# WIDEBAND SPECTRUM SENSING FOR DYNAMIC SPECTRUM SHARING 

## LUÍS MIGUEL GOMES TAVARES

Master in Electronics and Computer Engineering

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## LUÍS MIGUEL GOMES TAVARES

Master in Electronics and Computer engineering

Adviser: José António Beltran Gerald
Assistant Professor, University of Lisbon
Co-advisers: João Carlos da Palma Goes
Full Professor, School of Science and Technology - NOVA University Lisbon

## Examination Committee:

Chair: Paulo da Costa Luís da Fonseca Pinto, Full Professor, School of Science and Technology - NOVA University Lisbon

Rapporteurs: Vitor Manuel Mendes da Silva,
Assistant Professor, University of Coimbra
José Pedro Borrego,
Deputy Director-General, Autoridade Nacional de Comunicações

Adviser: José António Beltran Gerald,
Assistant Professor, University of Lisbon
Members: Paulo Gustavo Martins da Silva,
Assistant Professor, University of Algarve
Paulo da Costa Luís da Fonseca Pinto,
Full Professor, School of Science and Technology - NOVA University Lisbon
Paulo Miguel de Araújo Borges Montezuma de Carvalho,
Associate Professor, School of Science and Technology - NOVA University Lisbon

## Wideband Spectrum Sensing for Dynamic Spectrum Sharing

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Aos meus pais
Às minhas filhas

To my parents
To my daughters

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## Abstract

The proliferation of wireless devices grows exponentially, demanding more and more data communication capacity over wireless links. Radio spectrum is a scarce resource, and traditional wireless networks deployed by Mobile Network Operators (MNO) are based on an exclusive spectrum band allocation. However, underutilization of some licensed bands in time and geographic domains has been reported, especially in rural areas or areas away from high population density zones. This coexistence of increasingly high data communication needs and spectrum underutilization is an incomprehensible scenario. A more rational and efficient use of the spectrum is the possibility of Licensed Users (known as Primary Users - PU) to lease the spectrum, when not in use, to Unlicensed Users (known as Secondary Users - SU), or allowing the SU to opportunistically use the spectrum after sensing and verifying that the PU is idle. In this latter case, the SU must stop transmitting when the PU becomes active.

This thesis addresses the spectrum sensing task, which is essential to provide dynamic spectrum sharing between PUs and SUs. We show that the Spectral Correlation Function (SCF) and the Spectral Coherence Function (SCoF) can provide a robust signal detection algorithm by exploiting the cyclostationary characteristics of the data communication signal. We enhance the most used algorithm to compute de SCF - the FAM (FFT Accumulation Method) algorithm - to efficiently compute the SCF in a local/zoomed region of the support ( $f ; \alpha$ ) plane (frequency/cycle frequency plane). This will provide the quick identification of spectral bands in use by PUs or free, in a wideband sampling scenario.

Further, the characterization of the probability density of the estimates of the SCF and SCoF when only noise is present, using the FAM algorithm, will allow the definition of an adaptive threshold to develop a blind (with respect to the noise statistics) Constant False Alarm Rate (CFAR) detector (using the SCoF) and also a CFAR and a Constant Detection Rate (CDR) detector when that characterization is used to obtain an estimate of the background noise variance (using the SCF).

Keywords: Spectrum Sensing, Spectrum Sharing, 5G, Cognitive Radio, Cyclostationarity, FAM - FFT Accumulation Method

## Resumo

A proliferação de dispositivos sem fios cresce de forma exponencial, exigindo cada vez mais capacidade de comunicação de dados através de ligações sem fios. O espectro radioelétrico é um recurso escasso, e as redes sem fios tradicionais implantadas pelos Operadores de Redes Móveis baseiam-se numa atribuição exclusiva de bandas do espectro. No entanto, tem sido relatada a subutilização de algumas bandas licenciadas quer ao longo do tempo, quer na sua localização geográfica, especialmente em áreas rurais, e em áreas longe de zonas de elevada densidade populacional. A coexistência da necessidade cada vez maior de comunicação de dados, e a subutilização do espectro é um cenário incompreensível. Uma utilização mais racional e eficiente do espectro pressupõe a possibilidade dos Utilizadores Licenciados (conhecidos como Utilizadores Primários - Primary Users - $P U$ ) alugarem o espectro, quando este não está a ser utilizado, a Utilizadores Não Licenciados (conhecidos como Utilizadores Secundários - Secondary Users - SU), ou permitir ao SU utilizar oportunisticamente o espectro após a deteção e verificação de que o PU está inativo. Neste último caso, o SU deverá parar de transmitir quando o PU ficar ativo.

Nesta tese é abordada a tarefa de deteção espectral, que é essencial para proporcionar a partilha dinâmica do espectro entre PUs e SUs. Mostra-se que a Função de Correlação Espectral (Spectral Correlation Function - SCF) e a Função de Coerência Espectral (Spectral Coherence Function $S C o F$ ) permitem o desenvolvimento de um algoritmo robusto de deteção de sinal, explorando as características ciclo-estacionárias dos sinais de comunicação de dados. Propõe-se uma melhoria ao algoritmo mais utilizado para cálculo da SCF - o método FAM (FFT Accumulation Method) para permitir o cálculo mais eficiente da SCF numa região local/ampliada do plano de suporte $f / \alpha$ (plano de frequência/frequência de ciclo). Esta melhoria permite a identificação rápida de bandas espectrais em uso por PUs ou livres, num cenário de amostragem de banda larga.

Adicionalmente, é feita a caracterização da densidade de probabilidade das estimativas da SCF e SCoF quando apenas o ruído está presente, o que permite a definição de um limiar adaptativo, para desenvolver um detetor de Taxa de Falso Alarme Constante (Constant False Alarm Rate CFAR) sem conhecimento do ruído de fundo (usando a SCoF) e também um detetor CFAR e Taxa de Deteção Constante ( Constant Detection Rate - CDR ), quando se utiliza aquela caracterização para obter uma estimativa da variância do ruído de fundo (usando a SCF).

