On large orbits of subgroups of linear groups

H. Meng [*] A. Ballester-Bolinches[†] R. Esteban-Romero[‡]

Abstract

The main aim of this paper is to prove an orbit theorem and to apply it to obtain a result that can be regarded as a significant step towards the solution of Gluck's conjecture on large character degrees of finite solvable groups.

Mathematics Subject Classification (2010): 20C15, 20D20, 20D45

Keywords: finite groups, solvable groups, linear groups, regular orbits, representations of groups

1 Introduction

The main aim of this paper is to prove an orbit theorem (see Theorem A below) and to apply it to obtain a result that can be regarded as a significant step towards the solution of Gluck's conjecture on large character degrees of finite solvable groups (see Theorem B below). Hence all sets, groups, fields and modules to be considered here are finite, and we assume this without further comment.

Recall that if a group G is acting on a non-empty set Ω, an element w of Ω is in a *regular* orbit if $\mathrm{C}_G(w) = \{g \in G : wg = w\} = 1$, i.e., the orbit of w is as large as possible and it has size $|G|$. The study of regular orbits of linear groups actions, that is, regular orbits of actions of subgroups of $\mathrm{GL}(V)$ on a vector space V, plays an important role in many branches of group theory, particularly of that solvable groups. In fact, the solution of

[*]Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain, email: hangyangmenges@gmail.com

[†]Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain, email: Adolfo.Ballester@uv.es

[‡]Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain, email: Ramon.Esteban@uv.es; permanent address: Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València, Camí de Vera, s/n, 46022 València, Spain, email: resteban@mat.upv.es

some well-known open problems such as the so-called $k(GV)$-problem ([12]) depends on the existence of such orbits. Consequently, the problem of the existence of regular orbits has attracted the attention of several authors and it is an active and interesting research area in Group Theory.

In order to understand and motivate what is to follow, it is convenient to use some previous results as a model.

Espuelas (see [6, Theorem 3.1]) stated that if G is a group of odd order and V is a faithful and completely reducible G-module of odd characteristic, then G has a regular orbit on $V \oplus V$. Dolfi and Jabara ([5, Theorem 2]) extended Espuelas's result to the case where the Sylow 2-subgroups of the semidirect product $[V]G$ of V and the solvable group G are abelian, and Yang ([18, Theorem 2.3]) proved that the same is true if 3 does not divide the order of the solvable group G. A result of Wolf ([14, Theorem A]) shows that a similar result holds if G is supersolvable (see also [11, Theorem 3.1] for an improved result when G is nilpotent).

Dolfi ([4, Theorem 1.4]), reproving a result of Seress ([13, Theorem 2.1]), proved that any solvable group G has a regular orbit on $V \oplus V \oplus V$ and if either $(|V|, |G|) = 1$ or G is of odd order, then G has also a regular orbit on $V \oplus V$ ([4, Theorems 1.1 and 1.5]).

More recently, Yang ([19]) extends some of these results to the case when H is a subgroup of the solvable group G by proving that if V is a faithful completely reducible G-module (possibly of mixed characteristic) and if either H is nilpotent or 3 does not divide the order of H, then H has at least three regular orbits on $V \oplus V$. If the Sylow 2-subgroups of the semidirect product $[V]H$ are abelian, then H has at least two regular orbits on $V \oplus V$.

We prove that almost all previous results are consequences of the following surprising theorem.

Theorem A. *Let G be a solvable group and let V be a faithful completely reducible G-module (possibly of mixed characteristic). Suppose that H is a subgroup of G such that the semidirect product VH is S_4-free. Then H has at least two regular orbits on $V \oplus V$. Furthermore, if H is $\Gamma(2^3)$-free and $\mathrm{SL}(2,3)$-free, then H has at least three regular orbits on $V \oplus V$.*

Recall that if G and X are groups, then G is said to be X-*free* if X cannot be obtained as a quotient of a subgroup of G; $\Gamma(2^3)$ denotes the semilinear group of the Galois field of 2^3 elements.

The S_4-free hypothesis in Theorem A is not superfluous (see [5, Example 1]).

We now draw a series of conclusions from Theorem A.

Corollary 1 ([19]). *Let G be a solvable group acting completely reducibly and faithfully on an odd order module V. Suppose that H is a subgroup of*

G. If H is nilpotent or $3 \nmid |H|$, then H has at least three regular orbits on $V \oplus V$. If the Sylow 2-subgroup of the semidirect product VH is abelian, then H has at least two regular orbits on $V \oplus V$.

Corollary 2 (see [4, Theorem 1.1])**.** *Let G be a solvable group and V be a faithful completely reducible G-module. Suppose that $(|G|, |V|) = 1$. Then G has at least two regular orbits on $V \oplus V$.*

Proof. Arguing by induction on $|V| + |G|$, we may assume that V is an irreducible and faithful G-module over $\mathrm{GF}(p)$ for some prime p.

Applying Lemma 28, we may assume that $p = 2$ or 3. In both cases, VG is S_4-free. From Theorem 29, G has at least two regular orbits on $V \oplus V$ when $p = 2$, 3. □

Our next corollary shows that Theorem A of [14] holds for supersolvable subgroups of a solvable group provided that $|V|$ is odd.

Corollary 3. *Let G be a solvable group acting completely reducibly and faithfully on an odd order module V. If H is a supersolvable subgroup of G, then H has at least two regular orbits on $V \oplus V$.*

Proof. Note that H is S_4-free. Since V is of odd order, HV is S_4-free. By Theorem A, H has at least two regular orbits on $V \oplus V$. □

If G is a group, let $\mathrm{Irr}(G)$ denote the set of all irreducible complex characters of G and write $\mathrm{b}(G) = \max\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$, so that $\mathrm{b}(G)$ is the largest irreducible character degree of G. Gluck [7] conjectured that if G is solvable, then $|G : \mathrm{F}(G)| \leqslant \mathrm{b}(G)^2$, where $\mathrm{F}(G)$ is the Fitting subgroup of G. Gluck's conjecture is still open and has been studied extensively (see [14], [5] [11], [1]).

Our second main result not only extends almost all known results on Gluck's conjecture, but also it could also be very useful to solve Gluck's conjecture in the future.

The proof follows Gluck's strategy [7] to produce an irreducible character of large degree and it is a nice consequence of Theorem A.

Theorem B. *Let G be a solvable group satisfying one of the following conditions:*

1. *G is S_4-free;*
2. *$G/\mathrm{F}(G)$ is S_4-free and $\mathrm{F}(G)$ is of odd order;*
3. *$G/\mathrm{F}(G)$ is S_3-free.*

Then Gluck's conjecture is true for G.

Corollary 4 ([5, Theorem 1], [1, Corollary 2])**.** *Let G be a solvable group. If either the Sylow 2-subgroups of G are abelian or $|G/\mathrm{F}(G)|$ is not divisible by 6, then Gluck's conjecture is true for G.*

The paper is organized as follows. In Section 2, we introduce notation, terminology and background results. The primitive case of Theorem A is studied in Section 3, and the imprimitive one in Section 5, after establishing some key results about regular orbits on power sets in Section 4. Theorem B and some applications are showed in Section 5.

2 Preliminary results

Our first lemmas provide some useful characterizations of solvable S_4-free groups. Recall that a group G is said to be *p-nilpotent*, p a prime, if G has a normal Hall p'-subgroup.

Lemma 5. *Let G be a solvable group and let H be a* Hall *$\{2,3\}$-subgroup of G. Then G is S_3-free if and only if H is 3-nilpotent.*

Proof. If H is 3-nilpotent, then every $\{2,3\}$-subgroup of any section of G is 3-nilpotent. Consequently, G is S_3-free. Conversely, assume, arguing by contradiction, that G is S_3-free but H is not 3-nilpotent. Then H has a non-3-nilpotent subgroup K of minimal order. Then every proper subgroup of K is 3-nilpotent. Applying [8, Satz IV. 5.4], we have that K has a normal Sylow 3-subgroup P of exponent 3 and a Sylow 2-group Q of K is cyclic. Moreover, $\Phi(K) = \Phi(Q) \times \Phi(P)$, $P/\Phi(P) \cong P\Phi(K)/\Phi(K)$ and, by [2, Theorem VII.6.18], $Q\Phi(K)/\Phi(K)$ is a cyclic group of order 2 acting faithfully and irreducibly on $P/\Phi(P)$. It follows from [2, Theorem B.9.8] that $P/\Phi(P)$ is cyclic of order 3. Therefore $K/\Phi(K) \cong S_3$. This contradiction means that H is 3-nilpotent, as desired. □

Lemma 6. *Let G be a solvable group with $\mathrm{O}_{2'}(G) = 1$. Then G is S_3-free if and only if G is S_4-free.*

Proof. If G is S_3-free, then clearly G is S_4-free. Now assume that the converse is false and derive a contradiction. Let G be a counterexample of minimal order. Then G is S_4-free but not S_3-free.

Denote $X = \mathrm{O}_2(G)$. Then $X = \mathrm{F}(G)$ since $\mathrm{O}_{2'}(G) = 1$ and, by [2, Theorem A.10.6], $\mathrm{C}_G(X) \leqslant X$. Hence, for every subgroup S of G such that $X \leqslant S$, we have $\mathrm{O}_{2'}(S) = 1$ and so S satisfies the hypotheses of the lemma.

The minimal choice of G implies that S is S_3-free provided that S is a proper subgroup of G. In particular, by Lemma 5, G is a $\{2,3\}$-group and every proper subgroup of G/X is 3-nilpotent. If G/X were 3-nilpotent, then G would be 3-nilpotent and so S_3-free by Lemma 5. This would contradict our assumption. Consequently, G/X is a minimal non-3-nilpotent group. Denote with bars the images in $\overline{G} = G/X$. Then, by [8, Satz IV. 5.4], $\overline{G} = \overline{PQ}$ has a normal Sylow 3-subgroup $\overline{P}$ of exponent 3 and a cyclic Sylow 2-subgroup $\overline{Q}$. Moreover, since $\Phi(\overline{Q}) \leqslant \mathrm{O}_2(\overline{G}) = 1$, we have $\Phi(\overline{G}) = \Phi(\overline{Q}) \times \Phi(\overline{P}) = \Phi(\overline{P})$ and $\overline{Q}$ is of order 2. As in Lemma 5, $\overline{P}/\Phi(\overline{P})$ is of order 3. Thus $\overline{P}$ is of order 3 and $\Phi(\overline{P}) = 1$ since the exponent of $\overline{P}$ is 3. Therefore $G/X \cong S_3$.

Note that $\mathrm{O}_{2'}(G/\Phi(G)) = 1$ and so $G/\Phi(G)$ satisfies the hypotheses of the lemma. Hence, if $\Phi(G) \neq 1$, then $G/\Phi(G)$ is S_3-free and so it is 3-nilpotent by Lemma 5. Since $\Phi(G)$ is a 2-group, it follows that G is 3-nilpotent and so it is S_3-free by Lemma 5. This contradiction yields $\Phi(G) = 1$. By [2, Theorem A.10.6], $X = \mathrm{Soc}(G)$ is an abelian subgroup of G and there exists a subgroup M of G such that $G = XM$ and $X \cap M = 1$. Assume that X_1 and X_2 are two different minimal normal subgroups of G. Let $T_i/X_i = \mathrm{O}_{2'}(G/X_i)$. Since G/T_i is 3-nilpotent by the minimal choice of G, and $T_1 \cap T_2 \leqslant \mathrm{O}_{2'}(G) = 1$, it follows that G is 3-nilpotent. This contradicts our assumption. Consequently, X can be regarded as a faithful and irreducible M-module over the field of 2-elements. Recall that $M \cong G/X \cong S_3$, in this case, $|X| = 4$ and $G \cong S_4$. This final contradiction completes the proof of the lemma. □

Corollary 7. *Let G be a solvable group and let V be a faithful G-module over a field $\mathbb{F}$ of characteristic 2. Then the semidirect product VG is S_4-free if and only if G is S_3-free.*

Proof. Observe that $\mathrm{O}_{2'}(VG) \leqslant \mathrm{C}_G(V) = 1$. Thus if VG is S_4-free, then G is S_3-free by Lemma 6. Assume that G is S_3-free and there exist subgroups $A \lhd B \leqslant VG$ such that $B/A \cong S_4$. Then $VB/VA \cong B/A(B \cap V)$ has a section isomorphic to S_3 since $A(B \cap V)/A \leqslant \mathrm{O}_2(B/A)$. This means that $G \cong GV/V$ is not S_3-free. This contradiction implies that VG is S_4-free, as desired. □

The following lemma is elementary and it will be used without further reference.

Lemma 8. *Suppose that a group G acts on a non-empty set Ω. Then:*

1. *If $|\Omega| - |\bigcup_{1 \neq g \in G} \mathrm{C}_\Omega(g)| > k|G|$ for some non-negative integer k, then G has at least $k+1$ regular orbits on Ω. In particular, if $k = 0$, then G has at least one regular orbit on Ω.*

2. *If G has k regular orbits on Ω, then a subgroup H of G has at least $|G:H|k$ regular orbits on Ω.*

Let S be a permutation group on a set Ω. If K is a group, we denote by $K \wr S$ the wreath product of K with S with respect to the action of S on Ω, that is,

$$K \wr S = \{(f, \sigma) \mid f\colon \Omega \longrightarrow K, \sigma \in S\}$$

with the product $(f_1, \sigma_1)(f_2, \sigma_2) = (g, \sigma_1\sigma_2)$, where $g(w) = f_1(w)f_2(w^{\sigma_1})$ for all $w \in \Omega$.

