimensional irreducible Lie groups, since otherwise it reduces to the three-dimensional case which is discussed in Sect. 2 (see also [22]) as follows.

Theorem 1.2. *A simply connected non-Einstein four-dimensional irreducible Lie group G is an RG2 algebraic steady soliton if and only if it is homothetic to one of the Lie groups determined by the following Lie algebras, where $\{e_1,\dots,e_4\}$ is an orthonormal basis:*

(1) $\mathbb{R}\ltimes\mathfrak{e}(1,1)$, *for a coupling constant* $\alpha=\frac{2}{\kappa^2+1}$, *given by*

$$[e_1,e_3]=e_2,\quad [e_2,e_3]=e_1,\quad [e_1,e_4]=\kappa e_1,\quad [e_2,e_4]=\kappa e_2,$$

where $\kappa>0$, $\kappa\neq 1$.

(2) $\mathbb{R}\ltimes\mathfrak{h}^3$, *for a coupling constant* $\alpha=2$, *given by*

$$\begin{aligned}[e_1,e_2]&=e_3, & [e_1,e_4]&=\tfrac{\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_1,\\ [e_2,e_4]&=\tfrac{\kappa\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_2, & [e_3,e_4]&=\tfrac{(\kappa+1)\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_3,\end{aligned}$$

where $\kappa\in[-1,1)$.

(3) $\mathbb{R}\ltimes\mathfrak{h}^3$, *for a coupling constant* $\alpha=\frac{32\kappa^2}{16\kappa^4+1}$, *given by*

$$[e_1,e_2]=e_3,\quad [e_1,e_4]=\kappa e_1,\quad [e_2,e_4]=-\tfrac{1}{4\kappa}e_2,\quad [e_3,e_4]=\left(\kappa-\tfrac{1}{4\kappa}\right)e_3,$$

where $\kappa\in\left(0,\frac12\right]$, $\kappa\neq\frac12\sqrt{2-\sqrt{3}}$.

(4) $\mathbb{R}\ltimes\mathfrak{r}^3$, *for a coupling constant* $\alpha=\frac{2(\kappa^2+\delta^2+1)}{\kappa^4+\delta^4+1}$, *given by*

$$[e_1,e_4]=e_1,\quad [e_2,e_4]=\kappa e_2,\quad [e_3,e_4]=\delta e_3,$$

where $(\kappa,\delta)\in\{(x,y)\in\mathbb{R}^2; x\in(0,1], 0\neq y\leq x\}\backslash\{(1,1)\}$.

(5) $\mathbb{R}\ltimes\mathfrak{r}^3$, *for a coupling constant* $\alpha=\frac{2}{\kappa^2+p^2}$, *given by*

$$[e_1,e_4]=e_1,\quad [e_2,e_4]=\kappa e_2+he_3,\quad [e_3,e_4]=-he_2+pe_3,$$

where the parameters p and h are given by $p=\frac12\left(1+\sqrt{1-4\kappa(\kappa-1)}\right)$ *and* $h=\left(\frac{\kappa^2(2p^2+1)+p^2-1}{2(\kappa-p)^2}\right)^{\frac12}$, *for any* $\kappa\in(0,1)$.

The above result is in sharp contrast with the Ricci flow case where steady homogeneous Ricci solitons are Ricci flat and thus flat. Moreover, the Lie groups $(G,\langle\cdot,\cdot\rangle)$ corresponding to cases (2) and (4) are expanding (algebraic) Ricci solitons, while Lie groups corresponding to cases (1), (3) and (5) are not Ricci solitons. It follows from the analysis in Sects. 3–6 that all metrics in Theorem 1.2 represent different homothetical classes. Theorem 1.2 shows, therefore, a way in which the RG2 flow differs from the Ricci flow.

Let G be a semi-direct product $\mathbb{R}\ltimes\mathbb{R}^3$ corresponding to Assertion (4) in Theorem 1.2 for the special values $(\kappa,\delta)=(1,-1)$, which is an algebraic Ricci soliton, i.e., $\mathfrak{D}=Q+3\,\mathrm{Id}$ is a derivation (see [17,19,22]). The corresponding left-invariant metric $\langle\cdot,\cdot\rangle$ satisfies $\check{R}=\frac14\|R\|^2\langle\cdot,\cdot\rangle$ but it is not Einstein.

Moreover, $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q} - 3(\frac{\alpha}{2} - 1)\,\mathrm{Id}$ is a derivation and hence $(G, \langle\,\cdot\,,\,\cdot\,\rangle)$ is also an RG2 algebraic soliton. Therefore, there is a vector field ξ on G such that $\mathcal{L}_\xi g + \rho + \frac{\alpha}{4}\check{R} = 3(\frac{\alpha}{2} - 1)\langle\,\cdot\,,\,\cdot\,\rangle$ for any value of the coupling constant α, thus resulting in a steady, shrinking or expanding RG2 soliton depending on the value of α, in sharp contrast to Ricci solitons.

The paper is organized as follows. We recall some known facts about RG2 algebraic steady solitons from [22] (see also [15]) in the three-dimensional case and consider also the non-unimodular setting in Sect. 2. The proof of Theorem 1.2 follows after a case by case analysis developed through Sects. 3 to 6. Finally, the proof of Theorem 1.1 and Theorem 1.2 are given in Sect. 7. In particular, we show in Sect. 7.2 that all metrics in Theorem 1.2 represent different homothetical classes.

2. Three-dimensional RG2 algebraic steady solitons

2.1. Gröbner basis

Let $Q_i^j = \rho_{i\ell}g^{\ell j}$ and $\check{Q}_i^j = \check{R}_{i\ell}g^{\ell j}$ denote the corresponding $(1,1)$-tensor fields metrically equivalent to ρ and $\check{R}$, respectively. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{D}$ be the endomorphism of the Lie algebra determined by $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$. Then, $\mathfrak{D}$ defines an RG2 algebraic steady soliton if and only if it is a derivation (i.e., $\mathfrak{D}[x,y] - [\mathfrak{D}x,y] - [x,\mathfrak{D}y] = 0$) (see [22]). Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of $\mathfrak{g}$ and set $\mathfrak{D}_{ijk} = \langle \mathfrak{D}[e_i,e_j] - [\mathfrak{D}e_i,e_j] - [e_i,\mathfrak{D}e_j], e_k\rangle$. Hence, $\mathfrak{D}$ determines an RG2 algebraic steady soliton if and only if $\mathfrak{D}_{ijk} = 0$ for all $i,j,k \in \{1,\dots,n\}$.

The components $\mathfrak{D}_{ijk}$ determine a system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ on the structure constants which is rather involved, although it can be obtained from the expressions of the Ricci tensor ρ and the $\check{R}$-tensor. To obtain a full classification, one needs to solve the corresponding polynomial system of equations. When the system under consideration is simple, it is an elementary problem to find all common roots, but if the number of equations and their degrees increase, it may become a quite unmanageable assignment. There exist, however, some well-known strategies to approach this kind of problem.

Given a set $\mathcal{S}$ of polynomials $\mathfrak{P}_{ijk} \in \mathbb{R}[x_1,\dots,x_n]$, an n-tuple of real numbers $\vec{a} = (a_1,\dots,a_n)$ is a solution of the system of polynomial equations determined by $\mathcal{S}$ if and only if $\mathfrak{P}_{ijk}(\vec{a}) = 0$ for all i, j, k. It is immediate to see that $\vec{a}$ is a solution of the polynomial system of equations determined by $\mathcal{S}$ if and only if it is a solution of the system determined by all the polynomials in the ideal $\mathcal{I} = \langle \mathfrak{P}_{ijk}\rangle$ generated by $\mathcal{S}$: if two sets of polynomials generate the same ideal, the corresponding zero sets must be identical. Therefore, our strategy for solving the rather large polynomial systems consists of obtaining "better" polynomials that belong to the ideals generated by the corresponding polynomial systems. This is achieved by using the theory of Gröbner bases, whose construction we briefly recall below (see [8]).

Let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha \in \mathbb{Z}^n_{\geq 0}$ be a monomial in $\mathbb{R}[x_1, \ldots, x_n]$. A monomial ordering is any relation on the set of monomials x^α with $\alpha \in \mathbb{Z}^n_{\geq 0}$ satisfying

(1) It is a total ordering on $\mathbb{Z}^n_{\geq 0}$.
(2) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.
(3) $\mathbb{Z}^n_{\geq 0}$ is well ordered, so that every non-empty subset of $\mathbb{Z}^n_{\geq 0}$ has a smallest element with respect to the given ordering.

Establishing an ordering on $\mathbb{Z}^n_{\geq 0}$ will induce an ordering on the monomials. For our purposes, we will use the *lexicographical order* and the *graded reverse lexicographical order*. We say that $\alpha >_{\text{lex}} \beta$ if in the vector $\alpha - \beta \in \mathbb{Z}^n$, the leftmost non-zero entry is positive and we say that $\alpha >_{\text{grevlex}} \beta$ if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and the rightmost non-zero entry of $\alpha - \beta \in \mathbb{Z}^n$ is negative.

The basic bricks to introduce Gröbner bases are the leading terms of the polynomials, which are defined as follows. If $\mathfrak{P} = \sum_\alpha a_\alpha x^\alpha$ is a polynomial in $\mathbb{R}[x_1, \ldots, x_n]$, any monomial ordering orders the monomials of $\mathfrak{P}$. The *multidegree* of $\mathfrak{P}$ is the maximum $\alpha \in \mathbb{Z}^n_{\geq 0}$ so that $a_\alpha \neq 0$, where the maximum is taken with respect to the given monomial ordering. The corresponding monomial is called the *leading term*, i.e., $LT(\mathfrak{P}) = a_\alpha x^\alpha$.

Let $\mathcal{I} \subset \mathbb{R}[x_1, \ldots, x_n]$ be a non-zero ideal. Let $LT(\mathcal{I})$ be the set of leading terms of all elements of $\mathcal{I}$ and let $\langle LT(\mathcal{I})\rangle$ be the ideal generated by the elements of $LT(\mathcal{I})$. It is important to emphasize that if $\mathcal{I} = \langle \mathfrak{P}_{ijk}\rangle$, then $\langle LT(\mathcal{I})\rangle$ may be strictly larger than the ideal $\langle LT(\mathfrak{P}_{ijk})\rangle$. A finite subset $\mathcal{G} = \{\mathbf{g}_1, \ldots, \mathbf{g}_\nu\}$ of an ideal $\mathcal{I}$ is said to be a *Gröbner basis* with respect to some monomial order if the equality above holds, i.e., $\langle LT(\mathbf{g}_1), \ldots, LT(\mathbf{g}_\nu)\rangle = \langle LT(\mathcal{I})\rangle$.

The Hilbert Basis Theorem (see, for example [8, Chapter 2]) guarantees that any non-zero ideal $\mathcal{I} \subset \mathbb{R}[x_1, \ldots, x_n]$ has a Gröbner basis. Furthermore, any Gröbner basis for an ideal $\mathcal{I}$ is a basis of $\mathcal{I}$ (see [8] for more information). Therefore a strategy to analyze the solutions of a given system of polynomial equations consists in constructing a Gröbner basis of the ideal generated by the given polynomials and solving the polynomial equations (hopefully simpler) corresponding to the polynomials in the Gröbner basis.

Buchberger's algorithm (among others) provides a constructive algorithm to find one such basis (see, for example, [9]). We would like to emphasize that the Gröbner basis construction is very sensitive to the monomial order. For a certain ordering, simple Gröbner bases can be obtained with a reduced number of polynomials, while for other orderings both the number of polynomials and their form can be completely unmanageable. Lexicographical order is the most appropriate in most cases to get simple bases. However, it is not always possible to use such ordering by computational reasons, and other orderings must be taken into consideration. We therefore emphasize in each case the ordering under consideration.

2.2. The unimodular case

Three-dimensional RG2 algebraic steady solitons have been classified by Wears in the unimodular case [22] (see also [15]). Following Milnor [20], all left-invariant metrics on unimodular three-dimensional Lie groups are determined (up to orientation) by the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ so that

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2,$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis. Now, the result of Wears can be easily summarized as follows:

Lemma 2.1. *Let G be a three-dimensional unimodular Lie group. Then, G is a non-locally symmetric RG2 algebraic steady soliton if and only if it is homothetic to one of the following Lie groups:*

(1) *The Lie group $E(1,1)$ with a left-invariant metric given by:*
   (1.a) *the Lie algebra structure $(\lambda_1, \lambda_2, \lambda_3) = (1, -1, 0)$, where $\alpha = 2$, or*
   (1.b) *the Lie algebra structure $(\lambda_1, \lambda_2, \lambda_3) = (3, -1, 0)$, where $\alpha = \frac{1}{4}$.*
(2) *The Heisenberg group H^3 with a left-invariant metric given by the eigenvalues $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 1)$, where $\alpha = \frac{8}{3}$.*
(3) *The special unitary group $SU(2)$ with a left-invariant metric determined by $(\lambda_1, \lambda_2, \lambda_3) = (1, \frac{4}{3}, 1)$, where $\alpha = -\frac{9}{2}$.*

Remark 2.2. Metrics corresponding to case $(1.a)$ are algebraic Ricci solitons for $\lambda = -2$ (i.e., $Q + 2\,\mathrm{Id}$ is a derivation), while metrics corresponding to case (1.b) are not. Moreover, the Heisenberg Lie group is an algebraic Ricci soliton for $\lambda = -\frac{3}{2}$, while the special unitary group does not admit any non-Einstein Ricci soliton.

