Research Article

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On the spectral properties of real antitridiagonal Hankel matrices

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Abstract: In this article, we express the eigenvalues of real antitridiagonal Hankel matrices as the zeros of given rational functions. We still derive eigenvectors for these structured matrices at the expense of prescribed eigenvalues.

Keywords: antitridiagonal matrix, Hankel matrix, eigenvalue, eigenvector

MSC 2020: 15A18, 15B05

1 Introduction

Recently, some authors have computed the eigenvalues and eigenvectors for a sort of Hankel matrices, thus obtaining its eigendecomposition (see [7,8,12,15–18], among others). Moreover, the spectral analysis for a type of persymmetric antitridiagonal 2-Hankel matrices was also undertaken (see [1]). In contrast to the tridiagonal Toeplitz case where the eigenvalues and eigenvectors are well known (see, for instance, [12]), a possible closed-form expression for the eigenvalues and eigenvectors of general antitridiagonal Hankel matrices is yet to be found. In this quest, da Fonseca [5] gave a step forward by expressing explicitly the eigenvalues of odd order antitridiagonal Hankel matrices having null northeast-to-southwest diagonal. It must also be emphasized that, although antibidiagonal matrices or antitridiagonal matrices with null antidiagonal can be treated as matrices having tridiagonal structure in what concerns to spectral purposes or powers computation (see [3,6]), general antitridiagonal matrices, and particularly those considered here, cannot be reduced to tridiagonal ones (see Remark 3 of [3]).

The aim of this short note is to give a contribution for that research. Specifically, we shall present an eigenvalue localization tool for real antitridiagonal Hankel matrices, providing also associated eigenvectors. To achieve our purpose, we shall use eigendecompositions of anticirculant matrices available in [11] to ensure a decomposition for the matrices under study at first and results concerning to the sum of (rank one) matrices in a final stage to obtain all formulae.

The rational functions exhibited in this article to locate eigenvalues of real antitridiagonal Hankel matrices, as well as the expressions for its eigenvectors, are given in the explicit form, which, on the one hand, can be easily implemented in computer algebra systems, and, on the other hand, are useful for further theoretical investigations in this subject.

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2 Main results

Let n be a positive integer and consider the following $(n+2)\times(n+2)$ antitridiagonal Hankel matrix

$$\mathbf{H}_{n+2}=\begin{bmatrix} 0 & \dots & \dots & \dots & 0 & a & c \\ \vdots & & & \iddots & a & c & b \\ \vdots & & \iddots & \iddots & c & b & 0 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & a & c & \iddots & \iddots & & \vdots \\ a & c & b & \iddots & & & \vdots \\ c & b & 0 & \dots & \dots & \dots & 0 \end{bmatrix}, \tag{2.1}$$

where a, b, and c are real numbers. Throughout, we shall set

$$\omega := \mathrm{e}^{\frac{2\pi \mathrm{i}}{n+2}}, \tag{2.2}$$

where i denotes the imaginary unit.

2.1 Eigenvalue localization for $\mathbf{H}_{n+2}$

Our first statement is an eigenvalue localization theorem for matrices of the form (2.1).

Theorem 1. *Let n be a positive integer, a, b, and c be real numbers, and, ω is given by* (2.2),

$$\lambda_k := b + a\omega^{-nk} + c\omega^{-(n+1)k}, \quad k=0,1,\dots,n+1, \tag{2.3a}$$

$$\theta_k := \arg(\lambda_k), \quad k=0,1,\dots,n+1, \tag{2.3b}$$

$$F_n(t;\,\alpha,\beta) := \frac{1}{t-\lambda_0} + 2\sum_{k=1}^{n}\left[\frac{\cos\left(\frac{\theta_k}{2}-\alpha k\right)\cos\left(\frac{\theta_k}{2}-\beta k\right)}{t-|\lambda_k|} + \frac{\sin\left(\frac{\theta_k}{2}-\alpha k\right)\sin\left(\frac{\theta_k}{2}-\beta k\right)}{t+|\lambda_k|}\right], \tag{2.3c}$$

$$G_n(t;\,\alpha,\beta) := \frac{1}{t-\lambda_0} + \frac{1}{t-\lambda_{n+1}} + 2\sum_{k=1}^{n}\left[\frac{\cos\left(\frac{\theta_k}{2}-\alpha k\right)\cos\left(\frac{\theta_k}{2}-\beta k\right)}{t-|\lambda_k|} + \frac{\sin\left(\frac{\theta_k}{2}-\alpha k\right)\sin\left(\frac{\theta_k}{2}-\beta k\right)}{t+|\lambda_k|}\right]. \tag{2.3d}$$

(*a*) *If n is odd then the eigenvalues of $\mathbf{H}_{n+2}$ that are not of the form $\lambda_0, |\lambda_k|, -|\lambda_k| k=1,\dots,\frac{n+1}{2}$ are precisely the zeros of the function*

$$\begin{aligned} f(t) = 1 &+ \frac{a}{n+2}F_{\frac{n+1}{2}}\left(t;\,\frac{2\pi}{n+2},\frac{2\pi}{n+2}\right) + \frac{b}{n+2}F_{\frac{n+1}{2}}(t;\,0,0) \\ &+ \frac{ab}{(n+2)^2}F_{\frac{n+1}{2}}\left(t;\,\frac{2\pi}{n+2},\frac{2\pi}{n+2}\right)F_{\frac{n+1}{2}}(t;\,0,0) - \frac{ab}{(n+2)^2}F^2_{\frac{n+1}{2}}\left(t;\,0,\frac{2\pi}{n+2}\right). \end{aligned} \tag{2.4a}$$

Moreover, if $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_{n+2}$ are the eigenvalues of $\mathbf{H}_{n+2}$ and $\lambda_0, |\lambda_k|, -|\lambda_k| k=1,\dots,\frac{n+1}{2}$ are arranged in nondecreasing order as $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_{n+2}$, then

$$d_k + \min\{0,-a,-b\} \leqslant \mu_k \leqslant d_k + \max\{0,-a,-b\}, \quad k=1,\dots,n+2. \tag{2.4b}$$