If Y is a subgroup of K, we set $Y^\natural = \{(f, 1) \in K \wr S \mid f(w) \in Y$ for all $w \in \Omega\}$. It is clear that $Y^\natural$ is normalized by S and $Y^\natural S \cong Y \wr S$. In particular, $B = K^\natural$ is called the *base* group of $K \wr S$.

If W is a K-module, then we can consider $G \wr S$, where $G = [W]K$ is the semidirect product of W with K. In this case, $W^\natural$ is a $K \wr S$-module with the action given by $g^{(f,\sigma)}(w) = g(w^{\sigma^{-1}})^{f(w^{\sigma^{-1}})}$.

If H_1 and H_2 are permutation groups on the sets X_1 and X_2 respectively, then $H_1 \wr H_2 = \{(f, \sigma) \mid f\colon X_2 \longrightarrow H_1; \sigma \in H_2\}$ is a permutation group on $X_1 \times X_2$ with the action $(i, j)^{(f,\sigma)} = (i^{f(j)}, j^\sigma)$ (see [8, Satz I.15.3]).

We are interested here in regular orbits of a group G on completely reducible G-modules V over finite fields. Note that if $\mathbb{K}$ be a subfield of the field $\mathbb{F}$ and V is a completely reducible G-module over $\mathbb{F}$, then V is a completely reducible G-module over $\mathbb{K}$. Therefore, in looking for regular orbits of G on V, we can assume without loss of generality that $\mathbb{F}$ is a prime field.

Recall that an irreducible G-module V is called *imprimitive* if there is non-trivial decomposition of V into a direct sum of subspaces $V = V_1 \oplus \cdots \oplus V_n$ ($n > 1$) such that G permutes the set $\{V_1, \ldots V_n\}$. The irreducible G-module V is *primitive* if V is not imprimitive. A linear group $G \leqslant \mathrm{GL}(d, p^k)$, p a prime, is said to be primitive if the natural G-module is primitive.

Let G be a group and let V be a faithful G-module. Assume that $V = V_1 \oplus \cdots \oplus V_m$ ($m \geqslant 2$) is a decomposition of V into a direct sum of subspaces $\{V_1, \ldots, V_m\}$ which are permuted transitively by G. Write $L = \mathrm{N}_G(V_1)$. Then $|G : L| = m$. Let $g_1 = 1, \ldots,\ g_m$ be a right transversal of L in G. If $\Omega = \{1, \ldots, m\}$, there exists a homomorphism $\sigma\colon G \longrightarrow S_\Omega$ such that $Lg_ig = Lg_{i^{\sigma(g)}}$ for any $g \in G$. Let $K = L/\mathrm{C}_G(V_1)$ and $S = \sigma(G)$. Consider the map:

$$\begin{array}{rcl} \tau\colon G & \longrightarrow & K \wr S \\ g & \longmapsto & (h_g, \sigma_g), \end{array}$$

where $h_g \in K^\Omega$ is defined by $h_g(i) = g_igg^{-1}_{i^{\sigma(g)}}\mathrm{C}_G(V_1)$ for all $i \in \Omega$, and $\sigma_g = \sigma(g)$ for all $g \in G$. Write $\widehat{G} = K \wr S$. Then $V_1^\Omega = \{f \mid f\colon \Omega \longrightarrow V_1$ a map$\}$ is a $\widehat{G}$-module. Moreover:

Lemma 9. *1. τ is a monomorphism.*

2. The actions of G on V and $\tau(G)$ on V_1^{Ω} are equivalent.

3. $\widehat{G} = K^{\natural}\tau(G)$.

4. If $W_1 = \{f \in V_1^{\Omega} \mid f(i) = 0, \forall i \neq 1\}$, then $\mathrm{N}_{\tau(G)}(W_1)/\mathrm{C}_{\tau(G)}(W_1) \cong K$.

Proof. 1. It is straightforward to verify that τ is a homomorphism. Let $g \in G$ such that $\tau(g) = (h_g, \sigma_g) = 1$. Then $g_i = g_{i\sigma(g)}$. Since $h_g(i) = 1$ for each i, it follows that $g_i g = a(i,g)g_i$ for some $a(i,g) \in \mathrm{C}_G(V_1)$. Let $v \in V$ and assume that $v = \sum_i w_i g_i$, where $w_i \in V_1$, and $vg = \sum_i w_i(g_i g) = \sum_i w_i a(i,g) g_i = \sum_i w_i g_i = v$. This means that $g \in C_G(V) = 1$.

2. Let $v = \sum_i w_i g_i \in V$, where $w_i \in V_1$. If we set $\varphi\colon V \longrightarrow V_1^{\Omega}$, $v \longmapsto w$, where $w(i) = w_i$ for each $i \in \Omega$, it follows that φ is an isomorphism between the vector spaces V and V_1^{Ω} such that, for every $g \in G$,

$$\varphi(vg) = \varphi\left(\sum_i w_i g_i g\right) = \varphi\left(\sum_i w_i (g_i g g_{i\sigma(g)}^{-1}) g_{i\sigma(g)}\right) = w',$$

where $w'(i) = w_{i\sigma(g)^{-1}}(g_{i\sigma(g)^{-1}} g g_i^{-1})$. Bearing in mind the natural action of $\widehat{G}$ on V_1^{Ω}, we have that $\varphi(vg) = \varphi(v)\tau(g)$ for all $v \in V$ and $g \in G$.

3. Let $(f, \alpha) \in \widehat{G}$, $f \in K^{\natural}$, $\alpha \in S$. Since $S = \sigma(G)$, there exists $g \in G$ such that $\sigma_g = \alpha$. Then $(f, \alpha) = (f h_g^{-1}, 1)(h_g, \sigma_g) \in K^{\natural}\tau(G)$, as desired.

4. This follows directly from 2. □

Assume that V is a G-module as above. It is clear that if $V = V_1 \oplus \cdots \oplus V_m$ is a minimal decomposition of V into a direct sum of subspaces which are permuted transitively by G, it follows that L is a maximal subgroup of G and so S is a non-trivial primitive permutation group on Ω.

If V is a faithful imprimitive G-module, then we may assume further that V_1 is an irreducible L-module. Therefore if we are interested in regular orbits of the action of G on V, we may assume, by Lemma 9, that G is a subgroup of a wreath product $\widehat{G} = K \wr S$, where K is a group, W is a faithful K-module and S is a non-trivial primitive permutation group on a set Ω such that $\widehat{G} = K^{\natural}G$ and $V = W^{\Omega}$. In this context, a result of Wolf [15] that provides a formula to count the exact number of regular orbits $\widehat{G}$ on W^{Ω} is extremely useful.

Let $\Pi_l(\Omega, S)$ denote the set of all partitions of length l of Ω having the property that the subgroup $\{s \in S \mid \Delta_i^s = \Delta_i \text{ for all } i\}$ of S is trivial. Let k be the number of regular orbits of K on W. Then the number of regular orbits of $\widehat{G}$ on W^Ω is

$$\frac{1}{|S|} \sum_{2 \leqslant l \leqslant m} P(k,l) |\Pi_l(\Omega, S)|,$$

where $P(k,l) = k!/(k-l)!$ if $k \geqslant l$ and $P(k,l) = 0$ otherwise.

If V is the Galois field $\mathrm{GF}(q^m)$ for a prime power q, we write $\Gamma(q^m)$ for the semilinear group $\Gamma(V)$ of V (see [10, Section 2]).

Suppose that a group H acts on an abelian group A. Then H acts on the set $A^\star = \mathrm{Irr}(A)$ of all complex characters of A: for any $\chi \in A^\star$ and $h \in H$, χ^h is defined by setting $\chi^h(a) = \chi(a^{h^{-1}}), a \in A$.

The next result is proved in [10, Proposition 12.1].

Lemma 10. *Suppose that a group H acts on an abelian group A. Then*

1. $\mathrm{C}_H(A) = \mathrm{C}_H(A^\star)$.
2. *If $A = A_1 \times \cdots \times A_n$ and A_i is H-invariant, then $(A_1)^\star \times \cdots \times (A_n)^\star$ is H-isomorphic to $A^\star$.*
3. *If A is a completely reducible H-module, then $A^\star$ is a completely reducible H-module.*

The proof of Theorem B depends on the following lemma.

Lemma 11. *Assume that X is a group acting on an abelian group U and let $G = [U]X$. Then $|X : \mathrm{C}_X(\lambda)| \leqslant \mathrm{b}(G)$ for each $\lambda \in U^\star$. In particular, if X has a regular orbit on $U^\star \oplus U^\star$, then $|X| \leqslant \mathrm{b}(G)^2$.*

Proof. Suppose that $\lambda \in U^\star$. Let $\chi \in \mathrm{Irr}(G)$ such that λ is a constituent of χ_U. Then $\chi(1) \geqslant |G : \mathrm{C}_G(\lambda)|$ by [9, Theorem 19.3]. Clearly we have $U \subseteq \mathrm{C}_G(\lambda)$ and so $\mathrm{C}_G(\lambda) = U\,\mathrm{C}_X(\lambda)$. Thus $|X : \mathrm{C}_X(\lambda)| = |G : \mathrm{C}_G(\lambda)| \leqslant \chi(1) \leqslant \mathrm{b}(G)$.

Suppose that X has a regular orbit on $U^\star \oplus U^\star$. Then there exists $\lambda \in U^\star$ such that $|\mathrm{C}_X(\lambda)| \leqslant \sqrt{|X|}$. Consequently, $\sqrt{|X|} \leqslant |X : \mathrm{C}_X(\lambda)|$. Hence $|X| \leqslant \mathrm{b}(G)^2$. □

3 The primitive case

In attaining our first objective, which is to prove Theorem A for primitive modules, the following lemmas are crucial. The first one concerns primitive solvable linear groups over a field of characteristic two.

Lemma 12. *Let G be a solvable group and V be a faithful primitive G-module over a field $\mathbb{F}$ of characteristic 2. Assume that VG is S_4-free, then G has at least three regular orbits on $V \oplus V$ unless $|V| = 2^3$ and $G = \Gamma(V)$. In this case, G has exactly two regular orbits on $V \oplus V$.*

Proof. Let A be an abelian normal subgroup of G. Since V is a primitive G-module and A is normal in G, then V_A is a faithful and homogeneous A-module by Clifford's Theorem (see [10, Theorem 0.1]). By [10, Lemma 0.5], A is cyclic. Then [10, Corollary 1.10] applies. Let $F = \mathrm{F}(G)$ be the Fitting subgroup of G. Then F is of odd order since V is faithful for F, and it is a central product $F = ET$ of two normal subgroups E and T of G such that $Z = E \cap T = \mathrm{Soc}(\mathrm{Z}(F))$ and $1 \neq T = C_G(E)$ is cyclic. Hence $Z = \mathrm{Z}(E)$. Moreover, the Sylow subgroups of E are cyclic of prime order or extraspecial of prime exponent. Set $e^2 = |F/Z|$. Then 2 does not divide e.

Applying [19, Theorem 2.3], we have that G has at least four regular orbits on $V \oplus V$ unless $e = 1$, 3, 9.

Assume that $e = 1$. Then F is abelian. By [10, Corollary 2.3], G is isomorphic to a subgroup of $\Gamma(V) = \Gamma(2^n)$. If $n > 3$ and $0 \neq v \in V$, then $\mathrm{C}_G(v)$ has at least three regular orbits on V by [14, Proposition 9]. Hence G has at least three regular orbits on $V \oplus V$. If either $n = 1$ or G is of prime order, then G has at least three regular orbits on $V \oplus V$. Suppose that $1 \neq G$ is not of prime order. Then $n = 3$ since G is S_3-free and $\Gamma(2^2) \cong S_3$. In this case, $G \cong \Gamma(2^3)$ and so G has just two regular orbits on $V \oplus V$.

Suppose that either $e = 3$ or $e = 9$. Then every Hall $3'$-subgroup of E is contained in Z. Therefore $E/Z = LZ/Z$, where L is the Sylow 3-subgroup of E. Note that L is extra-especial since F is non-abelian.

Let $A = \mathrm{C}_G(T) \subseteq \mathrm{C}_G(Z)$. By [10, Corollary 1.10], E/Z is a completely reducible G/F-module and a faithful A/F-module over GF(3), the finite field of 3-elements. Hence $\mathrm{O}_3(A/F) = 1$. Let Q be a Sylow 2-subgroup of A. By Lemmas 5 and 6, every Hall $\{2, 3\}$-subgroup of G is 3-nilpotent. In particular, $QE/Z = E/Z \rtimes QZ/Z$ is nilpotent. Since $QF/F \leqslant A/F$ acts faithfully on E/Z, we have that $Q \leqslant F$. Consequently, $Q = 1$ and A is a $2'$-group. Furthermore, A preserves the non-degenerated symplectic form with respect to which E/Z is a symplectic space over GF(3) (see [8, Satz III.13.7]). Therefore A/F is either isomorphic to a completely reducible subgroup of $\mathrm{Sp}(2, 3) \cong \mathrm{SL}(2, 3)$ ($e = 3$) or a subgroup of $\mathrm{Sp}(4, 3)$ ($e = 9$). Applying [4, Lemma 3.2], we conclude that $|A/F|$ divides 3 or 5. In particular, $|A : F| \leqslant 5$.

