The structure group and the permutation group of a set-theoretic solution of the quantum Yang-Baxter equation

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Abstract. We describe the left brace structure of the structure group and the permutation group associated to an involutive, non-degenerate set-theoretic solution of the quantum Yang-Baxter equation by using the Cayley graph of its permutation group with respect to its natural generating system. We use our descriptions of the additions in both braces to obtain new properties of the structure and the permutation groups and to recover some known properties of these groups in a more transparent way.

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1. Introduction

The quantum Yang-Baxter equation (YBE) is a consistence equation of mathematical physics that plays a major role in the study of integrable systems in some scattering situations happening in statistical mechanics. It appeared for the first time in the paper of Yang [13]. One of the fundamental open problems related to this equation is to find all the solutions of the YBE. During the last years the non-degenerate involutive set-theoretic solutions of the YBE have received a lot of attention.

Given a non-empty set X, a map $r\colon X \times X \longrightarrow X \times X$ is called a *set-theoretic solution* of the YBE if

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23}, \tag{1}$$

where the maps r_{12}, $r_{23}\colon X \times X \times X \longrightarrow X \times X \times X$ are defined as $r_{12} = r \times 1_X$, $r_{23} = 1_X \times r$. For all x, $y \in X$, we denote by $f_x\colon X \longrightarrow X$ and $g_y\colon X \longrightarrow X$ the maps obtained by setting $r(x, y) = (f_x(y), g_y(x))$ for all x, $y \in X$.

The solution (X, r) is called *involutive* if $r^2 = 1_{X\times X}$ and *non-degenerate* if f_x, g_y are bijective maps for all x, $y \in X$. By a solution of the YBE we mean a non-degenerate involutive set-theoretic solution, as in [4, 6].

We can use techniques from group theory to study the solutions of the YBE by considering two fundamental groups associated to a solution (X, r) (see [8]): the *structure group*, with presentation

$$G(X, r) = \langle X \mid xy = f_x(y)g_y(x), \quad x, y \in X\rangle,$$

and the *permutation group*,

$$\mathcal{G}(X,r) = \langle f_x \mid x \in X \rangle \leq \mathrm{Sym}(X).$$

On the other hand, Rump introduced in [12] a new algebraic structure, called *left brace*, to study the solutions of the YBE. A *left brace* $(B,+,\cdot)$ is composed of a set B with two binary operations, $+$ and $\cdot$, for which $(B,+)$ is an abelian group and $(B,\cdot)$ is a group satisfying the distributivity-like condition

$$a \cdot (b+c) + a = a \cdot b + a \cdot c \quad \text{for all } a,\, b,\, c \in B.$$

In [6, Section 3], it is proved that, if (X,r) is a solution of the YBE, then it is possible to define additions in $G(X,r)$ and $\mathcal{G}(X,r)$ in such a way they become left braces. This construction was already devised in [5, 8, 9, 10, 11]. The paper [3] contains an interesting survey on left braces and properties of the structure group associated to a solution of the YBE.

The aim of this paper is to describe the left brace structures of $\mathcal{G}(X,r)$ and $G(X,r)$ by means of the Cayley graph of $(\mathcal{G}(X,r),\cdot)$ with respect to the natural generating set $\{f_x \mid x \in X\}$. We denote in this paper the composition of maps with $\cdot$ or simply with a juxtaposition. Recall that the *Cayley graph* $\Gamma(G,S)$ of a group G with respect to its generating set S has as vertices the elements of G and edges of the form $x \xrightarrow{s} xs$, labelled with s, for $x \in G$ and $s \in S$. For simplicity, in the Cayley graph of $\mathcal{G}(X,r)$ with respect to the natural generating set $\{f_x \mid x \in X\}$, the edge $\alpha \xrightarrow{f_x} \alpha f_x$ for $x \in X$ and $\alpha \in \mathcal{G}(X,r)$ will be represented as $\alpha \xrightarrow{x} \alpha f_x$, with label x instead of f_x. We warn the reader that different values of $x \in X$ might induce the same permutation f_x, that is, parallel edges are permitted; this happens in the so-called retractable solutions.

Our first main result gives a description of the left brace structure of the permutation group $\mathcal{G}(X,r)$.

Theorem A. *If in the Cayley graph of $(\mathcal{G}(X,r),\cdot)$ we replace the label of the edge $\alpha \xrightarrow{x} \alpha f_x$, $x \in X$, $\alpha \in \mathcal{G}(X,r)$, by $\alpha(x)$, then the labelled graph obtained in this way is the Cayley graph of an abelian group. Furthermore, if $+$ denotes the operation of this group, then $(\mathcal{G}(X,r),+,\cdot)$ is a left brace.*

This result says that the addition in $\mathcal{G}(X,r)$ is defined as $\alpha + f_{\alpha(x)} = \alpha f_x$ for $x \in X$, $\alpha \in \mathcal{G}(X,r)$, that is, $\alpha + f_z = \alpha f_{\alpha^{-1}(z)}$ for $z \in X$, $\alpha \in \mathcal{G}(X,r)$. The proof of this theorem will be given in Section 2.

Our second main result is a description of the structure group $G(X,r)$ of a solution (X,r) of the YBE in terms of the Cayley graph of the permutation group.

Theorem B. *Let (X,r) be a solution of the YBE. Let $e_{\alpha,x}$ denote the edge $\alpha \xrightarrow{x} \alpha f_{\alpha^{-1}(x)}$ in the Cayley graph of $(\mathcal{G}(X,r),+)$ constructed in Theorem* A. *Let E be the set of edges of this Cayley graph, then $(\mathcal{G}(X,r),\cdot)$ acts on E via $g * e_{\alpha,x} = e_{g\alpha,g(x)}$, $g \in \mathcal{G}(X,r)$, $x \in X$, and this action can be extended to an action of $(\mathcal{G}(X,r),\cdot)$ on the free $\mathbb{Z}$-module W with basis E. The $\mathbb{Z}$-submodule*

$$K = \langle e_{\alpha,y} - e_{\beta,y} \mid y \in X, \quad \alpha,\beta \in \mathcal{G}(X,r)\rangle$$

is invariant under the action of $(\mathcal{G}(X,r),\cdot)$ and so $(\mathcal{G}(X,r),\cdot)$ acts on W/K, which is a free $\mathbb{Z}$-module with basis $\{\bar{x} \mid x \in X\}$, where $\bar{x} = e_{1,x} + K$, $x \in X$. Let H be the subgroup of the semidirect product $[W/K]\mathcal{G}(X,r)$ with respect to this action generated by $\{(\bar{x}, f_x) \mid x \in X\}$. Then:

1. *H is isomorphic to the structure group $G(X,r)$.*
2. *$H = \{(\sum_{x\in X} a_x\bar{x}, \sum_{x\in X} a_x f_x) \mid a_x \in \mathbb{Z}, x \in X\}$.*

3. *The product of H has the form*

$$\left(\sum_{x\in X} a_x\bar{x},\alpha\right)\cdot\left(\sum_{x\in X} b_x\bar{x},\beta\right)=\left(\sum_{x\in X}(a_x+b_{\alpha^{-1}(x)})\bar{x},\alpha\beta\right),$$

where $\alpha=\sum_{x\in X} a_x f_x$, $\beta=\sum_{x\in X} b_x f_x$.

It is also possible to define an addition in the group of Theorem B such that $(H,+,\cdot)$ is a left brace.

Theorem C. *Let (X,r) be a solution of the YBE and let H be like in Theorem* B. *If we define in H an operation $+$ by means of*

$$\left(\sum_{x\in X} a_x\bar{x},\alpha\right)+\left(\sum_{x\in X} b_x\bar{x},\beta\right)=\left(\sum_{x\in X}(a_x+b_x)\bar{x},\alpha+\beta\right),$$

where $\alpha=\sum_{x\in X} a_x f_x$, $\beta=\sum_{x\in X} b_x f_x$, *then $(H,+,\cdot)$ is a left brace and the map* $\pi\colon H\longrightarrow\mathcal{G}(X,r)$ *given by*

$$\pi\left(\sum_{x\in X} a_x\bar{x},\alpha\right)=\alpha$$

is a left brace homomorphism.

The proofs of Theorems B and C will be presented in Section 3. A geometrical interpretation of the group H of Theorem B and an example will be given in Section 4. Our results provide a different approach to the results of [6] and [8]. In fact, we compare our results with theirs on Section 5 We expect that these descriptions can shed light on the structure of the permutation and structure groups of a solution of the YBE. Section 6 contain some applications of them.

2. The permutation group as a left brace

Let (X,r) be a non-degenerate, involutive solution of the YBE. Write

$$r(x,y)=(f_x(y),g_y(x))$$

for x, $y\in X$. Let $\mathcal{G}(X,r)=\langle f_x\mid x\in X\rangle$ be the corresponding permutation group.

The following two results are well-known. The first one comes from the fact that the solution is involutive and non-degenerate.

Lemma 1. *Given x, $y\in X$, we have that $f_{f_x(y)}(g_y(x))=x$, $g_{g_y(x)}(f_x(y))=y$. In particular, for every x, $y\in X$, $g_y(x)=f^{-1}_{f_x(y)}(x)$, $f_x(y)=g^{-1}_{g_y(x)}(y)$.*

The second one is an immediate consequence of Equation (1).