(*b*) *If n is even, then the eigenvalues of $\mathbf{H}_{n+2}$ that are not of the form $\lambda_0, |\lambda_k|, \lambda_{\frac{n}{2}+1}, -|\lambda_k| k=1,\dots,\frac{n}{2}$ are precisely the zeros of the function*:

$$\begin{aligned} g(t) = 1 &+ \frac{a}{n+2}G_{\frac{n}{2}}\left(t;\,\frac{2\pi}{n+2},\frac{2\pi}{n+2}\right) + \frac{b}{n+2}G_{\frac{n}{2}}(t;\,0,0) + \frac{ab}{(n+2)^2}G_{\frac{n}{2}}\left(t;\,\frac{2\pi}{n+2},\frac{2\pi}{n+2}\right)G_{\frac{n}{2}}(t;\,0,0) \\ &- \frac{ab}{(n+2)^2}\left[G_{\frac{n}{2}}\left(t;\,0,\frac{2\pi}{n+2}\right) - \frac{2}{t-\lambda_{\frac{n}{2}+1}}\right]^2. \end{aligned} \tag{2.5a}$$

Moreover, if $\nu_1 \leqslant \nu_2 \leqslant \ldots \leqslant \nu_{n+2}$ *are the eigenvalues of* $\mathbf{H}_{n+2}$ *and* $\lambda_0, |\lambda_k|, \lambda_{\frac{n}{2}+1}, -|\lambda_k|$ $k = 1, \ldots, \frac{n}{2}$ *are arranged in a nondecreasing order as* $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_{n+2}$, *then*

$$d_k + \min\{0, -a, -b\} \leqslant \nu_k \leqslant d_k + \max\{0, -a, -b\}, \quad k = 1, \ldots, n+2. \tag{2.5b}$$

2.2 Eigenvectors of $\mathbf{H}_{n+2}$

Owning the eigenvalues of $\mathbf{H}_{n+2}$ in (2.1), we are able to determine the corresponding eigenvectors.

Theorem 2. *Let* n *be an integer,* a, b, *and* c *be real numbers such that* $a \neq 0$, *and* λ_k, θ_k $(k = 0, 1, \ldots, n+1)$, $F_n(t; \alpha, \beta)$, *and* $G_n(t; \alpha, \beta)$ *be given by* (2.3a), (2.3b), (2.3c), *and* (2.3d), *respectively.*

(*a*) *If* n *is odd, the zeros* $\mu_1, \mu_2, \ldots, \mu_{n+2}$ *of* (2.4a) *are not of the form* $\lambda_0, |\lambda_k|, -|\lambda_k|$, $k = 1, \ldots, \frac{n+1}{2}$ *and* $bF_{\frac{n+1}{2}}(\mu_m, 0, 0) + n + 2 \neq 0$, *then*

$$\mathbf{u}(\mu_m) = \begin{bmatrix} F_{\frac{n+1}{2}}\left(\mu_m; 0, \frac{2\pi}{n+2}\right) - \dfrac{b\, F_{\frac{n+1}{2}}\left(\mu_m, 0, \frac{2\pi}{n+2}\right) F_{\frac{n+1}{2}}(\mu_m; 0, 0)}{b\, F_{\frac{n+1}{2}}(\mu_m, 0, 0) + n + 2} \\ F_{\frac{n+1}{2}}\left(\mu_m; -\frac{2\pi}{n+2}, \frac{2\pi}{n+2}\right) - \dfrac{b\, F_{\frac{n+1}{2}}\left(\mu_m, 0, \frac{2\pi}{n+2}\right) F_{\frac{n+1}{2}}\left(\mu_m; -\frac{2\pi}{n+2}, 0\right)}{b\, F_{\frac{n+1}{2}}(\mu_m, 0, 0) + n + 2} \\ \vdots \\ F_{\frac{n+1}{2}}\left(\mu_m; \frac{2(1-k)\pi}{n+2}, \frac{2\pi}{n+2}\right) - \dfrac{b\, F_{\frac{n+1}{2}}\left(\mu_m, 0, \frac{2\pi}{n+2}\right) F_{\frac{n+1}{2}}\left(\mu_m; \frac{2(1-k)\pi}{n+2}, 0\right)}{b\, F_{\frac{n+1}{2}}(\mu_m, 0, 0) + n + 2} \\ \vdots \\ F_{\frac{n+1}{2}}\left(\mu_m; -\frac{2(n+1)\pi}{n+2}, \frac{2\pi}{n+2}\right) - \dfrac{b\, F_{\frac{n+1}{2}}\left(\mu_m, 0, \frac{2\pi}{n+2}\right) F_{\frac{n+1}{2}}\left(\mu_m; -\frac{2(n+1)\pi}{n+2}, 0\right)}{b\, F_{\frac{n+1}{2}}(\mu_m, 0, 0) + n + 2} \end{bmatrix} \tag{2.6}$$

is an eigenvector of $\mathbf{H}_{n+2}$ *associated with* μ_m, $m = 1, 2, \ldots, n+2$.

(*b*) *If* n *is even, the zeros* $\nu_1, \nu_2, \ldots, \nu_{n+2}$ *of* (2.5a) *are not of the form* $\lambda_0, |\lambda_k|, \lambda_{\frac{n}{2}+1}, -|\lambda_k|$, $k = 1, \ldots, \frac{n}{2}$ *and* $bG_{\frac{n}{2}}(\nu_m, 0, 0) + n + 2 \neq 0$, *then*