Let W be an irreducible submodule of V_T. Then $V_T = sW$ for some positive integer s and $|G : A|$ divides $\dim W$ by [8, Hilfssatz II.3.11]. Since W is faithful for T and T is cyclic, we have that $|W| = 2^a$, where a is the smallest positive integer such that $|T| \mid 2^a - 1$ (see [10, Example 2.7]).

Applying [10, Corollary 2.6], we have that $\dim V$ is divisible by $e \cdot \dim W$. Therefore, $|V| = 2^{eab}$ for some $b > 0$.

Suppose that $a \leqslant 3$. Then $a = 2$ since $3 \mid |T|$ and T is of order 3. If $|G/A| = 2$, there exists an element $g \in G \setminus A$ of order 2 such that $G = A\langle g\rangle$ since A is $2'$-group. Then $T\langle g\rangle \cong S_3$, contrary to assumption. Hence $G = A$ is a $2'$-group. By [3, Theorem 2.2], we have G has a regular orbit on V. Hence G has at least $|V| \geqslant |W| = 4$ regular orbits on $V \oplus V$.

Assume that $a \geqslant 4$. We next prove that F has at least a regular orbit on V. It is enough to prove that

$$|V \setminus \bigcup_{S \in \mathcal{P}} \mathrm{C}_V(S)| > 0,$$

where $\mathcal{P}$ be the set of all subgroups of prime order of F.

Let $S \in \mathcal{P}$. Note that T acts fixed point freely on V so that $\mathrm{C}_V(S) = \{0\}$ if $S \leqslant T$. If S is not contained in T, then $|\mathrm{C}_V(S)| \leqslant 2^{\frac{1}{2}aeb}$ by [17, Lemma 2.4]. Note that every subgroup in $\mathcal{P}$ not contained in T has order 3 and the number of such subgroups is 12 if $e = 3$ and 120 if $e = 9$. Since $2^{3ab} - 12 \cdot 2^{\frac{3}{2}ab} > 0$ and $2^{9ab} - 120 \cdot 2^{\frac{9}{2}ab} > 0$ if $a \geqslant 4$ and $b \geqslant 1$, it follows that F has a regular orbit on V. Hence $\mathrm{C}_G(v) \cap F = 1$ for some $v \in V$.

Let $C = \mathrm{C}_G(v)$. We may assume that $C \neq 1$. Note that $|C| \leqslant |G/F| = |G : A||A : F| \leqslant 5a$. Since $|(C \cap A)| = |(C \cap A)F/F| \leqslant |A/F|$ and $|A/F|$ is of prime order, we can apply [17, Lemma 2.4] to conclude that there exists at most one subgroup S contained in $C \cap A$ such that $|C_V(S)| \leqslant 2^{\frac{3}{4}aeb}$. For a subgroup $S \subseteq C \setminus A$, we have $|C_V(S)| \leqslant 2^{\frac{1}{2}aeb}$.

Since $a \geqslant 4$, $eb \geqslant 3$, we have that $2^{aeb-1} > (5a-1)2^{\frac{1}{2}aeb}$, $2^{aeb-2} > 2^{\frac{3}{4}aeb}$ and $2^{aeb-2} > 10a$. Therefore

$$|V| - (|C| - 1)2^{\frac{1}{2}aeb} - 2^{\frac{3}{4}aeb} > 2|C|,$$

and then

$$|V \setminus \bigcup_{1 \neq g \in C} \mathrm{C}_V(g)| > 2|C|.$$

Consequently, $C = \mathrm{C}_G(v)$ has at least three regular orbits on V. This completes the proof of the lemma. □

Lemma 13. *Let G be a solvable primitive group of* $\mathrm{GL}(d,p)$*, p a prime number, and let V be the natural G-module. Assume that H is a subgroup of G such that the semidirect product VH is S_4-free. Then H has at least three regular orbits on $V \oplus V$ unless one of the following two cases occurs:*

1. *$d = 2$, $p = 3$ and $H = \mathrm{SL}(2,3)$.*

2. $d = 3$, $p = 2$ and $H = \Gamma(V) \cong \Gamma(2^3)$.

In both exceptional cases, H has just two regular orbits on $V \oplus V$.

Proof. Assume that p is odd. Then [4, Theorem 3.4] tells us that $H \leqslant G$ has at least $p \geqslant 3$ regular orbits on $V \oplus V$ unless one the following cases occurs:

1. $G = \mathrm{GL}(2,3)$. Then G has just one regular orbit on $V \oplus V$. Observe that $G/\mathrm{Z}(G) \cong \mathrm{PGL}(2,3) \cong S_4$, thus H is a proper subgroup of G since H is S_4-free. If $|G : H| \geqslant 3$, then H has at least three regular orbits on $V \oplus V$. Otherwise, $H = \mathrm{SL}(2,3)$ and the exceptional case 1 appears.

2. $G = \mathrm{SL}(2,3)$. Then G has just two regular orbits on $V \oplus V$. Hence if H is proper in G, H has at least four regular orbits on $V \oplus V$. Otherwise $H = G = \mathrm{SL}(2,3)$ and again the exceptional case 1 emerges.

3. $G = (Q_8 * Q_8)K \leqslant \mathrm{GL}(4,3)$, where K is isomorphic to a subgroup of index 1, 2 or 4 of $\mathrm{O}^+(4,2)$. If $\mathrm{O}_{2'}(H) = 1$, then H is 3-nilpotent by Lemmas 5 and 6. Using GAP, one can check that H has at least three regular orbits on $V \oplus V$.

If $\mathrm{O}_{2'}(H) \neq 1$, then $\mathrm{O}_{2'}(H)$ is isomorphic to C_3 or $C_3 \times C_3$. Then $H \leqslant \mathrm{N}_G(\mathrm{O}_{2'}(H))$. One checks by GAP that H has at least three regular orbits on $V \oplus V$.

Suppose that $p = 2$. If $H = G$, by Lemma 12, then H has at least three regular orbits on $V \oplus V$ unless $H = G = \Gamma(2^3) \leqslant \mathrm{GL}(3,2)$. In this exceptional case, H has just two regular orbits on $V \oplus V$.

Thus we can assume that H is a proper subgroup of G. By [4, Theorem 3.4], H has at least four regular orbits on $V \oplus V$ provided that G is not isomorphic to $\mathrm{GL}(2,2)$, $3^{1+2}.\mathrm{SL}(2,3)$ or $3^{1+2}.\mathrm{GL}(2,3)$.

If H is a proper subgroup of $G = \mathrm{GL}(2,2)$, then H is of prime order and there exists $v \in V$ such that $\mathrm{C}_H(v) = 1$. Hence H has at least $|V| = 4$ regular orbits on $V \oplus V$.

Suppose that G is isomorphic to $3^{1+2}.\mathrm{SL}(2,3)$ or $3^{1+2}.\mathrm{GL}(2,3)$ (as a subgroup of $\mathrm{GL}(6,2)$). By Corollary 7, H is S_3-free. In this case, one checks by GAP that H has at least three regular orbits on $V \oplus V$. □

Lemma 14. *Let G be a solvable primitive group of $\mathrm{GL}(d,p)$, p an odd prime, and let V be the natural G-module. If H is a subgroup of G of odd order, then H has at least five regular orbits on $V \oplus V$.*

Proof. If G is of odd order, then G has at least five regular orbits on $V \oplus V$ by [5, Proposition 3 (a)] and so does H. Thus we may assume that 2 divides $|G|$. Then $|G : H| \geqslant 2$. By [4, Theorem 3.4] that G has at least $p \geqslant 3$ regular orbits on $V \oplus V$, and so H has at least six regular orbits on $V \oplus V$, unless G is isomorphic to $\mathrm{GL}(2,3)$, $\mathrm{SL}(2,3)$ or $(Q_8 * Q_8)K \leqslant \mathrm{GL}(4,3)$, where K is isomorphic to a subgroup of index 1, 2 or 4 of $\mathrm{O}^+(4,2)$.

Assume that $G = \mathrm{GL}(2,3)$ or $\mathrm{SL}(2,3)$. Then G has at least one regular orbit on $V \oplus V$ and $|G : H| \geqslant 8$. It follows that H has at least eight regular orbits on $V \oplus V$.

Assume that $G = (Q_8 * Q_8)K \leqslant \mathrm{GL}(4,3)$, where K is isomorphic to a subgroup of index 1, 2 or 4 of $\mathrm{O}^+(4,2)$. Then H is isomorphic to a subgroup of $C_3 \times C_3$. Using GAP, one can check that H has a regular orbit on V and so H has at least $|V| = 3^4$ regular orbits on $V \oplus V$.

The proof of the lemma is complete. □

4 Regular orbits on the power set

The main goal of this section is to establish some results on regular orbits of permutation groups which play a crucial part in the proof of Theorem A.

Let S be a permutation group on a set Ω and consider the induced action of S on the power set $\mathcal{P}(\Omega)$ of Ω. Following [10, Chapter II, Section 5], we say that a regular orbit of S on $\mathcal{P}(\Omega)$ generated by $\Delta \subseteq \Omega$ is *strong* if the setwise stabilizer $\mathrm{Stab}_S(\Delta)$ is trivial, and $|\Delta| \neq \frac{|\Omega|}{2}$.

It is clear that a subset Δ of Ω generates a strong regular orbit of S on $\mathcal{P}(\Omega)$ if and only if so does $\Omega \setminus \Delta$. Then we conclude that the number of the strong regular orbits of S on $\mathcal{P}(\Omega)$ is even.

Gluck (see [10, Theorem 5.6]) proved that a primitive solvable permutation group S acting on a set Ω has an strong regular orbit on $\mathcal{P}(\Omega)$ if $|\Omega| > 9$. Zhang [20] proves that in this case S has at least 8 regular orbits on $\mathcal{P}(\Omega)$.

As a consequence, if S is a group of odd order, then S has at least two strong regular orbits on $\mathcal{P}(\Omega)$. We can push these ideas a bit further to show the following:

Lemma 15. *Let S be a primitive solvable permutation group of odd order on a set Ω. Then S has at least 18 strong regular orbits on $\mathcal{P}(\Omega)$, unless one of the following cases occurs:*

1. $|\Omega| = 3$ *and* $S \cong A_3$;
2. $|\Omega| = 5$ *and* $S \cong C_5$;
3. $|\Omega| = 7$ *and* $S \cong \Gamma(2^3)$.

In the exceptional cases 1 and 3, S has exactly two strong regular orbits on $\mathcal{P}(\Omega)$ and, in case 2, S has exactly 6 *strong regular orbits on $\mathcal{P}(\Omega)$.*

Proof. Assume that S is a primitive solvable permutation group of odd order on Ω such that $(S,\Omega) \neq (A_3,3), (C_5,5), (\Gamma(2^3),7)$. We shall prove that S has at least 18 strong regular orbits on $\mathcal{P}(\Omega)$.

Applying [8, Satz II.3.2], we conclude that S has a unique minimal normal subgroup, V say; $V = \mathrm{C}_S(V)$ and V is transitive and regular on Ω. Hence $|V| = |\Omega| = p^m$ for a prime p and a positive integer m. Moreover, if H is the stabilizer of an element of Ω, we have that $S = NH$ and $N \cap H = 1$. Furthermore, $|S| \leqslant \frac{1}{2}|\Omega|^{13/4}$ by [10, Corollary 3.6]. Let $n(g)$ be the number of cycles of $g \in S$ on Ω. Then $n(g) \leqslant 3|\Omega|/4$ by [10, Lemma 5.1] and g stabilizes exactly $2^{n(g)}$ subsets of Ω.

Next consider $X = \mathcal{P}(\Omega)$. We prove that

$$2^{|\Omega|} - \frac{1}{2}|\Omega|^{13/4}2^{3|\Omega|/4} \geqslant 18 \cdot \frac{1}{2}|\Omega|^{13/4} \geqslant 18|S|.$$

It is rather easy to see that the inequality holds if $|\Omega| \geqslant 81$. In this case, S has at least 18 regular orbits on X. Hence we assume in the sequel that $|\Omega| \leqslant 80$.

Suppose that $|\Omega| = p$. Then S is isomorphic to a subgroup of $[C_p]C_{p-1}$. If S is cyclic of order p, then $p \geqslant 7$ because $(S,|\Omega|) \neq (A_3,3)$ and $(C_5,5)$. In this case, every non-empty proper subset of Ω generates a strong regular orbit on $\mathcal{P}(\Omega)$. Thus S has exactly $(2^p-2)/p \geqslant 18$ strong regular orbits on $\mathcal{P}(\Omega)$. Assume that $1 \neq |H| \mid p-1$. Since $|S|$ is odd, we have $p \geq 7$. If $p=7$, then $|H| = 3$ and so $G \cong [C_7]C_3 \cong \Gamma(2^3)$, contrary to assumption. Therefore $p \geqslant 11$. Let q be a prime different from p and let T be a subgroup of S of order q. Then T is contained in some conjugate of H, and T fixes exactly $2^{1+(p-1)/q}$ subsets of Ω. Since S contains exactly p subgroups of order q, it follows that the number of non-regular orbits of S is at most $p\sum_{q|(p-1)} 2^{1+(p-1)/q}$. Then we have

$$2^p - p \sum_{3 \leqslant q|(p-1)} 2^{1+(p-1)/q} > 17p(p-1) \geqslant 17|S|.$$

Therefore S has at least 18 regular orbits on X.