2.3. The non-unimodular case

In addition to the previous RG2 algebraic steady solitons, there are some non-unimodular ones, which can be described as follows:

Lemma 2.3. *Let G be a three-dimensional non-unimodular Lie group. Then G is a non-locally symmetric RG2 algebraic steady soliton if and only if it is homothetic to a left-invariant metric determined by the Lie algebra $\mathfrak{g} = \mathrm{span}\{e_1, e_2, e_3\}$ given by*

$$[e_1, e_2] = (\xi + 1)e_2 + (\xi + 1)\eta e_3, \quad [e_1, e_3] = (\xi - 1)\eta e_2 - (\xi - 1)e_3,$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis and one of the following holds:

(1) *$\eta = 0$, $\xi > 0$ and $\xi \neq 1$, for a coupling constant $\alpha = \frac{2(\xi^2+1)}{(\xi^2+6)\xi^2+1}$.*
(2) *$\eta > 0$ and $\xi = 1 \pm \frac{\eta}{\sqrt{\eta^2+1}}$, for a coupling constant $\alpha = \frac{1}{2}\left(1 \mp \frac{\eta}{\sqrt{\eta^2+1}}\right)$.*

Proof. Following Milnor [20], any non-locally symmetric left-invariant metric on a non-unimodular Lie group is determined by Lie brackets

$$[e_1, e_2] = (\xi + 1)e_2 + (\xi + 1)\eta e_3, \quad [e_1, e_3] = (\xi - 1)\eta e_2 - (\xi - 1)e_3,$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis and $\eta \geq 0$, $\xi > 0$, excluding the case $\eta = 0$, $\xi = 1$. A straightforward calculation shows that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if the following polynomials vanish identically:

$$\begin{aligned}
\mathfrak{D}_{212} &= (\xi+1)(\alpha(\eta^2+1)^2\xi^4+2(\eta^2+1)(\alpha(2\eta^2+3)-1)\xi^2+\alpha-2),\\
\mathfrak{D}_{313} &= (1-\xi)(\alpha(\eta^2+1)^2\xi^4+2(\eta^2+1)(\alpha(2\eta^2+3)-1)\xi^2+\alpha-2),\\
\mathfrak{D}_{213} &= \eta(\xi+1)(\alpha(\eta^2+1)\xi^2+2\alpha(\eta^2+1)\xi+\alpha-2)((\eta^2+1)(\xi+2)\xi+1),\\
\mathfrak{D}_{312} &= \eta(\xi-1)(\alpha(\eta^2+1)\xi^2-2\alpha(\eta^2+1)\xi+\alpha-2)((\eta^2+1)(\xi-2)\xi+1).
\end{aligned}$$

Computing a Gröbner basis $\mathcal{G}$ of the ideal generated by the polynomials $\mathfrak{D}_{ijk} \in \mathbb{R}[\xi,\eta,\alpha]$ above with respect to the lexicographical order, one gets that such a basis $\mathcal{G}=\{\mathbf{g}_k\}$ consists of seven polynomials, among which one has the polynomials $\mathbf{g}_1=\eta(\alpha-2)(\xi^2-4(\alpha-1)^2)$ and $\mathbf{g}_2=\eta(\eta^2+1)(4\alpha(\alpha-1)(\eta^2+1)+1)\xi$. Since the polynomials $\mathbf{g}_k$ also belong to the ideal generated by the $\mathfrak{D}_{ijk} \in \mathbb{R}[\xi,\eta,\alpha]$, any solution of the system of equations $\{\mathfrak{D}_{ijk}=0\}$ must also be a solution of the equations $\{\mathbf{g}_k=0\}$. Hence, $\mathbf{g}_1$ leads to the following cases: $\alpha=2$, $\eta=0$ and $\xi^2=4(\alpha-1)^2$.

Setting $\alpha=2$, since $\xi>0$ one easily gets that $\mathfrak{D}=Q+\frac{\alpha}{4}\check{Q}$ is never a derivation of the Lie algebra. Assuming $\eta=0$, one has that $\mathfrak{D}=Q+\frac{\alpha}{4}\check{Q}$ is a derivation if and only if $((\xi^2+6)\xi^2+1)\alpha-2(\xi^2+1)=0$, which corresponds to Assertion (1).

Assume now that $\xi^2=4(\alpha-1)^2$ and $\eta>0$. In this case, the polynomial $\mathbf{g}_2$ leads to $4\alpha(\alpha-1)(\eta^2+1)+1=0$ and a straightforward calculation shows that these two conditions suffice for $\mathfrak{D}=Q+\frac{\alpha}{4}\check{Q}$ being a derivation. The first equation implies that $\alpha=1+\frac{\varepsilon}{2}\xi$, where $\varepsilon^2=1$. Then, the second equation above becomes $\varepsilon(\eta^2+1)(\varepsilon\xi+2)\xi+1=0$. If $\varepsilon=1$, then $\xi=-1\pm\frac{\eta}{\sqrt{\eta^2+1}}$ and thus $\xi<0$. If $\varepsilon=-1$, then $\xi=1\pm\frac{\eta}{\sqrt{\eta^2+1}}$ and Assertion (2) follows. □

Remark 2.4. Left-invariant metrics given in Lemma 2.3 define different homothetical classes. First, note that RG2 algebraic steady solitons corresponding to Assertion (1) are also algebraic Ricci solitons for a derivation $Q+2(\xi^2+1)\,\mathrm{Id}$, while RG2 algebraic steady solitons corresponding to Assertion (2) are not Ricci solitons (see, for example, [2]).

Let $(G_1,\langle\cdot,\cdot\rangle_1)$ and $(G_2,\langle\cdot,\cdot\rangle_2)$ be two Lie groups with negative scalar curvature τ_1 and τ_2, respectively. For $i=1,2$, let $\langle\cdot,\cdot\rangle_i^*=-\tau_i\langle\cdot,\cdot\rangle_i$ so that the scalar curvature of the normalized metric $\langle\cdot,\cdot\rangle_i^*$ is $\tau_i^*=-1$. Now, one has that $(G_1,\langle\cdot,\cdot\rangle_1)$ and $(G_2,\langle\cdot,\cdot\rangle_2)$ are homothetic if and only if the normalized metrics $\langle\cdot,\cdot\rangle_i^*$ are isometric. In this case, one has that $\|\rho_1^*\|=\|\rho_2^*\|$ and $\|R_1^*\|=\|R_2^*\|$, or equivalently, $\tau_1^{-2}\|\rho_1\|^2=\tau_2^{-2}\|\rho_2\|^2$ and $\tau_1^{-2}\|R_1\|^2=\tau_2^{-2}\|R_2\|^2$, where τ_i, ρ_i, R_i (resp., τ_i^*, ρ_i^*, R_i^*) are the scalar curvature, the Ricci tensor and the curvature tensor of $(G_i,\langle\cdot,\cdot\rangle_i)$ (resp., of the homothetic metric $\langle\cdot,\cdot\rangle_i^*$). The failure of any of these relations therefore implies that the left-invariant metrics $\langle\cdot,\cdot\rangle_i$ correspond to different homothetical classes.

Now, a standard calculation shows that left-invariant metrics in Assertion (1) corresponding to different values of the parameter ξ are never homothetical, since $\tau=-2(\xi^2+3)$ and $\|R\|^2=4(3\xi^4+10\xi^2+3)$. The same result holds for metrics in Assertion (2), where $\tau=-4\left(\eta^2+2-\eta\sqrt{\eta^2+1}\right)$ and $\|R\|^2=16(5\eta^2+4)\left(2\eta^2+1\pm2\eta\sqrt{\eta^2+1}\right)$.

3. The direct products $SL(2,\mathbb{R})\times\mathbb{R}$ and $SU(2)\times\mathbb{R}$

Let $\mathfrak{g}=\mathfrak{g}_3\times\mathbb{R}$ be a direct extension of the unimodular Lie algebra $\mathfrak{g}_3=\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{g}_3=\mathfrak{su}(2)$. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ and let $\langle\cdot,\cdot\rangle_3$ denote its restriction to $\mathfrak{g}_3$. Following the work of Milnor [20], there exists an orthonormal basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ of $\mathfrak{g}_3$ such that

$$[\mathbf{v}_2,\mathbf{v}_3]=\lambda_1\mathbf{v}_1,\qquad [\mathbf{v}_3,\mathbf{v}_1]=\lambda_2\mathbf{v}_2,\qquad [\mathbf{v}_1,\mathbf{v}_2]=\lambda_3\mathbf{v}_3,\tag{3}$$

where $\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}$ and $\lambda_1\lambda_2\lambda_3\neq 0$. Moreover, the associated Lie group corresponds to $SU(2)$ (resp., $SL(2,\mathbb{R})$) if $\lambda_1,\lambda_2,\lambda_3$ are all positive (resp., if any of $\lambda_1,\lambda_2,\lambda_3$ is negative).

Now, take $\mathbf{v}_4$ (not necessarily orthogonal to $\mathfrak{g}_3$) so that $[\mathbf{v}_4,\mathbf{v}_i]=0$, for all $i=1,2,3$. Finally, set $e_i=\mathbf{v}_i$ and $k_i=\langle\mathbf{v}_4,\mathbf{v}_i\rangle$ $(i=1,2,3)$, and normalize the vector $\hat{e}_4=\mathbf{v}_4-\sum_i k_i\mathbf{v}_i$ so that $\{e_1,\dots,e_4\}$ is an orthonormal basis with brackets given by

$$\begin{array}{ll}[e_1,e_2]=\lambda_3e_3,\quad [e_2,e_3]=\lambda_1e_1, & [e_3,e_1]=\lambda_2e_2,\\ [e_1,e_4]=\frac{1}{K}\{k_3\lambda_2e_2-k_2\lambda_3e_3\}, & [e_2,e_4]=\frac{1}{K}\{k_1\lambda_3e_3-k_3\lambda_1e_1\},\\ [e_3,e_4]=\frac{1}{K}\{k_2\lambda_1e_1-k_1\lambda_2e_2\}, & K=\|\hat{e}_4\|>0.\end{array}\tag{4}$$

Now, a straightforward calculation shows that the components ρ_{ij} of the Ricci tensor are

$$\begin{array}{l}2K^2\rho_{11}=\mathfrak{C}_{11},\,2K^2\rho_{12}=\mathfrak{C}_{12},\,2K^2\rho_{13}=\mathfrak{C}_{13},\,2K\rho_{14}=\mathfrak{C}_{14},\,2K^2\rho_{22}=\mathfrak{C}_{22},\\ 2K^2\rho_{23}=\mathfrak{C}_{23},\;\;2K\rho_{24}=\mathfrak{C}_{24},\,2K^2\rho_{33}=\mathfrak{C}_{33},\,2K\rho_{34}=\mathfrak{C}_{34},\,2K^2\rho_{44}=\mathfrak{C}_{44},\end{array}$$

where the coefficients $\mathfrak{C}_{ij}$ are polynomials on the structure constants given by

$$\begin{array}{lll}\mathfrak{C}_{12}=(\lambda_3^2-\lambda_1\lambda_2)k_1k_2, & \mathfrak{C}_{13}=(\lambda_2^2-\lambda_1\lambda_3)k_1k_3, & \mathfrak{C}_{14}=(\lambda_2-\lambda_3)^2k_1,\\ \mathfrak{C}_{23}=(\lambda_1^2-\lambda_2\lambda_3)k_2k_3, & \mathfrak{C}_{24}=(\lambda_1-\lambda_3)^2k_2, & \mathfrak{C}_{34}=(\lambda_1-\lambda_2)^2k_3,\end{array}$$
$$\mathfrak{C}_{11}=(\lambda_1^2-\lambda_3^2)k_2^2+(\lambda_1^2-\lambda_2^2)k_3^2+(\lambda_1^2-(\lambda_2-\lambda_3)^2)K^2,$$
$$\mathfrak{C}_{22}=(\lambda_2^2-\lambda_3^2)k_1^2-(\lambda_1^2-\lambda_2^2)k_3^2+(\lambda_2^2-(\lambda_1-\lambda_3)^2)K^2,$$
$$\mathfrak{C}_{33}=(\lambda_3^2-\lambda_2^2)k_1^2-(\lambda_1^2-\lambda_3^2)k_2^2-((\lambda_1-\lambda_2)^2-\lambda_3^2)K^2,$$
$$\mathfrak{C}_{44}=-(\lambda_2-\lambda_3)^2k_1^2-(\lambda_1-\lambda_3)^2k_2^2-(\lambda_1-\lambda_2)^2k_3^2.$$

Lemma 3.1. *Let G be a product $SL(2,\mathbb{R})\times\mathbb{R}$ or $SU(2)\times\mathbb{R}$. Then, G admits a non-symmetric RG2 algebraic steady soliton if and only if it is homothetic to the Lie group $SU(2)\times\mathbb{R}$ determined by*

$$[e_1,e_2]=e_3,\qquad [e_2,e_3]=e_1,\qquad [e_3,e_1]=\tfrac{4}{3}e_2,$$

for a coupling constant $\alpha=-\frac{9}{2}$, where $\{e_1,\dots,e_4\}$ is an orthonormal basis. Moreover, it is a fixed point for the RG2 flow.