$$\mathbf{v}(\nu_m) = \begin{bmatrix} G_{\frac{n}{2}}\left(\nu_m; 0, \frac{2\pi}{n+2}\right) - \dfrac{2}{\nu_m - \lambda_{\frac{n}{2}+1}} - \dfrac{b\, G_{\frac{n}{2}}(\nu_m; 0, 0)\left[G_{\frac{n}{2}}\left(\nu_m; 0, \frac{2\pi}{n+2}\right) - \frac{2}{\nu_m - \lambda_{\frac{n}{2}+1}}\right]}{b\, G_{\frac{n}{2}}(\nu_m; 0, 0) + n + 2} \\ G_{\frac{n}{2}}\left(\nu_m; -\frac{2\pi}{n+2}, \frac{2\pi}{n+2}\right) - \dfrac{b\left[G_{\frac{n}{2}}\left(\nu_m; -\frac{2\pi}{n+2}, 0\right) - \frac{2}{\nu_m - \lambda_{\frac{n}{2}+1}}\right]\left[G_{\frac{n}{2}}\left(\nu_m; 0, \frac{2\pi}{n+2}\right) - \frac{2}{\nu_m - \lambda_{\frac{n}{2}+1}}\right]}{b\, G_{\frac{n}{2}}(\nu_m; 0, 0) + n + 2} \\ \vdots \\ G_{\frac{n}{2}}\left(\nu_m; \frac{2(1-k)\pi}{n+2}, \frac{2\pi}{n+2}\right) - \dfrac{1 - (-1)^k}{\nu_m - \lambda_{\frac{n}{2}+1}} \\ - \dfrac{b\left[G_{\frac{n}{2}}\left(\nu_m; \frac{2(1-k)\pi}{n+2}, 0\right) - \frac{1+(-1)^k}{\nu_m - \lambda_{\frac{n}{2}+1}}\right]\left[G_{\frac{n}{2}}\left(\nu_m; 0, \frac{2\pi}{n+2}\right) - \frac{2}{\nu_m - \lambda_{\frac{n}{2}+1}}\right]}{b\, G_{\frac{n}{2}}(\nu_m; 0, 0) + n + 2} \\ \vdots \\ G_{\frac{n}{2}}\left(\nu_m; -\frac{2(n+1)\pi}{n+2}, \frac{2\pi}{n+2}\right) - \dfrac{b\left[G_{\frac{n}{2}}\left(\nu_m; -\frac{2(n+1)\pi}{n+2}, 0\right) - \frac{2}{\nu_m - \lambda_{\frac{n}{2}+1}}\right]\left[G_{\frac{n}{2}}\left(\nu_m; 0, \frac{2\pi}{n+2}\right) - \frac{2}{\nu_m - \lambda_{\frac{n}{2}+1}}\right]}{b\, G_{\frac{n}{2}}(\nu_m; 0, 0) + n + 2} \end{bmatrix} \tag{2.7}$$

is an eigenvector of $\mathbf{H}_{n+2}$ *associated with* ν_m, $m = 1, 2, \ldots, n+2$.

Remark. We point out that if $a = 0$, then expressions (2.6) and (2.7) should be replaced by

$$\mathbf{u}(\mu_m) = \begin{bmatrix} F_{\frac{n+1}{2}}(\mu_m;\ 0, 0) \\ F_{\frac{n+1}{2}}\left(\mu_m;\ -\dfrac{2\pi}{n+2}, 0\right) \\ \vdots \\ F_{\frac{n+1}{2}}\left(\mu_m;\ -\dfrac{2(n+1)\pi}{n+2}, 0\right) \end{bmatrix}$$

and

$$\mathbf{v}(\nu_m) = \begin{bmatrix} G_{\frac{n}{2}}(\nu_m;\ 0, 0) \\ G_{\frac{n}{2}}\left(\nu_m;\ -\dfrac{2\pi}{n+2}, 0\right) - \dfrac{2}{\nu_m - \lambda_{\frac{n}{2}+1}} \\ \vdots \\ G_{\frac{n}{2}}\left(\nu_m;\ \dfrac{2(1-k)\pi}{n+2}, 0\right) - \dfrac{1+(-1)^k}{\nu_m - \lambda_{\frac{n}{2}+1}} \\ \vdots \\ G_{\frac{n}{2}}\left(\nu_m;\ -\dfrac{2(n+1)\pi}{n+2}, 0\right) - \dfrac{2}{\nu_m - \lambda_{\frac{n}{2}+1}} \end{bmatrix},$$

respectively, provided that $b \neq 0$. Of course, if $a = b = 0$, then an eigenvalue decomposition of the exchange matrix is already known (see [7]).

3 Lemmas and proofs

Let n be a positive integer. Consider the following real anticirculant $(n+2) \times (n+2)$ matrix

$$\mathbf{A}_{n+2} := \begin{bmatrix} b & 0 & \dots & \dots & 0 & a & c \\ 0 & & & \iddots & a & c & b \\ \vdots & & \iddots & \iddots & c & b & 0 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & a & c & \iddots & \iddots & & \vdots \\ a & c & b & \iddots & & & 0 \\ c & b & 0 & \dots & \dots & 0 & a \end{bmatrix}, \tag{3.1}$$

and the $(n+2) \times (n+2)$ unitary discrete Fourier transform matrix $\mathbf{\Omega}_{n+2}$, that is, the matrix defined by

$$[\mathbf{\Omega}_{n+2}]_{k,\ell} := \frac{\omega^{(k-1)(\ell-1)}}{\sqrt{n+2}}, \tag{3.2}$$

where ω is given by (2.2). Our first auxiliary result is an orthogonal decomposition for (3.1). We shall denote by $*$ the conjugate transpose of any complex matrix.

Lemma 1. *Let n be a positive integer, a, b, and c be real numbers, and ω, λ_k, and θ_k, $k = 0, 1, \dots, n+1$ be given by* (2.2), (2.3a), (2.3b), *respectively.*

(*a*) *If n is odd be, then*

$$\mathbf{A}_{n+2} = \mathbf{P}_{n+2} \operatorname{diag}\left(\lambda_0, |\lambda_1|, \dots, |\lambda_{\frac{n+1}{2}}|, -|\lambda_{\frac{n+1}{2}}|, \dots, -|\lambda_1|\right)\mathbf{P}_{n+2}^\top, \tag{3.3a}$$

where $\mathbf{P}_{n+2}$ is the $(n+2) \times (n+2)$ orthogonal matrix defined by

$$[\mathbf{P}_{n+2}]_{k,\ell} = \begin{cases} \dfrac{1}{\sqrt{n+2}}, & \ell = 1 \\ \sqrt{\dfrac{2}{n+2}} \cos\left[\dfrac{\theta_{\ell-1}}{2} + \dfrac{2(k-1)(\ell-1)\pi}{n+2}\right], & 1 < \ell \leqslant \dfrac{n+3}{2} \\ \sqrt{\dfrac{2}{n+2}} \sin\left[\dfrac{\theta_{n+3-\ell}}{2} + \dfrac{2(k-1)(n+3-\ell)\pi}{n+2}\right], & \ell > \dfrac{n+3}{2}. \end{cases} \tag{3.3b}$$