Suppose that $|\Omega| = p^2$. Then $p = 5$ or 7 since $|S|$ is odd. Assume that $p = 5$. Since V is a faithful H-module, H is isomorphic to a subgroup of $\mathrm{GL}(2,5)$. Hence $|H| \leqslant 15$ and so $|S| \leqslant 5^3 \cdot 3$. In this case, $n(g) \leqslant 15$ for any $g \in S \setminus \{1\}$. Observe that

$$|X| - 2^{15} \cdot 5^3 \cdot 3 = 2^{25} - (2^{15} \cdot 5^3 \cdot 3) \geqslant 18 \cdot 5^3 \cdot 3 \geqslant 18|S|.$$

Now we assume that $p = 7$. Then $|S| \leqslant 3^2 \cdot 7^3$, $n(g) \leqslant 28$ for any $g \in S \setminus \{1\}$, and

$$|X| - 2^{28} \cdot 3^2 \cdot 7^3 = 2^{49} - (2^{28} \cdot 3^2 \cdot 7^3) \geqslant 18 \cdot 3^2 \cdot 7^3 \geqslant 18|S|.$$

In both cases, S has at least 18 regular orbits on X.

Suppose that $|\Omega| = p^3 \leqslant 80$. Then $p = 3$ and H is isomorphic to an irreducible subgroup of $\mathrm{GL}(3,3)$. By [10, Corollary 2.13], H can be considered as a subgroup of $\Gamma(3^3)$ or $C_2 \wr S_3$. Since H is of order odd and irreducible, the latter case is impossible. Thus H is a subgroup of $\Gamma(3^3)$ and $|H| \leqslant 3 \cdot 13$. Then $|S| \leqslant 3^4 \cdot 13$. Let $g \in S \setminus \{1\}$. Assume that g has not fixed points on Ω. Then g is either a product of a 13-cycle and some 3-cycles or a product of 3-cycles. Hence $n(g) \leqslant 27/3 = 9$. Suppose that g has at least one fixed point. Then g belongs to a conjugate of H. Since the action of H on Ω is equivalent to the action of H on V by conjugation, we have that the number of fixed points of g is just $|\mathrm{C}_V(g)|$. If order of g is 3, then $|\mathrm{C}_V(g)|$ and $n(g) \leqslant (27-3)/3+3 = 11$. If order of g is 13, then $n(g) \leqslant (27-1)/13+1 = 3$. Consequently, $n(g) \leqslant 11$ for any $g \in S \setminus \{1\}$. Note that

$$|X| - 2^{11} \cdot 3^4 \cdot 13 = 2^{27} - (2^{11} \cdot 3^4 \cdot 13) \geqslant 18 \cdot 3^4 \cdot 13 \geqslant 18|S|.$$

Hence S has at least 18 regular orbits on X.

If $|\Omega| = 3$ and $S \cong A_3$, then S has exactly two regular orbits on X. If $|\Omega| = 7$ and $S \cong \Gamma(2^3)$, each element of order 7 in S is a 7-cycle and each element of order 3 in S is the product of two disjoint 3-cycles. Thus every two-element subset and every five-element subset of Ω generate a strong regular orbit on X and S has exactly two strong regular orbits on X. If $|\Omega| = 5$ and $S \cong C_5$, then S has exactly $(2^5 - 2)/5 = 6$ strong regular orbits on X. This completes the proof of the lemma. □

Lemma 16. *Let S be a primitive solvable permutation group on a set Ω. Assume that $S^* \leqslant S$ and S^* acts non-transitively on Ω. Then one of the following occurs:*

1. *S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$; or*
2. *for each S^*-orbit Δ on Ω with $|\Delta| > 4$, we have $\mathrm{O}^{2'}(S^*)$ acts transitively on Δ and $|\Pi_3(\Delta, S^*)| \geqslant |S^*_\Delta|$, where S^*_Δ is the permutation group induced by the action of S^* on Δ.*

Proof. It is clear that we may assume that $|\Omega| \geqslant 5$ and $1 \neq S^*$ is a proper subgroup of S.

Since S is a primitive solvable permutation group on Ω, we can apply [8, Satz II.3.2] to conclude that S has a unique minimal normal subgroup, V

say; $V = C_S(V)$ and V is transitive and regular on Ω. Moreover, if H is the stabilizer of an element $\alpha \in \Omega$, we have that $S = VH$ and $V \cap H = 1$. Furthermore, the action of H on Ω is equivalent to the action of H on M by conjugation. In particular, if $\beta \in \Omega$, we have that $\mathrm{C}_H(\beta) := \mathrm{Stab}_H \beta = \mathrm{C}_H(v)$ for some $v \in V$.

Assume that $|V| = |\Omega|$ is a prime number, p say. Then V is a Sylow p-subgroup of S and so S^* is a p'-group. Without loss of generality, we may assume that S^* is contained in H. Let $\beta \in \Omega \setminus \{\alpha\}$. Then $\mathrm{C}_H(\beta) = \mathrm{C}_H(v)$ for some $1 \neq v \in V$. Therefore, $\mathrm{Stab}_H \beta = 1$. Then if $\Delta_1 = \{\beta\}$ and $\Delta_2 = \{\alpha, \beta\}$, it follows that $\mathrm{Stab}_{S^*} \Delta_i = 1$, $i = 1, 2$. Then Δ_1, Δ_2, $\Omega \setminus \Delta_1$ and $\Omega \setminus \Delta_2$ are in different regular orbits of S^* on $\mathcal{P}(\Omega)$. Thus S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$.

Consequently, we may suppose that $|\Omega|$ is not a prime. If S has a strong regular orbit on $\mathcal{P}(\Omega)$, then S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$ since $|S : S^*| \geqslant 2$. Then we may assume that S has no strong regular orbit on $\mathcal{P}(\Omega)$.

Therefore we only have to consider the exceptional cases (5) and (6) of [10, Theorem 5.6].

1. Suppose that $(S, |\Omega|) = (A\Gamma(2^3), 8)$.

   Since S^* is not transitive on Ω, the length of every orbit of S^* on Ω is at most 7.

   Assume that S^* has an orbit Δ on Ω such that $|\Delta| = 7$. Without loss of generality, we may suppose that α is fixed by all elements of S^* and so S^* is contained in H. By Lemma 15, $H \cong \Gamma(2^3)$ has a strong regular orbit on $\mathcal{P}(\Delta)$. Let Δ_1 is a two-element subset of Δ. Then $\mathrm{Stab}_{S^*}(\Delta_1) \leqslant \mathrm{Stab}_H(\Delta_1) = 1$. Denote $\Delta_2 = \{\alpha\} \cup \Delta_1$. Since $\mathrm{Stab}_{S^*}(\Delta_i){=}1$ for $i = 1$, 2, it follows that Δ_1, Δ_2, $\Omega \setminus \Delta_1$ and $\Omega \setminus \Delta_2$ lie in different regular orbits of S^* on $\mathcal{P}(\Omega)$. Thus S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$.

   Assume that S^* has an orbit Δ on Ω such that $|\Delta| = 6$. Then there exists $\beta \in \Delta$ with $|S^* : \mathrm{C}_{S^*}(\beta)| = 6$. Hence $|\mathrm{C}_{S^*}(\beta)|$ divides $2^2 \cdot 7$. On the other hand, $\mathrm{C}_{S^*}(\beta) \leqslant \mathrm{C}_S(\beta) \cong \Gamma(2^3)$. Thus $|\mathrm{C}_{S^*}(\beta)|$ divides 7. If $|\mathrm{C}_{S^*}(\beta)| = 7$, then $|S^*| = 2 \cdot 3 \cdot 7$. This is a contradiction since S has no subgroup of such order. Thus $\mathrm{C}_{S^*}(\beta) = 1$. Therefore if $\Delta_1 = \{\beta\}$ and $\Delta_2 = \{\gamma, \beta\}$ for some $\gamma \in \Omega \setminus \Delta$, it follows that $\mathrm{Stab}_{S^*} \Delta_i = 1$, $i = 1, 2$. Then Δ_1, Δ_2, $\Omega \setminus \Delta_1$ and $\Omega \setminus \Delta_2$ are in different regular orbits of S^* on $\mathcal{P}(\Omega)$. Thus S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$.

2. Suppose that $|\Omega| = 9$ and S is the semidirect product of $C_3 \times C_3$ with D_8, SD_{16}, $\mathrm{SL}(2, 3)$ or $\mathrm{GL}(2, 3)$.

In this case, we may assume that $V = C_3 \times C_3$ and S is a subgroup of $\mathrm{AGL}(2,3)$, the semidirect product of $C_3 \times C_3$ with $\mathrm{GL}(2,3)$. In particular, H is a subgroup of $A = \mathrm{GL}(2,3)$.

Since S^* is a $\{2,3\}$-group acting non-transitively on Ω and $|\Omega| = 9$, we have that the length of an orbit of S^* on Ω with more than 4 elements is either 6 or 8.

Suppose that S^* has an orbit Δ on Ω such that $|\Delta| = 8$. Without loss of generality, we may suppose that α is fixed by all elements of S^* and so S^* is contained in H. If $\beta \in \Delta$, we have that $\mathrm{C}_A(\beta)$ has two fixed points, β, γ say, and a orbit Γ of length 6 on Δ. Let $\mu \in \Gamma$ and let $\Delta_1 = \{\beta\}, \Delta_2 = \{\gamma, \mu\}$ and $\Delta_3 = \Gamma \setminus \{\mu\}$. Observe that $\bigcap_i \mathrm{Stab}_{S^*}(\Delta_i) \leqslant \bigcap_i \mathrm{Stab}_H(\Delta_i) = 1$. Thus $|\Pi_3(\Delta, S^*)| \geqslant |S^*_\Delta|$. Since $|S^* : \mathrm{C}_{S^*}(\beta)| = 8$, we have $\mathrm{O}^{2'}(S^*)$ acts transitively on Δ. In this case, 2 holds.

Suppose that S^* has an orbit Δ on Ω such that $|\Delta| = 6$. Put $\Gamma = \Omega \setminus \Delta$. Then S^* acts on Γ and $S^*/\mathrm{C}_{S^*}(\Gamma)$ is isomorphic to a subgroup of S_3. Note that $\mathrm{C}_{S^*}(\Gamma)$ is also isomorphic to a subgroup of S_3. Thus $|S^*|$ divides 36. Since $|\Delta| = 6$ divides $|S^*|$, we have that $|S^*| \in \{6, 12, 18, 36\}$.

If $|S^*| = 6$ then S^* has a strong regular orbit on Δ, and so S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$. If $|S^*| = 18$, one can check by GAP that S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$. If $|S^*| = 12$ or 36, one can check by GAP that S^* satisfies statement 2. □

Lemma 17. *Let S be a primitive solvable permutation group on a set Ω. Assume that $S^* \leqslant S$, S^* is transitive on Ω and S^* is S_4-free. Then either S^* has a strong regular orbit on $\mathcal{P}(\Omega)$ or S^* satisfies one of the following statements:*

1. $|\Omega| = 2$ *and* $S^* \cong S_2$;
2. $\mathrm{O}^{2'}(S^*)$ *acts transitively on the set* Ω *and there exists a* 3*-partition* $\{\Delta_1, \Delta_2, \Delta_3\}$ *of* Ω *such that* $\bigcap_i \mathrm{Stab}_{S^*} \Delta_i = 1$.

Proof. We may assume that $|\Omega| > 2$. If S has a strong regular orbit on $\mathcal{P}(\Omega)$, then so does S^*. Thus we may assume that $(S, |\Omega|)$ is one of the exceptional cases of [10, Theorem 5.6].

If $|\Omega| = 3$ and $S = S_3$, then either $S^* \cong C_3$ or $S^* \cong S_3$. If $S^* \cong C_3$, then S^* has a strong regular orbit on $\mathcal{P}(\Omega)$. If $S^* \cong S_3$, then S^* satisfies statement 2.

Assume that $|\Omega| = 4$ and $S = A_4$ or S_4. Since S^* is an S_4-free transitive subgroup of S, it follows that $S^* \cong A_4$, D_8 or $C_2 \times C_2$. Then $\mathrm{O}^{2'}(S^*)$ acts transitively on Ω and there exists a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of type $(1, 1, 2)$ of Ω such that $\bigcap_i \mathrm{Stab}_{S^*}(\Delta_i) = 1$. Thus statement 2 holds.

Assume that $|\Omega| \in \{5, 7, 8, 9\}$. In this case, by [15, Theorem 3.1], there exists a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of Ω such that $\bigcap_i \mathrm{Stab}_{S^*} \Delta_i = 1$.

Assume that $|\Omega| = 5$ and $S = F_{10}$ or F_{20}. Then $S^* \cong C_5$, F_{10} or F_{20}. If $S^* \cong C_5$, then S^* has a strong regular orbit on $\mathcal{P}(\Omega)$. If $S^* \cong F_{10}, F_{20}$, then S^* satisfies statement 2.

Assume that $|\Omega| = 7$ and $S = F_{42}$. Then $S^* \cong C_7, F_{21}$ or F_{42}. If $S^* \cong C_7$ or F_{21}, then S^* has a strong regular orbit on $\mathcal{P}(\Omega)$. If $S^* \cong F_{42}$, then S^* satisfies statement 2.