Palavras-chave: Deteção espectral, Partilha de espectro, 5G, Rádio Cognitivo, Cicloestacionariedade, FAM - FFT Accumulation Method

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## List of Symbols and Abbreviations

## Symbols

| $f$ | Frequency |
| :--- | :--- |
| $f_{s}$ | Sampling frequency $\left(f_{s}=1 / T_{s}\right)$ |
| $T_{s}$ | Sampling period $\left(T_{s}=1 / f_{s}\right)$ |
| $\alpha$ | Cyclic frequency |
| $(f ; \alpha)$ | Frequency/cyclic frequency plane |
| $s(n)$ | Transmitted signal |
| $w(n)$ | Noise signal |
| $r(n)$ | Received signal |
| $P_{D}$ | Probability of detection |
| $P_{M}$ | Probability of missing |
| $P_{F A}$ | Decision threshold |
| $\gamma$ | Normal distribution, with mean $\mu$ and variance $\sigma^{2}$ |
| $\mathcal{N}\left(\mu, \sigma^{2}\right)$ | Complementary error function |
| $e r f c(x)$ | erfc $(x)=1-\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}} \int_{x}^{+\infty} e^{-t^{2}} d t$ |
|  | Error function |
| $e r f(x)$ | decibel |
| dB | Likelihood ratio |
| $\Lambda(\boldsymbol{r})$ | (Central) chi-square distribution with N degrees of freedom |
| $\chi_{N}^{2}$ | Non-central chi-square distribution with N degrees of freedom and |
| $\chi_{N}^{2}(\lambda)$ | noncentrality parameter $\lambda$ |
| $Q_{m}(a, b)$ | Generalized Marcum Q-function |

$$
Q_{m}(a, b)=\frac{1}{a^{m-1}} \int_{b}^{+\infty} x^{m} e^{-\frac{x^{2}-a^{2}}{2}} \mathrm{I}_{m-1}(a x)
$$

| $\rho$ | Noise variance uncertainty |
| :---: | :---: |
| $\mathcal{T} \mathcal{W}_{1}$ | Tracy-Widom distribution of order 1 |
| $\lambda_{i}$ | i-th eigenvalue |
| $R_{x}^{\alpha}(\tau)$ | Cyclic Autocorrelation Function |
| $S_{x}^{\alpha}(f)$ | Spectral Correlation Function |
|  | $S_{x}^{\alpha}(f) \triangleq \int_{-\infty}^{+\infty} R_{x}^{\alpha}(\tau) e^{-j 2 \pi f \tau} d \tau$ |
| $C_{x}^{\alpha}(f)$ | Spectral Coherence Function |
| $R_{x}(t, \tau)$ | Autocorrelation function |
| $M(f, \alpha)$ | Fourier transform of the kernel of the Cyclic Spectrum Analyzer |
| ${ }_{2} F_{1}(a, b ; c ; z)$ | Gauss hypergeometric function |
|  | ${ }_{2} F_{1}(a, b ; c ; z)=\sum_{n=0}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}} \frac{z^{n}}{n!}$ |
| $(a){ }_{n}$ | Pochhammer symbol |
|  | $(a)_{n}=\left\{\begin{array}{c} 1 \quad n=0 \\ a(a+1) \ldots(a+n-1) \quad n>0 \end{array}\right.$ |
|  | or |
|  | $(a)_{n}=\frac{\Gamma(a+k)}{\Gamma(a)}$ |
| $\Gamma(z)$ | Gamma function |
|  | $\Gamma(z)=\int_{0}^{\infty} x^{z-1} e^{-x} d x$ |
| $M(s)$ | Moment Generating Function |
| $\varphi(w)$ | Characteristic Function |
| $K_{n}(x)$ | Modified Bessel function of the second kind, of order $n$ |
| $\odot$ | Hadamard (doted) product |
| $\otimes$ | Kronecker product |
| $1 \mathrm{~cm}(a, b)$ | Least Common Multiple between $a$ and $b$ |

## Abbreviations

| 3GPP | $3^{\text {rd }}$ Generation Partnership Project |
| :--- | :--- |
| ADC | Analog-to-Digital Converter |
| AGC | Automatic Gain Control |
| AGM | Arithmetic to Geometric Mean (Detector) |
| AP | Access Point |
| AWGN | Additive White Gaussian Noise |
| BPF | Bandpass filter |
| BPSK | Binary Phase-Shift Keying |
| CAF | Cyclic Autocorrelation Function |
| CBRS | Citizens Broadband Radio Service |
| cdf | Cumulative Distribution Function |
| CDR | Constant Detection Rate |
| CENELEC | Comité Européen de Normalisation Électrotechnique |
| CEPT | Conference of Postal and Telecommunications Administrations |
| CF | Characteristic Function |
| CFAR | Constant False Alarm Rate |
| CR | Cognitive Radio |
| CRN | Cognitive Radio Network |
| CSA | Concurrent Spectrum Access |
| CTA | Chirp Transform Algorithm |
| DAC | Digital-to-Analog Converter |
| DSA | Dynamic Spectrum Access |
| DTT | Digital Terrestrial Television |
| DTV | Digital TV Communications Committee |
| DYSPAN | Dynamic Spectrum Access Networks |
| EC | ECC |