Proof. Let $\mathfrak{D}=Q+\frac{\alpha}{4}\check{Q}$. Then, $\mathfrak{D}$ is a derivation of the Lie algebra if and only if all terms $\mathfrak{D}_{ijk}=\langle\mathfrak{D}[e_i,e_j]-[\mathfrak{D}e_i,e_j]-[e_i,\mathfrak{D}e_j],e_k\rangle$ vanish. The components $\mathfrak{D}_{ijk}$ can be obtained directly from the expressions of the Ricci tensor and the $\check{R}$-tensor, which are given by

$$\begin{array}{l}4K^4\check{R}_{11}=\mathfrak{R}_{11},4K^4\check{R}_{12}=\mathfrak{R}_{12},4K^4\check{R}_{13}=\mathfrak{R}_{13},4K^3\check{R}_{14}=\mathfrak{R}_{14},4K^4\check{R}_{22}=\mathfrak{R}_{22},\\ 4K^4\check{R}_{23}=\mathfrak{R}_{23},\,4K^3\check{R}_{24}=\mathfrak{R}_{24},4K^4\check{R}_{33}=\mathfrak{R}_{33},4K^3\check{R}_{34}=\mathfrak{R}_{34},4K^4\check{R}_{44}=\mathfrak{R}_{44},\end{array}$$

where the coefficients $\mathfrak{R}_{ij}$ are polynomials on the structure constants given by

$$\begin{aligned}
\mathfrak{R}_{11} = {} & (\lambda_1-\lambda_3)^2(\lambda_1^2+5\lambda_3^2+2\lambda_1\lambda_3)k_2^4 \\
& +(\lambda_1-\lambda_2)^2(\lambda_1^2+5\lambda_2^2+2\lambda_1\lambda_2)k_3^4 \\
& -(4\lambda_1\lambda_3^3-\lambda_1^2(\lambda_2^2+\lambda_3^2)-\lambda_3^2(\lambda_2^2+5\lambda_3^2-4\lambda_2\lambda_3))k_1^2k_2^2 \\
& -(4\lambda_1\lambda_2^3-\lambda_1^2(\lambda_2^2+\lambda_3^2)-\lambda_2^2(5\lambda_2^2+\lambda_3^2-4\lambda_2\lambda_3))k_1^2k_3^2 \\
& +2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1^2+\lambda_1(\lambda_2+\lambda_3)+5\lambda_2\lambda_3)k_2^2k_3^2 \\
& +(\lambda_2-\lambda_3)^2(\lambda_1^2+5\lambda_2^2+5\lambda_3^2-4(\lambda_2+\lambda_3)\lambda_1+6\lambda_2\lambda_3)K^2k_1^2 \\
& +2(\lambda_1-\lambda_3)^2(\lambda_1^2-\lambda_2^2+5\lambda_3^2+2(\lambda_1-\lambda_2)\lambda_3)K^2k_2^2 \\
& +2(\lambda_1-\lambda_2)^2(\lambda_1^2+5\lambda_2^2-\lambda_3^2+2(\lambda_1-\lambda_3)\lambda_2)K^2k_3^2 \\
& +\{\lambda_1^4+2\lambda_1^2(\lambda_2-\lambda_3)^2+(\lambda_2-\lambda_3)^2(5\lambda_2^2+5\lambda_3^2+6\lambda_2\lambda_3) \\
& \quad -8\lambda_1(\lambda_2-\lambda_3)^2(\lambda_2+\lambda_3)\}K^4,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{R}_{12} = {} & -(\lambda_2-\lambda_3)(\lambda_1(\lambda_2^2+2\lambda_3^2+\lambda_2\lambda_3)+(\lambda_2-5\lambda_3)\lambda_3^2)k_1^3k_2 \\
& -(\lambda_1-\lambda_3)(\lambda_1^2\lambda_2+(2\lambda_2-5\lambda_3)\lambda_3^2+\lambda_1\lambda_3(\lambda_2+\lambda_3))k_1k_2^3 \\
& -(\lambda_1^3\lambda_3-\lambda_1^2(2\lambda_2^2-2\lambda_3^2-\lambda_2\lambda_3)+\lambda_2^2\lambda_3(\lambda_2+2\lambda_3) \\
& +\lambda_1\lambda_2(\lambda_2-6\lambda_3)\lambda_3)k_1k_2k_3^2 \\
& +(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_1(2\lambda_2+\lambda_3)+(\lambda_2-5\lambda_3)\lambda_3)K^2k_1k_2, \\
\mathfrak{R}_{13} = {} & (\lambda_2-\lambda_3)(\lambda_1(2\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3)-\lambda_2^2(5\lambda_2-\lambda_3))k_1^3k_3 \\
& -(\lambda_1-\lambda_2)(\lambda_1^2\lambda_3-\lambda_2^2(5\lambda_2-2\lambda_3)+\lambda_1\lambda_2(\lambda_2+\lambda_3))k_1k_3^3 \\
& -(\lambda_1^3\lambda_2+\lambda_2(2\lambda_2+\lambda_3)\lambda_3^2+\lambda_1^2(2\lambda_2^2-2\lambda_3^2+\lambda_2\lambda_3) \\
& -\lambda_1\lambda_2(6\lambda_2-\lambda_3)\lambda_3)k_1k_2^2k_3 \\
& -(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1(\lambda_2+2\lambda_3)-\lambda_2(5\lambda_2-\lambda_3))K^2k_1k_3,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{R}_{14} = {} & -(\lambda_2-\lambda_3)^2(5\lambda_2^2+5\lambda_3^2-2\lambda_1(\lambda_2+\lambda_3)+6\lambda_2\lambda_3)k_1^3 \\
& +(\lambda_1-\lambda_3)(5\lambda_3^3-\lambda_1^2\lambda_2-3\lambda_2^2\lambda_3-\lambda_1(2\lambda_2^2+\lambda_3^2-2\lambda_2\lambda_3))k_1k_2^2 \\
& +(\lambda_1-\lambda_2)(5\lambda_2^3-\lambda_1^2\lambda_3-3\lambda_2\lambda_3^2-\lambda_1(\lambda_2^2+2\lambda_3^2-2\lambda_2\lambda_3))k_1k_3^2 \\
& -(\lambda_2-\lambda_3)^2(\lambda_1^2+5\lambda_2^2+5\lambda_3^2-6\lambda_1(\lambda_2+\lambda_3)+6\lambda_2\lambda_3)K^2k_1, \\
\mathfrak{R}_{22} = {} & (\lambda_2-\lambda_3)^2(\lambda_2^2+5\lambda_3^2+2\lambda_2\lambda_3)k_1^4 \\
& +(\lambda_1-\lambda_2)^2(5\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2)k_3^4 \\
& -(4\lambda_1\lambda_3^3-\lambda_1^2(\lambda_2^2+\lambda_3^2)-(\lambda_2^2+5\lambda_3^2-4\lambda_2\lambda_3)\lambda_3^2)k_1^2k_2^2 \\
& -2(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1(\lambda_2+5\lambda_3)+\lambda_2(\lambda_2+\lambda_3))k_1^2k_3^2 \\
& +(5\lambda_1^4-4\lambda_1^3(\lambda_2+\lambda_3)+\lambda_1^2(\lambda_2^2+\lambda_3^2)+\lambda_2^2\lambda_3^2)k_2^2k_3^2 \\
& -2(\lambda_2-\lambda_3)^2(\lambda_1^2-\lambda_2^2-5\lambda_3^2+2(\lambda_1-\lambda_2)\lambda_3)K^2k_1^2 \\
& +(\lambda_1-\lambda_3)^2(5\lambda_1^2+\lambda_2^2+5\lambda_3^2-4(\lambda_1+\lambda_3)\lambda_2+6\lambda_1\lambda_3)K^2k_2^2 \\
& +2(\lambda_1-\lambda_2)^2(5\lambda_1^2+\lambda_2^2-\lambda_3^2+2(\lambda_2-\lambda_3)\lambda_1)K^2k_3^2
\end{aligned}$$

$$
\begin{aligned}
&+\{5\lambda_1^4-4\lambda_1^3(2\lambda_2+\lambda_3)+2\lambda_1^2(\lambda_2^2-\lambda_3^2+4\lambda_2\lambda_3)\\
&\quad-4\lambda_1(\lambda_2-\lambda_3)^2\lambda_3+(\lambda_2-\lambda_3)^2(\lambda_2^2+5\lambda_3^2+2\lambda_2\lambda_3)\}K^4,\\
\mathfrak{R}_{23}=&-(5\lambda_1^4-2\lambda_1^3(\lambda_2+3\lambda_3)+\lambda_2\lambda_3^3+\lambda_1^2(\lambda_2+\lambda_3)\lambda_3)k_2^3k_3\\
&-(5\lambda_1^4-2\lambda_1^3(3\lambda_2+\lambda_3)+\lambda_2^3\lambda_3+\lambda_1^2\lambda_2(\lambda_2+\lambda_3))k_2k_3^3\\
&-(\lambda_1(\lambda_2^3+\lambda_3^3+\lambda_2^2\lambda_3+\lambda_2\lambda_3^2)-2\lambda_2^2\lambda_3^2+2\lambda_1^2(\lambda_2^2+\lambda_3^2-3\lambda_2\lambda_3))k_1^2k_2k_3\\
&-(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(5\lambda_1^2-\lambda_1(\lambda_2+\lambda_3)-2\lambda_2\lambda_3)K^2k_2k_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{24}=&-(\lambda_1-\lambda_3)^2(5\lambda_1^2+5\lambda_3^2-2\lambda_1\lambda_2+2(3\lambda_1-\lambda_2)\lambda_3)k_2^3\\
&-(\lambda_2-\lambda_3)(\lambda_1^2(2\lambda_2+3\lambda_3)+(\lambda_2-5\lambda_3)\lambda_3^2+\lambda_1\lambda_2(\lambda_2-2\lambda_3))k_1^2k_2\\
&-(\lambda_1-\lambda_2)(5\lambda_1^3-\lambda_1^2\lambda_2+\lambda_1(2\lambda_2-3\lambda_3)\lambda_3-\lambda_2(\lambda_2+2\lambda_3)\lambda_3)k_2k_3^2\\
&-(\lambda_1-\lambda_3)^2(5\lambda_1^2+\lambda_2^2+5\lambda_3^2-6\lambda_1\lambda_2+6(\lambda_1-\lambda_2)\lambda_3)K^2k_2,\\
\mathfrak{R}_{33}=&\,(\lambda_2-\lambda_3)^2(5\lambda_2^2+\lambda_3^2+2\lambda_2\lambda_3)k_1^4\\
&+(\lambda_1-\lambda_3)^2(5\lambda_1^2+\lambda_3^2+2\lambda_1\lambda_3)k_2^4\\
&+2(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_1(5\lambda_2+\lambda_3)+(\lambda_2+\lambda_3)\lambda_3)k_1^2k_2^2\\
&-(4\lambda_1\lambda_2^3-\lambda_1^2(\lambda_2^2+\lambda_3^2)-\lambda_2^2(5\lambda_2^2+\lambda_3^2-4\lambda_2\lambda_3))k_1^2k_3^2\\
&+(5\lambda_1^4-4\lambda_1^3(\lambda_2+\lambda_3)+\lambda_1^2(\lambda_2^2+\lambda_3^2)+\lambda_2^2\lambda_3^2)k_2^2k_3^2\\
&-2(\lambda_2-\lambda_3)^2(\lambda_1^2-5\lambda_2^2-\lambda_3^2+2(\lambda_1-\lambda_3)\lambda_2)K^2k_1^2\\
&+2(\lambda_1-\lambda_3)^2(5\lambda_1^2-\lambda_2^2+\lambda_3^2-2\lambda_1(\lambda_2-\lambda_3))K^2k_2^2\\
&+(\lambda_1-\lambda_2)^2(5\lambda_1^2+5\lambda_2^2+\lambda_3^2+6\lambda_1\lambda_2-4(\lambda_1+\lambda_2)\lambda_3)K^2k_3^2\\
&+\{5\lambda_1^4-4\lambda_1^3(\lambda_2+2\lambda_3)-2\lambda_1^2(\lambda_2^2-\lambda_3^2-4\lambda_2\lambda_3)\\
&\quad+(\lambda_2-\lambda_3)^2(5\lambda_2^2+\lambda_3^2+2\lambda_2\lambda_3)-4\lambda_1\lambda_2(\lambda_2-\lambda_3)^2\}K^4,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{34}=&-(\lambda_1-\lambda_2)^2(5\lambda_1^2+6\lambda_1\lambda_2+5\lambda_2^2-2\lambda_1\lambda_3-2\lambda_2\lambda_3)k_3^3\\
&+(\lambda_2-\lambda_3)(\lambda_1^2(3\lambda_2+2\lambda_3)-\lambda_2^2(5\lambda_2-\lambda_3)-\lambda_1(2\lambda_2-\lambda_3)\lambda_3)k_1^2k_3\\
&-(\lambda_1-\lambda_3)(5\lambda_1^3-\lambda_1^2\lambda_3-\lambda_1\lambda_2(3\lambda_2-2\lambda_3)-\lambda_2(2\lambda_2+\lambda_3)\lambda_3)k_2^2k_3\\
&-(\lambda_1-\lambda_2)^2(5\lambda_1^2+5\lambda_2^2+\lambda_3^2+6\lambda_1(\lambda_2-\lambda_3)-6\lambda_2\lambda_3)K^2k_3,\\
\mathfrak{R}_{44}=&\,(\lambda_2-\lambda_3)^2(5\lambda_2^2+5\lambda_3^2+6\lambda_2\lambda_3)k_1^4\\
&+(\lambda_1-\lambda_3)^2(5\lambda_1^2+5\lambda_3^2+6\lambda_1\lambda_3)k_2^4\\
&+(\lambda_1-\lambda_2)^2(5\lambda_1^2+5\lambda_2^2+6\lambda_1\lambda_2)k_3^4\\
&+2(\lambda_1^2(5\lambda_2^2-\lambda_3^2-2\lambda_2\lambda_3)-(\lambda_2^2-5\lambda_3^2+2\lambda_2\lambda_3)\lambda_3^2-2\lambda_1(\lambda_2^2+\lambda_3^2)\lambda_3)k_1^2k_2^2\\
&-2(\lambda_1^2(\lambda_2^2-5\lambda_3^2+2\lambda_2\lambda_3)-\lambda_2^2(5\lambda_2^2-\lambda_3^2-2\lambda_2\lambda_3)+2\lambda_1\lambda_2(\lambda_2^2+\lambda_3^2))k_1^2k_3^2\\
&+2(5\lambda_1^4-2\lambda_1^3(\lambda_2+\lambda_3)-\lambda_1^2(\lambda_2^2+\lambda_3^2)+5\lambda_2^2\lambda_3^2-2\lambda_1\lambda_2(\lambda_2+\lambda_3)\lambda_3)k_2^2k_3^2\\
&+(\lambda_2-\lambda_3)^2(\lambda_1^2+5\lambda_2^2+5\lambda_3^2-4\lambda_1(\lambda_2+\lambda_3)+6\lambda_2\lambda_3)K^2k_1^2\\
&+(\lambda_1-\lambda_3)^2(5\lambda_1^2+\lambda_2^2+5\lambda_3^2-4(\lambda_1+\lambda_3)\lambda_2+6\lambda_1\lambda_3)K^2k_2^2\\
&+(\lambda_1-\lambda_2)^2(5\lambda_1^2+5\lambda_2^2+\lambda_3^2+6\lambda_1\lambda_2-4(\lambda_1+\lambda_2)\lambda_3)K^2k_3^2.
\end{aligned}
$$

Since $\lambda_1\lambda_2\lambda_3\neq 0$, assume $\lambda_1=1$ and so we just work with the homothetic metric determined by $\tilde{e}_i=\frac{1}{\lambda_1}e_i$. The expressions of the Ricci tensor and the $\check{R}$-tensor imply that $\mathfrak{D}=Q+\frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ holds true, where $\mathfrak{P}_{ijk}$ are polynomials associated with the coefficients $\mathfrak{D}_{ijk}$ (which we omit for the sake of brevity). We consider separately the cases corresponding to different possibilities (up to rotation) for the constants k_1, k_2 and k_3 as follows.