(*b*) *If* n *is even, then*

$$\mathbf{A}_{n+2} = \mathbf{Q}_{n+2} \mathrm{diag}\Big(\lambda_0, |\lambda_1|, \dots, |\lambda_{\frac{n}{2}}|, \lambda_{\frac{n}{2}+1}, -|\lambda_{\frac{n}{2}}|, \dots, -|\lambda_1|\Big) \mathbf{Q}_{n+2}^{\top}, \tag{3.4a}$$

where $\mathbf{Q}_{n+2}$ *is the* $(n+2) \times (n+2)$ *orthogonal matrix whose entries are given by*

$$[\mathbf{Q}_{n+2}]_{k,\ell} = \begin{cases} \dfrac{1}{\sqrt{n+2}}, & \ell = 1 \\ \sqrt{\dfrac{2}{n+2}} \cos\left[\dfrac{\theta_{\ell-1}}{2} + \dfrac{2(k-1)(\ell-1)\pi}{n+2}\right], & 1 < \ell \leqslant \dfrac{n}{2} + 1 \\ \dfrac{(-1)^{k-1}}{\sqrt{n+2}}, & \ell = \dfrac{n}{2} + 2 \\ \sqrt{\dfrac{2}{n+2}} \sin\left[\dfrac{\theta_{n+3-\ell}}{2} + \dfrac{2(k-1)(n+3-\ell)\pi}{n+2}\right], & \ell > \dfrac{n}{2} + 2. \end{cases} \tag{3.4b}$$

Proof. Let n be a positive odd integer. According to Theorem 3.6 of [11], we have

$$\mathbf{A}_{n+2} = \mathbf{P}_{n+2} \mathrm{diag}\Big(\lambda_0, |\lambda_1|, \dots, |\lambda_{\frac{n+1}{2}}|, -|\lambda_{\frac{n+1}{2}}|, \dots, -|\lambda_1|\Big) \mathbf{P}_{n+2}^{*},$$

where

$$\mathbf{P}_{n+2} = \mathbf{\Omega}_{n+2}^{*} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{n+1} \end{bmatrix}$$

with $\mathbf{\Omega}_{n+2}$ given by (3.2) and $\mathbf{R}_{n+1}$ the following $(n+1) \times (n+1)$ matrix

$$\mathbf{R}_{n+1} = \frac{\sqrt{2}}{2} \begin{bmatrix} e^{-\frac{i\theta_1}{2}} & 0 & \dots & 0 & 0 & \dots & 0 & ie^{-\frac{i\theta_1}{2}} \\ 0 & e^{-\frac{i\theta_2}{2}} & \ddots & \vdots & \vdots & \iddots & ie^{-\frac{i\theta_2}{2}} & 0 \\ \vdots & \ddots & \ddots & 0 & 0 & \iddots & \iddots & \vdots \\ 0 & \dots & 0 & e^{-\frac{i\theta_{(n+1)/2}}{2}} & ie^{-\frac{i\theta_{(n+1)/2}}{2}} & 0 & \dots & 0 \\ 0 & \dots & 0 & e^{\frac{i\theta_{(n+1)/2}}{2}} & -ie^{\frac{i\theta_{(n+1)/2}}{2}} & 0 & \dots & 0 \\ \vdots & \iddots & \iddots & 0 & 0 & \ddots & \ddots & \vdots \\ 0 & e^{\frac{i\theta_2}{2}} & \iddots & \vdots & \vdots & \ddots & -ie^{\frac{i\theta_2}{2}} & 0 \\ e^{\frac{i\theta_1}{2}} & 0 & \dots & 0 & 0 & \dots & 0 & -ie^{\frac{i\theta_1}{2}} \end{bmatrix}.$$

Note that the first column of $\mathbf{P}_{n+2}$ has all components equal to $1/\sqrt{n+2}$; their next $(n+1)/2$ columns are

$$\sqrt{\frac{2}{n+2}} \begin{bmatrix} e^{-\frac{i\theta_\ell}{2}} + e^{\frac{i\theta_\ell}{2}} \\ \overline{\omega} e^{-\frac{i\theta_\ell}{2}} + \overline{\omega}^{(n+1)} e^{\frac{i\theta_\ell}{2}} \\ \overline{\omega}^2 e^{-\frac{i\theta_\ell}{2}} + \overline{\omega}^{2(n+1)} e^{\frac{i\theta_\ell}{2}} \\ \vdots \\ \overline{\omega}^{n+1} e^{-\frac{i\theta_\ell}{2}} + \overline{\omega}^{(n+1)^2} e^{\frac{i\theta_\ell}{2}} \end{bmatrix} \tag{3.5}$$

for each $\ell = 1, \dots, \frac{n+1}{2}$, and the last ones are

$$\sqrt{\frac{2}{n+2}}\begin{bmatrix} \mathrm{i}\mathrm{e}^{-\frac{\mathrm{i}\theta_\ell}{2}} - \mathrm{i}\mathrm{e}^{\frac{\mathrm{i}\theta_\ell}{2}} \\ \mathrm{i}\overline{\omega}\mathrm{e}^{-\frac{\mathrm{i}\theta_\ell}{2}} - \mathrm{i}\overline{\omega}^{(n+1)}\mathrm{e}^{\frac{\mathrm{i}\theta_\ell}{2}} \\ \mathrm{i}\overline{\omega}^2\mathrm{e}^{-\frac{\mathrm{i}\theta_\ell}{2}} - \mathrm{i}\overline{\omega}^{2(n+1)}\mathrm{e}^{\frac{\mathrm{i}\theta_\ell}{2}} \\ \vdots \\ \mathrm{i}\overline{\omega}^{n+1}\mathrm{e}^{-\frac{\mathrm{i}\theta_\ell}{2}} - \mathrm{i}\overline{\omega}^{(n+1)^2}\mathrm{e}^{\frac{\mathrm{i}\theta_\ell}{2}} \end{bmatrix} \tag{3.6}$$