Lemma 2. *If x, $y\in X$, then $f_xf_y=f_{f_x(y)}f_{g_y(x)}$ and $g_xg_y=g_{g_x(y)}g_{f_y(x)}$.*

In order to define an addition in $\mathcal{G}(X,r)$ with Cayley graph as in Theorem A, we start by defining the addition of elements of $\mathcal{G}(X,r)$ and its generators f_x, $x\in X$, or their opposites. Given $\alpha\in\mathcal{G}(X,r)$ and $x\in X$, we write

$$\alpha+f_x=\alpha f_{\alpha^{-1}(x)},\qquad \alpha+f^{-1}_{g_x^{-1}(x)}=\alpha f^{-1}_{g^{-1}_{\alpha^{-1}(x)}(\alpha^{-1}(x))}.$$

Let us note that $f^{-1}_{g_x^{-1}(x)} + f_x = 1_X = f_x + f^{-1}_{g_x^{-1}(x)}$. In light of these equalities, we will write εf_x, for $x \in X$, to denote f_x if $\varepsilon = 1$ and $-f_x = f^{-1}_{g_x^{-1}}(x)$ if $\varepsilon = -1$. The following result gathers the information about this addition needed for our purposes.

Lemma 3. *The following properties hold:*

1. *If $\alpha \in \mathcal{G}(X, r)$ and x, $y \in X$, then $(\alpha + f_x) + f_y = (\alpha + f_y) + f_x$.*
2. *If $\alpha \in \mathcal{G}(X, r)$ and $x \in X$, then $(\alpha + f_x) - f_x = (\alpha - f_x) + f_x = \alpha$.*
3. *If $\alpha \in \mathcal{G}(X, r)$ and x, $y \in X$, then $(\alpha + f_x) - f_y = (\alpha - f_y) + f_x$.*
4. *If $\alpha \in \mathcal{G}(X, r)$ and x, $y \in X$, then $(\alpha - f_x) - f_y = (\alpha - f_y) - f_x$.*
5. *Let σ be a permutation of $\{1, \dots, m\}$, $\varepsilon_j \in \{-1, 1\}$, $x_j \in X$, $1 \le j \le m$. Then*

$$(\cdots((\varepsilon_1 f_{x_1} + \varepsilon_2 f_{x_2}) + \varepsilon_3 f_{x_3}) + \cdots) + \varepsilon_m f_{x_m} = (\cdots((\varepsilon_{\sigma(1)} f_{x_{\sigma(1)}} + \varepsilon_{\sigma(2)} f_{x_{\sigma(2)}}) + \varepsilon_{\sigma(3)} f_{x_{\sigma(3)}}) + \cdots) + \varepsilon_{\sigma(m)} f_{x_{\sigma(m)}}.$$

6. *If $(\cdots(\varepsilon_1 f_{x_1} + \varepsilon_2 f_{x_2}) + \cdots) + \varepsilon_m f_{x_m} = (\cdots(\mu_1 f_{z_1} + \mu_2 f_{z_2}) + \cdots) + \mu_t f_{z_t}$ and $(\cdots(\eta_1 f_{y_1} + \eta_2 f_{y_2}) + \cdots) + \eta_s f_{y_s} = (\cdots(\nu_1 f_{w_1} + \nu_2 f_{w_2}) + \cdots) + \nu_u f_{w_u}$, with $\varepsilon_i \in \{-1, 1\}$, $x_i \in X$, $1 \le i \le m$; $\mu_j \in \{-1, 1\}$, $z_j \in X$, $1 \le j \le t$; $\eta_k \in \{-1, 1\}$, $y_k \in X$, $1 \le k \le s$; $\nu_h \in \{-1, 1\}$, $w_h \in X$, $1 \le h \le u$, then*

$$\begin{aligned}(\cdots((((\cdots(\varepsilon_1 f_{x_1} + \varepsilon_2 f_{x_2}) + \cdots) + \varepsilon_m f_{x_m}) \\ + \eta_1 f_{y_1}) + \eta_2 f_{y_2}) + \cdots) + \eta_s f_{y_s} \\ = (\cdots((((\cdots(\mu_1 f_{z_1} + \mu_2 f_{z_2}) + \cdots) \\ + \mu_t f_{z_t}) + \nu_1 f_{w_1}) + \nu_2 f_{w_2}) + \cdots) + \nu_u f_{w_u}.\end{aligned}$$

7. *If $x \in X$, then $f_x^{-1} = -f_{f_x^{-1}(x)}$.*
8. *If $\alpha \in \mathcal{G}(X, r)$ and $x \in X$, then $\alpha f_x^{-1} = \alpha - f_{\alpha(f_x^{-1}(x))}$ and $\alpha f_x = \alpha + f_{\alpha(x)}$.*
9. *If $\alpha \in \mathcal{G}(X, r)$, then $\alpha = 1_X$ or there exist $t \in \mathbb{N}$, $x_i \in X$ and $\varepsilon_i \in \{-1, 1\}$, $1 \le i \le t$, such that*

$$\alpha = \sum_{i=1}^{t} \varepsilon_i f_{x_i} = (\cdots(\varepsilon_1 f_{x_1} + \varepsilon_2 f_{x_2}) + \cdots) + \varepsilon_t f_{x_t}.$$

Proof. 1. We have that

$$\begin{aligned}(\alpha + f_x) + f_y &= \alpha f_{\alpha^{-1}(x)} f_{f^{-1}_{\alpha^{-1}(x)}(\alpha^{-1}(y))} \\ &= \alpha f_{f_{\alpha^{-1}(x)}(f^{-1}_{\alpha^{-1}(x)}(\alpha^{-1}(y)))} f_{g_{f^{-1}_{\alpha^{-1}(x)}(\alpha^{-1}(y))}(\alpha^{-1}(x))} && \text{(by Lemma 2)} \\ &= \alpha f_{\alpha^{-1}(y)} f_{f^{-1}_{f_{\alpha^{-1}(x)}(f^{-1}_{\alpha^{-1}(x)}(\alpha^{-1}(y)))}(\alpha^{-1}(x))} && \text{(by Lemma 1)} \\ &= \alpha f_{\alpha^{-1}(y)} f_{f^{-1}_{\alpha^{-1}(y)}(\alpha^{-1}(x))} \\ &= (\alpha + f_y) + f_x.\end{aligned}$$

2. Let us call $u = \alpha^{-1}(x)$. By Lemma 1, $g_{f_u^{-1}(u)} = f_u^{-1}(u)$ and so
$$\begin{aligned}(\alpha + f_x) - f_x &= \alpha f_{\alpha^{-1}(x)} f^{-1}_{g^{-1}_{f^{-1}_{\alpha^{-1}(x)}(\alpha^{-1}(x))}(f^{-1}_{\alpha^{-1}(x)}(\alpha^{-1}(x)))} \\ &= \alpha f_u f^{-1}_{g^{-1}_{f_u^{-1}(u)}(f_u^{-1}(u))} \\ &= \alpha f_u f^{-1}_{g^{-1}_{f_u^{-1}(u)}(g_{f_u^{-1}(u)}(u))} \\ &= \alpha f_u f_u^{-1} = \alpha.\end{aligned}$$
Again by Lemma 1, $f_{g_u^{-1}(u)}(u) = g_u^{-1}(u)$ and so
$$\begin{aligned}(\alpha - f_x) + f_x &= \alpha f^{-1}_{g^{-1}_{\alpha^{-1}(x)}(\alpha^{-1}(x))} f_{f_{g^{-1}_{\alpha^{-1}(x)}(\alpha^{-1}(x))}(\alpha^{-1}(x))} \\ &= \alpha f^{-1}_{g_u^{-1}(u)} f_{f_{g_u^{-1}(u)}(u)} \\ &= \alpha f^{-1}_{g_u^{-1}(u)} f_{g_u^{-1}(u)} = \alpha.\end{aligned}$$
3. We have that
$$\begin{aligned}(\alpha + f_x) - f_y &= (((\alpha - f_y) + f_y) + f_x) - f_y && \text{(by Statement 2)} \\ &= (((\alpha - f_y) + f_x) + f_y) - f_y && \text{(by Statement 1)} \\ &= (\alpha - f_y) + f_x && \text{(by Statement 2).}\end{aligned}$$
4. We also apply Statements 2 and 3.
$$\begin{aligned}(\alpha - f_x) - f_y &= (((\alpha - f_x) - f_y) + f_x) - f_x \\ &= (((\alpha - f_x) + f_x) - f_y) - f_x \\ &= (\alpha - f_y) - f_x.\end{aligned}$$
5. This follows as a consequence the previous statements, the facts that $1 + \varepsilon_1 f_{x_1} = \varepsilon_1 f_{x_1}$ and $1 + \varepsilon_{\sigma(1)} f_{x_{\sigma(1)}} = \varepsilon_{\sigma(1)} f_{x_{\sigma(1)}}$ and the fact that the symmetric group of degree m is generated by the transpositions $(i, i+1)$, $1 \le i \le m-1$.
6. We use Statement 5:
$$\begin{aligned}&(\cdots((((\cdots(\varepsilon_1 f_{x_1} + \varepsilon_2 f_{x_2}) + \cdots) + \varepsilon_m f_{x_m}) \\ &\qquad + \eta_1 f_{y_1}) + \eta_2 f_{y_2}) + \cdots) + \eta_s f_{y_s} \\ &= (\cdots((((\cdots(\mu_1 f_{z_1} + \mu_2 f_{z_2}) + \cdots) + \mu_t f_{z_t}) \\ &\qquad + \eta_1 f_{y_1}) + \eta_2 f_{y_2}) + \cdots) + \eta_s f_{y_s} \\ &= (\cdots((((\cdots(\eta_1 f_{y_1} + \eta_2 f_{y_2}) + \cdots) + \eta_s f_{y_s}) \\ &\qquad + \mu_1 f_{z_1}) + \mu_2 f_{z_2}) + \cdots) + \mu_t f_{z_t} \\ &= (\cdots((((\cdots(\nu_1 f_{w_1} + \nu_2 f_{w_2}) + \cdots) + \nu_u f_{w_u}) \\ &\qquad + \mu_1 f_{z_1}) + \mu_2 f_{z_2}) + \cdots) + \mu_t f_{z_t} \\ &= (\cdots((((\cdots(\mu_1 f_{z_1} + \mu_2 f_{z_2}) + \cdots) + \mu_t f_{z_t}) \\ &\qquad + \nu_1 f_{w_1}) + \nu_2 f_{w_2}) + \cdots) + \nu_u f_{w_u}\end{aligned}$$