3.1. $k_1 k_2 k_3 \neq 0$

Since all the k_is and λ_is are different from zero, we simplify (when possible) the polynomials $\{\mathfrak{P}_{ijk}\} \subset \mathbb{R}[k_1, k_2, k_3, K, \alpha, \lambda_2, \lambda_3]$. Constructing a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\{\mathfrak{P}_{ijk}\}$ with respect to the graded reverse lexicographical order, we get that the polynomials $\mathbf{g}_{11} = (\lambda_3 - 1)K^4$ and $\mathbf{g}_{12} = (\lambda_2 - 1)K^4$ belong to $\mathcal{G}_1$. Thus, $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and hence the manifold is symmetric.

3.2. $k_1 = 0$ and $k_2 k_3 \neq 0$

Proceeding as in the previous case, compute a Gröbner basis $\mathcal{G}_2$ of the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\} \subset \mathbb{R}[k_2, k_3, K, \alpha, \lambda_2, \lambda_3]$ with respect to the lexicographical order. Since the polynomials $\mathbf{g}_{21} = (\lambda_3 - 1)^2 K^4$ and $\mathbf{g}_{22} = (\lambda_2 + \lambda_3 - 2)K^4$ belong to $\mathcal{G}_2$, one has that $\lambda_3 = \lambda_2 = 1$, which corresponds to the situation in § 3.1.

3.3. $k_1 = k_2 = 0$

Simplifying the polynomials $\{\mathfrak{P}_{ijk}\}$ when possible as in the previous cases and computing a Gröbner basis $\mathcal{G}_3$ of the ideal generated by $\{\mathfrak{P}_{ijk}\} \subset \mathbb{R}[k_3, K, \alpha, \lambda_2, \lambda_3]$ with respect to the graded reverse lexicographical order, one gets that the polynomial $\mathbf{g}_{31} = k_3^3(\lambda_2 - 1)^2 K^2$ belongs to $\mathcal{G}_3$. Hence, either $k_3 = 0$ or $\lambda_2 = 1$ and, in both cases, e_4 determines a parallel left-invariant vector field. Now, a direct calculation shows that, in this case, any non-symmetric RG2 algebraic steady soliton is determined by Lemma 2.1-(3), obtaining the case given in Lemma 3.1. Finally, the tensor field $RG[g]$ vanishes, which finishes the proof. □

4. The semi-direct products $\mathbb{R} \ltimes E(1,1)$ and $\mathbb{R} \ltimes E(2)$

Let $\mathfrak{g}_3$ be either the Poincaré algebra $\mathfrak{e}(1,1)$ or the Euclidean algebra $\mathfrak{e}(2)$ and let $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{g}_3$ be a semi-direct extension. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle_3$ its restriction to $\mathfrak{g}_3$. Following the work of Milnor [20], there exists an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{g}_3$ such that

$$[\mathbf{v}_2, \mathbf{v}_3] = \lambda_1 \mathbf{v}_1, \qquad [\mathbf{v}_3, \mathbf{v}_1] = \lambda_2 \mathbf{v}_2, \qquad [\mathbf{v}_1, \mathbf{v}_2] = 0, \tag{5}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \lambda_2 \neq 0$. Moreover, $\mathfrak{g}_3 = \mathfrak{e}(2)$ (resp., $\mathfrak{g}_3 = \mathfrak{e}(1,1)$) if $\lambda_1 \lambda_2 > 0$ (resp., $\lambda_1 \lambda_2 < 0$). The algebra of derivations of $\mathfrak{g}_3$ is given by

$$\operatorname{der}(\mathfrak{g}_3) = \left\{ \begin{pmatrix} b & a & c \\ -\frac{\lambda_2}{\lambda_1} a & b & d \\ 0 & 0 & 0 \end{pmatrix} ; \, a, b, c, d \in \mathbb{R} \right\}.$$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis of $\mathfrak{g}$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are given by Eq. (5) and $\mathfrak{g} = \mathbb{R}\mathbf{v}_4 \oplus \mathfrak{g}_3$. Since $\mathbf{v}_4$ is not necessarily orthogonal to $\mathfrak{g}_3$, set $k_i =$

$\langle \mathbf{v}_i, \mathbf{v}_4\rangle$, for $i = 1,2,3$. Let $\hat{e}_4 = \mathbf{v}_4 - \sum_i k_i\mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1,\dots,e_4\}$ of $\mathfrak{g} = \mathbb{R}\oplus\mathfrak{g}_3$ so that

$$\begin{aligned}
&[e_2,e_3] = \lambda_1 e_1,\ [e_3,e_1] = \lambda_2 e_2,\\
&[e_4,e_1] = \tfrac{1}{K}\{be_1 - \lambda_2(\tfrac{a}{\lambda_1}+k_3)e_2\},\ [e_4,e_2] = \tfrac{1}{K}\{(a+k_3\lambda_1)e_1 + be_2\}, \quad (6)\\
&[e_4,e_3] = \tfrac{1}{K}\{(c-k_2\lambda_1)e_1 + (d+k_1\lambda_2)e_2\},
\end{aligned}$$

where $K = \|\hat{e}_4\| > 0$.

To simplify the notation, set $A = \frac{a}{\lambda_1}+k_3$, $C = c-k_2\lambda_1$ and $D = d+k_1\lambda_2$. Now, a straightforward calculation shows that the components ρ_{ij} of the Ricci tensor become

$$\begin{aligned}
&2K^2\rho_{11} = \mathfrak{C}_{11},\ 2K^2\rho_{12} = \mathfrak{C}_{12},\ 2K^2\rho_{13} = \mathfrak{C}_{13},\ 2K\rho_{14} = \mathfrak{C}_{14},\ 2K^2\rho_{22} = \mathfrak{C}_{22},\\
&2K^2\rho_{23} = \mathfrak{C}_{23},\ \ 2K\rho_{24} = \mathfrak{C}_{24},\ 2K^2\rho_{33} = \mathfrak{C}_{33},\ 2K\rho_{34} = \mathfrak{C}_{34},\ 2K^2\rho_{44} = \mathfrak{C}_{44},
\end{aligned}$$

where the coefficients $\mathfrak{C}_{ij}$ are polynomials on the structure constants given by

$$\begin{aligned}
&\mathfrak{C}_{12} = 2Ab(\lambda_2-\lambda_1) + CD, \qquad \mathfrak{C}_{13} = AD\lambda_2 - 3bC, \qquad \mathfrak{C}_{14} = D\lambda_2,\\
&\mathfrak{C}_{23} = -AC\lambda_1 - 3bD, \qquad \mathfrak{C}_{24} = -C\lambda_1, \qquad \mathfrak{C}_{34} = A(\lambda_1-\lambda_2)^2,\\
&\mathfrak{C}_{11} = (A^2+K^2)(\lambda_1^2-\lambda_2^2) - 4b^2 + C^2,\\
&\mathfrak{C}_{22} = (A^2+K^2)(\lambda_2^2-\lambda_1^2) - 4b^2 + D^2,\\
&\mathfrak{C}_{33} = -K^2(\lambda_1-\lambda_2)^2 - C^2 - D^2,\\
&\mathfrak{C}_{44} = -A^2(\lambda_1-\lambda_2)^2 - 4b^2 - C^2 - D^2.
\end{aligned}$$

Recall that any Einstein metric is a genuine fixed point of the RG2 flow. Moreover, the product manifold $\mathbb{R}\times E(1,1)$ is an RG2 algebraic steady soliton just considering the RG2 algebraic steady solitons in Lemma 2.1-(1). Henceforth, we focus on the irreducible non-Einstein case.

Lemma 4.1. *Let G be a semi-direct product $\mathbb{R}\ltimes E(1,1)$ or $\mathbb{R}\ltimes E(2)$. Then, G admits a non-Einstein irreducible RG2 algebraic steady soliton if and only if it is homothetic to the Lie group $\mathbb{R}\ltimes E(1,1)$ determined by*

$$[e_1,e_3] = e_2, \quad [e_2,e_3] = e_1, \quad [e_1,e_4] = \kappa e_1, \quad [e_2,e_4] = \kappa e_2,$$

where $\kappa > 0$, $\kappa \neq 1$ and for a coupling constant $\alpha = \frac{2}{\kappa^2+1}$. Here, $\{e_1,e_2,e_3,e_4\}$ is an orthonormal basis. Moreover, these metrics are never algebraic Ricci solitons.

Remark 4.2. Left-invariant metrics determined by Lemma 4.1 define different homothetical classes for any $\kappa > 0$, $\kappa \neq 1$. This is obtained proceeding as in Remark 2.4, since $\tau = -(6\kappa^2+2)$ and $\|R\|^2 = 4(3\kappa^4 + 2\kappa^2 + 3)$.

Proof. Let $\langle\cdot,\cdot\rangle$ be a left-invariant metric as described in (6). A standard calculation shows that the components of the $\check{R}$-tensor are given by

$$\begin{aligned}
&4K^4\check{R}_{11} = \mathfrak{R}_{11}, 4K^4\check{R}_{12} = \mathfrak{R}_{12}, 4K^4\check{R}_{13} = \mathfrak{R}_{13},\ 4K^3\check{R}_{14} = \mathfrak{R}_{14}, 4K^4\check{R}_{22} = \mathfrak{R}_{22},\\
&4K^4\check{R}_{23} = \mathfrak{R}_{23}, 4K^3\check{R}_{24} = \mathfrak{R}_{24}, 4K^4\check{R}_{33} = \mathfrak{R}_{33},\ 4K^3\check{R}_{34} = \mathfrak{R}_{34}, 4K^4\check{R}_{44} = \mathfrak{R}_{44},
\end{aligned}$$

where the coefficients $\mathfrak{R}_{ij}$ are polynomials on the structure constants given by

$$
\begin{aligned}
\mathfrak{R}_{11} =\ & (A^2+K^2)^2(\lambda_1^4+5\lambda_2^4-8\lambda_1\lambda_2^3+2\lambda_1^2\lambda_2^2)+(2C^2+D^2)(A^2+K^2)\lambda_1^2\\
&+(16A^2b^2-2C^2(A^2+K^2)+5D^2(A^2+K^2))\lambda_2^2\\
&-(16A^2b^2+4D^2(A^2+K^2))\lambda_1\lambda_2-2AbCD(3\lambda_1+\lambda_2)\\
&+16b^4+b^2(6C^2+4D^2)+C^2(C^2+D^2),\\
\mathfrak{R}_{12} =\ & 4Ab(A^2+K^2)(\lambda_1^3-\lambda_2^3-3\lambda_1^2\lambda_2+3\lambda_1\lambda_2^2)+Ab(16b^2+5C^2+D^2)\lambda_1\\
&-Ab(16b^2+C^2+5D^2)\lambda_2-2CD(A^2+K^2)\lambda_1\lambda_2+CD(2b^2+C^2+D^2),\\
\mathfrak{R}_{13} =\ & -AD(A^2+K^2)(5\lambda_2^3+\lambda_1^2\lambda_2-6\lambda_1\lambda_2^2)+bC(3A^2+4K^2)\lambda_1^2+bC(7A^2-2K^2)\lambda_2^2\\
&-2bC(5A^2+K^2)\lambda_1\lambda_2+AD(8b^2-C^2+2D^2)\lambda_1-AD(12b^2+2C^2+5D^2)\lambda_2\\
&+4bC(3b^2+C^2+D^2),\\
\mathfrak{R}_{14} =\ & -D(A^2+K^2)(5\lambda_2^3+\lambda_1^2\lambda_2-6\lambda_1\lambda_2^2)-AbC(\lambda_1^2-9\lambda_2^2+8\lambda_1\lambda_2)\\
&+D(2b^2-C^2+2D^2)\lambda_1-D(2b^2+2C^2+5D^2)\lambda_2,\\
\mathfrak{R}_{22} =\ & (A^2+K^2)^2(5\lambda_1^4+\lambda_2^4-8\lambda_1^3\lambda_2+2\lambda_1^2\lambda_2^2)\\
&+(A^2(16b^2+5C^2-2D^2)+(5C^2-2D^2)K^2)\lambda_1^2+(C^2+2D^2)(A^2+K^2)\lambda_2^2\\
&-(16A^2b^2+4C^2(A^2+K^2))\lambda_1\lambda_2+AbCD(2\lambda_1+6\lambda_2)\\
&+16b^4+b^2(4C^2+6D^2)+D^2(C^2+D^2),\\
\mathfrak{R}_{23} =\ & AC(A^2+K^2)(5\lambda_1^3-6\lambda_1^2\lambda_2+\lambda_1\lambda_2^2)+bD(7A^2-2K^2)\lambda_1^2+bD(3A^2+4K^2)\lambda_2^2\\
&-2bD(5A^2+K^2)\lambda_1\lambda_2+AC(12b^2+5C^2+2D^2)\lambda_1-AC(8b^2+2C^2-D^2)\lambda_2\\
&+4bD(3b^2+C^2+D^2),\\
\mathfrak{R}_{24} =\ & C(A^2+K^2)(5\lambda_1^3-6\lambda_1^2\lambda_2+\lambda_1\lambda_2^2)+AbD(9\lambda_1^2-\lambda_2^2)+C(2b^2+5C^2+2D^2)\lambda_1\\
&-C(2b^2+2C^2-D^2)\lambda_2-8AbD\lambda_1\lambda_2,\\
\mathfrak{R}_{33} =\ & K^2(A^2+K^2)(5\lambda_1^4+5\lambda_2^4-4\lambda_1^3\lambda_2-4\lambda_1\lambda_2^3)-2K^2(A^2+K^2)\lambda_1^2\lambda_2^2\\
&+(A^2(5C^2+D^2)+2(2b^2+5C^2-D^2)K^2)\lambda_1^2\\
&+(A^2(C^2+5D^2)+2(2b^2-C^2+5D^2)K^2)\lambda_2^2\\
&-4(A^2(C^2+D^2)+(2b^2+C^2+D^2)K^2)\lambda_1\lambda_2+12AbCD(\lambda_1-\lambda_2)\\
&+5(C^2+D^2)(2b^2+C^2+D^2),\\
\mathfrak{R}_{34} =\ & -A(A^2+K^2)(5\lambda_1^4+5\lambda_2^4-4\lambda_1^3\lambda_2-4\lambda_1\lambda_2^3)+2A(A^2+K^2)\lambda_1^2\lambda_2^2\\
&-A(12b^2+5C^2-3D^2)\lambda_1^2-A(12b^2-3C^2+5D^2)\lambda_2^2+24Ab^2\lambda_1\lambda_2\\
&-6bCD(\lambda_1-\lambda_2),\\
\mathfrak{R}_{44} =\ & A^2(A^2+K^2)(5\lambda_1^4+5\lambda_2^4-4\lambda_1^3\lambda_2-4\lambda_1\lambda_2^3)-2A^2(A^2+K^2)\lambda_1^2\lambda_2^2\\
&+(2A^2(12b^2+5C^2-D^2)+(4b^2+5C^2+D^2)K^2)\lambda_1^2\\
&+(2A^2(12b^2-C^2+5D^2)+(4b^2+C^2+5D^2)K^2)\lambda_2^2\\
&-4(A^2(12b^2+C^2+D^2)+(2b^2+C^2+D^2)K^2)\lambda_1\lambda_2+24AbCD(\lambda_1-\lambda_2)\\
&+16b^4+12b^2(C^2+D^2)+5(C^2+D^2)^2.
\end{aligned}
$$