for $\ell = \frac{n+1}{2}, \dots, 1$. Since

$$\overline{\omega}^k\mathrm{e}^{-\frac{\mathrm{i}\theta_\ell}{2}} + \overline{\omega}^{(n+1)k}\mathrm{e}^{\frac{\mathrm{i}\theta_\ell}{2}} = \cos\left(\frac{\theta_\ell}{2} + \frac{2k\ell\pi}{n+2}\right), \quad k = 0, 1, \dots, n+1, \tag{3.7}$$

$$\mathrm{i}\overline{\omega}^k\mathrm{e}^{-\frac{\mathrm{i}\theta_\ell}{2}} - \mathrm{i}\overline{\omega}^{(n+1)k}\mathrm{e}^{\frac{\mathrm{i}\theta_\ell}{2}} = \sin\left(\frac{\theta_\ell}{2} + \frac{2k\ell\pi}{n+2}\right), \quad k = 0, 1, \dots, n+1, \tag{3.8}$$

and we obtain that the entries of $\mathbf{P}_{n+2}$ are given by (3.3b), which leads to (3.3a). Supposing a positive even integer n, Theorem 3.7 in [11] ensures

$$\mathbf{A}_{n+2} = \mathbf{Q}_{n+2}\mathrm{diag}\Big(\lambda_0, |\lambda_1|, \dots, |\lambda_{\frac{n}{2}}|, \lambda_{\frac{n}{2}+1}, -|\lambda_{\frac{n}{2}}|, \dots, -|\lambda_1|\Big)\mathbf{Q}_{n+2}^\top,$$

where

$$\mathbf{Q}_{n+2} = \mathbf{\Omega}_{n+2}^*\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{n+1} \end{bmatrix}$$

with $\mathbf{\Omega}_{n+2}$ given by (3.2) and $\mathbf{S}_{n+1}$ by the $(n+1)\times(n+1)$ matrix

$$\mathbf{S}_{n+1} = \frac{\sqrt{2}}{2}\begin{bmatrix}
\mathrm{e}^{-\frac{\mathrm{i}\theta_1}{2}} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \mathrm{i}\mathrm{e}^{-\frac{\mathrm{i}\theta_1}{2}} \\
0 & \mathrm{e}^{-\frac{\mathrm{i}\theta_2}{2}} & \ddots & \vdots & \vdots & \vdots & \iddots & \mathrm{i}\mathrm{e}^{-\frac{\mathrm{i}\theta_2}{2}} & 0 \\
\vdots & \ddots & \ddots & 0 & 0 & 0 & \iddots & \iddots & \vdots \\
0 & \dots & 0 & \mathrm{e}^{-\frac{\mathrm{i}\theta_{n/2}}{2}} & 0 & \mathrm{i}\mathrm{e}^{-\frac{\mathrm{i}\theta_{n/2}}{2}} & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & \frac{2}{\sqrt{2}} & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & \mathrm{e}^{\frac{\mathrm{i}\theta_{n/2}}{2}} & 0 & -\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}\theta_{n/2}}{2}} & 0 & \dots & 0 \\
\vdots & \iddots & \iddots & 0 & 0 & 0 & \ddots & \ddots & \vdots \\
0 & \mathrm{e}^{\frac{\mathrm{i}\theta_2}{2}} & \iddots & \vdots & \vdots & \vdots & \ddots & -\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}\theta_2}{2}} & 0 \\
\mathrm{e}^{\frac{\mathrm{i}\theta_1}{2}} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}\theta_1}{2}}
\end{bmatrix}.$$

The first column of $\mathbf{Q}_{n+2}$ has all its components equal to $1/\sqrt{n+2}$. The next $n/2$ columns are given by (3.5) for $\ell = 1, \dots, \frac{n}{2}$, and the $n/2$ last ones are defined by (3.6) for each $\ell = \frac{n}{2}, \dots, 1$; the $\left(\frac{n}{2}+1\right)$th column of $\mathbf{Q}_{n+2}$ is

$$\sqrt{\frac{1}{n+2}}\begin{bmatrix} 1 \\ \overline{\omega}^{\frac{n}{2}+1} \\ \overline{\omega}^{2\left(\frac{n}{2}+1\right)} \\ \vdots \\ \overline{\omega}^{(n+1)\left(\frac{n}{2}+1\right)} \end{bmatrix} = \sqrt{\frac{1}{n+2}}\begin{bmatrix} 1 \\ -1 \\ (-1)^2 \\ \vdots \\ (-1)^{n+1} \end{bmatrix}.$$

From identities (3.7) and (3.8), we obtain (3.4a). The proof is completed. □

The following statement is a decomposition for the matrices $\mathbf{H}_{n+2}$ and plays a central role in the main results.

Lemma 2. *Let n be a positive integer, a, b, and c bereal numbers, and ω, λ_k, θ_k, $k = 0, 1, \dots, n+1$ be given by* (2.2), (2.3a), *and* (2.3b), *respectively.*

(*a*) *If n is odd,*

$$\mathbf{x} = \sqrt{\frac{2}{n+2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \cos\left(\frac{\theta_1}{2}\right) \\ \vdots \\ \cos\left[\frac{\theta_{(n+1)/2}}{2}\right] \\ \sin\left[\frac{\theta_{(n+1)/2}}{2}\right] \\ \vdots \\ \sin\left(\frac{\theta_1}{2}\right) \end{bmatrix}, \quad \mathbf{y} = \sqrt{\frac{2}{n+2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \cos\left(\frac{\theta_1}{2} - \frac{2\pi}{n+2}\right) \\ \vdots \\ \cos\left[\frac{\theta_{(n+1)/2}}{2} - \frac{(n+1)\pi}{n+2}\right] \\ \sin\left[\frac{\theta_{(n+1)/2}}{2} - \frac{(n+1)\pi}{n+2}\right] \\ \vdots \\ \sin\left(\frac{\theta_1}{2} - \frac{2\pi}{n+2}\right) \end{bmatrix}, \tag{3.9}$$

then

$$\mathbf{H}_{n+2} = \mathbf{P}_{n+2}\left[\operatorname{diag}\left(\lambda_0, |\lambda_1|, \ldots, |\lambda_{\frac{n+1}{2}}|, -|\lambda_{\frac{n+1}{2}}|, \ldots, -|\lambda_1|\right) - b\mathbf{x}\mathbf{x}^\top - a\mathbf{y}\mathbf{y}^\top\right]\mathbf{P}_{n+2}^\top,$$

where $\mathbf{P}_{n+2}$ *is the* $(n+2)\times(n+2)$ *matrix defined by* (3.3b).