7. Note that $-f_{f_x^{-1}(x)} = f^{-1}_{g^{-1}_{f_x^{-1}(x)}(f_x^{-1}(x))}$. It suffices to check that
$$g^{-1}_{f_x^{-1}(x)}(f_x^{-1}(x)) = x,$$
that is, $f_x(g_{f_x^{-1}(x)}(x)) = x$. But by Lemma 1,
$$f_x(g_{f_x^{-1}(x)}(x)) = f_x(f^{-1}_{f_x(f_x^{-1}(x))}(x)) = f_x(f_x^{-1}(x)) = x$$
and so the equality holds.
8. Now consider
$$\alpha - f_{\alpha(f_x^{-1}(x))} = \alpha f^{-1}_{g^{-1}_{\alpha^{-1}(\alpha(f_x^{-1}(x)))}(\alpha^{-1}(\alpha(f_x^{-1}(x))))} = \alpha f^{-1}_{g^{-1}_{f_x^{-1}(x)}(f_x^{-1}(x))} = \alpha f_x^{-1}$$
by the argument of Statement 7. Since $\alpha + f_{\alpha(x)} = \alpha f_{\alpha^{-1}(\alpha(x))} = \alpha f_x$, the second equality is clear.
9. This last statement is immediate by Statement 8, because every element of $\mathcal{G}(X, r)$ can be expressed as a finite product of elements of the form f_x or f_x^{-1}, with $x \in X$. $\square$

Theorem A appears as a consequence of the next result, in which the previous addition is extended to all elements of $\mathcal{G}(X, r)$.

Theorem 4. *Given*
$$\alpha = (\cdots(\varepsilon_1 f_{x_1} + \varepsilon_2 f_{x_2}) + \cdots) + \varepsilon_m f_{x_m} \in \mathcal{G}(X, r),$$
$$\beta = (\cdots(\eta_1 f_{y_1} + \eta_2 f_{y_2}) + \cdots) + \eta_s f_{y_s} \in \mathcal{G}(X, r),$$
with $\varepsilon_i \in \{-1, 1\}$, $x_i \in X$, $1 \le i \le m$; $\eta_j \in \{-1, 1\}$, $y_j \in X$, $1 \le j \le s$, *the assignment*
$$\alpha + \beta = (\cdots((((\cdots(\varepsilon_1 f_{x_1} + \varepsilon_2 f_{x_2}) + \cdots) + \varepsilon_m f_{x_m}) + \eta_1 f_{y_1}) + \eta_2 f_{y_2}) + \cdots) + \eta_s f_{y_s},$$
$\alpha+1 = 1+\alpha = \alpha$, $1+1 = 1$ *defines an internal binary operation in* $\mathcal{G}(X, r)$ *such that* $(\mathcal{G}(X, r), +, \cdot)$ *is a left brace.*

Proof. By Lemma 3 (9), we have that all elements of $\mathcal{G}(X, r)$ different from 1 can be expressed as a sum of generators of $\mathcal{G}(X, r)$ or their opposites, and by Lemma 3 (6), we have that $+$ is an internal binary operation. An immediate consequence of Lemma 3 (5) is the commutativity of $+$. By definition, 1 is the neutral element of $+$.

Next we prove that $+$ is associative. Note first that if $f = f_x$ or $f = -f_x$ for $x \in X$, then $\alpha + (\beta + f) = (\alpha + \beta) + f$. We must prove that $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ when $\alpha, \beta, \gamma \in \mathcal{G}(X, r)$. If $\gamma = 1$, the result is clear. If $\gamma \ne 1$, then there exist $t \in \mathbb{N}$, $x_i \in X$ and $\varepsilon_i \in \{-1, 1\}$, $1 \le i \le t$, such that $\gamma = \sum_{i=1}^{t} \varepsilon_i f_{x_i}$. We argue by induction and assume that $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$ when δ can be expressed as a sum of $t - 1$ terms $\varepsilon_i f_{x_i}$ (when $t = 1$, this sum is understood to be 1). We express $\gamma = \delta + \varepsilon_t f_{x_t}$, where $\delta = \sum_{i=1}^{t-1} \varepsilon_i f_{x_i}$ is a sum of $t - 1$ terms. By the induction hypothesis, $(\alpha+\beta)+\delta=\alpha+(\beta+\delta)$. Then

$$\begin{aligned}
(\alpha+\beta)+\gamma &= (\alpha+\beta)+(\delta+\varepsilon_t f_{x_t}) \\
&= ((\alpha+\beta)+\delta)+\varepsilon_t f_{x_t} && \text{(by the previous remark)} \\
&= (\alpha+(\beta+\delta))+\varepsilon_t f_{x_t} && \text{(by the inductive hypothesis)} \\
&= \alpha+((\beta+\delta)+\varepsilon_t f_{x_t}) && \text{(by the previous remark)} \\
&= \alpha+(\beta+(\delta+\varepsilon_t f_{x_t})) && \text{(by the previous remark)} \\
&= \alpha+(\beta+\gamma).
\end{aligned}$$

Hence the addition is associative.

By Lemma 3 (2), the commutativity, and the associativity, we have that if $\alpha=\sum_{i=1}^t \varepsilon_i f_{x_i}$, with $\varepsilon_i\in\{-1,1\}$ and $x_i\in X$, $1\leq i\leq t$, and $\gamma=\sum_{i=1}^t(-\varepsilon_j)f_{x_i}$, then $\alpha+\gamma=\gamma+\alpha=1$ and γ becomes the symmetric element of α.

We conclude that $(\mathcal{G}(X,r),+)$ is an abelian group. To conclude the proof, we must show that if α, β, $\gamma\in\mathcal{G}(X,r)$, then $\alpha(\beta+\gamma)+\alpha=\alpha\beta+\alpha\gamma$. The result is clear when $\gamma=1$, because $\alpha(\beta+1)+\alpha=\alpha\beta+\alpha 1$.

We prove now the result when $\gamma=f_x$, $x\in X$.

$$\begin{aligned}
\alpha(\beta+f_x)+\alpha &= \alpha(\beta f_{\beta^{-1}(x)})+\alpha \\
&= (\alpha\beta)f_{\beta^{-1}(x)}+\alpha \\
&= (\alpha\beta+f_{\alpha(\beta(\beta^{-1}(x)))})+\alpha \\
&= \alpha\beta+(f_{\alpha(\beta(\beta^{-1}(x)))}+\alpha) \\
&= \alpha\beta+(f_{\alpha(x)}+\alpha) \\
&= \alpha\beta+(\alpha+f_{\alpha(x)}) \\
&= \alpha\beta+\alpha f_{\alpha^{-1}(\alpha(x))} \\
&= \alpha\beta+\alpha f_x.
\end{aligned}$$

Now we prove the result for $\gamma=-f_x$, $x\in X$.

$$\begin{aligned}
\alpha(\beta-f_x)+a &= \alpha(\beta f^{-1}_{g^{-1}_{\beta^{-1}(x)}(\beta^{-1}(x))})+\alpha \\
&= (\alpha\beta)f^{-1}_{g^{-1}_{\beta^{-1}(x)}(\beta^{-1}(x))}+\alpha \\
&= (\alpha\beta-f_{\alpha(x)})+\alpha \\
&= \alpha\beta+(-f_{\alpha(x)}+\alpha) \\
&= \alpha\beta+(\alpha-f_{\alpha(x)}) \\
&= \alpha\beta+\alpha f^{-1}_{g^{-1}_{\alpha^{-1}(\alpha(x))}(\alpha^{-1}(\alpha(x)))} \\
&= \alpha\beta+\alpha f^{-1}_{g_x^{-1}(x)} \\
&= \alpha\beta+\alpha(-f_x).
\end{aligned}$$

Now we suppose that $\gamma=\sum_{i=1}^t\varepsilon_i f_{x_i}$ with $t\in\mathbb{N}$, $\varepsilon_i\in\{-1,1\}$, $x_i\in X$, $1\leq i\leq t$. We argue by induction on t and we may suppose that $\alpha\cdot(\beta+\delta)+\alpha=\alpha\cdot\beta+\alpha\cdot\delta$ for $\delta=\sum_{i=1}^{t-1}\varepsilon_i f_{x_i}$ (when

$t-1=0$, we agree that $\delta=1$). Let $\phi=\varepsilon_t f_{x_t}$. Then

$$\begin{aligned}\alpha(\beta+\gamma)+\alpha &= \alpha(\beta+(\delta+\phi))+a\\ &= \alpha((\beta+\delta)+\phi)+a\\ &= \alpha(\beta+\delta)+\alpha\phi\\ &= \alpha\beta+\alpha\delta-\alpha+\alpha\phi\\ &= \alpha\beta+(\alpha\delta+\alpha\phi)-\alpha\\ &= \alpha\beta+\alpha(\delta+\phi)\\ &= \alpha\beta+\alpha\gamma.\end{aligned}$$

This shows that $(\mathcal{G}(X,r),+,\cdot)$ is a left brace. □

3. The structure group as a left brace

The following construction can be regarded as an analogue of the one described in [2]. Consider now the Cayley graph of $(\mathcal{G}(X,r),+)$ with edge set E. Let W be the free $\mathbb{Z}$-module with basis E. The group $(\mathcal{G}(X,r),\cdot)$ acts on the left on E as follows: if $g\in\mathcal{G}(X,r)$ and $(\alpha\xrightarrow{\alpha(x)}\alpha f_x)\in E$, then

$$g*\left(\alpha\xrightarrow{\alpha(x)}\alpha f_x\right)=\left(g\alpha\xrightarrow{g\alpha(x)}g\alpha f_x\right)\in E$$

and we can extend the action to W: if $\sum_{e\in E}m_e e\in W$, with $m_e\in\mathbb{Z}$ for $e\in E$, then

$$g*\left(\sum_{e\in E}m_e e\right)=\sum_{e\in E}m_e(g*e)\in W\text{ for all }g\in\mathcal{G}(X,r).$$

Therefore, we can construct the semidirect product $[W]\mathcal{G}(X,r)$. Now we identify all edges in $(\mathcal{G}(X,r),+)$ with the same label. This is equivalent to taking the quotient modulo

$$K=\langle e_{\alpha,y}-e_{\beta,y}\mid y\in X,\ \alpha,\beta\in\mathcal{G}(X,r)\rangle\,,$$

where $e_{\alpha,y}$ denotes the edge $\alpha\xrightarrow{y}\alpha f_{\alpha^{-1}(y)}$.