Since $\lambda_1\lambda_2 \neq 0$, we work with a homothetic basis $\tilde{e}_i = \frac{1}{\lambda_1}e_i$ so that we may assume $\lambda_1 = 1$. The expressions of the Ricci tensor and the $\check{R}$-tensor imply that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ holds true, where $\mathfrak{P}_{ijk} \in \mathbb{R}[A, b, \lambda_2, C, D, K, \alpha]$ are the polynomials associated with the coefficients $\mathfrak{D}_{ijk}$ (which we omit for the sake of brevity). We compute a Gröbner basis $\mathcal{G}$ of the ideal $\mathcal{I} = \langle \mathfrak{P}_{ijk} \rangle$ with respect to the graded reverse lexicographical order and a detailed analysis of that basis shows that the polynomials

$$\mathbf{g}_1 = D^3(4b^2 + 25K^2)K^4, \qquad \mathbf{g}_2 = C(D^2 + C^2\lambda_2)K^4,$$
$$\mathbf{g}_3 = b\left\{3CDK^2 + Ab((\alpha+2)D^2 + 4(\lambda_2 - 1)K^2)\right\}K^2$$

belong to $\mathcal{G}$. Thus, $C = D = 0$ and $4Ab^2(\lambda_2 - 1)K^4 = 0$; so we have three different possibilities corresponding to $b = 0$, $A = 0$ or $\lambda_2 = 1$. We consider the three situations separately.

4.1. $b = 0$

Constructing a Gröbner basis $\mathcal{G}_1$ of the ideal $\mathcal{G} \cup \{b\} \subset \mathbb{R}[A, b, \lambda_2, K, \alpha]$ with respect to the lexicographical order, one gets that the polynomial

$$\mathbf{g}_{11} = (\lambda_2 - 1)(\lambda_2 + 1)(\lambda_2 + 3)(3\lambda_2 + 1)\lambda_2(A^2 + K^2)K^2$$

belongs to $\mathcal{G}_1$. This shows that λ_2 must take one of the different values $\lambda_2 = 1$, $\lambda_2 = -1$, $\lambda_2 = -3$ or $\lambda_2 = -\frac{1}{3}$. If $\lambda_2 = 1$, then the metric is Einstein. We analyze the other three cases separately.

4.1.1. $\boldsymbol{\lambda_2 = -1}$. Considering the coefficient $K^4\mathfrak{D}_{312} = (A^2\alpha + (\alpha-2)K^2)(A^2 + K^2)$, one has that $\alpha = \frac{2K^2}{A^2+K^2}$, and a straightforward calculation shows that, in this case, $\mathfrak{D}$ is a derivation of the Lie algebra. Moreover, setting $\gamma = -\frac{A}{K}$, one has the Lie algebra structure

$$[e_1, e_3] = e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = \gamma e_2, \quad [e_2, e_4] = \gamma e_1,$$

with $\alpha = \frac{2}{\gamma^2+1}$. A standard calculation shows that $v = e_4 - \gamma e_3$ determines a parallel left-invariant vector field on G. Therefore, G is a reducible RG2 algebraic steady soliton and one easily checks that it is obtained as a product extension of Lemma 2.1-(1.a).

4.1.2. $\boldsymbol{\lambda_2 = -3}$. Since $K^4\mathfrak{D}_{312} = 48(4A^2\alpha + (4\alpha - 1)K^2)(A^2 + K^2)$, we have $\alpha = \frac{K^2}{4(A^2+K^2)}$, and a straightforward calculation shows that, in this case, $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra. In this situation, setting $\kappa = -\frac{A}{K}$ one has

$$[e_1, e_3] = 3e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = 3\kappa e_2, \quad [e_2, e_4] = \kappa e_1,$$

with $\alpha = \frac{1}{4(\kappa^2+1)}$. Now, a direct calculation shows that $\ker \mathrm{Q} = \operatorname{span}\{e_4 - \kappa e_3\}$ and $v = e_4 - \kappa e_3$ is a parallel left-invariant vector field on G. Therefore, G is a reducible RG2 algebraic steady soliton, which is obtained as a product extension of Lemma 2.1–(1.b).

4.1.3. $\boldsymbol{\lambda_2 = -\frac{1}{3}}$. The coefficient $81K^4\mathfrak{D}_{321} = 16(4A^2\alpha + (4\alpha - 9)K^2)(A^2 + K^2)$ implies that $\alpha = \frac{9K^2}{4(A^2+K^2)}$ and a straightforward calculation shows that, in this case, $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra. Setting $\kappa = -\frac{A}{K}$, in the previous case just consider the homothety determined by $(e_1, e_2, e_3, e_4) \mapsto 3(e_2, e_1, e_3, e_4)$.

4.2. $A = 0$ and $b \neq 0$

Compute a Gröbner basis $\mathcal{G}_2$ of the ideal $\mathcal{G} \cup \{A\}$ with respect to the lexicographical order in $\mathbb{R}[K, A, b, \alpha, \lambda_2]$. We get that the polynomial $\mathbf{g}_{21} = (\lambda_2 - 1)(\lambda_2 + 1)^2 \lambda_2 \alpha^2 b^7$ belongs to $\mathcal{G}_2$ and thus $(\lambda_2 - 1)(\lambda_2 + 1) = 0$. If $\lambda_2 = 1$, then the manifold is symmetric and isometric to a product $\mathbb{R} \times N(c)$, where $N(c)$ is a space of constant negative curvature. On the other hand, if $\lambda_2 = -1$, then the coefficient $K^2 \mathfrak{D}_{312} = (\alpha - 2)K^2 + b^2 \alpha$ and thus $\alpha = \frac{2K^2}{b^2 + K^2}$. A straightforward calculation shows that $\mathfrak{D} = Q + \frac{\alpha}{4} \check{Q}$ defines an RG2 algebraic steady soliton where, setting $\kappa = -\frac{b}{K} \neq 0$, the left-invariant metric is determined by

$$[e_1, e_3] = e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = \kappa e_2,$$

with $\alpha = \frac{2}{\kappa^2 + 1}$. Note that the replacement $e_4 \mapsto -e_4$ defines an isometry which interchanges κ and $-\kappa$. Hence, one may assume $\kappa > 0$ without loss of generality. Also, observe that the Ricci operator has eigenvalues $Q = -2\,\mathrm{diag}[\kappa^2, \kappa^2, 1, \kappa^2]$ and thus the metric is Einstein if and only if $\kappa^2 = 1$. Moreover, a direct calculation shows that these metrics are irreducible. Furthermore, the metric is a Ricci soliton if and only $Q + 2\,\mathrm{Id}$ is a derivation, which may occur if and only if $\kappa(\kappa^2 - 1) = 0$. Hence, it is a Ricci soliton if and only if it is Einstein. We conclude that these metrics correspond to the ones given in Lemma 4.1.

4.3. $\lambda_2 = 1$ and $bA \neq 0$

In this case, the manifold is symmetric and isometric to a product $\mathbb{R} \times N(c)$, where $N(c)$ is a space of constant negative curvature, which finishes the proof. □

5. The semi-direct product $\mathbb{R} \ltimes H^3$

Let $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{h}^3$ be a semi-direct product of $\mathbb{R}$ with the Heisenberg algebra $\mathfrak{h}^3$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal basis of $\mathfrak{h}^3$ so that

$$[\mathbf{v}_1, \mathbf{v}_2] = \gamma \mathbf{v}_3, \quad [\mathbf{v}_2, \mathbf{v}_3] = 0, \quad [\mathbf{v}_1, \mathbf{v}_3] = 0, \qquad \gamma \neq 0.$$

The algebra of derivations of $\mathfrak{h}^3$ with respect to a rotated basis that we also denote by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is given by (see [5])

$$\mathrm{der}(\mathfrak{h}^3) = \left\{ \begin{pmatrix} a & c & 0 \\ -c & d & 0 \\ h & f & a + d \end{pmatrix} ; a, c, d, h, f \in \mathbb{R} \right\}.$$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis of $\mathfrak{g}$ where $\mathrm{ad}(e_4)$ is determined by a derivation as above. After normalization, as in the previous sections, there is an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ where the non-zero Lie brackets are given as follows:

$$\begin{array}{ll} [e_1, e_2] = \gamma e_3, & [e_4, e_1] = \frac{1}{K}\{a e_1 - c e_2 + (h + k_2 \gamma) e_3\}, \\ [e_4, e_3] = \frac{1}{K}(a + d) e_3, & [e_4, e_2] = \frac{1}{K}\{c e_1 + d e_2 + (f - k_1 \gamma) e_3\}, \quad K > 0. \end{array} \tag{7}$$

We use the notation $F = f - k_1\gamma$ and $H = h + k_2\gamma$. Then the non-zero components of the Ricci tensor are given by

$$2K^2\rho_{11} = \mathfrak{C}_{11},\ 2K^2\rho_{12} = \mathfrak{C}_{12},\ 2K^2\rho_{13} = \mathfrak{C}_{13},\ \ 2K\rho_{14} = \mathfrak{C}_{14},\ 2K^2\rho_{22} = \mathfrak{C}_{22},$$
$$2K^2\rho_{23} = \mathfrak{C}_{23},\ \ 2K\rho_{24} = \mathfrak{C}_{24},\ 2K^2\rho_{33} = \mathfrak{C}_{33},\ 2K^2\rho_{44} = \mathfrak{C}_{44},$$

where the coefficients $\mathfrak{C}_{ij}$ are determined by the structure constants as follows:

$$\begin{aligned}
\mathfrak{C}_{11} &= -4a^2 - 4ad - H^2 - \gamma^2K^2, & \mathfrak{C}_{22} &= -4d^2 - 4ad - F^2 - \gamma^2K^2,\\
\mathfrak{C}_{33} &= -4a^2 - 4d^2 - 8ad + F^2 + H^2 + \gamma^2K^2, & \mathfrak{C}_{44} &= -4a^2 - 4d^2 - 4ad - F^2 - H^2,\\
\mathfrak{C}_{12} &= -2ac + 2cd - FH, & \mathfrak{C}_{14} &= -\gamma F, \qquad \mathfrak{C}_{24} = \gamma H,\\
\mathfrak{C}_{13} &= -2Ha + Fc - 3Hd, & \mathfrak{C}_{23} &= -3Fa - Hc - 2Fd.
\end{aligned}$$

In addition to Einstein metrics and symmetric products, $\mathbb{R} \times H^3$ is an RG2 algebraic steady soliton considering the RG2 algebraic steady solitons in Lemma 2.1-(2). Henceforth, we focus on the irreducible non-Einstein case.

Lemma 5.1. *Let G be a semi-direct product $\mathbb{R} \ltimes H^3$. Then, G admits an irreducible non-Einstein RG2 algebraic steady soliton if and only if it is homothetic to one of the following, where $\{e_1, \dots, e_4\}$ is an orthonormal basis:*

(1) *The left-invariant metric determined by*

$$[e_1, e_2] = e_3, \qquad [e_1, e_4] = \tfrac{\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_1,$$
$$[e_2, e_4] = \tfrac{\kappa\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_2,\ [e_3, e_4] = \tfrac{(\kappa+1)\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_3,$$

where $\kappa \in [-1, 1)$ and for a coupling constant $\alpha = 2$.

(2) *The left-invariant metric determined by*

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = -\tfrac{1}{4\kappa}e_2, \quad [e_3, e_4] = \left(\kappa - \tfrac{1}{4\kappa}\right)e_3,$$

where $\kappa \in \left(0, \frac{1}{2}\right]$, $\kappa \neq \frac{1}{2}\sqrt{2 - \sqrt{3}}$, and for a coupling constant $\alpha = \frac{32\kappa^2}{16\kappa^4+1}$.

Moreover, metrics in case (1) are algebraic Ricci solitons, whereas left-invariant metrics (2) are not Ricci solitons.

Remark 5.2. Left-invariant metrics in Lemma 5.1 corresponding to different values of the parameter κ determine different homothetical classes. The scalar curvature and the norm of the Ricci tensor of left-invariant metrics in Assertion (1) are given by $\tau = -\frac{5\kappa^2+8\kappa+5}{\kappa^2+\kappa+1}$ and $\|\rho\|^2 = -\frac{3}{2}\tau$, while for metrics in Assertion (2) one has $\tau = -\frac{48\kappa^4-16\kappa^2+3}{8\kappa^2}$ and $\|\rho\|^2 = \frac{768\kappa^8-512\kappa^6+224\kappa^4-32\kappa^2+3}{64\kappa^4}$. Now, proceeding as in Remark 2.4, a standard calculation shows that left-invariant metrics in Assertion (1) corresponding to different values of κ are never homothetic and the same holds true for left-invariant metrics in Assertion (2).