If $|\Omega| = 8$ and $S = A\Gamma(2^3)$, then one can check by GAP that $\mathrm{O}^{2'}(S^*)$ acts transitively on Ω. Therefore S^* satisfies statement 2.

Assume that $|\Omega| = 9$ and $S = \mathrm{AGL}(2, 3)$. If $\mathrm{O}^{2'}(S^*)$ is not transitive on Ω, then one can check by GAP that S^* has a strong regular orbit on $\mathcal{P}(\Omega)$. $\square$

Corollary 18. *Let S be a primitive solvable permutation group on a set Ω. Assume that $S^* \leqslant S$ is of odd order and S^* is transitive on Ω. Then S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$, unless one of the following cases occurs:*

1. *$|\Omega| = 3$ and $S^* \cong A_3$;*
2. *$|\Omega| = 7$ and $S^* \cong \Gamma(2^3)$.*

In the exceptional cases, S^* has just two strong regular orbits on $\mathcal{P}(\Omega)$.

Proof. Assume that S has a strong regular orbit on $\mathcal{P}(\Omega)$. If S is of odd order, then by Lemma 15, then S has at least four strong regular orbits on $\mathcal{P}(\Omega)$ unless $(S, |\Omega|) = (A_3, 3)$ or $(\Gamma(2^3), 7)$. Then S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$ unless $(S^*, |\Omega|) = (A_3, 3)$ or $(\Gamma(2^3), 7)$. If S is of order even, then $|S : S^*| \geqslant 2$. Since S has at least two strong regular orbits on $\mathcal{P}(\Omega)$, S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$.

If S has no strong regular orbit on $\mathcal{P}(\Omega)$, then $(S, |\Omega|)$ is one of exceptional cases (2)–(9) of [10, Theorem 5.6].

If $|\Omega| = 3$ and $S = S_3$, then $S^* \cong A_3$. We are in case (1). If $|\Omega| = 4$ and $S = A_4$ or S_4, then S has no odd order subgroups which are transitive on Ω. If $|\Omega| = 5$ and $S = F_{10}$ or F_{20}, then $S^* \cong C_5$ and S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$.

Assume that $|\Omega| = 7$ and $S = F_{42}$. Then $S^* \cong C_7$ or $\Gamma(2^3)$. If $S^* \cong C_7$, then S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$. If $S^* \cong \Gamma(2^3)$, we are in case (2). If $|\Omega| = 8$ and $S = A\Gamma(2^3)$, then S has no subgroup of odd order which is transitive on Ω.

Assume that $|\Omega| = 9$ and $S = \mathrm{AGL}(2,3)$. Then S^* is a subgroup of a Sylow 3-subgroup of S. It can be proved, using GAP, that S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$. $\square$

Lemma 19. *Let H_1 and H_2 be permutation groups on the sets X_1 and X_2 respectively. If H_1 has $2s$ strong regular orbits on $\mathcal{P}(X_1)$ and H_2 has $2t$ strong regular orbits on $\mathcal{P}(X_2)$. Then $H = H_1 \wr H_2$ has at least $2st$ strong regular orbits on $\mathcal{P}(X_1 \times X_2)$. If $s = 1$, then $H_1 \wr H_2$ has exactly $2t$ strong regular orbits on $\mathcal{P}(X_1 \times X_2)$.*

Proof. Assume that $\Delta_1, \ldots, \Delta_s$, $X_1 \setminus \Delta_1, \ldots, X_1 \setminus \Delta_s$ belong to different strong regular orbits of H_1 on $\mathcal{P}(X_1)$ and that $\Gamma_1, \ldots, \Gamma_t$, $X_2 \setminus \Gamma_1, \ldots, X_2 \setminus \Gamma_t$ belong to different strong regular orbits of H_2 on $\mathcal{P}(X_2)$. Let us denote

$$\Sigma_{ij} = \Delta_i \times \Gamma_j \cup (X_1 \setminus \Delta_i) \times (X_2 \setminus \Gamma_j),$$

for $1 \leqslant i \leqslant s, 1 \leqslant j \leqslant t$.

We prove first that $\mathrm{Stab}_H(\Sigma_{ij}) = 1$. Let $y \in X_2$, we denote $\varepsilon(y) = |\{(x_1, x_2) \in \Sigma_{ij} \mid x_2 = y\}|$. Since $|\Delta_i| \neq |X_1 \setminus \Delta_i|$, it is clear that $\varepsilon(y) = |\Delta_i|$ (respectively, $|X_1 \setminus \Delta_i|$) if and only if $y \in \Gamma_j$ (respectively, $y \in X_2 \setminus \Gamma_j$).

Let $(f, \sigma) \in \mathrm{Stab}_H(\Sigma_{ij})$ and $y \in \Gamma_j$. Then $(\Delta_i \times \{y\})^{(f,\sigma)} = \Delta_i^{f(y)} \times \{y^\sigma\} \subseteq \Sigma_{ij}$. Observe that $\varepsilon(y^\sigma) = |\Delta_i^{f(y)}| = |\Delta_i|$, which implies that $y^\sigma \in \Gamma_j$. Thus $\sigma \in \mathrm{Stab}_{H_2}(\Gamma_j) = 1$. We also have $\Delta_i^{f(y)} = \Delta_i$ and so $f(y) \in \mathrm{Stab}_{H_1}(\Delta_i) = 1$. Now we can argue similarly with $y \in X_2 \setminus \Gamma_j$ and conclude that $f = 1$. Thus $\mathrm{Stab}_H(\Sigma_{ij}) = 1$.

Observe that $|\Sigma_{ij}| \neq \frac{|X_1||X_2|}{2}$ and so Σ_{ij} generates a strong regular orbit of H on $\mathcal{P}(X_1 \times X_2)$.

Assume that there exists $(f, \sigma) \in H$ such that $\Sigma_{ij}^{(f,\sigma)} = \Sigma_{uv}$ for some indices $1 \leqslant i, u \leqslant s, 1 \leqslant j, v \leqslant t$. If $y \in X_2$, then $(\Delta_i \times \{y\})^{(f,\sigma)} = \Delta_i^{f(y)} \times y^\sigma \in \Sigma_{uv}$ and $\Delta_i^{f(y)} = \Delta_u$ or $X_1 \setminus \Delta_u$. This implies that $i = u$. Analogously, $j = v$. By using a similar argument, we can prove Σ_{ij} is not H-conjugate to $X_1 \times X_2 \setminus \Sigma_{uv}$. Thus $\Sigma_{ij}, X_1 \times X_2 \setminus \Sigma_{ij}$ belong to different strong regular orbits of H on $\mathcal{P}(X_1 \times X_2)$. Then we conclude that H has at least $2st$ strong regular orbits on $\mathcal{P}(X_1 \times X_2)$.

Assume that $s = 1$. We prove that the orbits generated by $\Sigma_{1j}, X_1 \times X_2 \setminus \Sigma_{1j}$ are exactly the strong regular orbits of H on $\mathcal{P}(X_1 \times X_2)$.

Let $\Phi \in \mathcal{P}(X_1 \times X_2)$ such that $\mathrm{Stab}_H(\Phi) = 1$. Then $\Phi = \bigcup_{y \in X_2} \Phi_y \times \{y\}$, where $\Phi_y = \{x \in X_1 \mid (x, y) \in \Phi\}$. Assume there exists $y_0 \in X_2$ such that $\mathrm{Stab}_{H_1}(\Phi_{y_0}) \neq 1$. Take $1 \neq u \in \mathrm{Stab}_{H_1}(\Phi_{y_0})$ and let $f \in H_1^{X_2}$ such that $f(y) = u$ if $y = y_0$ and $f(y) = 1$ otherwise. Then it follows that $1 \neq (f, 1) \in \mathrm{Stab}_H(\Phi) = 1$. This contradiction yields $\mathrm{Stab}_{H_1}(\Phi_y) = 1$ for each $y \in X_2$.

Since all H_1-regular orbits are generated by Δ_1 and $X_1 \setminus \Delta$, it follows that Φ_y is H_1-conjugate to Δ_1 or $X_1 \setminus \Delta_1$ for each $y \in X_2$. Let $B_1 = \{y \in X_2 \mid \Phi_y$ is H_1-conjugate to $\Delta_1\}$ and $B_2 = \{y \in X_2 \mid \Phi_y$ is H_1-conjugate to $X_1 \setminus \Delta_1\}$. Observe that $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = X_2$.

For each $y \in X_2$, there exists $u_y \in H_1$ such that $\Phi_y^{u_y} = \Delta_1$ (if $y \in B_1$) or $= X_1 \setminus \Delta_1$ (if $y \in B_2$). Let $g \in H_1^{X_2}$ such that $g(y) = u_y$ for each $y \in X_2$. Write $\widetilde{\Phi} = \Phi^{(g,1)} = (\bigcup_{y\in B_1} \Phi_y^{g(y)} \times \{y\}) \cup (\bigcup_{y\in B_2} \Phi_y^{g(y)} \times \{y\}) = (\bigcup_{y\in B_1} \Delta_1 \times \{y\}) \cup (\bigcup_{y\in B_2} (X_1 \setminus \Delta_1) \times \{y\}) = (\Delta_1 \times B_1) \cup ((X_1 \setminus \Delta_1) \times B_2)$.

Assume that $\mathrm{Stab}_{H_2}(B_1) \neq 1$, and let $1 \neq \sigma \in \mathrm{Stab}_{H_2}(B_1)$. Since $B_2 = X_2 \setminus B_1$, we have $\sigma \in \mathrm{Stab}_{H_2}(B_2)$. Thus $1 \neq (1,\sigma) \in \mathrm{Stab}_H(\widetilde{\Phi}) = 1$, which is a contradiction. Therefore B_1 generates a regular orbit of H_2 on X_2. Without loss of generality, we may assume that $B_1^\alpha = \Gamma_j$ for some $\alpha \in H_2$. Then $B_2^\alpha = (X_2 \setminus B_1)^\alpha = X_2 \setminus \Gamma_j$. So we have $\widetilde{\Phi}^{(1,\alpha)} = (\Delta_1 \times \Gamma_j) \cup ((X_1 \setminus \Delta_1) \times (X_2 \setminus \Gamma_j)) = \Sigma_{1j}$. Thus Φ is H-conjugate to Σ_{1j}, as desired. □

Remark 20. If $s \neq 1$, $H = H_1 \wr H_2$ has not exactly $2st$ strong regular orbits on the power set of $X_1 \times X_2$ in general. Let $(H_1 = \langle(1,2,3,4,5)\rangle$, $X_1 = \{1,2,3,4,5\})$ and $(H_2 = \langle(1,2,3)\rangle$, $X_2 = \{1,2,3\})$.

Note that the regular orbits generated by $\Delta_1 = \{1\}$, $\Delta_2 = \{1,2\}$, $\Delta_3 = \{1,3\}$, $X_1 \setminus \Delta_1$, $X_1 \setminus \Delta_2$, $X_1 \setminus \Delta_3$ are exactly the strong regular orbits of H_1 on $\mathcal{P}(X_1)$. It is also clear that H_2 has exactly two strong regular orbits on $\mathcal{P}(X_2)$, namely the ones generated by $\Gamma_1 = \{1\}$ and $X_2 \setminus \Gamma_1$.

According to Lemma 19, we have that the subsets $\Sigma_{i1} = \Delta_i \times \Gamma_j \cup (X_1 \setminus \Delta_i) \times (X_2 \setminus \Gamma_1)$, for $1 \leqslant i \leqslant 3$, generate 6 strong regular orbits of H on $\mathcal{P}(X_1 \times X_2)$. The subset

$$\Phi = \Delta_1 \times \{1\} \cup \Delta_2 \times \{2\} \cup \Delta_3 \times \{3\}$$

also generates a strong regular orbit on $\mathcal{P}(X_1 \times X_2)$ and Φ does not belong to the orbits generated by Σ_{i1}, $1 \leqslant i \leqslant 3$.

Definition 21. Let $\mathcal{K}$ denote the class of all pairs $(S, \mathrm{d}(S))$ satisfying the following conditions:

1. S is a permutation group of degree $\mathrm{d}(S)$, and

2. $S \cong H_1 \wr \cdots \wr H_n$, where H_i is either $H_i \cong A_3$ (of degree $\mathrm{d}(H_i) = |X_i| = 3$) or $H_i \cong \Gamma(2^3)$ (of degree $\mathrm{d}(H_i) = |X_i| = 7$) for each i, and $n \geqslant 1$.

Applying Lemmas 15 and 19, we have:

Corollary 22. *If S is a permutation group on Ω such that $(S, |\Omega|) \in \mathcal{K}$, then S has exactly two regular orbits on $\mathcal{P}(\Omega)$.*

5 The imprimitive case

Lemma 23. *Let K be a group and let W a faithful K-module over a field of prime characteristic, p say. Let S be a primitive solvable permutation group on an m-element set Ω, and assume that $S^* \leqslant S$ is transitive on Ω. Let $\widehat{G} = K \wr S^*$ and $V = W^{\Omega}$. Let G be a subgroup of $\widehat{G}$ such that $\widehat{G} = K^{\natural}G$ and VG is S_4-free. Then:*

1. *If K has at least five regular orbits on $W \oplus W$, then G has at least five regular orbits on $V \oplus V$.*
2. *If K is of even order, K has at least three regular orbits on $W \oplus W$ and $p \neq 2$, then G has at least three regular orbits on $V \oplus V$.*
3. *If K has at least three regular orbits on $W \oplus W$ and $p = 2$, then G has at least three regular orbits on $V \oplus V$.*

Proof. 1. It follows from [16, Proposition 3.2(3)] since G is a subgroup of $K \wr S$.