Lemma 5. $K\cong K\times 1\trianglelefteq[W]\mathcal{G}(X,r)$

Proof. For each generator $e_{\alpha,y}-e_{\beta,y}$ of K, we have that

$$g*(e_{\alpha,y}-e_{\beta,y})=g*e_{\alpha,y}-g*e_{\beta,y}=e_{g\alpha,g(y)}-e_{g\beta,g(y)},$$

which is also one of the generators of K. Thus, K is invariant for the action of $(\mathcal{G}(X,r),\cdot)$.

Next, let us see that $K \trianglelefteq [W]\mathcal{G}(X,r)$. Let $e_{\alpha,y} - e_{\beta,y}$ be a generator of K and let $\left(\sum_{e\in E} m_e e, g\right)$ be an element of $[W]\mathcal{G}(X,r)$. Then

$$\begin{aligned}
&\left(\sum_{e\in E} m_e e, g\right)^{-1} (e_{\alpha,y} - e_{\beta,y}, 1) \left(\sum_{e\in E} m_e e, g\right)\\
&= \left(\sum_{e\in E}(-m_e)(g^{-1} * e), g^{-1}\right)\left(e_{\alpha,y} - e_{\beta,y} + 1 * \left(\sum_{e\in E} m_e e\right), 1\cdot g\right)\\
&= \left(\sum_{e\in E}(-m_e)(g^{-1} * e) + g^{-1} * \left(e_{\alpha,y} - e_{\beta,y} + \sum_{e\in E} m_e e\right), g^{-1}\cdot g\right)\\
&= \left(\sum_{e\in E}(-m_e)(g^{-1} * e) + \left(e_{g^{-1}\alpha,g^{-1}(y)} - e_{g^{-1}\beta,g^{-1}(y)}\right) + \sum_{e\in E} m_e(g^{-1} * e), 1\right)\\
&= \left(e_{g^{-1}\alpha,g^{-1}(y)} - e_{g^{-1}\beta,g^{-1}(y)}, 1\right) \in K.
\end{aligned}$$

□

With this, we can construct the quotient group

$$[W]\mathcal{G}(X,r)/K \cong [W/K]\mathcal{G}(X,r)$$

and take the subgroup

$$H = \langle (e_{1,x} + K, f_x) \mid x \in X\rangle \leq [W/K]\mathcal{G}(X,r).$$

Our next goal is to prove that this group H we have just constructed by means of the Cayley graph of $(\mathcal{G}(X,r),+)$ is isomorphic to the structure group $G(X,r) = \langle X \mid xy = f_x(y)g_y(x),\quad x,y \in X\rangle$. To simplify the notation, as we have identified all the edges with the same label by taking quotients modulo K, we can regard the group H as

$$H = \langle (\bar{x}, f_x) \mid x \in X\rangle \leq [\mathbb{Z}^{(X)}]\mathcal{G}(X,r),$$

where $\bar{x} = e_{1,x} + K$ and $W/K \cong \mathbb{Z}^{(X)}$ is a free abelian group with basis X and the action of $\mathcal{G}(X,r)$ over W becomes the following action of $\mathcal{G}(X,r)$ over $\mathbb{Z}^{(X)}$:

$$\sigma * \left(\sum_{x\in X} a_x \bar{x}\right) = \sum_{x\in X} a_x \overline{\sigma(x)},\qquad \sigma \in \mathcal{G}(X,r).$$

From now on, we will omit the bars in the elements of $W/K \cong \mathbb{Z}^{(X)}$ to simplify the notation.

Proof of Theorem B. Note that, by the definition of the semidirect product, if $(\sum_{x\in X} a_x x, \alpha)$, $(\sum_{x\in X} b_x x, \beta) \in H$, with a_x, $b_x \in \mathbb{Z}$ for each $x \in X$, their product is

$$\begin{aligned}
\left(\sum_{x\in X} a_x x, \alpha\right)\left(\sum_{x\in X} b_x x, \beta\right) &= \left(\sum_{x\in X} a_x x + \sum_{x\in X} b_x \alpha(x), \alpha\beta\right)\\
&= \left(\sum_{x\in X} a_x x + \sum_{x\in X} b_{\alpha^{-1}(x)} x, \alpha\beta\right)\\
&= \left(\sum_{x\in X} (a_x + b_{\alpha^{-1}(x)}) x, \alpha\beta\right). \qquad (2)
\end{aligned}$$

Let $x \in X$ and let (x, f_x) be a generator of H. Note that $(x, f_x)^{-1} = (-f_x^{-1}(x), f_x^{-1})$, because, clearly,

$$(x, f_x)(-f_x^{-1}(x), f_x^{-1}) = (x - f_x(f_x^{-1}(x)), f_x f_x^{-1}) = (0, 1)$$

and

$$(-f_x^{-1}(x), f_x^{-1})(x, f_x) = (-f_x^{-1}(x) + f_x^{-1}(x), f_x^{-1} f_x) = (0, 1).$$

Let F be the free group on the set of generators X. Then there exists an epimorphism $\beta\colon F \longrightarrow G(X, r)$ sending each generator of F to the corresponding generator of $G(X, r)$. Note that the kernel of β is the normal closure of $\langle y^{-1}x^{-1}f_x(y)g_y(x) \mid x, y \in X\rangle$ in F. Moreover, as H is also an X-generated group, there exists an epimorphism $\gamma\colon F \longrightarrow H$ given by $\gamma(x) = (x, f_x)$. Call $V = \operatorname{Ker}\gamma$, $N = \operatorname{Ker}\beta$. We will prove now that $N \leq V$. It is enough to check that $y^{-1}x^{-1}f_x(y)g_y(x) \in V$ for x, $y \in X$.

$$\begin{aligned}
&\gamma(y^{-1}x^{-1}f_x(y)g_y(x)) \\
&\qquad = (-f_y^{-1}(y), f_y^{-1})(-f_x^{-1}(x), f_x^{-1})(f_x(y), f_{f_x(y)})(g_y(x), f_{g_y(x)}) \\
&\qquad = (-f_y^{-1}(y) - f_y^{-1}f_x^{-1}(x) + f_y^{-1}f_x^{-1}f_x(y) + f_y^{-1}f_x^{-1}f_{f_x(y)}(g_y(x)), \\
&\qquad\qquad\quad f_y^{-1}f_x^{-1}f_{f_x(y)}f_{g_y(x)}).
\end{aligned}$$

We have that $f_x f_y = f_{f_x(y)}f_{g_y(x)}$ by Lemma 2 and then $f_y^{-1}f_x^{-1}f_{f_x(y)}f_{g_y(x)} = 1$. For the first component, it is clear that $-f_y^{-1}(y)+f_y^{-1}f_x^{-1}f_x(y) = 0$, and $f_{f_x(y)}(g_y(x)) = x$ by Lemma 1. Thus, we obtain that $y^{-1}x^{-1}f_x(y)g_y(x) \in V$. It follows that there exists an epimorphism $\eta\colon G(X, r) \longrightarrow H$ such that $\eta \cdot \beta = \gamma$, that is, the following diagram is commutative.

$$\begin{tikzcd}
F \arrow[r, "\beta"] \arrow[dr, "\gamma"'] & G(X,r) \arrow[d, "\eta"] \\
& H
\end{tikzcd}$$

We will prove now that $N = V$. Let $w = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \in V$ with $x_i \in X$, $\varepsilon_i \in \{-1, 1\}$, $1 \leq i \leq n$. We prove by induction on n that $w \in N$. If $w = 1$, that is, w has no letters, then it is clear that $w \in N$. Suppose that if a word with less than n letters or their inverses belongs to V, then it belongs to N.

Since the positive exponents contribute as positive coefficients in the free abelian group generated by X and the negative exponents contribute as negative coefficients, we have that the number of positive exponents coincides with the number of negative exponents and n is even, $n = 2m$, say. Since r is non-degenerate, given x, $z \in X$ there exists $y \in X$ such that $z = f_x(y)$ and so $y = f_x^{-1}(z)$. Therefore $n_{x,z} = (f_x^{-1}(z))^{-1}x^{-1}zg_{f_x^{-1}(z)}(x) \in N$. It follows that if

$$w = x_1^{\varepsilon_1} \cdots x_{i-1}^{\varepsilon_{i-1}} x^{-1} z x_{i+2}^{\varepsilon_{i+2}} \cdots x_r^{\varepsilon_r}$$

and

$$u = x_1^{\varepsilon_1} \cdots x_{i-1}^{\varepsilon_{i-1}} f_x^{-1}(z)(g_{f_x^{-1}(z)}(x))^{-1} x_{i+2}^{\varepsilon_{i+2}} \cdots x_r^{\varepsilon_r},$$

then $u^{-1}w = n_{x,z}^{\left(g_{f_x^{-1}(z)}(x)\right)^{-1} x_{i+2}^{\varepsilon_{i+2}} \cdots x_r^{\varepsilon_r}} \in N$, so $w \in N$ if, and only if, $u \in N$. Note also that $f_x^{-1}(z)$ and $g_{f_x^{-1}(z)}(x)$ are two elements of X. Therefore, in order to prove that all words in V which are products of $2m$ elements, m of them in X and m of them inverses of elements of X, belong to N, it is enough to do it for all words of the form $w_0 = x_1 \cdots x_m y_m^{-1} \cdots y_1^{-1} \in V$, with $x_j, y_j \in X$, $1 \leq j \leq m$.