Proof. Let $\langle\cdot,\cdot\rangle$ be a left-invariant metric on $\mathbb{R} \ltimes H^3$ determined by the Lie algebra inner product (7). A straightforward calculation shows that the components of the $\check{R}$-tensor are given by

$$4K^4\check{R}_{11}=\mathfrak{R}_{11}, 4K^4\check{R}_{12}=\mathfrak{R}_{12}, 4K^4\check{R}_{13}=\mathfrak{R}_{13},\ 4K^3\check{R}_{14}=\mathfrak{R}_{14}, 4K^4\check{R}_{22}=\mathfrak{R}_{22},$$
$$4K^4\check{R}_{23}=\mathfrak{R}_{23}, 4K^3\check{R}_{24}=\mathfrak{R}_{24}, 4K^4\check{R}_{33}=\mathfrak{R}_{33},\ 4K^3\check{R}_{34}=\mathfrak{R}_{34}, 4K^4\check{R}_{44}=\mathfrak{R}_{44},$$

where the coefficients $\mathfrak{R}_{ij}$ are polynomials on the structure constants given by

$$\begin{aligned}
\mathfrak{R}_{11} &= 16a^4+16a^3d+8a^2c^2+16a^2d^2+8c^2d^2-16ac^2d+4(F^2+4H^2)a^2\\
&\quad +2F^2c^2+2(5H^2+2\gamma^2K^2)d^2+4FHac+12(H^2+\gamma^2K^2)ad-20FHcd\\
&\quad +5(H^2+\gamma^2K^2)(F^2+H^2+\gamma^2K^2),\\
\mathfrak{R}_{12} &= 8a^3c-8cd^3-8a^2cd+8acd^2+12FHa^2-2FHc^2+12FHd^2\\
&\quad +2(F^2+5H^2)ac+10FHad-2(5F^2+H^2)cd+5FH(F^2+H^2+\gamma^2K^2),\\
\mathfrak{R}_{13} &= 4H(4a^3+3d^3+8a^2d+ac^2-c^2d+8ad^2)-4F(3cd^2+2acd)\\
&\quad +H(5F^2+4(H^2+\gamma^2K^2))a+F(F^2+H^2)c+(3F^2H+4H(H^2+\gamma^2K^2))d,\\
\mathfrak{R}_{14} &= 2\gamma F(a^2+6d^2+8ad)+2\gamma H(4ac-cd)+5\gamma F(F^2+H^2+\gamma^2K^2),\\
\mathfrak{R}_{22} &= 16d^4+16ad^3+8a^2c^2+16a^2d^2+8c^2d^2-16ac^2d+2(5F^2+2\gamma^2K^2)a^2\\
&\quad +2H^2c^2+4(4F^2+H^2)d^2+2FH(10ac-2cd)+12(F^2+\gamma^2K^2)ad\\
&\quad +5(F^2+\gamma^2K^2)(F^2+H^2+\gamma^2K^2),\\
\mathfrak{R}_{23} &= 4F(3a^3+4d^3+8a^2d-ac^2+c^2d+8ad^2)+4H(3a^2c+2acd)\\
&\quad +F(4F^2+3H^2+4\gamma^2K^2)a-H(F^2+H^2)c+F(4F^2+5H^2+4\gamma^2K^2)d,\\
\mathfrak{R}_{24} &= -2\gamma H(6a^2+d^2)-2\gamma F(ac-4cd)-16\gamma Had-5\gamma H(F^2+H^2+\gamma^2K^2),\\
\mathfrak{R}_{33} &= 16a^4+16d^4+48a^3d+48ad^3+64a^2d^2+6F^2a^2+2(F^2+H^2)c^2+6H^2d^2\\
&\quad +4(F^2+H^2-\gamma^2K^2)ad+(F^2+H^2+\gamma^2K^2)^2,\\
\mathfrak{R}_{34} &= 4\gamma(a^2c+cd^2-2acd)-\gamma FH(a-d)-\gamma(F^2+H^2)c,\\
\mathfrak{R}_{44} &= 16a^4+16d^4+32a^3d+32ad^3+16a^2c^2+48a^2d^2+16c^2d^2-32ac^2d\\
&\quad +4(3F^2+6H^2+\gamma^2K^2)a^2+4(F^2+H^2)c^2+4(6F^2+3H^2+\gamma^2K^2)d^2\\
&\quad +24FH(ac-cd)+4(6F^2+6H^2+\gamma^2K^2)ad+5(F^2+H^2)(F^2+H^2+\gamma^2K^2).
\end{aligned}$$

Note that since $\gamma\neq 0$, one may work with a homothetic basis $\tilde{e}_i=\frac{1}{\gamma}e_i$, so that we may assume $\gamma=1$. It follows from the expressions obtained for the Ricci tensor and for the $\check{R}$-tensor that $\mathfrak{D}=Q+\frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if the system of polynomial equations $\{\mathfrak{P}_{ijk}=0\}$ holds true, where $\mathfrak{P}_{ijk}\in\mathbb{R}[a,c,d,H,F,K,\alpha]$ are the polynomials associated with the coefficients $\mathfrak{D}_{ijk}$ (which we omit for the sake of brevity). We construct a Gröbner basis $\mathcal{G}$ of the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\}$ with respect to the lexicographical order and we get that the polynomial $\mathbf{g}_1=d^4FHK^2$ is in the basis. Therefore, we have three possibilities which we analyze separately.

5.1. $d=0$

Constructing a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\mathcal{G}\cup\{d\}\subset\mathbb{R}[a,c,d,H,F,K,\alpha]$ with respect to the lexicographical order, one has that the polynomials $\mathbf{g}_{11}=aHK^4$ and $\mathbf{g}_{12}=aFK^4$ are in $\mathcal{G}_1$. Thus, $a=0$ or $F=H=0$, $a\neq 0$.

5.1.1. $a=0$. We construct another Gröbner basis $\mathcal{G}_1'$ of the ideal generated by $\mathcal{G}_1\cup\{a\}\subset\mathbb{R}[a,c,d,H,F,K,\alpha]$ with respect to the lexicographical order and the polynomials $\mathbf{g}_{11}'=cFK^2$ and $\mathbf{g}_{12}'=cHK^2$ belong to $\mathcal{G}_1'$. Hence, either $c=0$ or $F=H=0$ and a standard calculation shows that $v=-\frac{F}{K}e_1+\frac{H}{K}e_2+e_4$ is a left-invariant parallel vector field on G in any case.

Therefore, in this case, any RG2 algebraic steady soliton is reducible and one easily checks that it is obtained as a product extension of Lemma 2.1-(2).

5.1.2. $F = H = 0$ and $a \neq 0$. Since $4K^4\mathfrak{D}_{131} = a^3 c\alpha$, we get $c = 0$ and thus $4K^5\mathfrak{D}_{343} = a^3(4a^2\alpha + (\alpha - 8)K^2)$, which shows that $\alpha = \frac{8K^2}{4a^2+K^2}$. Now, a straightforward calculation shows that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if $a = \varepsilon\frac{\sqrt{3}}{2}K$, with $\varepsilon^2 = 1$. In this case, $\alpha = 2$ and the left-invariant metric is determined by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\varepsilon\tfrac{\sqrt{3}}{2}e_1, \quad [e_3, e_4] = -\varepsilon\tfrac{\sqrt{3}}{2}e_3.$$

Note that the replacement $e_4 \mapsto -e_4$ defines an isometry which interchanges $\varepsilon = -1$ with $\varepsilon = 1$. Moreover, a direct calculation shows that this metric is never Einstein and that it is irreducible. Furthermore, a straightforward calculation shows that $Q + \frac{3}{2}\operatorname{Id}$ is a derivation of the Lie algebra and thus an algebraic Ricci soliton. Thus, taking $\varepsilon = -1$, the above left-invariant metric determines an RG2 algebraic steady soliton which corresponds to Assertion (1) with $\kappa = 0$.

5.2. $H = 0$, $d \neq 0$

Computing a Gröbner basis $\mathcal{G}_2$ of the ideal generated by $\mathcal{G} \cup \{H\} \subset \mathbb{R}[a, c, d, H, F, K, \alpha]$ with respect to the lexicographical order, one has that the polynomials $\mathbf{g}_{21} = dFK^4(12F^2 + 7K^2)$ and $\mathbf{g}_{22} = (a - d)c^3K^4$ are in $\mathcal{G}_2$. Hence, $F = 0$ and either $a = d$ or $c = 0$, $a \neq d$.

5.2.1. $F = 0$, $a = d$. Construct a new Gröbner basis $\mathcal{G}_2'$ of the ideal generated by $\mathcal{G}_2 \cup \{F, a - d\} \subset \mathbb{R}[a, c, d, H, F, K, \alpha]$ with respect to the lexicographical order. We get that the polynomial $\mathbf{g}_{21}' = (\alpha+4)(\alpha-2)(3\alpha-8)^2K^8$ is in $\mathcal{G}_2'$ and hence either $\alpha = -4$, $\alpha = 2$ or $\alpha = \frac{8}{3}$. In the first case, $\alpha = -4$, we get $K^5\mathfrak{D}_{141} = -9a^3(4a^2 + K^2)$ which cannot vanish. If $\alpha = 2$, then we get $2K^5\mathfrak{D}_{141} = 9a^3(4a^2 - K^2)$, from where $a = \pm\frac{1}{2}K$ and the metric is Einstein. If $\alpha = \frac{8}{3}$, then $K^5\mathfrak{D}_{411} = 4a^3(K^2 - 6a^2)$ from where $a = \pm\frac{1}{\sqrt{6}}K$. Then $\mathfrak{D}_{123} = -\frac{5}{9}$, which shows that no RG2 algebraic steady soliton may exist in this setting.

5.2.2. $F = 0$, $c = 0$ and $a \neq d$. First, we determine α using the component $\mathfrak{D}_{242}$. In particular, $4K^5\mathfrak{D}_{242} = d(a^2 + d^2 + ad)(4(a^2 + d^2 + ad)\alpha + K^2(\alpha - 8))$, which implies that $\alpha = \frac{8K^2}{4(a^2+d^2+ad)+K^2}$. A straightforward calculation shows that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if $(4ad + K^2)(4(a^2 + d^2 + ad) - 3K^2) = 0$ and thus $K = \frac{2}{\sqrt{3}}\sqrt{a^2 + d^2 + ad}$ or $a = -\frac{K^2}{4d}$.

In the first case, $K = \frac{2}{\sqrt{3}}\sqrt{a^2 + d^2 + ad}$, the left-invariant metric determined by the Lie algebra structure

$$\begin{aligned}
&[e_1, e_2] = e_3, && [e_1, e_4] = -\tfrac{a\sqrt{3}}{2\sqrt{a^2+d^2+ad}}e_1,\\
&[e_2, e_4] = -\tfrac{d\sqrt{3}}{2\sqrt{a^2+d^2+ad}}e_2, && [e_3, e_4] = -\tfrac{(a+d)\sqrt{3}}{2\sqrt{a^2+d^2+ad}}e_3
\end{aligned}$$

is an RG2 algebraic steady soliton with $\alpha = 2$. Recall that $d \neq 0$ and note that the replacement $e_4 \mapsto -e_4$ defines an isometry between (a, d) and $(-a, -d)$. Hence, assuming $d > 0$, setting $\kappa = \frac{a}{d}$ and applying the homothety determined by $(e_1, e_2, e_3, e_4) \mapsto (e_2, e_1, -e_3, -e_4)$, we obtain

$$[e_1, e_2] = e_3, \qquad [e_1, e_4] = \tfrac{\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_1,$$
$$[e_2, e_4] = \tfrac{\kappa\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_2, \; [e_3, e_4] = \tfrac{(\kappa+1)\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_3.$$

Since $a \neq d$, we have $\kappa \neq 1$. Moreover, the metrics corresponding to the parameters κ and $\frac{1}{\kappa}$ are isometric. Indeed, $(e_1, e_2, e_3, e_4) \mapsto (e_2, e_1, -e_3, e_4)$ if $\kappa > 0$ and $(e_1, e_2, e_3, e_4) \mapsto (e_2, e_1, -e_3, -e_4)$ if $\kappa < 0$ determine the corresponding isometries. Hence, we may assume $\kappa \in [-1, 1)$. Furthermore, a direct calculation shows that these metrics are never Einstein and that they are irreducible. Finally, a straightforward calculation shows that $Q + \frac{3}{2}\,\mathrm{Id}$ is a derivation of the Lie algebra and thus these metrics are algebraic Ricci solitons. We conclude that these metrics correspond to Assertion (1).

In the second case above, assuming $a = -\frac{K^2}{4d}$, we set $\kappa = \frac{K}{4d}$. Thus, $\alpha = \frac{32\kappa^2}{16\kappa^4+1}$ and the left-invariant metric determined by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = -\tfrac{1}{4\kappa} e_2, \quad [e_3, e_4] = \left(\kappa - \tfrac{1}{4\kappa}\right) e_3$$

is an RG2 algebraic steady soliton. Note that $\kappa \neq 0$. Moreover, replacing $e_4 \mapsto -e_4$, we may assume $\kappa > 0$, and $(e_1, e_2, e_3, e_4) \mapsto (e_2, -e_1, e_3, -e_4)$ defines an isometry interchanging κ and $\frac{1}{4\kappa}$, which shows that, without loss of generality, one may restrict the parameter to $\kappa \in \left(0, \frac{1}{2}\right]$. A direct calculation shows that these metrics are never Einstein and that they are irreducible. Finally, the metrics above are algebraic Ricci solitons if and only if $16\kappa^4 - 16\kappa^2 + 1 = 0$ (i.e., $\kappa = \frac{1}{2}\sqrt{2-\sqrt{3}}$), in which case $Q + \frac{3}{2}\,\mathrm{Id}$ is a derivation. A straightforward calculation shows that, taking the homothetical case $\kappa = -\frac{1}{2}\sqrt{2-\sqrt{3}}$, it corresponds to the special case of Assertion (1) for the value $\kappa = -(2+\sqrt{3})^{-1}$. Therefore, these metrics correspond to Assertion (2).