2. By [16, Proposition 3.2(2)], we may assume that $m \leqslant 4$. If S has a regular orbit on the power set of Ω, then $|\Pi_2(\Omega, S)| \geqslant |S|/2$. Thus, in this case, $K \wr S$ has at least three regular orbits on $V \oplus V$ by Wolf's formula and so does G. Therefore we may assume that S has not any regular orbit on $\mathcal{P}(\Omega)$ and so S is one of the first two exceptional cases of [10, Theorem 5.6]. Note that $S^* \cong \widehat{G}/K^{\natural}$ is isomorphic to a quotient of G. Hence S^* is S_4-free.

   Assume that $|\Omega| = 4$ and $S \cong A_4$ or S_4. Since S^* is a transitive on Ω, it follows that S^* is either isomorphic to a subgroup of A_4 or D_8. It suffices to consider that $S^* \cong A_4$ or D_8.

   If $S^* \cong A_4$, we have $|\Pi_3(\Omega, S^*)| = 6$. Thus $\widehat{G}$ (and so G) has at least three regular orbits on $V \oplus V$.

   If $S^* \cong D_8$, we have $|\Pi_3(\Omega, S^*)| = 4$. Thus $\widehat{G}$ (and so G) has at least three regular orbits on $V \oplus V$.

   Assume that $|\Omega| = 3$ and $S \cong S_3$. Since S^* is transitive on Ω, it follows that $S \cong C_3$ or S_3. If $S^* \cong C_3$, we have $|\Pi_2(\Omega, S^*)| = 3$ and so H has at least three regular orbits on $V \oplus V$.

   Assume that $S^* = S \cong S_3$. In this case, we have that $|\Pi_2(\Omega, S^*)| = 0$ and $|\Pi_3(\Omega, S^*)| = 1$. Thus $\widehat{G}$ has at least one regular orbit on $V \oplus V$.

   Since K is of even order, $\widehat{G}$ has a subgroup isomorphic to $C_2 \wr S_3$ and so $\widehat{G}$ is not S_4-free. Since G is S_4-free, we have that G is a proper subgroup of $\widehat{G}$. Suppose that $|\widehat{G} : G| = 2$. Then $G \lhd \widehat{G}$ and $B = K^\natural$ is not contained in G. Let $N = B \cap G$. Then N is normal in $\widehat{G}$ and $|B : N| = 2$. In particular, there exists a direct factor K_1 of B which is not contained in N. Then $B = K_1 N$ and $|K_1 : K_1 \cap N| = 2$. Note that $C = (K_1 \cap N)^\natural$ is a normal subgroup of $\widehat{G}$ contained in B such that $\widehat{G}/C \cong C_2 \wr S_3$. Thus there exists a normal subgroup L of $\widehat{G}$ contained in B such that $\widehat{G}/L \cong S_4$. Therefore $\widehat{G} = LG$ and $G/G \cap L \cong \widehat{G}/L \cong S_4$, contrary to assumption. Consequently, $|\widehat{G} : G| \geqslant 3$ and so G has at least three regular orbits on $V \oplus V$.

3. If $p = 2$, we have that G is S_3-free by Corollary 7. Arguing as in case 2, we conclude that G has at least three regular orbits on $V \oplus V$. □

Definition 24. Let G be a group and let V a G-module such that the action of G on V is equivalent to the action of a subgroup X of $U \wr S = U^\natural X$ on W^Ω, where U is a group, W is a U-module and S is a permutation group on a set Ω such that $(S, |\Omega|) \in \mathcal{K}$ (see Definition 21) or $(S, |\Omega|) = (1, 1)$.

1. We say that V of type (**I**) if $|W| = 2^3$ and $U = \Gamma(W)$.

2. V is said to be of type (**II**) if $|W| = 3^2$ and $U = \mathrm{SL}(2, 3)$.

Lemma 25. *Suppose that V is a G-module of type (**I**) or type (**II**) (see Definition 24). There exist $0 \neq x \in V$ and y_1, y_2, z_1, $z_2 \in V$ lying in different $\mathrm{C}_G(x)$-orbits satisfying the following conditions:*

1. $\mathrm{C}_G(x) \cap \mathrm{C}_G(y_i) = 1$ for each i; and

2. $\mathrm{C}_G(x) \cap \mathrm{C}_G(z_i)$ is a 3-group for each i.

Moreover, G has exactly two regular orbits on $V \oplus V$.

Proof. Without loss of generality, we may suppose that $G = U \wr S$ and $V = W^\Omega$, U is a group, W is a U-module and S is a permutation group on a set Ω such that $(S, |\Omega|) = (1, 1)$ or $(S, |\Omega|) \in \mathcal{K}$, and either $|W| = 2^3$ and $U = \Gamma(W)$ or $|W| = 3^2$ and $U = \mathrm{SL}(2, 3)$. Let $0 \neq w \in W$. Then $\mathrm{C}_U(w)$ is a 3-group and has exactly two regular orbits on W. Then we assume that u_1, u_2 belong to different regular orbits of $\mathrm{C}_U(w)$ on W. In particular, $\mathrm{C}_U(w) \cap \mathrm{C}_U(u_i) = 1$ for each i. Write $v_1 = 0, v_2 = w$. Then $\mathrm{C}_U(w) \cap \mathrm{C}_U(v_i) = \mathrm{C}_U(w)$ is a 3-group. Observe that u_1, u_2, v_1, v_2 belong to four different $\mathrm{C}_U(w)$-orbits. Thus the lemma holds when $(S, |\Omega|) = (1, 1)$.

Now we may assume that $(S, |\Omega|) \in \mathcal{K}$. Applying Corollary 22, we get that S has exactly two strong regular orbits on $\mathcal{P}(\Omega)$. Hence, by Wolf's formula, G has exactly two regular orbits on $V \oplus V$. Let $\Delta \subseteq \Omega$ such that $\mathrm{Stab}_S(\Delta) = 1$ and $x \in V = W^{\Omega}$ such that $x(i) = w$ for all $i \in \Omega$. Assume that y_1, y_2, z_1, $z_2 \in V$ satisfy

$$\begin{aligned} y_1(i) &= u_1, \quad i \in \Delta; & y_1(i) &= u_2, \quad i \in \Omega \setminus \Delta; \\ y_2(i) &= u_2, \quad i \in \Delta; & y_2(i) &= u_1, \quad i \in \Omega \setminus \Delta; \\ z_1(i) &= v_1, \quad i \in \Delta; & z_1(i) &= v_2, \quad i \in \Omega \setminus \Delta; \\ z_2(i) &= v_2, \quad i \in \Delta; & z_2(i) &= v_1, \quad i \in \Omega \setminus \Delta. \end{aligned}$$

It is not difficult to see that y_1, y_2, z_1, z_2 belong to different regular orbits of $\mathrm{C}_G(x)$ on V. We first show that $\mathrm{C}_G(x) \cap \mathrm{C}_G(y_j) = 1$ for each j. Let $(f, \sigma) \in \mathrm{C}_G(x) \cap C_G(y_j)$, where $f \in U^{\Omega}$ and $\sigma \in S$. Then

$$x(i^{\sigma^{-1}})^{f(i^{\sigma^{-1}})} = x(i); y_j(i^{\sigma^{-1}})^{f(i^{\sigma^{-1}})} = y_j(i), \forall i \in \Omega.$$

Hence $f(i) \in \mathrm{C}_U(w)$ for each i. Since u_1, u_2 lie in different orbits of $\mathrm{C}_U(w)$ on W, we have $\Delta^{\sigma} = \Delta$ and thus $\sigma \in \mathrm{Stab}_S(\Delta) = 1$. Then $u_1^{f(i)} = u_1$ or $u_2^{f(i)} = u_2$ for each i and so $f(i) \in \mathrm{C}_U(w) \cap \mathrm{C}_U(u_1) = 1$ or $f(i) \in \mathrm{C}_U(w) \cap \mathrm{C}_U(u_2) = 1$. In any case, $f = 1$, as desired.

Now take $(f, \sigma) \in \mathrm{C}_G(x) \cap C_G(z_j)$ for each j. Arguing in a similar way, we have $f(i) \in \mathrm{C}_U(w)$ for each i and $\sigma = 1$. Then $v_1^{f(i)} = y$ or $v_2^{f(i)} = z$ for each i and so $f(i) \in \mathrm{C}_U(w) \cap \mathrm{C}_U(v_1)$ or $f(i) \in \mathrm{C}_U(w) \cap \mathrm{C}_U(v_2)$. Note that $\mathrm{C}_U(w) \cap \mathrm{C}_U(v_1)$ is a 3-group. Then $(f, \sigma) = (f, 1)$ is a 3-element and thus $\mathrm{C}_G(x) \cap C_G(z_j)$ is a 3-group for each j, as desired. $\square$

Let G be a group and let V a faithful G-module. Assume that there $V = V_1 \oplus \cdots \oplus V_m$ ($m \geqslant 2$) is a direct sum of subspaces which are permuted transitively by G. Write $\Omega = \{1, \ldots, m\}$, $L = \mathrm{N}_G(V_1)$ and $N = \mathrm{Core}_G(L)$. Then $m = |G : L|$ and $S = G/N$ is a permutation group on Ω induced by the action of G on a right transversal of L in G. We have the following:

Lemma 26. *Assume that G is solvable and VG is S_4-free. Assume further that V_1, as a $L/\mathrm{C}_G(V_1)$-module, is of type (**I**) or type (**II**) (see Definition 24).*

1. *Suppose that $\mathrm{O}^{2'}(S)$ acts transitively on Ω and there exists a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of Ω such that $\bigcap_i \mathrm{Stab}_S \Delta_i = 1$. Then G has at least three regular orbits on $V \oplus V$.*

2. *If $m \leqslant 4$, then G has at least three regular orbits on $V \oplus V$ unless $m = 3$ and $G/N \cong C_3$; in this case, G has at least two regular orbits on $V \oplus V$.*

Proof. Applying Lemma 9, we may assume without loss of generality G is a subgroup of $\widehat{G} = U \wr S$, where $U = L/\mathrm{C}_G(V_1)$. Moreover, we have that $\widehat{G} = U^\natural G$, $N = G \cap U^\natural$ and $\mathrm{N}_G(W_j)/\mathrm{C}_G(W_j) \cong U$, where $W_j = \{f \in V \mid f(i) = 0, \forall i \neq j\}$, $j \in \Omega$.

Applying Lemma 25 to the pair (U, V_1) allows us to conclude that there exists $0 \neq x \in V_1$ such that $\mathrm{C}_U(x)$ has four different orbits on V_1 with representatives y_1, y_2, z_1, z_2 satisfying $\mathrm{C}_U(x) \cap \mathrm{C}_U(y_i) = 1$ and $\mathrm{C}_U(x) \cap \mathrm{C}_U(z_i)$ is a 3-group for each i.

Assume that $\mathrm{O}^{2'}(S)$ acts transitively on Ω and there exists 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of Ω such that $\bigcap_i \mathrm{Stab}_S \Delta_i = 1$. Then $1 \neq \mathrm{O}^{2'}(S)$.

Our first goal will be to prove the following statement:

$(*)$ *Let $(f, 1)$ be a 3-element of N for $f \in U^\natural$. Suppose that $f(i_0) = 1$ for some $i_0 \in \Omega$. Then $f = 1$.*

Let $P \in \mathrm{Syl}_3(N)$ such that $(f, 1) \in P$. By the Frattini Argument, $G = N\,\mathrm{N}_G(P)$. Let $\rho \in S$ be a 2-element. Then ρ determines a 2-element $(g, \rho) \in \mathrm{N}_G(P)$. Let $T = N\langle (g, \rho)\rangle$.

We show that T is S_3-free. If U-module V_1 is of type (**I**), then $p = 2$ and G is S_3-free by Corollary 7. Hence T is S_3-free. If U-module V_1 is of type (**II**), then $p = 3$ and $\mathrm{O}_{2'}(U) = 1$. Since $N\,\mathrm{C}_G(W_j)/\mathrm{C}_G(W_j) \trianglelefteq \mathrm{N}_G(W_j)/\mathrm{C}_G(W_j) \cong U$, we have $\mathrm{O}_{2'}(N) \leqslant \cap_j \mathrm{C}_G(W_j) = \mathrm{C}_G(V) = 1$. Then we have $\mathrm{O}_{2'}(T) \leqslant \mathrm{O}_{2'}(N) = 1$ since T/N is a 2-group. By Lemma 6, T is S_3-free.

Since $P\langle (g, \rho)\rangle$ is $\{2, 3\}$-subgroup of T, we can apply Lemma 5 to conclude that $P\langle (g, \rho)\rangle$ is 3-nilpotent. Hence $(f, 1)(g, \rho) = (g, \rho)(f, 1)$, that is,

$$f(i)g(i) = g(i)f(i^\rho), \forall i \in \Omega,$$

Therefore $f(i) = 1$ if and only if $f(i^\rho) = 1$.