(*b*) *If n is even,*

$$\mathbf{x} = \sqrt{\frac{2}{n+2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \cos\left(\frac{\theta_1}{2}\right) \\ \vdots \\ \cos\left(\frac{\theta_{n/2}}{2}\right) \\ \frac{1}{\sqrt{2}} \\ \sin\left(\frac{\theta_{n/2}}{2}\right) \\ \vdots \\ \sin\left(\frac{\theta_1}{2}\right) \end{bmatrix}, \quad \mathbf{y} = \sqrt{\frac{2}{n+2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \cos\left(\frac{\theta_1}{2} - \frac{2\pi}{n+2}\right) \\ \vdots \\ \cos\left(\frac{\theta_{n/2}}{2} - \frac{n\pi}{n+2}\right) \\ -\frac{1}{\sqrt{2}} \\ \sin\left(\frac{\theta_{n/2}}{2} - \frac{n\pi}{n+2}\right) \\ \vdots \\ \sin\left(\frac{\theta_1}{2} - \frac{2\pi}{n+2}\right) \end{bmatrix}, \tag{3.10}$$

then

$$\mathbf{H}_{n+2} = \mathbf{Q}_{n+2}\left[\operatorname{diag}\left(\lambda_0, |\lambda_1|, \ldots, |\lambda_{\frac{n}{2}}|, \lambda_{\frac{n}{2}+1}, -|\lambda_{\frac{n}{2}}|, \ldots, -|\lambda_1|\right) - b\mathbf{x}\mathbf{x}^\top - a\mathbf{y}\mathbf{y}^\top\right]\mathbf{Q}_{n+2}^\top,$$

where $\mathbf{Q}_{n+2}$ *is the* $(n+2)\times(n+2)$ *whose the entries are given by* (3.4b).

Proof. We only prove (a) since (b) can be proven in the same way. Consider a positive odd integer n and the following matrices

$$\mathbf{K}_{n+2} = \begin{bmatrix} b & 0 & \ldots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 \end{bmatrix},$$

$$\mathbf{G}_{n+2} = \begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & 0 & 0 \\ 0 & \dots & 0 & a \end{bmatrix}.$$

From Lemma 1,

$$\mathbf{P}_{n+2}^\top \mathbf{H}_{n+2} \mathbf{P}_{n+2} = \mathbf{P}_{n+2}^\top (\mathbf{A}_{n+2} - \mathbf{K}_{n+2} - \mathbf{G}_{n+2}) \mathbf{P}_{n+2} = \operatorname{diag}\left(\lambda_0, |\lambda_1|, \dots, |\lambda_{\frac{n+1}{2}}|, -|\lambda_{\frac{n+1}{2}}|, \dots, -|\lambda_1|\right) - b\mathbf{x}\mathbf{x}^\top - a\mathbf{y}\mathbf{y}^\top,$$

where $\mathbf{P}_{n+2}$ is the matrix defined by (3.3b), $\mathbf{A}_{n+2}$ is the matrix (3.1), and $\mathbf{x}$ is the first row of $\mathbf{P}_{n+2}$, i.e.,

$$\mathbf{x} = \sqrt{\frac{2}{n+2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \cos\left(\frac{\theta_1}{2}\right) \\ \vdots \\ \cos\left[\frac{\theta_{(n+1)/2}}{2}\right] \\ \sin\left[\frac{\theta_{(n+1)/2}}{2}\right] \\ \vdots \\ \sin\left(\frac{\theta_1}{2}\right) \end{bmatrix},$$

and $\mathbf{y}$ is the last row of $\mathbf{P}_{n+2}$,

$$\mathbf{y} = \sqrt{\frac{2}{n+2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \cos\left[\frac{\theta_1}{2} + \frac{2(n+1)\pi}{n+2}\right] \\ \vdots \\ \cos\left[\frac{\theta_{(n+1)/2}}{2} + \frac{(n+1)^2}{2} \cdot \frac{2\pi}{n+2}\right] \\ \sin\left[\frac{\theta_{(n+1)/2}}{2} + \frac{(n+1)^2}{2} \cdot \frac{2\pi}{n+2}\right] \\ \vdots \\ \sin\left[\frac{\theta_1}{2} + \frac{2(n+1)\pi}{n+2}\right] \end{bmatrix} = \sqrt{\frac{2}{n+2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \cos\left(\frac{\theta_1}{2} - \frac{2\pi}{n+2}\right) \\ \vdots \\ \cos\left[\frac{\theta_{(n+1)/2}}{2} - \frac{n+1}{2} \cdot \frac{2\pi}{n+2}\right] \\ \sin\left[\frac{\theta_{(n+1)/2}}{2} - \frac{n+1}{2} \cdot \frac{2\pi}{n+2}\right] \\ \vdots \\ \sin\left(\frac{\theta_1}{2} - \frac{2\pi}{n+2}\right) \end{bmatrix}.$$