| ECMA | European Computer Manufacturers Association |
| :---: | :---: |
| ED | Energy Detector |
| ESC | Environment Sensing Capability |
| ETSI | European Telecommunications Standards Institute |
| EU | European Union |
| FAM | FFT Accumulation method |
| FCC | Federal Communications Commission |
| FFT | Fast Fourier Transform |
| FLOP | (real) Floating-point Operation |
| FPGA | Field-Programmable Gate Array |
| FSA | Fixed Spectrum Allocation |
| FT | Fourier Transform |
| GAA | General Authorized Access |
| GEV | Generalized Extreme Value |
| i.i.d. | Independent and Identically Distributed |
| IEEE | Institute of Electrical and Electronics Engineers |
| IFFT | Inverse Fast Fourier Transform |
| IMT | International Mobile Telecommunications |
| ISM | Industrial Scientific and Medical |
| ITU | International Telecommunication Union |
| LAA | Licensed Assisted Access |
| LAN | Local Area Network |
| LBT | Listen-Before-Talk |
| LNA | Low-Noise Amplifier |
| LO | Local Oscillator |
| LPF | Lowpass Filter |
| LSA | Leased Shared Access |
| LTE | Long-Term Evolution |
| MAC | Medium Access Control |


| ME/MED | Maximum Eigenvalue (Detector) |
| :---: | :---: |
| MEAM/MEAMD | Maximum Eigenvalue to Arithmetic Mean (Detector) |
| MFD | Matched Filter Detector |
| MGF | Moment Generating Function |
| ML | Maximum Likelihood |
| MME/MMED | Maximum-Minimum Eigenvalue (Detector) |
| MNO | Mobile Network Operator |
| NR | New Radio |
| NRA | National Regulatory Authority |
| OFDM | Orthogonal Frequency-Division Multiplexing |
| OSA | Opportunistic Spectrum Access |
| PA | Power Amplifier |
| PAL | Priority Access License |
| pdf | Probability Density Function |
| PMSE | Programme Making and Special Events |
| PSD | Power Spectral Density |
| PU | Primary User |
| QoS | Quality-of-Service |
| QPSK | Quadrature Phase-Shift Keying |
| RAN | Regional Area Network |
| RF | Radio Frequency |
| ROC | Receiver Operating Characteristic |
| RSPG | Radio Spectrum Policy Group |
| RSPP | Radio Spectrum Policy Programme |
| r.v. | Random Variable |
| SAS | Spectrum Access System |
| SCF | Spectral Correlation Function |
| SCoF | Spectral Coherence Function |
| SCP | Smoothed Cyclic Periodogram |


| SDR | Software Defined Radio |
| :--- | :--- |
| SMLE/SMLED | Sum of Multiple Large Eigenvalues (Detector) |
| SNR/snr | Signal-to-Noise Ratio (logarithmic/linear) |
| SoI | Signal-of-Interest |
| SSE/SSED | Signal-Subspace Eigenvalues (Detector) |
| SSCA | Strip Spectral Correlation Analyzer |
| SU | Secondary User |
| TCAM | Telecommunication Conformity Assessment and Market Committee |
| TVWS | TV White Spaces |
| UHF | Ultra-High Frequency |
| VHF | Very-High Frequency |
| VNI | Visual Network Index (Cisco) |
| WLAN | Wireless Local Area Network |
| WRAN | Wireless Regional Area Network |

## Chapter 1

## 1 Introduction

In today's world, the proliferation of wireless devices grows exponentially, demanding more and more data communication capacity over wireless links. According to the Cisco Visual Networking Index (VNI) (Cisco, 2017) - one of the most cited sources in the world for global IP traffic, and updated in 2020 (Cisco, 2020) - the global mobile data traffic is expected to grow around 7 -fold from 2016 to 2023, representing a compound annual growth rate of $46 \%$ (from 7,200 PByte/month in 2016 to 48,270 PByte/month in 2021, and similar growth rate till 2023). Figure 1.1 shows this forecast evolution, detailing the application type of data traffic (although referenced in the report, the File Sharing application data is too small to be visible in the graphic, being around $0.3 \%$ of global data traffic). This figure shows the combined Cisco forecast for the decade 2011-2021, merging the VNI 2011 and VNI 2017 sets. The evolution is quite impressive, showing the exponential growth in data communication needs. Cisco started discontinued the online VNI tool, doing annual update reports, with the latest issued in 2020, as mentioned.


Figure 1.1 - Data traffic forecast in PByte/month, CISCO VNI (Cisco, 2017).

Also, the current proliferation of the Internet-of-Things (IoT) will scale up communications needs. Several studies predict an almost exponential growth in the number of IoT devices, totaling
between 20 and 30 billion by 2020/2023, e.g. (Calsoft, 2018)(Cisco, 2020), with a forecast of 500 billion devices connected to the Internet by 2030 (Cisco, 2016).

This continuous increase in demand for new and faster wireless services and applications creates a natural shortage in the available wireless spectrum. Furthermore, traditional wireless networks deployed by telecommunication service providers (licensed users) are based on an exclusive allocation of spectrum bands, bought in national auctions, where the regulation process guarantees that any other radio service does not create harmful interference on those bands. This model is naturally the one preferred by network operators, allowing a middle to long-term predictable and sustainable network investment with financial return. This model also has the benefit of avoiding any digital divide, as network operators are obliged to invest in low economic interest areas, by being granted exclusivity in high economic interest areas through coverage obligations in spectrum licenses.

However, this licensing model, long-established and in use today, is being questioned worldwide for two main reasons: underutilization of licensed bands both in time and geographical domains, and the aforementioned growing demand for radio communications services and broadband ubiquitous access.

Several spectrum surveys have been done in the past, mainly in the USA. The most cited report on spectrum utilization was carried out in the USA by the Federal Communications Commission - Spectrum Policy Task Force (R. Engelman, K. Abrokwah, 2002) in several cities (Atlanta, New Orleans, San Diego and New York) and for approximately 700 MHz below 1 GHz . This report showed that large portions of the licensed spectrum were underused, reaching, on average, only $5 \%$ to $12 \%$ usage. Nevertheless, during peak hours, $85 \%$ usage could be reached.