Since $\mathrm{O}^{2'}(S)$ acts transitively on Ω, it follows that for each $i \in \Omega$, there exist 2-elements $\rho_1, \dots, \rho_s$ such that $i_0^{\rho_1 \cdots \rho_s} = i$. Since $f(i_0) = 1$, we have

$$1 = f(i_0) = f(i_0^{\rho_1}) = \cdots = f(i_0^{\rho_1 \cdots \rho_s}) = f(i),$$

thus $f(i) = 1$ for each $i \in \Omega$ and the statement is proved.

Let $v \in V$ such that $v(i) = x$ for each $i \in \Omega$ and consider the elements u_1, u_2, $u_3 \in V$ defined by

$$\begin{array}{llllll} u_1(i) = z_1, & i \in \Delta_1; & u_1(i) = y_1, & i \in \Delta_2; & u_1(i) = y_2, & i \in \Delta_3; \\ u_2(i) = z_2, & i \in \Delta_1; & u_2(i) = y_1, & i \in \Delta_2; & u_2(i) = y_2, & i \in \Delta_3; \\ u_3(i) = z_1, & i \in \Delta_1; & u_3(i) = z_2, & i \in \Delta_2; & u_3(i) = y_2, & i \in \Delta_3. \end{array}$$

Then we will show $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j) = 1$ for all $j \in \{1, 2, 3\}$, and u_1, u_2 and u_3 belong to different regular orbits of $\mathrm{C}_G(v)$ on V.

Let $(f, \sigma) \in \mathrm{C}_G(v) \cap \mathrm{C}_G(u_j)$, where $f \in U^\natural$ and $\sigma \in S$. Then

$$v(i^{\sigma^{-1}})^{f(i^{\sigma^{-1}})} = v(i); u_j(i^{\sigma^{-1}})^{f(i^{\sigma^{-1}})} = u_j(i), \forall i \in \Omega.$$

Hence $f(i) \in \mathrm{C}_U(x)$ for each i. Then we have $u_j(i^{\sigma^{-1}}), u_j(i)$ lie in the same orbit of $\mathrm{C}_U(x)$ on V_1, $\forall i \in \Omega$. Since y_1, y_2, z_1, z_2 lie in different orbits of $\mathrm{C}_U(x)$ on V_1, it implies that $\sigma \in \bigcap_i \mathrm{Stab}_S(\Delta_i) = 1$. For each $i \in \Omega$, $f(i) \in \mathrm{C}_U(x) \cap \mathrm{C}_U(y_i)$ or $\mathrm{C}_U(x) \cap \mathrm{C}_U(z_i)$ for $i = 1$ or 2. Thus $f(i)$ is a 3-element for each i and clearly $(f, \sigma) = (f, 1)$ is a 3-element. Let $i_0 \in \Delta_3$. Then $y_2^{f(i_0)} = y_2$, and so $f(i_0) \in \mathrm{C}_U(x) \cap \mathrm{C}_U(y_2) = 1$. Thus $f(i_0) = 1$.

Since $(f, \sigma) = (f, 1)$ is a 3-element of N and $f(i_0) = 1$ for some $i_0 \in \Omega$, it follows from statement $(*)$ that $f = 1$. Thus $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j) = 1$, $j = 1$, 2, 3. Similar arguments allows us to conclude that u_1, u_2 and u_3 belong to different regular orbits of $\mathrm{C}_G(v)$ on V. Consequently, G has at least three regular orbits on $V \oplus V$, and the statement 1 holds.

Suppose that $|\Omega| \leqslant 4$. If $|\Omega| = 4$, then S is isomorphic to A_4, D_8, $C_2 \times C_2$ since S is transitive and S_4-free. In these cases, $\mathrm{O}^{2'}(S)$ acts transitively on Ω and S has a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of type $(1, 1, 2)$ of Ω such that $\bigcap_i \mathrm{Stab}_S \Delta_i = 1$. By statement 1, G has at least three regular orbits on $V \oplus V$.

If $|\Omega| = 3$, we have that S is isomorphic to S_3 or C_3. Suppose that $S \cong C_3$. Then S has exactly two regular orbits on $\mathcal{P}(\Omega)$. Hence G has two regular orbits on $V \oplus V$. If $S \cong S_3$, it follows that $\mathrm{O}^{2'}(S)$ acts transitively on Ω and S has a 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of type $(1, 1, 1)$ of Ω such that $\bigcap_i \mathrm{Stab}_S \Delta_i = 1$. By statement 1, G has at least three regular orbits on $V \oplus V$.

If $|\Omega| = 2$, then $S \cong S_2$. Let $v \in V$ such that $v(i) = x$ for each $i \in \Omega$ and consider the elements u_1, u_2, $u_3 \in V$ defined by

$$\begin{aligned} u_1(1) &= z_1, & u_1(2) &= y_1; \\ u_2(1) &= z_2, & u_2(2) &= y_1; \\ u_3(1) &= y_2, & u_3(2) &= y_1. \end{aligned}$$

With similar arguments to those used above, one can show that u_1, u_2 and u_3 belong to different regular orbits of $\mathrm{C}_G(v)$ on V. Consequently, G has at least three regular orbits on $V \oplus V$. □

6 Proof of Theorem A

Our proof of Theorem A depends heavily on some results which are of independent interest. The first one concerns the odd case.

Theorem 27. *Let G be a solvable group and let V be an irreducible and faithful G-module over $\mathrm{GF}(p)$, p an odd prime. If $H \leqslant G$ and H is of odd order, then H has at least five regular orbits on $V \oplus V$.*

Proof. We argue by induction on $|G|$. By Lemma 14, we may assume that V is an imprimitive G-module. Assume that $V = V_1 \oplus \cdots \oplus V_m$ ($m \geqslant 2$) is a direct sum of subspaces which are permuted transitively by G. Write $\Omega = \{1, \ldots, m\}$, $L = \mathrm{N}_G(V_1)$ and $N = \mathrm{Core}_G(L)$. Then $m = |G : L|$ and $S = G/N$ is a permutation group on Ω induced by the action of G on a right transversal of L in G. By Lemma 9, we may assume without loss of generality G is a subgroup of $\widehat{G} = U \wr S$, where $U = \mathrm{N}_G(V_1)/\mathrm{C}_G(V_1)$ and $L = \mathrm{N}_G(V_1)$ is a maximal subgroup of G and $V = V_1^{\Omega}$. Since V is G-irreducible, we may also assume that V_1 is L-irreducible.

Let $A = (L \cap H)\,\mathrm{C}_G(V_1)/\mathrm{C}_G(V_1)$. Then the triple (L, A, V_1) satisfies the hypotheses of the theorem. By induction, A has at least five regular orbits on $V_1 \oplus V_1$.

Assume that $\{V_{11}, \ldots, V_{1t}\}$ is the H-orbit of V_1 in $\{V_1, \ldots, V_m\}$, $t = |H : L \cap H|$. Let $W = V_{11} \oplus \cdots \oplus V_{1t}$. It is clear that we may assume $t \geqslant 2$. Therefore, by Lemma 9, $H/\mathrm{C}_H(V_1)$ is isomorphic to a subgroup X of the wreath product $A \wr T = A^{\natural}X$, where T is a transitive permutation group on $\Omega_1 = \{1, \ldots, t\}$ and the action $H/\mathrm{C}_H(V_1)$ on W is equivalent to the action of X on $V_1^{\Omega_1}$. By [10, Corollary 5.7], T has an strong regular orbit on $\mathcal{P}(\Omega_1)$. By Lemma 23, H has at least five regular orbits on $W \oplus W$. Thus H has at least five regular orbits on $V \oplus V$. □

Lemma 28. *Let G be a solvable group and V be an irreducible and faithful G-module over $\mathrm{GF}(p)$, where p is a prime and $p \geqslant 5$. Then G has at least five regular orbits on $V \oplus V$.*

Proof. We suppose that the theorem is false and derive a contradiction. Let G be a counterexample of minimal order. If V is a primitive G-module, it follows from [4, Theorem 3.4] that either G has at least $p \geqslant 5$ regular orbits on $V \oplus V$. Now we assume V is an imprimitive G-module. Let $V = V_1 \oplus \cdots \oplus V_m$ ($m \geqslant 2$) and G permutes $\{V_1, \ldots, V_m\}$. Without loss of generality, G is a subgroup of $\widehat{G} = U \wr S$, where $U = \mathrm{N}_G(V_1)/\mathrm{C}_G(V_1)$ and $L = \mathrm{N}_G(V_1)$ is a maximal subgroup of G, $S \cong G/N$ is a primitive permutation group on $\Omega = \{1, \ldots, m\}$, where $N = \mathrm{Core}_G(L)$, and $V = V_1^{\Omega}$. Moreover, V_1 is an irreducible and faithful U-module. By induction, U has at least five regular orbits on $V_1 \oplus V_1$. It follows from [16, Proposition 3.2(3)] that G has at least five regular orbits on $V \oplus V$. □

The following important result provides the key to prove Theorem A.

Theorem 29. *Let G be a solvable group and V be an irreducible and faithful, G-module over* $\mathrm{GF}(p)$. *If $H \leqslant G$ and VH is S_4-free, then either H has at least three regular orbits on $V \oplus V$ or V, as H-module, is of type (**I**) or type (**II**) (see Definition 24).*

Proof. We suppose that the theorem is false and derive a contradiction. Let G be a counterexample of minimal order. If V is a primitive G-module, it follows from Lemma 13 that either H has at least three regular orbits on $V \oplus V$ or the H-module V of type (**I**) or type (**II**). This contradicts the choice of G. Consequently, V is an imprimitive G-module. Then, repeating the arguments of the first part of the proof of Theorem 27 and using the same notation, we may assume without loss of generality G is a subgroup of $\widehat{G} = U \wr S$, where $U = \mathrm{N}_G(V_1)/\mathrm{C}_G(V_1)$ and $L = \mathrm{N}_G(V_1)$ is a maximal subgroup of G, $S \cong G/N$, $N = \mathrm{Core}_G(L)$, and $V = V_1^{\Omega}$. Moreover, V_1 is an irreducible L-module.

Let $A = (L \cap H)\,\mathrm{C}_G(V_1)/\mathrm{C}_G(V_1)$. Then the triple (L, A, V_1) satisfies the hypotheses of the theorem. The minimal choice of G implies that either A at least three regular orbits on $V_1 \oplus V_1$ or V_1, as A-module, is of type (**I**) or type (**II**).

Let $\{V_{11}, \ldots, V_{1t}\}$ be the H-orbit of V_1 in $\{V_1, \ldots, V_m\}$, $t = |H : L \cap H|$. Let $W = V_{11} \oplus \cdots \oplus V_{1t}$. If we may assume $t \geqslant 2$, then, by Lemma 9, $H/\mathrm{C}_H(W)$ is isomorphic to a subgroup X of the wreath product $A \wr T = A^{\natural} X$, where T is a transitive permutation group on $\Omega_1 = \{1, \ldots, t\}$ and the action $H/\mathrm{C}_H(W)$ on W is equivalent to the action of X on $V_1^{\Omega_1}$.

Write $S^* = HN/N \leqslant S$. Assume that S^* is not transitive on Ω. Our next aim is to prove that in this case S^* has at least four strong regular orbits on $\mathcal{P}(\Omega)$. Suppose not. By Lemma 16, either $|\Omega_1| \leqslant 4$ or $\mathrm{O}^{2'}(S^*)$ acts transitively on Ω_1 and $\Pi_3(\Omega_1, T) \geqslant |T|$.

If $|\Omega_1| = 1$, then $W = V_1$ and $H/\mathrm{C}_H(W)$ has at least two regular orbits on $W \oplus W$. Now we may assume that $|\Omega_1| = t \geqslant 2$.

If the A-module V_1 is of type (**I**) or type (**II**), then, by Lemma 26, we have that $H/\mathrm{C}_H(W)$ has at least two regular orbits on $W \oplus W$.

Assume that A at least three regular orbits on $V_1 \oplus V_1$. If $\Pi_3(\Omega_1, T) \geqslant |T|$, then $H/\mathrm{C}_H(W)$ has at least three regular orbits on $W \oplus W$ by Wolf's formula. Assume that $|\Omega_1| \leqslant 4$. If $p = 2$ or, $p \neq 2$ and A is of order even, then $H/\mathrm{C}_H(W)$ has three regular orbits on $W \oplus W$ by Lemma 23. If $p \neq 2$ and A is of order odd, then $H/\mathrm{C}_H(W)$ has five regular orbits on $W \oplus W$ by Lemma 27.

Consequently, in both cases, $H/\mathrm{C}_H(W)$ has at least two regular orbits on $W \oplus W$. This implies that H has at least four regular orbits on $V \oplus V$, contrary to assumption.

Thus S^* has at least four regular orbits on $\mathcal{P}(\Omega)$. Let $L_i = \mathrm{N}_G(L_i)$ and $H_i = (L_i \cap H)\mathrm{C}_G(V_i)/\mathrm{C}_G(V_i)$ for all $i \in \{1, \dots, m\}$. Note that $A = H_1$ and $L = L_1$. Arguing as before, we conclude that H_i has at least two regular orbits on $V_i \oplus V_i$ for all $i \in \{1, \dots, m\}$.