5.3. $F = 0$, $dH \neq 0$

Construct a Gröbner basis $\mathcal{G}_3$ of the ideal generated by $\mathcal{G} \cup \{F\} \subset \mathbb{R}[a, c, d, H, F, K, \alpha]$ with respect to the lexicographical order. Since the polynomial $\mathbf{g}_{31} = dH(12\,H^2 + 7K^2)K^4$ belongs to $\mathcal{G}_3$, it follows that no RG2 algebraic steady solitons may exist in this setting, finishing the proof. □

6. The semi-direct product $\mathbb{R} \ltimes \mathbb{R}^3$

Let $\mathfrak{r}^3$ be the Abelian algebra. The corresponding algebra of derivations is $\mathfrak{gl}(3, \mathbb{R})$. For any $D \in \mathfrak{gl}(3, \mathbb{R})$, decomposing it into its symmetric and skew-symmetric part, one has (see [5])

$$\mathrm{der}(\mathfrak{r}^3) = \left\{ \begin{pmatrix} a & -b & -c \\ b & f & -h \\ c & h & p \end{pmatrix}; a, b, c, f, h, p \in \mathbb{R} \right\}.$$

The corresponding semi-direct product $\mathbb{R} \oplus \mathfrak{r}^3$ expresses in an orthonormal basis $\{e_1, \ldots, e_4\}$ as

$$\begin{array}{l}[e_4, e_1] = \frac{1}{K}(ae_1 + be_2 + ce_3), \quad [e_4, e_2] = \frac{1}{K}(-be_1 + fe_2 + he_3), \\ [e_4, e_3] = \frac{1}{K}(-ce_1 - he_2 + pe_3), \, K > 0.\end{array} \tag{8}$$

Now, the non-zero components of the Ricci tensor are

$$\begin{array}{l}K^2\rho_{11} = \mathfrak{C}_{11}, \, K^2\rho_{12} = \mathfrak{C}_{12}, \, K^2\rho_{13} = \mathfrak{C}_{13}, \, K^2\rho_{22} = \mathfrak{C}_{22}, \\ K^2\rho_{23} = \mathfrak{C}_{23}, \, K^2\rho_{33} = \mathfrak{C}_{33}, \, K^2\rho_{44} = \mathfrak{C}_{44},\end{array}$$

where the coefficients $\mathfrak{C}_{ij}$ are given in terms of the structure constants as follows:

$$\begin{array}{ll}\mathfrak{C}_{11} = -a^2 - (f+p)a, & \mathfrak{C}_{22} = -fa - f(f+p), \\ \mathfrak{C}_{33} = -pa - (f+p)p, & \mathfrak{C}_{44} = -a^2 - f^2 - p^2, \\ \mathfrak{C}_{12} = ab - fb, & \mathfrak{C}_{13} = ac - pc, \quad \mathfrak{C}_{23} = (f-p)h.\end{array}$$

In addition to Einstein metrics and symmetric products, $\mathbb{R} \ltimes \mathbb{R}^3$ is an RG2 algebraic steady soliton just considering the RG2 algebraic steady solitons in Lemma 2.3. Henceforth, we focus on the irreducible non-Einstein case.

Lemma 6.1. *Let G be a semi-direct product $\mathbb{R} \ltimes \mathbb{R}^3$. Then, G admits an irreducible non-Einstein RG2 algebraic steady soliton if and only if it is homothetic to one of the following, where $\{e_1, \ldots, e_4\}$ is an orthonormal basis:*

(1) *The left-invariant metric determined by*
$$[e_1, e_4] = e_1, \quad [e_2, e_4] = fe_2, \quad [e_3, e_4] = pe_3, \text{ with } \alpha = \frac{2(f^2+p^2+1)}{f^4+p^4+1},$$
where $(f, p) \in \{(x, y) \in \mathbb{R}^2; x \in (0, 1], 0 \neq y \leq x\} \backslash \{(1, 1)\}$.

(2) *The left-invariant metric determined by*
$$[e_1, e_4] = e_1, \quad [e_2, e_4] = fe_2 + he_3, \quad [e_3, e_4] = -he_2 + pe_3,$$
where the parameters p and h are given by $p = \frac{1}{2}\left(1 + \sqrt{1 - 4f(f-1)}\right)$ and $h = \left(\frac{f^2(2p^2+1)+p^2-1}{2(f-p)^2}\right)^{\frac{1}{2}}$, with coupling constant $\alpha = \frac{2}{f^2+p^2}$ and $f \in (0, 1)$.

Furthermore, Lie groups in case (1) *are algebraic Ricci solitons, whereas left-invariant metrics* (2) *are never Ricci solitons.*

Remark 6.2. Left-invariant metrics in Lemma 6.1 define distinct homothetic classes for different values of the parameters in each assertion. For left-invariant metrics in Assertion (1), we have

$$\begin{array}{r}\tau = -2(f^2 + p^2 + fp + f + p + 1), \\ \|\rho\|^2 = -(f^2 + p^2 + 1)\tau, \\ \|R\|^2 = 4(f^4 + p^4 + f^2p^2 + f^2 + p^2 + 1).\end{array}$$

Proceeding as in Remark 2.4, a straightforward calculation shows that any left-invariant metric in Assertion (1) with $p \neq -f - 1$ is never homothetic to any other metric in Assertion (1). For $p = -f - 1$, we cannot use the same argument since $\tau^{-2}\|\rho\|^2 = 1$ and $\tau^{-2}\|R\|^2 = 3$. Nevertheless, considering the third-order Riemannian scalar curvature invariant $\check{\mathfrak{R}} = R_{ijk\ell}\, R^{k\ell pq}\, R_{pq}{}^{ij}$ and setting $p = -f - 1$, one has that

$$\tau^{-3}\check{\mathfrak{R}} = \frac{f(f+1)\left(f(f+1)\left(f^2+f+9\right)+3\right)+1}{\left(f^2+f+1\right)^3},$$

from where it follows that two different left-invariant metrics in Assertion (1) with $p = -f - 1$ are never homothetic since $0 < f \leq 1$.

For Assertion (2), we get that two different left-invariant metrics are never homothetic proceeding as in Remark 2.4 and using that

$$\begin{aligned} \tau &= -5f - 4 - (f+2)\sqrt{1-4f(f-1)}, \\ \|\rho\|^2 &= -2f^4 + 4f^2 + 17f + \tfrac{13}{2} + (2f^2 + 6f + \tfrac{11}{2})\sqrt{1-4f(f-1)}. \end{aligned}$$

Proof. The non-zero components of the $\check{R}$-tensor are given by

$$\begin{aligned} &\tfrac{1}{2}K^4\check{R}_{11} = \mathfrak{R}_{11},\ \tfrac{1}{2}K^4\check{R}_{12} = \mathfrak{R}_{12},\ \tfrac{1}{2}K^4\check{R}_{13} = \mathfrak{R}_{13},\ \tfrac{1}{2}K^4\check{R}_{22} = \mathfrak{R}_{22}, \\ &\tfrac{1}{2}K^4\check{R}_{23} = \mathfrak{R}_{23},\ \tfrac{1}{2}K^4\check{R}_{33} = \mathfrak{R}_{33},\ \tfrac{1}{2}K^4\check{R}_{44} = \mathfrak{R}_{44}, \end{aligned}$$

where the coefficients $\mathfrak{R}_{ij}$ are polynomials on the structure constants given by

$$\begin{aligned} \mathfrak{R}_{11} &= a^4 + a^2b^2 + a^2c^2 - 2fab^2 - 2pac^2 + (f^2+p^2)a^2 + f^2b^2 + p^2c^2, \\ \mathfrak{R}_{12} &= -a^3b + fa^2b - f^2ab + h(f-p)ac + f^3b - hp(f-p)c, \\ \mathfrak{R}_{13} &= -a^3c + pa^2c + h(f-p)ab - p^2ac - fh(f-p)b + p^3c, \\ \mathfrak{R}_{22} &= a^2b^2 - 2fab^2 + f^2(a^2+b^2) + f^2(f^2+p^2) + (f-p)^2h^2, \\ \mathfrak{R}_{23} &= a^2bc - (f+p)abc + fpbc - (f^3 - p^3 - f^2p + fp^2)h, \\ \mathfrak{R}_{33} &= a^2c^2 - 2pac^2 + p^2(a^2+c^2) + p^4 + f^2(h^2+p^2) - (2f-p)h^2p, \\ \mathfrak{R}_{44} &= a^4 + 2a^2b^2 + 2a^2c^2 - 4fab^2 - 4pac^2 + 2f^2b^2 + 2p^2c^2 \\ &\quad + f^4 + p^4 + 2(f-p)^2h^2. \end{aligned}$$

Let $\langle\cdot,\cdot\rangle$ be a left-invariant metric on $\mathbb{R} \ltimes \mathbb{R}^3$ determined by the Lie algebra inner product (8). We consider the diagonal matrix $\operatorname{diag}[a, f, p]$ in the decomposition of elements of $\operatorname{der}\left(\mathfrak{r}^3\right)$ and we analyze separately the cases of the determinant being null and non-null.

6.1. $afp = 0$

In this case, at least one of a, f and p must be zero. Thus, without loss of generality, we may assume $a = 0$. Moreover, one may work with a homothetic basis $\tilde{e}_i = Ke_i$ so that we may assume $K = 1$. A key observation in this case is that if $b = c = 0$, then e_1 determines a parallel left-invariant vector field. Hence, if $b = c = 0$ and G admits an RG2 algebraic steady soliton, then G splits as a product $\mathbb{R} \times H$, where H corresponds to the non-unimodular Lie group determined by the Lie algebra $\mathfrak{h} = \operatorname{span}\{e_2, e_3, e_4\}$ with

$$[e_2, e_4] = -fe_2 - he_3, \quad [e_3, e_4] = he_2 - pe_3,$$

and the RG2 algebraic steady solitons are determined by Lemma 2.3.

Otherwise, the expressions obtained for the Ricci tensor and for the $\check{R}$-tensor imply that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ holds true, where $\mathfrak{P}_{ijk} = \mathfrak{D}_{ijk} \in \mathbb{R}[f, b, c, h, p, \alpha]$ are the polynomials given by the components $\mathfrak{D}_{ijk}$ (which we omit for the sake of brevity). We start with a Gröbner basis $\mathcal{G}_1$ of the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\}$ with respect to the lexicographical order and we get that the polynomial $\mathbf{g}_{11} = p^2(p^2\alpha - 2)c^3$ belongs to $\mathcal{G}_1$. Therefore, we have three possibilities which we analyze separately.

6.1.1. $\boldsymbol{p = 0}$. Constructing a Gröbner basis $\mathcal{G}_1'$ of the ideal generated by $\mathcal{G}_1 \cup \{p\} \subset \mathbb{R}[f, b, c, h, p, \alpha]$ with respect to the lexicographical order, one has that the polynomials $\mathbf{g}_{11}' = fh(b^2 + h^2)$ and $\mathbf{g}_{12}' = bf(b^2 + h^2)$ belong to $\mathcal{G}_1'$. If $f = 0$, then the metric is Einstein. If $f \neq 0$, then $b = h = 0$ and we get $\mathfrak{D}_{422} = -\frac{1}{2}f^3(f^2\alpha - 2)$. Thus, $\alpha = \frac{2}{f^2}$ and this case is symmetric, and thus reducible since it is not Einstein.

6.1.2. $\boldsymbol{\alpha = \frac{2}{p^2}, p \neq 0}$. We construct a Gröbner basis $\mathcal{G}_1''$ of the ideal generated by $\mathcal{G}_1 \cup \{p^2\alpha - 2\} \subset \mathbb{R}[c, b, h, p, \alpha, f]$ with respect to the graded reverse lexicographical order and the polynomials $\mathbf{g}_{11}'' = cf(b^2 + c^2)$ and $\mathbf{g}_{12}'' = c(b^2 f + c^2 p - fh^2 + h^2 p)$ belong to $\mathcal{G}_1''$. Hence, necessarily $c = 0$. Moreover, the polynomials $\mathbf{g}_{13}'' = bf^2(b^2 + c^2)$ and $\mathbf{g}_{14}'' = b(f - p)(f^2 - 2h^2 + fp)$ also belong to $\mathcal{G}_1''$. Thus, $b = 0$ and G is reducible or otherwise $f = h = 0$ and the manifold is symmetric.

6.1.3. $\boldsymbol{c = 0, p \neq 0, \alpha \neq \frac{2}{p^2}}$. Constructing a Gröbner basis $\mathcal{G}_1'''$ of the ideal generated by $\mathcal{G}_1 \cup \{c\} \subset \mathbb{R}[f, b, c, h, p, \alpha]$ with respect to the lexicographical order, one has that the polynomial $\mathbf{g}_{11}''' = b^2p^2(p^2\alpha - 2)$ belongs to $\mathcal{G}_1'''$. Thus, necessarily $b = 0$ and G is reducible.

6.2. $\boldsymbol{afp \neq 0}$

Without loss of generality, one may work with a homothetic basis $\tilde{e}_i = \frac{K}{a} e_i$ so that we may assume $K = a = 1$. A key observation in this case is that the cases $b = c = 0$, $c = h = 0$ and $b = h = 0$ are homothetic. Indeed, considering $(e_1, e_2, e_3, e_4) = \frac{1}{p}(e_3, e_2, e_1, e_4)$ the case $c = h = 0$ reduces to $b = c = 0$. Analogously, considering $(e_1, e_2, e_3, e_4) = \frac{1}{f}(e_2, e_1, e_3, e_4)$ the case $b = h = 0$ reduces to $b = c = 0$.

Using the expressions obtained for the Ricci tensor and for the $\check{R}$-tensor, it follows that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ holds true, where $\mathfrak{P}_{ijk} \in \mathbb{R}[b, c, f, h, p, \alpha]$ are the polynomials given by the components $\mathfrak{D}_{ijk}$ (which we omit for the sake of brevity). Now, we construct a Gröbner basis $\mathcal{G}_2$ of the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\}$ with respect to the lexicographical order and we get that the polynomial $\mathbf{g}_{21} = ch(\alpha - 2)^4(3\alpha - 2)(\alpha^2 - 2\alpha + 4)$ belongs to $\mathcal{G}_2$. Therefore, we have four possibilities which we analyze separately.

6.2.1. $\boldsymbol{c = 0}$. Constructing a Gröbner basis $\mathcal{G}_2'$ of the ideal generated by $\{\mathfrak{P}_{ijk}\} \cup \{c\} \subset \mathbb{R}[h, b, c, p, \alpha, f]$ with respect to the lexicographical order, one has that the polynomials $\mathbf{g}_{21}' = bfh(f - 1)$ and $\mathbf{g}_{22}' = bh(\alpha - 2)$ belong to $\mathcal{G}_2'$. Hence, we are led to the cases $b = 0$, $h = 0$ and $f = 1$, $\alpha = 2$.