In another study, carried out from 2004 to 2005 in several USA locations, and for the 30 MHz to 3 GHz band (Mchenry and McCloskey, 2006), similar conclusions were reached, indicating an average of $5.2 \%$ occupancy. In addition, the maximum average occupancy was measured in New York ( $13.1 \%$ ) and the minimum ( $1 \%$ ) in a radio quiet zone (the National Radio Astronomy Observatory). Surprisingly, from this study, the occupancy ratio between a densely populated area and a radio-quiet zone is only about thirteen.

Similar surveys were undertaken in other parts of the world, resulting in similar underutilization conclusions, e.g., in Europe for the $470-790 \mathrm{MHz}$ band (Achtzehn et al., 2012), Singapore for the $80-5,859 \mathrm{MHz}$ band (Islam et al., 2008). Also, for more in-depth results, several references can be found in (Chen and Oh, 2014). Spectrum surveys continued to be held around the world for several purposes. As an example, the Software Defined Receiver IoT Spectrum Survey (SDRIOTSS) project, started in 2021, captures Internet of Things (IoT) signals in the Radio Frequency (RF) spectrum to improve awareness of IoT communications in congested and contested RF environments (Ritchie, 2021).

The license-exempt Industrial Scientific and Medical (ISM) (ITU, 2016) bands are typically used for personal local wireless networks. These bands are reserved internationally for industrial, scientific and medical applications other than telecommunications. Equipment operating in these bands must deal with harmful interference generated by other applications, without any regulatory protection. As these bands are being used by growing unlicensed devices, they too are becoming
overcrowded, with estimates of 75.38 billion connected license-exempt devices in the EU by 2030 (Tech4i2 Limited and Real Wireless, 2016). The 2.4 GHz already suffers congestion today, which is expected to increase; the 5 GHz band is expected to be further congested as devices move from the 2.4 to the 5 GHz band. Underused spectrum was found in several bands, harmonized in the EU for short-range devices ${ }^{1}$, namely: the $401-406 \mathrm{MHz}$ band (Active medical implant devices); the $433.05-434.79 \mathrm{MHz}$ band (Non-specific short-range devices); the $868.0-869.65 \mathrm{MHz}$ band (Non-specific short-range devices and Low duty cycle/high-reliability devices), and the 2.4835 2.5 GHz band (Active medical implant devices). The $5.725-5.875 \mathrm{GHz}$ band (Non-specific shortrange devices) is currently lightly used with a slow take-up towards 2030 (Tech4i2 Limited and Real Wireless, 2016).

Regulatory approaches along the EU (and worldwide) are being carried out to allow more rational use of RF spectrum, namely by leasing spectrum to other users (Secondary Users - SU) when not in use by the owner (Incumbent, Licensee, or Primary User - PU) or opportunistic use by some SU after sensing the spectrum and verifying that the PU is idle, stopping transmitting when the PU becomes active.

Allowing a more rational and efficient (re)use of the RF spectrum calls for the development of radios capable of performing a (possibly wide) spectrum sensing to detect used and idle bands. This knowledge would enable an opportunist usage of the detected idle bands, therefore improving the overall spectrum usage efficiency when compared to a fixed spectrum allocation method.

### 1.1 Implementations of Dynamic Spectrum Access

The concept of Dynamic Spectrum Access (DSA) will be addressed thoroughly in the next chapter as it is considered the enabling strategy to achieve spectral usage efficiency. In the following subsections, we will list some implementations using DSA, some of which naturally rely on spectrum sensing.

We consider that spectrum sensing is the key enabler to a successful DSA strategy, providing efficient radio spectrum usage and sharing. Although incumbent licensees prefer a fixed spectrum allocation, to easily achieve a desired Quality-of-Service (QoS), only a fine-grained temporal or geographic shared access will provide a fair, rational, and efficient use of the scarce spectrum available. The RF spectrum is a national resource, and governments also prefer a fixed spectrum allocation, resulting from an auction system to sell the rights of usage of specific bands, therefore securing a high revenue in the process. However, the need to improve the spectrum usage should lead to new spectrum sharing mechanisms, where the RF band rights owners will be called to cooperate to cope with the ever-increasing data traffic requirements.

[^0]
## TV White Spaces (TVWS)

As referenced by the Dynamic Spectrum Alliance ${ }^{2}$, and although the Alliance supports several technologies, the TV White Spaces (TVWS) is the more mature one. The regulation and standards already exist, and broadband services have already been tested in the field (Sum et al., 2013). The switch from Analog Television to the Digital Terrestrial Television (DTT) led to large portions of the VHF/UHF TV spectrum to become vacant, on a geographic basis, named TV White Spaces. Many countries have already auctioned parts of the spectrum released by the digital switch-over to private broadband operators. However, the spectrum exclusively assigned to the DTT is not entirely used, with spectrum white spaces available at different locations. However, re(use) of these TVWS must rely on accurate detection of occupied bands by licensed/primary users (PU), namely DTT channels or Programme Making and Special Events (PMSE) users.

The two algorithms more widely considered to guarantee the protection of PUs are the Geolocation Database and Spectrum Sensing, as adopted by the FCC in the United States, Ofcom in the United Kingdom, and proposed to the EU in the Electronic Communication Committee (ECC) Report 159 (European Electronic Communications Committee, 2011). In the geolocation algorithm, an official geolocation database can list and assign the available/free frequency bands and maximum transmit power to any secondary user (SU) willing to use the available spectrum, based on the SU location and PUs band occupancy (Wang, Gao and Evans, 2015). However, for unlicensed use of TVWS, such as PMSE services in Europe, the spectrum sensing algorithm must be used, where SUs will sense the RF spectrum and use an idle/free band as long as the PU is absent. Should the PU become active, the SU must stop transmitting on that band, and start searching for a new free one, thus providing dynamic access to the TVWS.