Choose $u_i, v_i \in V_i \oplus V_i$ generating two different regular H_i-orbits on $V_i \oplus V_i$ for all $i \in \{1, \dots, m\}$. Note that these elements can be chosen to satisfy the following property: if $V_i = V_j^h$ for some $h \in H$, then $u_i = u_j^h$ and $v_i = v_j^h$. In particular, we have that u_i, v_j are not H-conjugate for all $i, j \in \{1, \dots, m\}$.

Assume that $\Delta \subseteq \Omega$ lies in a regular orbit of S^* on $\mathcal{P}(\Omega)$. This means that $\mathrm{Stab}_{S^*}(\Delta) = 1$. We may assume that $\Delta = \{1, \dots, s\}$, $s < m$. Let $x = u_1 + \cdots + u_s + v_{s+1} + \cdots + v_n$. Then $\mathrm{C}_H(x) \leqslant \mathrm{Stab}_H(\Delta) \leqslant N$ since $\mathrm{Stab}_{S^*}(\Delta) = 1$. This implies that $\mathrm{C}_H(x) \leqslant \mathrm{C}_N(u_i) \leqslant \mathrm{C}_H(V_i)$, $1 \leqslant i \leqslant s$, and $\mathrm{C}_H(x) \leqslant \mathrm{C}_N(v_j) \leqslant \mathrm{C}_H(V_j)$, $s+1 \leqslant j \leqslant m$. Hence $\mathrm{C}_H(x) \subseteq \bigcap_i \mathrm{C}_G(V_i) = 1$ and x lies in an H-regular orbit on $V \oplus V$.

Therefore every regular orbit of S^* on $\mathcal{P}(\Omega)$ determines a regular orbit of H on $V \oplus V$. In particular, H has at least four regular orbits on $V \oplus V$. This contradicts the choice of G.

Consequently, S^* acts transitively on Ω. Then $\Omega = \Omega_1$, $S^* = T$, $V = W$. We may assume that $X = H$ and so H is a subgroup of $\widehat{H} = A \wr T = A^\natural H$.

If A had at least three regular orbits on $V_1 \oplus V_1$, then H would have at least three regular orbits on $V \oplus V$ by Lemmas 23 and 27. This would contradict the choice of G. Therefore, V_1 is an A-module of type (**I**) or (**II**).

Assume that T has a strong regular orbit on $\mathcal{P}(\Omega)$. Since, by Lemma 25, A has two regular orbits on $V_1 \oplus V_1$, it follows that $\widehat{H}$ has at least two regular orbits on $V \oplus V$ by Wolf's formula. If $|\widehat{H} : H| \geqslant 2$, then H would have at least four regular orbits on $V \oplus V$, against the choice of G. Thus $H = \widehat{H}$.

Assume that T has even order. If V_1 is of type (**I**), 3 divides $|A|$ and so H has a subgroup isomorphic to $C_3 \wr C_2$. In particular, H is not S_3-free. This contradicts our assumption since H is S_3-free by Lemma 7. If V_1 is of type (**II**), then H has a subgroup isomorphic to $\mathrm{SL}(2,3) \wr C_2$ which has a section isomorphic to S_4, which is not the case. Therefore $|T|$ is odd. In this case, we can apply Corollary 18 to conclude that T has at least four strong regular orbits on $\mathcal{P}(\Omega)$, and so H has at least four regular orbits on $V \oplus V$ by Wolf's formula, unless $(T, \mathrm{d}(T)) = (A_3, 3)$ or $(\Gamma(2^3), 7)$. In any case we have $(T, \mathrm{d}(T)) \in \mathcal{K}$ and the H-module V is of type (**I**) or (**II**), a conclusion which contradicts our choice of G.

Consequently, T has not strong regular orbits on $\mathcal{P}(\Omega)$. By Lemma 17, either $|\Omega| = 2$, $T \cong S_2$ or $\mathrm{O}^{2'}(S^*)$ acts transitively on Ω and there exists 3-partition $\{\Delta_1, \Delta_2, \Delta_3\}$ of Ω such that $\bigcap_i \mathrm{Stab}_S \Delta_i = 1$. We can then apply Lemma 26 to conclude that H has at least three regular orbits on $V \oplus V$.

This is the desired contradiction. □

Proof of Theorem A. We argue by induction on $|G|+|H|+|V|$. Assume that V is not an irreducible G-module. Then there exist non-zero G-submodules V_1 and V_2 such that $V = V_1 \oplus V_2$. Clearly, V_i is a faithful, completely reducible $G/\mathrm{C}_G(V_i)$-module, $i = 1, 2$. Since $H\mathrm{C}_G(V_i)/\mathrm{C}_G(V_i)$ satisfies the hypotheses of the theorem, we conclude that $H\,\mathrm{C}_G(V_i)/\mathrm{C}_G(V_i)$ has at least two regular orbits on $V_i \oplus V_i$, $i = 1, 2$. Moreover, if H is $\Gamma(2^3)$-free and $\mathrm{SL}(2,3)$-free, then $H\mathrm{C}_G(V_i)/\mathrm{C}_G(V_i)$ has three regular orbits on $V_i \oplus V_i$ for each i. Therefore we may assume that V is an irreducible G-module over $\mathrm{GF}(p)$ for some prime p. Applying Theorem 29 we conclude that either H has at least three regular orbits on $V \oplus V$ or V, as H-module, is of type (**I**) or type (**II**). In the latter case, H has at least two regular orbits on $V \oplus V$ by Lemma 25. Note that if H is $\Gamma(2^3)$-free and $\mathrm{SL}(2,3)$-free, then H-module V is not of type (**I**) or type (**II**), and so H has at least three regular orbits on $V \oplus V$ by Theorem 29. □

7 Proof of Theorem B

Before launching into the proof of our second main result, we prepare the way with some previous considerations.

Let G be a solvable group. Set $U = \mathrm{F}(G)/\Phi(G)$ and $V = U^\star = \mathrm{Irr}(U)$. According to [2, Theorem A.10.6], there exists a subgroup X of $\overline{G} = G/\Phi(G)$ such that $\overline{G} = UX$ and $U \cap X = 1$ and U is a faithful completely reducible X-module. By Lemma 10, V is a faithful completely reducible X-module. Let U_1 be the Hall $2'$-subgroup and let U_2 be the Sylow 2-subgroup of U. Then $U = U_1 \times U_2$. Applying Lemma 10, we have that $W = W_1 \oplus W_2$, where $W_i = (U_i)^\star$, is X-isomorphic to V, and $\mathrm{C}_X(W_i) = \mathrm{C}_X(U_i)$, $i = 1, 2$.

Proof of Theorem B. In order to prove our second main Theorem and related results we need to establish some observations and notation.

Assume that G is a solvable group satisfying one of the statements of the theorem. With the above observations in mind, the burden lies in proving that X has a regular orbit on $W \oplus W$.

Assume that G is S_4-free. Since $U_2X/\mathrm{C}_X(U_2)$ is S_4-free, we have that $X/\mathrm{C}_X(W_2) = X/\mathrm{C}_X(U_2)$ is S_3-free by Corollary 7. Applying again this lemma, we have that $W_2X/\mathrm{C}_X(W_2)$ is S_4-free. Since $X/\mathrm{C}_X(W_1)$ is S_4-free, we have $W_1X/\mathrm{C}_X(W_1)$ is S_4-free as W_1 is $2'$-group. By Theorem A that $X/\mathrm{C}_X(W_i)$ has at least two regular orbits on $W_i \oplus W_i$, $i = 1, 2$. This implies that X has a regular orbit on $W \oplus W$.

If G satisfies statement (2), then $W_2 = 1$ and $W = W_1$. Since X is S_4-free, WX is S_4-free. It follows from Theorem A that X has a regular orbit on $W \oplus W$.

Assume that G satisfies statement (3). Since X is S_3-free, we have $X/\mathrm{C}_X(W_2)$ is S_3-free. It follows from Corollary 7 that $W_2X/\mathrm{C}_X(W_2)$ is S_4-free. Since $X/\mathrm{C}_X(W_1)$ is S_4-free, we have $W_1X/\mathrm{C}_X(W_1)$ is S_4-free since W_1 is $2'$-group. Thus $W_iX/\mathrm{C}_X(W_i)$ is S_4-free for both $i = 1, 2$. It follows from Theorem A that $X/\mathrm{C}_X(W_i)$ has at least two regular orbits on $W_i \oplus W_i$, $i = 1, 2$. Thus X has a regular orbit on $W \oplus W$.

Thus X has a regular orbit on $V \oplus V$ in all cases. By Lemma 11, $|G : \mathrm{F}(G)| \leqslant \mathrm{b}(G)^2$. □

We derive now some results related to Gluck's conjecture. The first one is part of [1, Theorem 7].

Corollary 30. *Let G be a solvable group and let H be a π-Hall subgroup of G, where $\pi = \pi(\mathrm{F}(G))$. Then $|G : H| \leqslant \mathrm{b}(G)^2$.*

Proof. Let K be a Hall π'-subgroup of G. Since $(|K|, |U|) = 1$, we have $\mathrm{C}_K(U) \leqslant \mathrm{C}_K(\mathrm{F}(G)) \leqslant K \cap \mathrm{F}(G) = 1$. Thus U and V are faithful completely reducible K-modules. By Lemma 11, $|G : H| = |K| \leqslant \mathrm{b}(G)^2$. □

Our second result is a direct consequence of Theorem A.

Corollary 31 ([19, Theorem 4.6])**.** *If G is a solvable group, then $|G : \mathrm{F}(G)|_{3'} \leqslant \mathrm{b}(G)^2$.*

Proof. Let K be a $3'$-Hall subgroup of X. Clearly KV is S_4-free, by Theorem A, K has a regular orbit on $V \oplus V$. By Lemma 11, $|G : \mathrm{F}(G)|_{3'} = |K| \leqslant \mathrm{b}(G)^2$. □

Acknowledgements

The research of this paper has been supported by the grant MTM2014-54707-C3-1-P from the Ministerio de Economía y Competitividad, Spain, and FEDER, European Union and by the grant PROMETEO/2017/057 from the Generalitat, Valencian Community, Spain. The first author is supported by the predoctoral grant 201606890006 from the China Scholarship Council.

References

[1] J. P. Cossey, Z. Halasi, A. Maróti, and H. N. Nguyen. On a conjecture of Gluck. *Math. Z.*, 279:1067–1080, 2015.

[2] K. Doerk and T. Hawkes. *Finite soluble groups*, volume 4 of *De Gruyter Expositions in Mathematics.* Walter de Gruyter & Co., Berlin, 1992.

[3] S. Dolfi. Intersections of odd order Hall subgroups. *Bull. London Math. Soc.*, 37(1):61–66, 2005.

[4] S. Dolfi. Large orbits in coprime actions of solvable groups. *Trans. Amer. Math. Soc.*, 360(1):135–152, 2008.

[5] S. Dolfi and E. Jabara. Large character degrees of solvable groups with abelian Sylow 2-subgroups. *J. Algebra*, 313(2):687–694, 2007.

[6] A. Espuelas. Large character degrees of groups of odd order. *Illinois J. Math.*, 35(3):499–505, 1991.

[7] D. Gluck. The largest irreducible character degree of a finite group. *Canad. J. Math.*, 37(3):442–415, 1985.

[8] B. Huppert. *Endliche Gruppen I*, volume 134 of *Grund. Math. Wiss.* Springer Verlag, Berlin, Heidelberg, New York, 1967.

[9] B. Huppert. *Character theory of finite groups*, volume 25 of *De Gruyter Expositions in Mathematics.* Walter de Gruyter, 1998.

[10] O. Manz and T.R. Wolf. *Representations of Solvable Groups*, volume 185 of *Mathematical Society Lecture Note Series.* Cambridge University Press, London, 1993.

[11] A. Moretó and T. R. Wolf. Orbit sizes, character degrees and Sylow subgroups. *Adv. Math.*, 184(1):18–36, 2004. Erratum: *ibid.*, no. 2, page 409.

[12] P. Schmid. *The solution of the $k(GV)$-problem*, volume 4 of *ICP Advanced Texts in Mathematics.* Imperial College Press, London, 2007.

[13] Á. Seress. The minimal base size of primitive solvable permutation groups. *J. London Math. Soc.*, 53(2):243–255, 2006.

[14] T .Wolf. Large orbits of supersolvable linear groups. *J. Algebra*, 215:235–247, 1999.

[15] T. R. Wolf. Regular orbits of induced modules of finite groups. In C. Y. Ho, P. Sin, P. H. Tiep, and A. Turull, editors, *Finite Groups 2003. Proceedings of the Gainesville conference on finite groups. March 6–12, 2003. In honour of John Thompson to this 70th birthday*, pages 389–399, Berlin, 2004. Walter de Gruyter.

[16] Y. Yang. Orbits of the actions of finite solvable groups. *J. Algebra*, 321:2012–2021, 2009.

[17] Y. Yang. Regular orbits of finite primitive solvable groups. *J. Algebra*, 323:2735–2755, 2010.

[18] Y. Yang. Large character degrees of solvable $3'$-groups. *Proc. Amer. Math. Soc.*, 139(9):3171–3173, 2011.

[19] Y. Yang. Large orbits of subgroups of solvable linear groups. *Israel J. Math.*, 199(1):345–362, 2014.

[20] J. Zhang. Finite groups with few regular orbits on the power set. *Algebra Colloq.*, 4(4):471–480, 1997.