$\underline{b=0}$. In this case, we construct a Gröbner basis $\widetilde{\mathcal{G}}_2'$ of the ideal generated by $\mathcal{G}_2' \cup \{b\} \subset \mathbb{R}[h,b,c,p,\alpha,f]$ with respect to the graded reverse lexicographical order. We get that the polynomial $\widetilde{\mathbf{g}}_{21}' = h(f-p)^2(f^2+p^2-f-p)$ belongs to $\widetilde{\mathcal{G}}_2'$.

If $h=0$, then we get $2\,\mathfrak{D}_{411} = -(f^4+p^4+1)\alpha + 2(f^2+p^2+1)$. Therefore, $\alpha = \frac{2(f^2+p^2+1)}{f^4+p^4+1}$ and the left-invariant metric given by

$$[e_1,e_4]=-e_1, \quad [e_2,e_4]=-fe_2, \quad [e_3,e_4]=-pe_3$$

is an RG2 algebraic steady soliton. The metric is Einstein if and only if $f=p=1$. Since the isometry $(e_1,e_2,e_3,e_4)\mapsto(e_1,e_3,e_2,e_4)$ transforms (f,p) into (p,f), we may assume that $p\le f$. Moreover, $(e_1,e_2,e_3,e_4)\mapsto \frac{1}{f}(e_2,e_1,e_3,e_4)$ defines an homothety between (f,p) and $(\frac{1}{f},\frac{p}{f})$. Therefore, we may assume without loss of generality that (f,p) belongs to the set $\{(x,y)\in\mathbb{R}^2; x\in(0,1], 0\neq y\le x\}\backslash\{(1,1)\}$. Furthermore, a direct calculation shows that these metrics are irreducible and a straightforward calculation shows that $Q+(f^2+p^2+1)\,\mathrm{Id}$ is a derivation of the Lie algebra and thus an algebraic Ricci soliton. Finally, the isometry $e_4\mapsto -e_4$ shows that these metrics correspond to Assertion (1).

If $p=f$ and $h\neq 0$, then we get $2\,\mathfrak{D}_{411} = -(2f^4+1)\alpha + 2(2f^2+1)$. Therefore, $\alpha=\frac{2(2f^2+1)}{2f^4+1}$ and the left-invariant metric given by

$$[e_1,e_4]=-e_1, \quad [e_2,e_4]=-fe_2-he_3, \quad [e_3,e_4]=he_2-fe_3$$

is an RG2 algebraic steady soliton. The metric is Einstein if and only if $f=1$. Moreover, a straightforward calculation shows that $Q+(2f^2+1)\,\mathrm{Id}$ is a derivation of the Lie algebra and thus an algebraic Ricci soliton. A direct calculation shows that the curvature tensor of type $(1,3)$ does not depend on h and hence it follows from the work of Kulkarni [18] that this case is homothetic (although not homothetically isomorphic) to the case in Assertion (1) when $p=f$.

If $f^2+p^2-f-p=0$ and $p\neq f$, $h\neq 0$, then we get

$$2\,\mathfrak{D}_{411} = -(f^4+p^4+2(f-p)^2h^2+1)\alpha + 2(f^2+p^2+1),$$

which implies $\alpha=\frac{2(f^2+p^2+1)}{f^4+p^4+2(f-p)^2\,h^2+1}$. Now, a straightforward calculation shows that

$$\tfrac{f^2+p^2+1}{\alpha}\,\mathfrak{D}_{422} = h^2(f-p)(2(f-p)^2h^2 - f^2(2p^2+1) - p^2+1).$$

Since $h\neq 0$ and $p\neq f$, it follows that $h=\tilde{\varepsilon}\left(\frac{f^2(2p^2+1)+p^2-1}{2(f-p)^2}\right)^{\frac{1}{2}}$, with $\tilde{\varepsilon}^2=1$. On the other hand, since $f^2+p^2-f-p=0$, we get $p=\frac{1}{2}\left(1+\varepsilon\sqrt{1-4f(f-1)}\right)$, with $\varepsilon^2=1$. For this choice of h and p, we have $\alpha=\frac{2}{f^2+p^2}$ and the left-invariant metric given by

$$[e_1,e_4]=-e_1, \quad [e_2,e_4]=-fe_2-he_3, \quad [e_3,e_4]=he_2-pe_3 \tag{9}$$

is an RG2 algebraic steady soliton. A direct calculation shows that these metrics are never Einstein. Note that a substitution of $e_3\mapsto -e_3$ is an isometry which interchanges $\tilde{\varepsilon}=-1$ by $\tilde{\varepsilon}=1$. Hence, we take $\tilde{\varepsilon}=1$ and to ensure that the structure constants are real, we take $f \in \left(0, \frac{1+\sqrt{2}}{2}\right] \setminus \{1\}$ if $\varepsilon = 1$, and $f \in \left(1, \frac{1+\sqrt{2}}{2}\right]$ if $\varepsilon = -1$. Consider now a pair (ε, f) so that $\varepsilon = -1$ and define a corresponding pair $\left(\varepsilon = 1, \frac{1}{2}(1 - \sqrt{1 - 4f(f-1)}\right)$. It now follows that $(e_1, e_2, e_3, e_4) \mapsto (e_1, -e_3, e_2, e_4)$ determines an isometry between the two cases above, which shows that one may assume $\varepsilon = 1$ without loss of generality. Moreover, one can specialize $f \in (0, 1)$. To do this, if $f \in \left(1, \frac{1+\sqrt{2}}{2}\right]$, one has that $\frac{1}{2}(1 + \sqrt{1 - 4f(f-1)}) \in (0, 1)$ and repeating the same change of basis as above, we get that both cases are isometric. Finally, a straightforward calculation shows that these metrics are irreducible and that they are never an algebraic Ricci soliton. Thus, we conclude that this case corresponds to Assertion (2) after the replacement $e_4 \mapsto -e_4$.

$h = 0, b \neq 0$. Since $c = h = 0$, this case reduces to the case $c = b = 0$.

$f = 1, \alpha = 2, bh \neq 0$. In this case, we have $\mathfrak{D}_{413} = bhp^2(p-1)$. Since $bhp \neq 0$, it follows that $p = 1$ and the metric is Einstein.

6.2.2. $h = 0, c \neq 0$. We construct a Gröbner basis $\mathcal{G}_2''$ of the ideal generated by $\mathcal{G}_2 \cup \{h\} \subset \mathbb{R}[b, c, f, h, p, \alpha]$ with respect to the lexicographical order and we get that the polynomials $\mathbf{g}_{21}'' = bcp(p-1)$ and $\mathbf{g}_{22}'' = bcf(f-1)$ belong to $\mathcal{G}_2''$. If $b \neq 0$, then $f = p = 1$ and the corresponding metric is Einstein. Otherwise, $b = h = 0$, which reduces to the case $c = b = 0$ in §6.2.1.

6.2.3. $\alpha = 2, ch \neq 0$. Constructing a Gröbner basis $\mathcal{G}_2'''$ of the ideal generated by $\mathcal{G}_2 \cup \{\alpha - 2\} \subset \mathbb{R}[b, c, f, h, p, \alpha]$ with respect to the lexicographical order, one has that the polynomials $\mathbf{g}_{21}''' = cp^2(p-1)^2$ and $\mathbf{g}_{22}''' = hp(f-p)(f+p-1)$ belong to $\mathcal{G}_2'''$. Hence, it follows that $p = f = 1$ and the corresponding metric is Einstein.

6.2.4. $\alpha = \frac{2}{3}, ch \neq 0$. Constructing a Gröbner basis $\mathcal{G}_2''''$ of the ideal generated by $\mathcal{G}_2 \cup \{3\alpha - 2\} \subset \mathbb{R}[b, c, f, h, p, \alpha]$ with respect to the lexicographical order, one has that the polynomial $\mathbf{g}_{21}'''' = ch^2$ belongs to $\mathcal{G}_2''''$. Since $ch \neq 0$, there is no solution in this case, which finishes the proof.

□

7. The proof of Theorem 1.1 and Theorem 1.2

7.1. The proof of Theorem 1.1

First of all, recall that if the symmetric $(0, 2)$-tensor field $RG = -2\rho - \frac{\alpha}{2}\check{R}$, then the coupling constant α necessarily satisfies $\tau + \frac{\alpha}{4}\|R\|^2 = 0$. Hence, the manifold is flat or otherwise $\alpha = -4\tau\|R\|^{-2}$.

Let (M, g) be a complete and simply connected homogeneous four-dimensional manifold. Then it is isometric to a symmetric space or to a Lie group with a left-invariant metric [3]. The analysis of left-invariant metrics on Lie groups was carried out through Sects. 3 to 6. In each case, all possible derivations of the form $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ are given, showing that $\mathfrak{D} = 0$ if and only if the metric is Einstein or a product $\mathbb{R}^k \times N^{4-k}(c)$ for $k = 1, 2$, unless it corresponds to the left-invariant metric on $SU(2)\times\mathbb{R}$ given in Lemma 3.1 and determined by the Lie algebra

$$[e_1,e_2]=e_3, \qquad [e_2,e_3]=e_1, \qquad [e_3,e_1]=\tfrac{4}{3}e_2,$$

where $\{e_1,\dots,e_4\}$ is an orthonormal basis of $\mathfrak{su}(2)\times\mathbb{R}$.

On the other hand, if (M,g) is a non-Einstein symmetric space, then it splits as a product $N_1^k(c_1)\times N_2^{4-k}(c_2)$, where $k=1,2$, and $N_i^\ell(c_i)$ is a space of constant curvature c_i. If $k=1$, the resulting manifold is isometric to $\mathbb{R}\times N^3(c)$ and it satisfies $RG[g]=0$. If $k=2$, we compute the tensor field $RG[g]$ for a product $N_1^2(c_1)\times N_2^2(c_2)$, with coupling constant $\alpha=-\frac{4\tau}{\|R\|^2}$. An explicit calculation shows that the $(1,1)$-tensor field $Q-\frac{\tau}{\|R\|^2}\check{Q}$ takes the form

$$\frac{c_1c_2}{c_1^2+c_2^2}\operatorname{diag}[c_2-c_1,c_2-c_1,c_1-c_2,c_1-c_2].$$

Hence, assuming $c_1\neq c_2$, one has that $\rho-\frac{\tau}{\|R\|^2}\check{R}=0$ if and only if $c_1c_2=0$, which finishes the proof.

Remark 7.1. Products $\mathbb{R}^k\times N(c)$ are rigid gradient Ricci solitons [21]. In contrast, the product Lie group $SU(2)\times\mathbb{R}$, although it is an RG2 steady soliton, it is not a Ricci soliton (see, for example, [2]).

7.2. The proof of Theorem 1.2

The result follows at once from Lemmas 3.1, 4.1, 5.1, and 6.1. Moreover, all metrics corresponding to each assertion in Theorem 1.2 represent different homothetical classes as shown in Remarks 4.2, 5.2, and 6.2. Next we show that no metrics corresponding to different assertions in Theorem 1.2 may be homothetic.

Cases (1) and (3). In Case (1), in addition to τ and $\|R\|^2$ determined in Remark 4.2, one has $\|\rho\|^2=12\kappa^4+4$. In Case (3), in addition to τ and $\|\rho\|^2$ already computed in Remark 5.2, we have $\|R\|^2=\frac{16\kappa^2(48\kappa^6-16\kappa^4+14\kappa^2-1)+3}{64\kappa^4}$.

Now, a straightforward calculation following Remark 2.4 and using the invariants τ, $\|\rho\|^2$ and $\|R\|^2$ shows that left-invariant metrics corresponding to cases (1) and (3) in Theorem 1.2 are never homothetic.

Cases (1) and (5). In Case (1), we consider the invariants τ and $\|R\|^2$ determined in Remark 4.2 and in Case (5) we consider τ determined in Remark 6.2 and

$$\|R\|^2=-2\left(2(\kappa-1)\kappa^3-\kappa^2-8\kappa-3\right)+(2(\kappa+2)\kappa+6)\sqrt{1-4(\kappa-1)\kappa}.$$

A straightforward calculation following Remark 2.4 now shows that left-invariant metrics corresponding to cases (1) and (5) in Theorem 1.2 are never homothetic.

Cases (3) and (5). We proceed as in Remark 2.4 using the invariants τ and $\|\rho\|^2$ previously determined and a straightforward calculation shows that left-invariant metrics corresponding to cases (3) and (5) in Theorem 1.2 are never homothetic.

Secondly, we analyze the cases in Theorem 1.2 which are Ricci solitons, i.e., cases (2) and (4). Considering the second-order homothetic invariants $\tau^{-2}\|R\|^2$ and $\tau^{-2}\|\rho\|^2$ for metrics in cases (2) and (4), one has

$$\begin{aligned} \tau^{-2}\|R\|^2 &= \tfrac{(\kappa^2+\kappa+1)(\kappa(11\kappa+14)+11)}{(5\kappa^2+8\kappa+5)^2}, &\text{ and } \tau^{-2}\|R\|^2 &= \tfrac{\delta^4+\delta^2(\kappa^2+1)+\kappa^4+\kappa^2+1}{(\delta^2+\delta\kappa+\delta+\kappa^2+\kappa+1)^2}, \\ \tau^{-2}\|\rho\|^2 &= \tfrac{3(\kappa^2+\kappa+1)}{2(5\kappa^2+8\kappa+5)}, &\text{ and } \tau^{-2}\|\rho\|^2 &= \tfrac{\delta^2+\kappa^2+1}{2(\delta^2+\delta\kappa+\delta+\kappa^2+\kappa+1)}, \end{aligned}$$

respectively. Moreover, the third-order homothetic invariant $\tau^{-3}\check{\mathfrak{R}} =$ for metrics in case (2) is given by

$$\tau^{-3}\check{\mathfrak{R}} = \frac{80\kappa^6+330\kappa^5+741\kappa^4+938\kappa^3+741\kappa^2+330\kappa+80}{4\left(5\kappa^2+8\kappa+5\right)^3},$$

while it becomes $\tau^{-3}\check{\mathfrak{R}} = \frac{\delta^6+\delta^3(\kappa^3+1)+\kappa^6+\kappa^3+1}{(\delta^2+\delta\kappa+\delta+\kappa^2+\kappa+1)^3}$ for metrics in case (4). Now, proceeding as in the previous cases, one has that no metric corresponding to case (2) may be homothetic to a metric in case (4).

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