As referenced in (Sum et al., 2013), possible usage scenarios for TVWS include (just the ones considered more important):

- Rural Broadband Access. Although wireless systems are the natural solution for broadband access in rural environments, the high frequency of conventional systems (2.4 and 5 GHz ) implies high infrastructure and maintenance costs for acceptable coverage. The long-reaching of lower frequency TVWS makes them the natural solution for suitable coverage;
- Smart Utility and IoT. Connectivity of smart meters from electricity, gas, and water, along with other IoT devices in a wireless network, is a possible reality based on the unlicensed TVWS technology;
- Public Safety. Most police, fire, and civil protection wireless safety networks provide only voice or low-speed data. Again, the long-reaching frequencies of TVWS would provide a broadband public safety network with advanced services;
- Multimedia in Transportation. TVWS could guarantee establishing a large area network connectivity, to give internet access to people using public/private transportation like buses or trains.

[^1]However, TVWS did not evolve as one would initially assume, mainly due to the rapid evolution of WiFi networks and lack of regulation (Song, 2022).

## Citizens Broadband Radio Service (CBRS)

The Dynamic Spectrum Alliance also supports the Citizens Broadband Radio Service (CBRS) in full use in the United States since January 2020 (FCC, 2020) and with pilot tests in Europe (Kokkinen, 2017). The three-tier model in the US allows the sharing of the 5G mid-band (3.553.7 GHz ) among Incumbent, Priority Access License (PAL) and General Authorized Access (GAA) users. Access to spectrum channels uses the geolocation database concept, administered by some Spectrum Access System administrators, known as the CBRS Spectrum Access System (CBRSSAS $)^{3}$.

The Incumbent user is in the upper tier, with guaranteed protection from any other users' interference. In the US, Incumbent User services include high-powered defense radar systems on fixed, mobile shipborne, and airborne platforms, as well as fixed satellite service earth stations. To guarantee interference protection, the SAS uses the information provided by the Environment Sensing Capability (ESC) sensor network that monitors the Incumbent radio activity along the US. When the ESC sensors detect an Incumbent transmission, it alerts the SAS, which in turn activates a protection zone for that area by updating the geolocation database. For the shipborne platform, there are 41 non-overlapping coastal areas ${ }^{4}$ to detect Incumbent naval radio activity and signaling the SAS accordingly. When activity is detected, a protection zone is established; when no activity is detected, channels are marked free for other users' usage (Sahoo et al., 2018).

Priority Access License users can use spectrum bands as long as they do not interfere with incumbent users. The General Authorized Access user can also use the available bands, but without interfering with either Incumbent or Priority users. Priority users obtain a Priority Access Licensee (PAL) in a specific auction for 10 MHz bands, granting them precedence over the General users. General users do not need a license but must meet FCC technical, financial, character, and citizenship qualifications to be eligible as that type of user.

The users who want to use some CBRS band first have to request it to the cloud-based CBRSSAS in order to proceed to the reservation of unused channels in that particular geographic area. If channels are free, SAS will grant the requests. When devices that have been granted permission to transmit on CBRS channels have finished usage, the channels are put back into the pool of free channels that the SAS can choose from to grant further requests.

The CBRS technology aims to be used in various scenarios, namely:

- A complement to Mobile Networks. CBRS spectrum can provide Mobile Network Operators the possibility to enhance their small cell layer without fixed band allocation, increasing network capacity, especially in urban areas. Also, the use of CBRS bands can provide carrier aggregation, thus boosting network throughput;

[^2]- Fixed Wireless Access. Technology at 3.5 GHz can deliver network speed close to the fixed fiber;
- Private Networks. CBRS can enable a property owner or enterprise to deploy private networks using LTE or 5G technology, providing reliability and high QoS, where before they could probably only use Wi-Fi technology;
- Industrial IoT. 5G networks on CBRS channels can ensure an optimized and robust onpremise network solution for IoT connectivity with high reliability, high throughput, and low latency.

Although CBRS took time to initial deployment, it started enabling connectivity for rural broadband, sports venues, or remote industrial (private) networks in 2019 in the EUA. As referenced, CBRS can also be the enabling technology for the short-term deployment of private LTE/5G networks (JuniperNetworks, 2021).

## MulteFire

MulteFire technology ${ }^{5}$, in Release 1.1 since 2018, has similar objectives as CBRS, allowing enterprises to deploy LTE private networks in unlicensed regional and global spectrum bands worldwide. Using the global unlicensed $5 \mathrm{GHz}, 2.4 \mathrm{GHz}$, and regional $800 / 900 \mathrm{MHz}$ and 1.9 GHz bands, MulteFire networks need no approval or allocated spectrum for deployment. MulteFire networks can co-exist or overlap with other MulteFire or Wi-Fi networks, given the provided "Listen-Before-Talk" (LBT) mechanism (with spectrum sensing). The LBT aims to guarantee a friendly coexistence, naturally using a spectrum sensing procedure. This graceful coexistence is defined as assuring that the performance of an individual node in a network with $m$ Wi-Fi Access Points (AP) and $n$ MulteFire stations is not worse than under a network with $m+n$ Wi-Fi APs.

### 1.1.1 TV White Spaces (TVWS) standards: IEEE 802.22, IEEE 802.11af and ECMA-392

TVWS standards have been mainly developed by two organizations: IEEE and ECMA. The IEEE 802.22 (IEEE International, 2019) standard for Wireless Regional Area Networks (WRANs) and the IEEE 802.11af (IEEE International, 2013) standard for Wireless Local Area Networks (WLANs) use the concept of spectrum management, namely the geolocation database, and dynamic spectrum access to provide wireless access services (or low-powered wireless microphones) using the same radio spectrum allocated to analog and digital TV service, on TV White Spaces. Specifically, the 802.22 standard addresses this wireless access service for a large coverage area, whereas the 802.11af addresses a small coverage one. The geolocation database enables the operation at different zones worldwide, providing information on free channels available and for

[^3]how long, and their respective bandwidth and maximum transmit power allowed (Khattab and Bayoumi, 2015).