Call $F_{i,t} = f_{x_i} \cdots f_{x_{t-1}}(x_t)$ for $1 \leq i < t$, with $F_{t,t} = x_t$, and $G_s = f_{x_1} \cdots f_{x_m} f_{y_m}^{-1} \cdots f_{y_s}^{-1}(y_s)$ for $1 \leq s \leq m$. Then

$$\begin{aligned}\gamma(w_0) &= (F_{1,1} + F_{1,2} + F_{1,3} + \cdots + F_{1,m} - G_m - G_{m-1} - \cdots - G_1, \\ &\qquad f_{x_1} \cdots f_{x_m} f_{y_m}^{-1} \cdots f_{y_1}^{-1}) \\ &= (0, 1).\end{aligned}$$

We conclude that $f_{x_1} \cdots f_{x_m} f_{y_m}^{-1} \cdots f_{y_1}^{-1} = 1$ and so $G_1 = y_1$. Since all $F_{1,j}$ for $1 \leq j \leq m$ and G_k for $1 \leq k \leq m$ are elements of X, there exists a t with $1 \leq t \leq m$ such that $y_1 = F_{1,t}$. Note that for $1 \leq k \leq t-1$, $n_k = F_{k,t} g_{F_{k+1,t}}(x_k) F_{k+1,t}^{-1} x_k^{-1} \in N$. Call

$$w_k = x_k^{-1} w_{k-1} n_k x_k$$

for $1 \leq k \leq t-1$. Then $w_k \in V$ for $1 \leq k \leq t-1$ and $w_k \in N$ if and only if $w_{k-1} \in N$. We check by induction on k that

$$w_k = x_{k+1} \cdots x_m y_m^{-1} \cdots y_2^{-1} g_{F_{2,t}}(x_1) \cdots g_{F_{k+1,t}}(x_k) F_{k+1,t}^{-1}$$

for $1 \leq k \leq t$. For $k = 1$, since $y_1 = F_{1,t}$, we have that

$$w_1 = x_1^{-1} w_0 (F_{1,t} g_{F_{2,t}}(x_1) F_{2,t}^{-1} x_1^{-1}) x_1 = x_2 \cdots x_m y_m^{-1} \cdots y_2^{-1} g_{F_{2,t}}(x_1) F_{2,t}^{-1}.$$

Suppose that $w_{k-1} = x_k \cdots x_m y_m^{-1} \cdots y_2^{-1} g_{F_{2,t}}(x_1) \cdots g_{F_{k,t}}(x_{k-1}) F_{k,t}^{-1}$. Then

$$\begin{aligned}w_k &= x_k^{-1} w_{k-1} (F_{k,t} g_{F_{k+1,t}}(x_k) F_{k+1,t}^{-1} x_k^{-1}) x_k \\ &= x_{k+1} \cdots x_m y_m^{-1} \cdots y_2^{-1} g_{F_{2,t}}(x_1) \cdots g_{F_{k+1,t}}(x_k) F_{k+1,t}^{-1}.\end{aligned}$$

We conclude that

$$w_{t-1} = x_t \cdots x_m y_m^{-1} \cdots y_2^{-1} g_{F_{2,t(x_1)}} \cdots g_{F_{t,t}}(x_{t-1}) F_{t,t}^{-1}.$$

Since $F_{t,t} = x_t$, we have that

$$x_t^{-1} w_{t-1} x_t = x_{t+1} \cdots x_m y_m^{-1} \cdots y_2^{-1} g_{F_{2,t}}(x_1) \cdots g_{F_{t,t}}(x_{t-1}),$$

so that $w_{t-1} \in V$ and $w_{t-1} \in N$ if and only if $x_t^{-1} w_{t-1} x_t \in N$. We conclude that $w_0 \in N$ if and only if $x_t^{-1} w_{t-1} x_t \in N$. But $x_t^{-1} w_{t-1} x_t$ can be expressed as a word with $2m-2$ elements of X or their inverses. By induction, $x_t^{-1} w_{t-1} x_t \in N$. We conclude that $V = N$.

Then we have proved that the homomorphism $\eta\colon G(X,r) \longrightarrow H$ is in fact an isomorphism. This proves the first statement.

In order to prove the second statement, first we note that, due to the associativity and the commutativity of the additions in $\mathcal{G}(X,r)$ and in $\mathbb{Z}^{(X)}$, it is enough to show that

$$H = \left\{ \left(\sum_{i=1}^{r} \varepsilon_i x_i, \sum_{i=1}^{r} \varepsilon_i f_{x_i} \right) \,\middle|\, r \in \mathbb{N} \cup \{0\}, \varepsilon_i \in \{-1,1\}, x_i \in X, 1 \leq i \leq r \right\}, \tag{3}$$

where the element corresponding to $r = 0$ is the neutral element $(0,1)$. Let us call K the right hand side of Equation (3).

We prove first that $H \subseteq K$ by induction on the number r of factors in $T \cup T^{-1}$, where $T = \{(x, f_x) \mid x \in X\}$ is the natural generating set for H, appearing in an element of H. Clearly, the generators (x, f_x) and their inverses $(x, f_x)^{-1} = (-f_x^{-1}(x), f_x^{-1}) = (-f_x^{-1}(x), -f_{f_x^{-1}(x)})$ belong to K for each $x \in X$. Suppose that $(w, \alpha) = \prod (x_i, f_{x_i})^{\varepsilon_i} \in K$. Then

$$(w, \alpha)(x, f_x) = (w + \alpha(x), \alpha f_x) = (w + \alpha(x), \alpha + f_{\alpha(x)}) \in K$$

and

$$\begin{aligned}(w, \alpha)(x, f_x)^{-1} &= (w, \alpha)(-f_x^{-1}(x), f_x^{-1}) \\ &= (w - \alpha f_x^{-1}(x), \alpha f_x^{-1}) = (w - \alpha f_x^{-1}(x), \alpha - f_{\alpha f_x^{-1}(x)}) \in K.\end{aligned}$$

We conclude that $H \subseteq K$.

We prove now that $K \subseteq H$. We argue by induction on the number r of terms in $(v, \beta) = (\sum_{i=1}^r \varepsilon_i x_i, \sum_{i=1}^r \varepsilon_i f_{x_i}) \in H$. Let

$$(v_0, \beta_0) = \left(\sum_{i=1}^{r-1} \varepsilon_i x_i, \sum_{i=1}^{r-1} \varepsilon_i f_{x_i} \right) \in K.$$

By the inductive hypothesis, $(v_0, \beta_0) \in H$. Assume that $\varepsilon_r = 1$, then

$$\begin{aligned}(v_0, \beta_0)(\beta_0^{-1}(x_r), f_{\beta_0^{-1}(x_r)}) &= (v_0 + x_r, \beta_0 f_{\beta_0^{-1}(x_r)}) \\ &= (v_0 + x_r, \beta_0 + f_{x_r}) = (v, \beta) \in H.\end{aligned}$$

Assume now that $\varepsilon_r = -1$. Then

$$(v, \beta)(\beta^{-1}(x_r), f_{\beta^{-1}(x_r)}) = (v + x_r, \beta f_{\beta^{-1}(x_r)}) = (v + x_r, \beta + f_{x_r}) = (v_0, \beta_0),$$

which implies that $(v, \beta) = (v_0, \beta_0)(\beta^{-1}(x_r), f_{\beta^{-1}(x_r)})^{-1} \in H$. This completes the proof of Statement 2.

Statement 3 follows from Statement 2 and Equation 2. $\square$

Proof of Theorem C. Since the additions in $\mathcal{G}(X, r)$ and $\mathbb{Z}^{(X)}$ make them abelian groups, only the distributivity-like condition is in doubt. Consider three elements $\left(\sum_{x \in X} a_x x, \alpha\right)$, $\left(\sum_{x \in X} b_x x, \beta\right)$, $\left(\sum_{x \in X} c_x x, \gamma\right)$ of H. Then, bearing in mind that $(\mathcal{G}(X, r), +, \cdot)$ is a left brace and Theorem B (3), we obtain:

$$\begin{aligned}&\left(\sum_{x \in X} a_x x, \alpha\right) \left(\left(\sum_{x \in X} b_x x, \beta\right) + \left(\sum_{x \in X} c_x x, \gamma\right) \right) + \left(\sum_{x \in X} a_x x, \alpha\right) \\ &= \left(\sum_{x \in X} a_x x, \alpha\right) \left(\sum_{x \in X} (b_x + c_x) x, \beta + \gamma\right) + \left(\sum_{x \in X} a_x x, \alpha\right) \\ &= \left(\sum_{x \in X} (a_x + b_{\alpha^{-1}(x)} + c_{\alpha^{-1}(x)}) x, \alpha(\beta + \gamma)\right) + \left(\sum_{x \in X} a_x x, \alpha\right) \\ &= \left(\sum_{x \in X} (a_x + b_{\alpha^{-1}(x)} + c_{\alpha^{-1}(x)} + a_x) x, \alpha(\beta + \gamma) + \alpha\right) \\ &= \left(\sum_{x \in X} (a_x + b_{\alpha^{-1}(x)} + a_x + c_{\alpha^{-1}(x)}) x, \alpha\beta + \alpha\gamma\right)\end{aligned}$$

$$= \left(\sum_{x\in X}(a_x + b_{\alpha^{-1}(x)})x, \alpha\beta\right) + \left(\sum_{x\in X}(a_x + c_{\alpha^{-1}(x)})x, \alpha\gamma\right)$$
$$= \left(\sum_{x\in X} a_x x, \alpha\right)\left(\sum_{x\in X} b_x x, \beta\right) + \left(\sum_{x\in X} a_x x, \alpha\right)\left(\sum_{x\in X} c_x x, \gamma\right).$$

It follows that $(H, +, \cdot)$ is a left brace. The fact that π is a left brace epimorphism is clear. □

Therefore, given a finite involutive non-degenerate set-theoretic solution (X, r) of the YBE, we can construct its structure group $G(X, r)$ using the Cayley graph of the group $(\mathcal{G}(X, r), +)$.