□

Proof of Theorem 1. Consider a positive odd integer n, $\mathbf{x}, \mathbf{y}$ given by (3.9), and $d_1 = \lambda_0$, $d_2 = |\lambda_1|, \dots,$ $d_{\frac{n+3}{2}} = |\lambda_{\frac{n+1}{2}}|$, $d_{\frac{n+5}{2}} = -|\lambda_{\frac{n+1}{2}}|, \dots, d_{n+2} = -|\lambda_1|$. According to Lemma 2, it should be noted that the matrix $\mathbf{H}_{n+2}$ and

$$\operatorname{diag}(d_1, d_2, \dots, d_{n+2}) - b\mathbf{x}\mathbf{x}^\top - a\mathbf{y}\mathbf{y}^\top \tag{3.11}$$

share the same eigenvalues. Let us adopt the notations of [2] by denoting $\mathscr{S}(k, m)$ the collection of all k-element subsets of $\{1, 2, \dots, m\}$ written in the increasing order; in addition, for any rectangular matrix $\mathbf{M}$, we shall indicate by $\det(\mathbf{M}[I, J])$ the minor determined by the subsets $I = \{i_1 < i_2 < \dots < i_k\}$ and $J = \{j_1 < j_2 < \dots < j_k\}$. Setting

$$\mathbf{X} = \begin{bmatrix} -b\sqrt{\frac{2}{n+2}} & -a\sqrt{\frac{2}{n+2}} \\ -b\sqrt{\frac{2}{n+2}}\cos\left(\frac{\theta_1}{2}\right) & -a\sqrt{\frac{2}{n+2}}\cos\left(\frac{\theta_1}{2}-\frac{2\pi}{n+2}\right) \\ \vdots & \vdots \\ -b\sqrt{\frac{2}{n+2}}\cos\left[\frac{\theta_{(n+1)/2}}{2}\right] & -a\sqrt{\frac{2}{n+2}}\cos\left[\frac{\theta_{(n+1)/2}}{2}-\frac{(n+1)\pi}{n+2}\right] \\ -b\sqrt{\frac{2}{n+2}}\cos\left[\frac{\theta_{(n+1)/2}}{2}\right] & -a\sqrt{\frac{2}{n+2}}\cos\left[\frac{\theta_{(n+1)/2}}{2}-\frac{(n+1)\pi}{n+2}\right] \\ \vdots & \vdots \\ -b\sqrt{\frac{2}{n+2}}\cos\left(\frac{\theta_1}{2}\right) & -a\sqrt{\frac{2}{n+2}}\cos\left(\frac{\theta_1}{2}-\frac{2\pi}{n+2}\right) \end{bmatrix}^\top$$

and

$$\mathbf{Y} = \begin{bmatrix} \sqrt{\frac{2}{n+2}} & \sqrt{\frac{2}{n+2}} \\ \sqrt{\frac{2}{n+2}}\cos\left(\frac{\theta_1}{2}\right) & \sqrt{\frac{2}{n+2}}\cos\left(\frac{\theta_1}{2}-\frac{2\pi}{n+2}\right) \\ \vdots & \vdots \\ \sqrt{\frac{2}{n+2}}\cos\left[\frac{\theta_{(n+1)/2}}{2}\right] & \sqrt{\frac{2}{n+2}}\cos\left[\frac{\theta_{(n+1)/2}}{2}-\frac{(n+1)\pi}{n+2}\right] \\ \sqrt{\frac{2}{n+2}}\cos\left[\frac{\theta_{(n+1)/2}}{2}\right] & \sqrt{\frac{2}{n+2}}\cos\left[\frac{\theta_{(n+1)/2}}{2}-\frac{(n+1)\pi}{n+2}\right] \\ \vdots & \vdots \\ \sqrt{\frac{2}{n+2}}\cos\left(\frac{\theta_1}{2}\right) & \sqrt{\frac{2}{n+2}}\cos\left(\frac{\theta_1}{2}-\frac{2\pi}{n+2}\right) \end{bmatrix}^\top,$$

we have from Theorem 1 of [2] that ζ is an eigenvalue of (3.11) if and only if

$$1 + \sum_{k=1}^{n+2} \sum_{J\in\mathscr{S}(k,n+2)} \sum_{I\in\mathscr{S}(k,2)} \frac{\det(\mathbf{X}[I,J])\det(\mathbf{Y}[I,J])}{\prod_{j\in J}(d_j-\zeta)} = 0$$

provided that ζ is not an eigenvalue of $\operatorname{diag}(d_1,\ldots,d_{n+2})$. Since

$$\begin{aligned} 1 &+ \sum_{k=1}^{n+2} \sum_{J\in\mathscr{S}(k,n+2)} \sum_{I\in\mathscr{S}(k,2)} \frac{\det(\mathbf{X}[I,J])\det(\mathbf{Y}[I,J])}{\prod_{j\in J}(d_j-\zeta)} \\ &= 1 - b\sum_{k=1}^{n+2}\frac{[\mathbf{Y}]_{1,k}^2}{d_k-\zeta} - a\sum_{k=1}^{n+2}\frac{[\mathbf{Y}]_{2,k}^2}{d_k-\zeta} + ab\sum_{1\leqslant k<\ell\leqslant n+2}\frac{([\mathbf{Y}]_{1,k}[\mathbf{Y}]_{2,\ell}-[\mathbf{Y}]_{1,\ell}[\mathbf{Y}]_{2,k})^2}{(d_k-\zeta)(d_\ell-\zeta)}, \end{aligned}$$

we obtain (2.4a). Let $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_{n+2}$ be the eigenvalues of $\mathbf{H}_{n+2}$ and $d_{\tau(1)} \leqslant d_{\tau(2)} \leqslant \ldots \leqslant d_{\tau(n+2)}$ be arranged in nondecreasing order by some bijection τ defined in $\{1, 2, \ldots, n+2\}$. Thus,

$$\lambda_{\tau(k)} + \lambda_{\min}(-b\mathbf{x}\mathbf{x}^\top - a\mathbf{y}\mathbf{y}^\top) \leqslant \mu_k \leqslant \lambda_{\tau(k)} + \lambda_{\max}(-b\mathbf{x}\mathbf{x}^\top - a\mathbf{y}\mathbf{y}^\top) \tag{3.12}$$

for each $k = 1, 2, \ldots, n+2$ (see [9], page 242). By using Miller's formula for the determinant of the sum of matrices (see [13], page 70), we can compute the characteristic polynomial of $-b\mathbf{x}\mathbf{x}^\top - a\mathbf{y}\mathbf{y}^\top$,

$$\begin{aligned} &\det(t\mathbf{I}_{n+2} + b\mathbf{x}\mathbf{x}^\top + a\mathbf{y}\mathbf{y}^\top) \\ &\quad= [1 + bt^{-1}\mathbf{x}^\top\mathbf{x} + at^{-1}\mathbf{y}^\top\mathbf{y} + abt^{-2}(\mathbf{x}^\top\mathbf{x})(\mathbf{y}^\top\mathbf{y}) - abt^{-2}(\mathbf{x}^\top\mathbf{y})^2]\det(t\mathbf{I}_{n+2}) \\ &\quad= t^n[t^2 + (a+b)t + ab] \end{aligned}$$