The ECMA-392 standard was first published in 2009, now in its second edition (ECMA International, 2012), targets a Wireless Local Area Network (WLAN) aiming to provide internet access in homes, campus, parks, etc. This standard is similar to IEEE 802.11af, where both provide the information on TVWS available from a geolocation database. However, ECMA-392 additionally incorporates the spectrum sensing functionality to periodically check the existence of possible incumbent PU on the channel in use, forcing the SU to stop transmitting (Um, Hwang and Jeong, 2012).

### 1.1.2 4G Long-Term Evolution - Unlicensed (LTE-U), Licensed Assisted Access (LAA) and 5G New Radio - Unlicensed (NR-U)

As the number of mobile subscribers grows steadily, and without the 5G network established and fully functioning worldwide, the Third Generation Partnership Project (3GPP) developed extensions to the Long-Term Evolution (LTE) wireless standard to allow cellular operators to use the unlicensed 5 GHz band $^{6}$, therefore offloading part of their data traffic to that band, in a Carrier Aggregation scenario.

## LTE-U

Although developed outside the 3GPP partnership, and mainly led by Qualcomm, the proprietary technology known as LTE-U is being evaluated in several countries. As it does not incorporate the Listen-Before-Talk (LBT) mechanism, it does not meet the regulatory requirements in several parts of the world (e.g., Europe and Japan). Nevertheless, this technology can be used in countries like the USA, Korea or China, where the LBT mechanism is not mandatory. Nevertheless, some mechanisms are included to allow a fair coexistence in the unlicensed bands (Qualcomm Research, 2014):

- Channel Selection, where an energy detector is used to assess channel utilization and to provide the selection of a channel not in use (clean channel);
- Carrier-Sensing Adaptive Transmission, where when no clean channel can be found, LTE-U user will transmit on an occupied channel, sharing it with some other Wi-Fi network. The channel is sensed for a long period (from 10 ms to 200 ms ), and, based on the channel occupancy, it will stop transmitting in a defined time cycle, proportionally to the detection of other Wi-Fi users in the sensing period.

[^4]
## LAA

The Licensed Assisted Access (LAA) technology is standardized by the 3GPP, starting in Release 13 (3GPP, 2015b), and it includes the Listen-Before-Talk mechanism, so that LAA should not impact Wi-Fi more than any other Wi-Fi network on the same channel. In 3GPP Release 13, LAA was defined only for the downlink; Release 14 (3GPP, 2015a) defined the enhanced-LAA (eLAA), including the uplink. Future releases continued to develop the technology referred to as Further enhanced LAA (FeLAA) (3GPP, 2017).

## NR-U

Starting in 3GPP Release 15 (3GPP, 2017), the development of New Radio (NR) set the path to fulfill the 5 G requirements established by the International Telecommunication Union (ITU) in the International Mobile Telecommunications 2020 Standard (IMT-2020) ${ }^{7}$. The Dynamic Spectrum Sharing solution was chosen to efficiently enable a smooth transition from 4 G to 5 G , allowing LTE and NR to share the same frequency bands.

As with LTE, the 3GPP addressed the NR unlicensed (NR-U) in Release 16 (3GPP, 2018). NRU's main goal is to extend NR operation to the unlicensed spectrum bands available, offloading part of their data traffic to those bands, but keeping a fair coexistence between different networks and radio access technologies. Unlike the LAA technology that addressed the 5 GHz band, NR-U now considers multiple bands, including millimeter wavelength bands: $2.4 \mathrm{GHz}, 5 \mathrm{GHz}, 6 \mathrm{GHz}$, and 60 GHz unlicensed bands, as well as 3.5 GHz and 37 GHz bands (Zhen et al., 2020). Moreover, and differently from LTE, a standalone operation in unlicensed bands was defined for 5 G NR-U (Lagen et al., 2020) (Naik et al., 2020).

### 1.1.3 Licensed Shared Access

The only spectrum sharing method so far recognized by CEPT in Europe is the Licensed Shared Access (LSA) regulation approach (Kalliovaara et al., 2018), as defined by the ECC (ECC, 2014), for administrations wishing to introduce Mobile/Fixed Communications Networks while maintaining the incumbent use. The ECC issued a decision in June 2014, with harmonized conditions, using LSA, for the $2,300-2,400 \mathrm{MHz}$ band $^{8}$. This is the pioneer band used in Europe to test LSA, with pioneer tests already taking place in Spain in 2015; Italy, Finland and France in 2016; The Netherlands in 2017 and Portugal in 2019.

ETSI, which produces globally applicable standards for Information and Communications Technologies, including fixed, mobile, radio, broadcast and Internet technologies, has completed

[^5]the specification for the support of LSA in $2018^{9}$. This provides a means to enable spectrum sharing coordination between LSA licensees and existing spectrum licensees, thereby ensuring Quality of Service (QoS).

With LSA licensee, incumbents can temporarily lease their licensed bands to other operators or Secondary Users, guaranteeing a predictable Quality of Service (QoS). The key enablers of LSA are the LSA Repository and the LSA Controller. The LSA Repository is the database containing the information of spectrum bands available from incumbents and the spectrum sharing rules. The LSA Controller guarantees an interference-free operation on the available spectrum by the incumbents or by the Secondary Users. A new concept called LSA evolution is still under development, which considers the use of spectrum sensing to provide a more dynamic version of traditional LSA, needed for 5G usage scenarios (Yrjölä and Kokkinen, 2017) (Frascolla et al., 2016).