4. A geometrical interpretation of the structure group in terms of the Cayley graph of the permutation group

In this section we present a geometrical interpretation of Theorem B in the line of [2]. Let G be a group with a generating set S and let F be the free group on S. There exists a unique epimorphism $\beta\colon F \longrightarrow G$ that sends the generators of F to the corresponding generators of G. Let $w \in F$, then w is a word on $S \cup S^{-1}$, $w = s_1^{\varepsilon_1} \dots s_r^{\varepsilon_r}$ with $r \geq 0$, $\varepsilon_i \in \{-1, 1\}$, $s_i \in S$, $1 \leq i \leq r$, say. If we have the Cayley graph of G with respect to S, we can consider a path of length r starting from 1 and following the edges labelled s_i, in the same sense if $\varepsilon_i = 1$ and in the opposite sense if $\varepsilon_i = -1$, for $1 \leq i \leq r$. The other end of this path is $\beta(w)$.

According to Theorem A, if we draw the Cayley graph of the permutation group $(\mathcal{G}(X, r), \cdot)$ with respect to the natural generating set $S = \{f_x \mid x \in X\}$ and we replace in each edge of the form $\alpha \xrightarrow{x} \alpha f_x$, $x \in X$, $\alpha \in \mathcal{G}(X, r)$ the label x by $\alpha(x)$, then we obtain the Cayley graph of $(\mathcal{G}(X, r), +)$ with respect to the same generating set. We can use these Cayley graphs to obtain the images of elements of the free group on S in $(\mathcal{G}(X, r), \cdot)$ and $(\mathcal{G}(X, r), +)$.

Suppose now that we want to obtain an element of $(G(X, r), +)$. We can identify $G(X, r)$ with the subgroup H of Theorem B with the generating set $T = \{(x, f_x) \mid x \in X\}$ identified in the obvious way with X. We can follow in the Cayley graph of $(\mathcal{G}(X, r), +)$ a path labelled with the terms as before. The last end of the path corresponds to the second component. If we take into account the number of signed traversals of edges labelled by each element of X in the Cayley graph of $(\mathcal{G}(X, r), +)$, we also obtain the coefficients $a_x \in \mathbb{Z}$ of the first component $\sum_{x\in X} a_x\bar{x}$ of the sum.

Finally, suppose that we want to obtain an element of $(G(X, r), \cdot)$, identified again with H with generating set T as in the previous paragraph. If we follow the path in the Cayley graph of $(\mathcal{G}(X, r), \cdot)$ starting from 1 with edges labelled with the corresponding elements of X, the last end corresponds to the second component of the product. In order to find the first component of the product, we can follow the same path in the Cayley graph of $(\mathcal{G}(X, r), +)$, with the new assignments of labels, and take into account the number of signed traversals of edges labelled with $x \in X$ to obtain the coefficient $b_x \in \mathbb{Z}$ of the first component $\sum_{x\in X} b_x\bar{x}$ of the product.

Example 6. Let (X, r) be the solution of the YBE with $X = \{1, 2, 3, 4, 5\}$ and $f_1 = f_2 = f_3 = 1$, $f_4 = (1, 2)(4, 5)$, and $f_5 = (1, 3)(4, 5)$. The left hand side of Figure 1 shows the Cayley graph of $(\mathcal{G}(X, r), \cdot)$, in which we have drawn with just one loop the three loops, corresponding to the edges labelled 1, 2, and 3, around each vertex. The right hand side of Figure 1 shows the Cayley graph of $(\mathcal{G}(X, r), +)$.

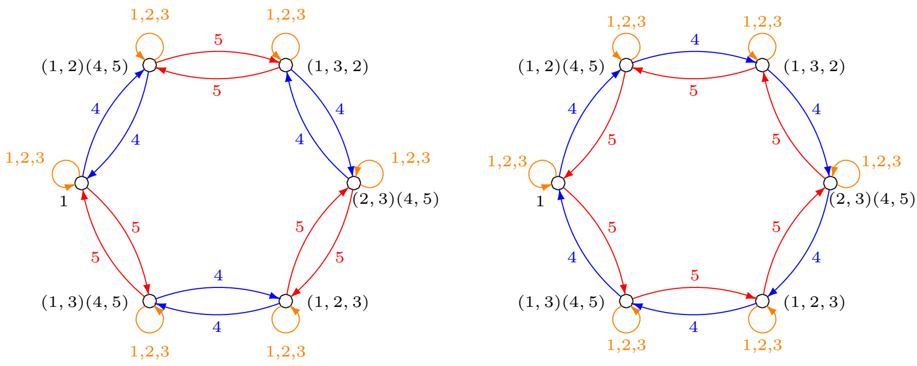

FIGURE 1. Cayley graphs of $(\mathcal{G}(X,r),\cdot)$ and $(\mathcal{G}(X,r),+)$ (Example 6)

Consider now the free group F with basis X and the word $w = 445^{-1} \in F$. Its image in $(\mathcal{G}(X,r),+)$ will be $f_4 + f_4 - f_5$. This can be obtained by following in the Cayley graph of $(\mathcal{G}(X,r),+)$ the path starting from 1 and with edges labelled 4, 4, and 5 (the last one reversed). The path is drawn in Figure 2.

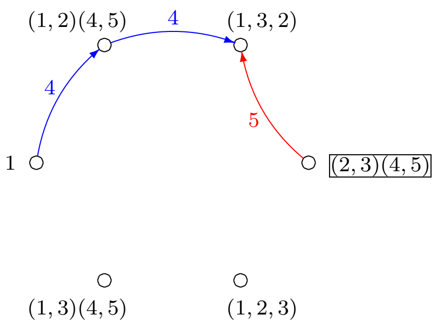

FIGURE 2. Path in the Cayley graph of $(\mathcal{G}(X,r),+)$ (Example 6)

We obtain that $f_4 + f_4 - f_5 = (2,3)(4,5)$. We see that this path contains two arcs labelled 4 and an arc labelled 5 traversed in the opposite direction. This means that w maps to $(\bar{4}, f_4) + (\bar{4}, f_4) - (\bar{5}, f_5) = (2 \cdot \bar{4} + (-1) \cdot \bar{5}, (2,3)(4,5))$ in $(G(X,r),+)$.

Now we compute the image of the same word $w = 445^{-1} \in F$ in the multiplicative group $(G(X,r),\cdot)$, that is, $(\bar{4}, f_4) \cdot (\bar{4}, f_4) \cdot (\bar{5}, f_5)^{-1}$. This can be obtained by following in the Cayley graph of $(\mathcal{G}(X,r),\cdot)$ the path starting from 1 and with edges labelled 4, 4, and 5 (the last one reversed). The path is drawn on the left hand side of Figure 3.

We see that $f_4 \cdot f_4 \cdot f_5^{-1} = (1,3)(4,5)$. Now we can consider the same path in the Cayley graph of $(\mathcal{G}(X,r),+)$, with the labels changed according to Theorem B. The new labels for these edges are now 4, 5, 4, respectively, the first two ones traversed in the direction of the edges and last one reversed. The path appears on the right hand side of Figure 3. The edges labelled 4 cancel because there is one traversed positively and another one traversed negatively, and

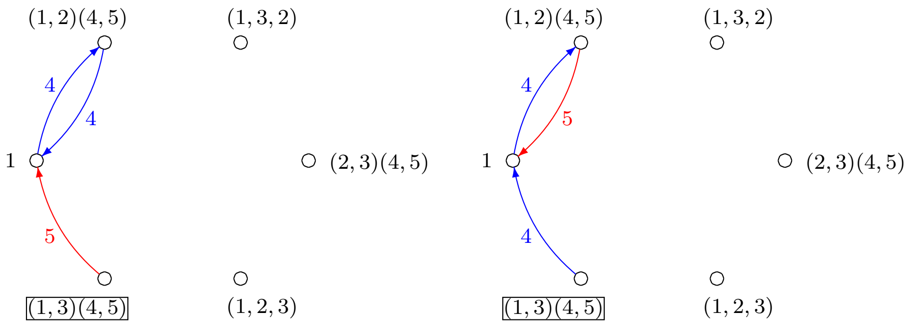

FIGURE 3. Paths in the Cayley graphs of $(\mathcal{G}(X, r), \cdot)$ and $(\mathcal{G}(X, r), +)$ (Example 6)

there is an edge labelled 5 traversed positively. Consequently, the image of w in $(G(X, r), \cdot)$ is $(\bar{4}, f_4) \cdot (\bar{4}, f_4) \cdot (\bar{5}, f_5)^{-1} = (0 \cdot \bar{4} + 1 \cdot \bar{5}, (1, 3)(4, 5))$.

5. A comparison with other definitions of the addition

Bachiller, Cedó, and Jespers defined in [1] an addition in the structure group of a solution of the YBE which induces an addition in the permutation group such that both groups acquire brace structures. The aim of this section is to prove that both additions coincide, respectively, with our additions in the structure group and in the permutation group.