because

$$\mathbf{x}^\top\mathbf{x} = \frac{2}{n+2}\left[\frac{1}{2} + \sum_{k=1}^{\frac{n+1}{2}}\cos^2\left(\frac{\theta_k}{2}\right) + \sum_{k=1}^{\frac{n+1}{2}}\sin^2\left(\frac{\theta_k}{2}\right)\right] = 1,$$

$$\mathbf{y}^\top\mathbf{y} = \frac{2}{n+2}\left[\frac{1}{2} + \sum_{k=1}^{\frac{n+1}{2}}\cos^2\left(\frac{\theta_k}{2} - \frac{2k\pi}{n+2}\right) + \sum_{k=1}^{\frac{n+1}{2}}\sin^2\left(\frac{\theta_k}{2} - \frac{2k\pi}{n+2}\right)\right] = 1,$$

and

$$\begin{aligned}\mathbf{x}^\top\mathbf{y} &= \frac{2}{n+2}\left[\frac{1}{2} + \sum_{k=1}^{\frac{n+1}{2}}\cos\left(\frac{\theta_k}{2}\right)\cos\left(\frac{\theta_k}{2} - \frac{2k\pi}{n+2}\right) + \sum_{k=1}^{\frac{n+1}{2}}\sin\left(\frac{\theta_k}{2}\right)\sin\left(\frac{\theta_k}{2} - \frac{2k\pi}{n+2}\right)\right] \\ &= \frac{2}{n+2}\left[\frac{1}{2} + \sum_{k=1}^{\frac{n+1}{2}}\cos\left(\frac{2k\pi}{n+2}\right)\right] \\ &= \frac{2}{n+2}\left\{\frac{1}{2} - \frac{1}{2} + \frac{\sin\left[\left(\frac{n+1}{2} + \frac{1}{2}\right)\frac{2\pi}{n+2}\right]}{2\sin\left(\frac{2\pi}{n+2}\right)}\right\} \\ &= 0.\end{aligned}$$

Hence, $\operatorname{Spec}(-b\mathbf{x}\mathbf{x}^\top - a\mathbf{y}\mathbf{y}^\top) = \{0, -a, -b\}$ and (3.12) yields (2.4b). The proof of the remaining assertion is performed in the same way and so will be omitted. □

Proof of Theorem 2. Since both assertions can be proven in the same way, we only prove (a). Let n be a positive odd integer, $a \neq 0$, and $\mu_1, \mu_2, \ldots, \mu_{n+2}$ the eigenvalues of $\mathbf{H}_{n+2}$. We can rewrite the matricial equation $(\mu_m \mathbf{I}_{n+2} - \mathbf{H}_{n+2})\mathbf{z} = \mathbf{0}$ as follows:

$$\mathbf{P}_{n+2}\Big[\mu_m\mathbf{I}_{n+2} - \operatorname{diag}\Big(\lambda_0, |\lambda_1|, \ldots, |\lambda_{\frac{n+1}{2}}|, -|\lambda_{\frac{n+1}{2}}|, \ldots, -|\lambda_1|\Big) + b\mathbf{x}\mathbf{x}^\top + a\mathbf{y}\mathbf{y}^\top\Big]\mathbf{P}_{n+2}^\top\mathbf{z} = \mathbf{0} \tag{3.13}$$

where $\mathbf{x}$ and $\mathbf{y}$ are defined in (3.9) and $\mathbf{P}_{n+2}$ is the matrix whose the entries are given by (3.3b). Thus,

$$\Big[\mu_m\mathbf{I}_{n+2} - \operatorname{diag}\Big(\lambda_0, |\lambda_1|, \ldots, |\lambda_{\frac{n+1}{2}}|, -|\lambda_{\frac{n+1}{2}}|, \ldots, -|\lambda_1|\Big) + b\mathbf{x}\mathbf{x}^\top + a\mathbf{y}\mathbf{y}^\top\Big]\mathbf{w} = \mathbf{0}, \quad \mathbf{w} = \mathbf{P}_{n+2}^\top\mathbf{z},$$

that is,

$$\mathbf{w} = \xi\Big[\mu_m\mathbf{I}_{n+2} - \operatorname{diag}\Big(\lambda_0, |\lambda_1|, \ldots, |\lambda_{\frac{n+1}{2}}|, -|\lambda_{\frac{n+1}{2}}|, \ldots, -|\lambda_1|\Big) + b\mathbf{x}\mathbf{x}^\top\Big]^{-1}\mathbf{y},$$

for $\xi \neq 0$ (see Theorem 5 of [4], page 41) and

$$\mathbf{z} = \xi\mathbf{P}_n\Big[\mu_m\mathbf{I}_{n+2} - \operatorname{diag}\Big(\lambda_0, |\lambda_1|, \ldots, |\lambda_{\frac{n+1}{2}}|, -|\lambda_{\frac{n+1}{2}}|, \ldots, -|\lambda_1|\Big) + b\mathbf{x}\mathbf{x}^\top\Big]^{-1}\mathbf{y},$$

is a nontrivial solution of (3.13). Thus, choosing $\xi = 1$, we conclude that the vector having components (2.6) is an eigenvector of $\mathbf{H}_{n+2}$ associated with the eigenvalue μ_m. □

Remark. From computational point of view, our approach still carries advantages; indeed, since both matrices $\mathbf{H}_{n+2}$ and $\operatorname{diag}(d_1, d_2, \ldots, d_{n+2}) - b\,\mathbf{x}\mathbf{x}^\top - a\,\mathbf{y}\mathbf{y}^\top$ possess the same eigenvalues (see Lemma 2), the algorithm proposed in [14] for rank–2 modification may be used to approximate them, enjoying from the benefits therein mentioned. On the other hand, the corresponding eigenvectors can be immediately computed by using (2.6) and (2.7), dismissing the algorithm also presented in [14] for eigenvectors update.

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