The LSA test pilot results in Portugal were reported in 2020 and presented a different approach from traditional solutions; instead of sensing the PU, it envisions the sensing of a known SU (an LTE base station). In May 2021, the Portuguese regulatory authority for postal communications and electronic communications - Autoridade Nacional de Comunicasões - ANACOM - organized a webinar aimed at highlighting the results obtained and the main conclusions, as well as hearing the opinion of all the project's stakeholders, collecting comments from other countries that have promoted similar studies, as well as from international organizations with direct involvement in the issues underlying the LSA, such as the European Commission (EC), the European Conference of Postal and Telecommunications Administrations (CEPT), and the European Telecommunications Standards Institute (ETSI) ${ }^{10}$.

As the $2,300-2,400 \mathrm{MHz}$ band in Portugal is licensed to the three national TV broadcasters (RTP, SIC, TVI) for Programme Making and Special Events (PMSE), ANACOM's goal was to develop a system that could allow the implementation of the LSA concept (using an LTE Base station as SU) without predefining safety areas or introducing a notification in case of a PMSE signal being transmitted somewhere. The solution devised had to include and develop an LTE spectrum sensing device to be used by the PU, to sense this SU and alert the LSA controller to force the LTE SU to stop transmitting (Milheiro et al., 2020)(Kokkinen et al., 2019).

### 1.2 Spectrum Regulation in the EU

At a national level, the radio spectrum is managed by National Administrations, responsible for defining the national table of radio spectrum allocations ${ }^{11}$, establishing the framework for radio spectrum usage, and assignment of the radio spectrum to different users.

[^6]At the European Union level, the EC, ETSI, and the Electronic Communications Committee (ECC) of the CEPT cooperate on aspects related to the regulatory environment for radio equipment and spectrum, both at the EU level and at a broader intergovernmental level across Europe (see Figure 1.2).

The Radio Spectrum Policy Group (RSPG), established under EC decision 2002/622/EC, is a high-level advisory group that assists the European Commission in the development of radio spectrum policy at a strategic level, namely on:

- Radio spectrum policy issues;
- Coordination of policy approaches;
- Harmonized conditions, where appropriate, regarding the availability and efficient use of radio spectrum necessary for the establishment and functioning of the internal market.

The Telecommunication Conformity Assessment and Market Committee (TCAM), established under EC decision 1999/5/EC, assists the EC as an advisory and a regulatory committee in conformity assessment of radio equipment in the EU (Directive 2014/53/EU).

The CEPT is a European cooperative body of 48 national regulatory administrations in the field of posts and telecommunications. It is a recognized regional organization acting in accordance with pan-European goals set up by CEPT.


Figure 1.2 - Relationship between entities involved in the European spectrum regulation.
The ECC brings together 48 countries to develop common policies and non-binding regulations in electronic communications and related applications for Europe, to provide the focal point for information on spectrum use. Its primary objective is to harmonize the efficient use of the radio spectrum, satellite orbits and numbering resources across Europe. It takes an active role at the international level, preparing common European proposals to represent European interests in the ITU and other international organizations.

The ECC undertakes compatibility studies and establishes conditions and parameters under which sharing the spectrum between different users may occur. The ECC recommendations are crucial to the EC decisions and establishment of EU directives, which are produced as a response to mandates of the EC on specific topics, namely the RF spectrum harmonization within the EU.

The European Union officially recognizes ETSI as a European Standards Organization under the Regulation (EU) No. 1025/2012. ETSI is an independent, not-for-profit association with more than 900 members (including national administrations, companies and international organizations), drawn from 65 countries across five continents worldwide ${ }^{12}$, participating directly in its work. ETSI was created under the auspices of CEPT in 1988, which transferred all of its telecommunication standardization activities to ETSI. The ECC has strong cooperation with ETSI to ensure coherence between ECC decisions and ETSI Harmonized Standards. For reference, Figure 1.3 shows the relation between European Standard Organizations and International Standards Organizations (including not only ETSI but CEN and CENELEC).


Figure 1.3 - Relation between European and International Standards Organizations.

### 1.2.1 Harmonized Radio Spectrum in the EU

In 2012 the EU Parliament approved the multi-annual Radio Spectrum Policy Programme $(R S P P)^{13}$, aiming to achieve a " 1200 MHz " total bandwidth, harmonized in all EU countries by 2015 for wireless broadband. This decision requires the Member States to allow transfers or leases of "rights of use" of spectrum in specified harmonized bands. This would allow enough spectrum available for wireless applications and services, including the 4G wireless broadband, entertainment/culture (mobile TV or wireless electronic books), transport systems, health (such as medical appliances and devices to assist disabled persons), research, civil protection, the environment and energy (including smart energy grids and smart metering systems), without

[^7]affecting the requirements of other policies such as defense.
This decision was considered crucial to reducing the digital divide in the EU, contributing to the Digital Single Market ${ }^{14}$ as envisioned by the EC. The RSPP defined frequency bands ${ }^{15}$ and specific steps to be taken by the EU member states and the EC, namely:

- By the end of 2012, Member States should have authorized the use of the harmonized 2.52.69 GHz, 3.4-3.8 GHz, and $900 / 1800 \mathrm{MHz}$ bands for use by wireless broadband communications, including 3rd and 4th generation mobile communication services
- By 1 January 2013, all Member States (unless an individual exemption has been obtained before that date) should have authorized the use of the 800 MHz band for wireless broadband communications. One of the main objectives here is to cover sparsely populated areas
- By mid-2013 at the latest, the Commission, in cooperation with the Member States, will set out the details for an inventory to analyze efficient spectrum use in the EU, in the 400 MHz to 6 GHz range. This will form the basis of possible further action on the coordinated allocation of spectrum bands to specific uses, such as wireless broadband
- By 2015 at the latest, EU members will provide Spectrum trading between spectrum users in a set of harmonized bands where flexible use has already been introduced
- By 2015 at the latest, the Commission and the Member States must ensure sufficient harmonized spectrum becomes available for safety services and civil protection.