The definitions of the additions in [1] depend strongly on the isomorphism between the structure group and the subgroup of the semidirect product of the free abelian group $\mathbb{Z}^X$ with basis X and the symmetric group on X given by Etingof, Schedler, and Soloviev in [8]. For the reader's convenience, we summarise the arguments of [8] and we adapt their notation to the left actions we are considering here.

In [8], their authors prove that $G(X, r)$ is isomorphic to a subgroup of the semidirect product $M_X = [\mathbb{Z}^X]\mathrm{Sym}(X)$ associated with the natural action of $\mathrm{Sym}(X)$ on X, extended by linearity to $\mathbb{Z}^X$. The product in M_X is defined by

$$(a, \sigma)(b, \tau) = (a + \sigma(b), \sigma\tau), \quad \text{for all } a,\, b \in \mathbb{Z}^X,\, \sigma,\, \tau \in \mathrm{Sym}(X).$$

They define a group homomorphism $\psi\colon G(X, r) \longrightarrow M_X$ by means of $\psi(x) = (x, f_x) \in M_X$ for each $x \in X$. Write $\psi(g) = (\pi(g), \phi(g)) \in M_X$ for each $g \in G$. Then $\phi\colon G(X, r) \longrightarrow \mathrm{Sym}(X)$ is a group homomorphism with image $\mathcal{G}(X, r)$ and $\pi\colon G(X, r) \longrightarrow \mathbb{Z}^X$ is a derivation or 1-cocycle with respect to the action of $G(X, r)$ on $\mathbb{Z}^X$ given by $g \bullet \sum_{x \in X} a_x x = \sum_{x \in X} a_x \phi(g)(x)$, that is, $\pi(g_1 g_2) = \pi(g_1) + \phi(g_1)(\pi(g_2))$ for g_1, $g_2 \in G(X, r)$.

They prove that π is bijective by showing that it possesses an inverse $h\colon \mathbb{Z}^X \longrightarrow G(X, r)$. Call $\mathbb{Z}^X_k$ the set of elements of $\mathbb{Z}^X$ that can be expressed as a sum of at most k terms of the form x or $-x$ with $x \in X$, in such a way $\mathbb{Z}^X = \bigcup_{k \geq 1} \mathbb{Z}^X_k$. The inverse h of π is defined for elements of $\mathbb{Z}^X_1$ as $h(0) = 1$, $h(x) = x$ and $h(-x) = (g_x^{-1}(x))^{-1}$ for $x \in X$. If h has been already defined for elements of $\mathbb{Z}^X_{k-1}$ and $\eta \in \mathbb{Z}^X_k$, then $\eta = a + \xi$ for $a \in \mathbb{Z}^X_{k-1}$, $\xi \in \{x, -x\}$ for some $x \in X$. In this case, we consider the right action of $G(X, r)$ on $\mathbb{Z}^X$ defined by $a \star x = f_x^{-1}(a)$ for $a \in \mathbb{Z}^X$, $x \in X$, and define $h(\eta) = h(a)h(\xi \star h(a))$. In this case, $h(x \star h(a)) = h(\phi(h(a))^{-1}(x)) = \phi(h(a))^{-1}(x)$ and $h((-x) \star h(a)) = h(\phi(h(a))^{-1}(-x)) = h(-\phi(h(a))^{-1}(x)) = (g_{\phi(h(a))^{-1}(x)}(\phi(h(a))^{-1}(x)))^{-1}$.

This construction is used by Bachiller, Cedó, and Jespers in [1] to define additions in $G(X, r)$ and $\mathcal{G}(X, r)$. The addition in $G(X, r)$ is defined by means of

$$g_1 + g_2 = h(\pi(g_1) + \pi(g_2)) \quad \text{for } g_1,\ g_2 \in G(X, r).$$

Given $g \in G(X, r)$ and $x \in X$, we obtain that

$$\begin{aligned} g + x &= h(\pi(g) + \pi(x)) = h(\pi(g) + x) = h(\pi(g))h(x \star h(\pi(g))) \\ &= gh(x \star g) = g\phi(g)^{-1}(x), \end{aligned}$$

and so

$$\begin{aligned} \psi(g + x) &= (\pi(g) + x, \phi(g\phi(g)^{-1}(x))) = (\pi(g) + x, \phi(g)\phi(\phi(g)^{-1}(x))) \\ &= (\pi(g) + x, \phi(g)f_{\phi(g)^{-1}(x)}), \end{aligned}$$

while

$$\begin{aligned} g - x &= h(\pi(g) - \pi(x)) = h(\pi(g) - x) = h(\pi(g))h((-x) \star h(\pi(g))) \\ &= gh((-x) \star g) = g(g_{\phi(g)^{-1}(x)}(\phi(g)^{-1}(x)))^{-1} \end{aligned}$$

and so

$$\begin{aligned} \psi(g - x) &= (\pi(g) - x, \phi(g(g_{\phi(g)^{-1}(x)}(\phi(g)^{-1}(x)))^{-1})) \\ &= (\pi(g) - x, \phi(g)f^{-1}_{g_{\phi(g)^{-1}(x)}(\phi(g)^{-1}(x))}). \end{aligned}$$

The addition in $\mathcal{G}(X, r)$, that coincides with the image of ϕ, is defined by $\phi(g_1) + \phi(g_2) = \phi(g_1 + g_2)$, where $g_1,\ g_2 \in G(X, r)$. Let $\alpha \in \mathcal{G}(X, r)$ and $x \in X$. Then $\alpha = \phi(g)$ for a certain $g \in G(X, r)$ and $f_x = \phi(x)$. Then $\alpha + f_x = \phi(g) + \phi(x) = \phi(g + x) = \phi(g)f_{\phi(g)^{-1}(x)} = \alpha f_{\alpha^{-1}(x)}$ and $\alpha - f_x = \phi(g) - \phi(x) = \phi(g - x) = \phi(g)f^{-1}_{g_{\phi(g)^{-1}(x)}(\phi(g)^{-1}(x))} = \alpha f^{-1}_{g_{\alpha^{-1}(x)}(\alpha^{-1}(x))}$. We conclude that the additions obtained with the arguments of [8] and [1] coincide with our additions.

6. Some applications

In this section we present some applications of our main theorems.

The fixed points of the natural action of r on $X \times X$ are immediately identified in the Cayley graph. They were called *frozen pairs* in [7].

Proposition 7. *Let $x \in X$. Consider in the Cayley graph of the additive group of $\mathcal{G}(X, r)$ the path of length two starting at 1 with both edges labelled with x and consider the corresponding edges on the Cayley graph of the multiplicative group of $\mathcal{G}(X, r)$, with labels x, y, respectively. Then $r(x, y) = (x, y)$. Moreover, (x, y) is the unique pair of the form (x, z) with $z \in X$ such that $r(x, z) = (x, z)$.*

Proof. We have that $f_x \cdot f_y = f_x + f_{f_x(y)} = f_x + f_x$ and $x = f_x(y)$. Hence $y = f_x^{-1}(x)$. Now $r(x, y) = (f_x(y), f^{-1}_{f_x(y)}(x)) = (x, f_x^{-1}(x)) = (x, y)$.

Moreover, if $r(x, z) = (x, z)$, then $f_x(z) = x$ and so $z = f_x^{-1}(x) = y$, hence the unicity holds. $\square$

We see that the relations explicitly mentioned in the definition of the structure group and the trivial ones of the form $xy = xy$ are the unique relations that can be found in this group involving equalities of products of two generators. The proof is already implicit in the proof of Theorem B, but it becomes more evident from our description of the structure group.

Theorem 8. *Let x, y, z, $t \in X$ be regarded as elements of the structure group $G(X, r)$. Then $xy = zt$ if and only if $x = z$ and $y = t$ or $r(x, y) = (z, t)$.*

Proof. The element xy corresponds to a path of length 2 in the Cayley graph of $\mathcal{G}(X, r)$ starting at 1 and with labels x and y, and the element zt corresponds to a path of length 2 in the Cayley graph of $\mathcal{G}(X, r)$ starting at 1 and with labels z and t. They correspond to paths in the Cayley graph of the additive group of $\mathcal{G}(X, r)$ starting at 1 and with labels x, $f_x(y)$ for the first one, and z, $f_z(t)$ for the second one. Hence $\{x, f_x(y)\} = \{z, f_z(t)\}$. If $x = z$, then $f_x(y) = f_z(t) = f_x(t)$ and, since f_x is bijective, $y = t$ and we are in the first case. Now suppose that $x = f_z(t)$, $z = f_x(y)$. Then $r(x, y) = (f_x(y), f^{-1}_{f_x(y)}(x)) = (z, f_z^{-1}(x)) = (z, t)$ and we are in the second case. The converse is clear. $\square$

Recall that if G is a permutation group acting on Ω, a *block* of this action is a subset $B \subseteq \Omega$ such that for each $g \in G$, $gB = B$ or $gB \cap B = \emptyset$. Note that we are not requiring the action to be transitive. Suppose that (X, r) is a solution of the YBE. The *retract* relation $\sim$ on X given by $x \sim y$ if and only if $f_x = f_y$, where, as usual, $r(x, y) = (f_x(y), g_y(x))$, x, $y \in X$, defines an equivalence relation on X. We see that the equivalence classes for $\sim$ form a block system for the action of $\mathcal{G}(X, r)$ on X.

Proposition 9. *Let (X, r) be a solution of the YBE. The equivalence classes for the retract relation $\sim$ on X are blocks for the natural action of $\mathcal{G}(X, r)$ on X.*

Proof. Suppose that $x \sim \bar{x}$, that is, $f_x = f_{\bar{x}}$. Then, given $\alpha \in \mathcal{G}(X, r)$, $\alpha f_x = \alpha f_{\bar{x}}$, that is, $\alpha + f_{\alpha(x)} = \alpha + f_{\alpha(\bar{x})}$, which implies that $f_{\alpha(x)} = f_{\alpha(\bar{x})}$. $\square$

Proposition 9 gives an immediate justification for the construction of the solution associated to the retraction.

Proposition 10 ([8, Section 3.2], **see also** [6])**.** *If (X, r) is a solution of the YBE, then $\tilde{r}\colon (X/{\sim}) \times (X/{\sim})$ given by $\tilde{r}([x], [y]) = ([f_x(y)], [g_y(x)])$ is a map such that $(X/{\sim}, \tilde{r})$ is a solution of the YBE and the natural surjection $\varphi\colon X \longrightarrow X/{\sim}$ induces a homomorphism of solutions of the YBE.*

Recall that the *socle* of a brace $(B, +, \cdot)$ is

$$\begin{aligned}\operatorname{Soc}(B) &= \{a \in B \mid \text{for all } b \in B,\ a + b = ab\}\\ &= \{a \in B \mid \text{for all } b \in B,\ ab - a = b\}.\end{aligned}$$

Then $\lambda_a\colon B \longrightarrow B$ given by $\lambda_a(b) = ab - a$ defines an automorphism of $(B, +)$. In order to prove that $\lambda_a(b) = b$ for all $b \in B$, it is enough to check the condition for the elements of a generating set of $(B, +)$. Assume now that $B = \mathcal{G}(X, r)$. We obtain that

$$\operatorname{Soc}(\mathcal{G}(X, r)) = \{\alpha \in \mathcal{G}(X, r) \mid \text{for all } x \in X,\ \alpha + f_x = \alpha f_x\}.$$

Since $\alpha f_x = \alpha + f_{\alpha(x)}$, we have that

$$\begin{aligned}\operatorname{Soc}(\mathcal{G}(X, r)) &= \{\alpha \in \mathcal{G}(X, r) \mid \text{for all } x \in X,\ \alpha + f_x = \alpha + f_{\alpha(x)}\}\\ &= \{\alpha \in \mathcal{G}(X, r) \mid \text{for all } x \in X,\ f_x = f_{\alpha(x)}\}.\end{aligned}$$

Hence

$$\mathrm{Soc}(\mathcal{G}(X,r)) = \{\alpha \in \mathcal{G}(X,r) \mid \text{for all } x \in X,\ x \sim \alpha(x)\}.$$

We conclude the following result.

Proposition 11. *If (X,r) is a solution of the YBE, then*

$$\mathrm{Soc}(\mathcal{G}(X,r)) = \{\alpha \in \mathcal{G}(X,r) \mid \alpha \textit{ induces the identity on } X/{\sim}\}.$$

The permutation group $\mathcal{G}(X/{\sim},\tilde{r})$ of the retraction $(X/{\sim},\tilde{r})$ of (X,r) is

$$\mathcal{G}(X/{\sim},\tilde{r}) = \langle \tilde{f}_{[x]} \mid x \in X\rangle,$$

where $\tilde{f}_{[x]}\colon X/{\sim} \longrightarrow X/{\sim}$ is given by $\tilde{f}_{[x]}([y]) = [f_x(y)]$. It is clear that all relations of $\mathcal{G}(X,r)$ are satisfied by $\mathcal{G}(X/{\sim},\tilde{r})$, since if a product of elements of the form f_x or f_x^{-1} acts trivially on X, then it acts trivially on the blocks of $X/{\sim}$. By von Dyck's theorem, there exists a group epimorphism $\eta\colon \mathcal{G}(X,r) \longrightarrow \mathcal{G}(X/{\sim},\tilde{r})$ such that $\eta(f_x) = \tilde{f}_{[x]}$. Then

$$\operatorname{Ker}\eta = \{\alpha \in \mathcal{G}(X,r) \mid \alpha \text{ induces the identity on } X/{\sim}\} = \mathrm{Soc}(\mathcal{G}(X,r))$$

by Proposition 11. Therefore we have the following result, that can be compared with [12, Proposition 7]:

Proposition 12. *If (X,r) is a solution of the YBE and $(X/{\sim},\tilde{r})$ is its retraction, then*

$$\mathcal{G}(X,r)/\mathrm{Soc}(\mathcal{G}(X,r)) \cong \mathcal{G}(X/{\sim},\tilde{r}).$$

We can use Proposition 12 to obtain the Cayley graph of the permutation group associated to the retraction of a solution of the YBE. We identify in X the elements related with respect to the retract relation. The arcs in the Cayley graph corresponding to the same retraction class will be identified and the vertices will be replaced by the permutation that this vertex induces on $X/{\sim}$. The elements of $\mathrm{Soc}(\mathcal{G}(X,r))$ will be mapped to $1_{X/\sim}$. We identify all vertices with the same labels, that will correspond to the same element of $\mathcal{G}(X,r)/\mathrm{Soc}(\mathcal{G}(X,r))$. This new graph will be the Cayley graph of $\mathcal{G}(X/{\sim},\tilde{r}) \cong \mathcal{G}(X,r)/\mathrm{Soc}(\mathcal{G}(X,r))$.

Example 13. Let $X = \{1,2,3,4,5\}$ and let r be the solution of the YBE given by $f_1 = f_2 = f_3 = 1_X$, $f_4 = (1,2)(4,5)$, $f_5 = (1,3)(4,5)$. The Cayley graph of $(\mathcal{G}(X,r),\cdot)$ is given in Figure 4, where each loop in the figure represents three loops with labels 1, 2, and 3, respectively. The retraction classes are $\{1,2,3\}$, $\{4\}$, and $\{5\}$. We identify the arcs corresponding in each retraction class and we replace the vertices by the result of the action of $\mathcal{G}(X,r)$ on the blocks of $X/{\sim}$. The result is shown on Figure 5. We see that the vertices corresponding in Figure 4 to 1, $(1,3,2)$, and $(1,2,3)$ are replaced by 1 in Figure 5, because they are the elements of $\mathrm{Soc}(\mathcal{G}(X,r))$. Now the vertices with the same labels must be identified. This gives the graph with two vertices corresponding to $\mathcal{G}(X/{\sim},\tilde{r}) = \{1,([4],[5])\}$ that is drawn on Figure 6.

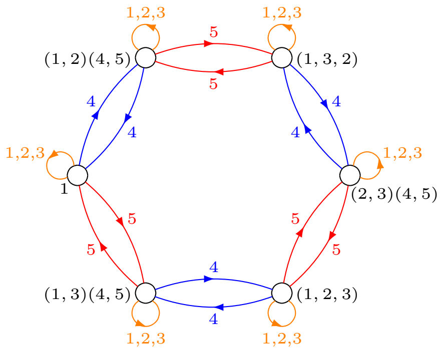

FIGURE 4. Cayley graph of the multiplicative group of $\mathcal{G}(X, r)$ in Example 13

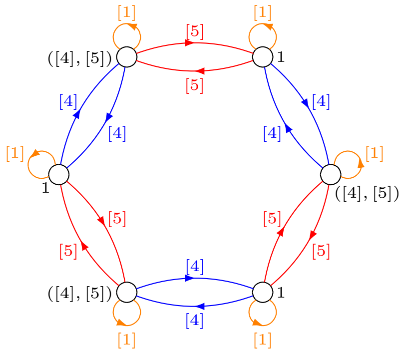

FIGURE 5. Identification in the Cayley graph of $\mathcal{G}(X, r)$ of the arcs in the same retraction class and substitution of the vertices by the result of the action on the blocks in Example 13

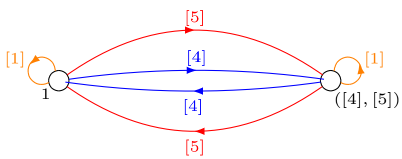

FIGURE 6. Identification of equal vertices in the retraction in Example 13

We can also repeat the process to find that (X, r) is a multipermutation solution with multipermutation level 3. If we call $X_1 = X/\!\sim\, = \{[1] = \{1, 2, 3\}, [4] = \{4\}, [5] = \{5\}\}$, or $X_1 = \{1, 4, 5\}$ for shorter, and $r_1 = \tilde{r}$, its retraction is $X_2 = X_1/\!\sim\, = \{[1] = \{1\}, [4] = \{4, 5\}\}$ and Figure 7 shows the Cayley graph of $\mathcal{G}(X_2, r_2 = \tilde{r}_1)$. Finally, the retraction of $X_2 = \{1, 4\}$ is

$X_3 = \{[1] = \{1, 4\}\}$ which has only one element. Then, (X, r) has multipermutation level 3. The Cayley graph of $\mathcal{G}(X_3, r_3 = \tilde{r}_2)$ is drawn in Figure 8.

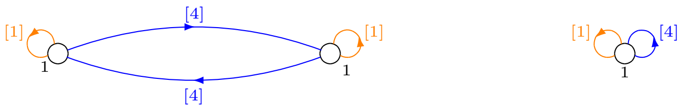

FIGURE 7. Cayley graph of $\mathcal{G}(X_1/{\sim}, \tilde{r}_1)$ in Example 13

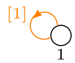

FIGURE 8. Cayley graph of $\mathcal{G}(X_2/{\sim}, \tilde{r}_2)$ in Example 13

A brace is trivial whenever it coincides with its socle. Proposition 11 can be used to give a characterisation of when the permutation group of a solution of the YBE is a trivial brace.

Proposition 14. *The following statements are equivalent for a solution (X, r) of the YBE.*

1. *The permutation group $\mathcal{G}(X, r)$ is a trivial brace.*
2. *For every x, $y \in X$, if there exists $\alpha \in \mathcal{G}(X, r)$ such that $\alpha(x) = y$, then $f_x = f_y$ (in other words, x and y are related by the retract relation).*

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