In recent years, several actions have been taken regarding the inventory in the 400 MHz to 6 GHz band, leading to accomplishing the " 1200 MHz " objective of the RSPP. By 2018 a total of 1260 MHz bandwidth had already been harmonized ${ }^{16}$, as shown in Table 1.1 (these bands include uplink and downlink channels).

| Frequency Band [MHz] | Bandwidth [MHz] | EU decision |
| :--- | :---: | :--- |
| $694-790$ <br> $(703-733 \& 758-788 \mathrm{MHz})$ | 80 | $2016 / 687 / \mathrm{EU}$ |
|  |  | Amendment: <br> $2017 / 899 / \mathrm{EU}$ |
| $790-862$ <br> $(791-821 \& 832-862 \mathrm{MHz})$ | 60 | $2010 / 267 / \mathrm{EU}$ |
| $880-915$ | 35 | $2009 / 114 / \mathrm{EC}$ |

[^8]| Frequency Band [MHz] | Bandwidth [MHz] | EU decision |
| :--- | :---: | :--- |
| $925-960$ | 35 | $2009 / 114 / \mathrm{EC}$ |
| $1,427-1,452$ | 25 | $2018 / 661 / \mathrm{EU}$ |
| $1,452-1,492$ | 40 | $2015 / 750 / \mathrm{EU}$ <br> Amendment: <br> $2018 / 661 / \mathrm{EU}$ |
| $1,492-1,517$ |  | $2018 / 661 / \mathrm{EU}$ |
| $1,710-1,785$ | 25 | $2009 / 766 / \mathrm{EC}$ |
| $1,805-1,880$ | 75 | $2009 / 766 / \mathrm{EC}$ |
|  |  | Amendments: |
| $1,920-1,980$ | 60 | $2011 / 251 / \mathrm{EU}$, |
| $2,110-2,170$ | 60 | $2018 / 637 / \mathrm{EU}$ (IoT) |
| $2,300-2,400$ | 100 | $2012 / 688 / \mathrm{EU}$ |
| $2,500-2,690$ | 190 | $2012 / 688 / \mathrm{EU}$ |
|  |  | Under consideration |
|  |  | $2008 / 477 / \mathrm{EC}$ |
|  |  | derogation |
|  |  | Bulgaria $2009 / 1 / \mathrm{EC} ;$ |
|  |  | France |

Table 1.1 - Frequency bands harmonized within the EU.
These bands are harmonized in the EU on a non-exclusivity basis, thus opening the way to the possibility of spectrum sharing and reuse. Also, these bands can be used in a technology-free perspective, where harmonized technical usage conditions establish a predictable sharing environment through the combination of a frequency band with transmit power limit/field strength limit/power density limit, as well as some additional parameters and usage restrictions, based on underlying compatibility studies. Therefore, such conditions should prevent harmful interference, foster reliable and efficient use of frequency bands, allow flexibility for a variety of applications, and, consequently, make it possible for devices in most Member States to be operated on a nonexclusive and shared basis. Nevertheless, when harmonized technical conditions or general authorization conditions would not be sufficient by themselves to ensure appropriate quality of service, Member States would remain free to apply individual rights for shared non-exclusive use of these bands.

### 1.2.2 Present and Future Harmonized Radio Spectrum in the EU for 5G

In July 2018, the European Commission issued a mandate to CEPT to review the harmonized technical conditions for EU-harmonized bands and to develop less restrictive technical conditions
suitable for next-generation 5G, terrestrial wireless systems. Specifically, under consideration, are the bands: $880-915 \mathrm{MHz}$ and $925-960 \mathrm{MHz}$ (' 900 MHz band'); $1,710-1,785 \mathrm{MHz}$ and $1,805-1,880$ MHz (' 1800 MHz band'); $1.920-1.980 \mathrm{GHz}$ and $2.110-2.170 \mathrm{GHz}$ ('paired terrestrial 2 GHz band') and $2.500-2.690 \mathrm{MHz}$ ('2.6 GHz band').

This mandate is a follow-up of the December 2016 mandate, to develop harmonized technical conditions for spectrum use to support the introduction of 5 G terrestrial wireless systems. The $3.400-3.800 \mathrm{GHz}$ band was considered the primary band for 5G-based services in Europe (even before 2020) and the pioneer band in $24.25-27.5 \mathrm{GHz}$ (alongside the already harmonized bands below 1 GHz , namely the 700 MHz band). Other bands for long-term usage are also under consideration, namely $40.5-43.5 \mathrm{GHz}$, and $66-71 \mathrm{GHz}^{17}$.

### 1.3 Research Question and Approach

As seen in the previous sections, spectrum sharing is critical for achieving a desirable enhancement in occupancy efficiency, with a more rational (re)use of the RF spectrum. Opportunistic use of spectral sub-bands/channels by interested Secondary Users (SUs) in an almost standalone scenario calls for a robust spectrum sensing to detect when no Primary Users (PUs) are transmitting on licensed sub-bands so that they can be used by the SU. Further, spectrum sensing must also be carried out to detect the possible start of activity of the licensed PU, which implies the SU to stop transmitting and start looking for some other available channel.

With these challenges taken into consideration, the main research question for this work is the following:

## Research Question

How can wideband spectrum sensing for Dynamic Spectrum Sharing in a Cognitive Radio-based environment be best achieved?

The proposed hypothesis to address this research question is:

## Hypothesis and Approach

After a detailed literature survey and critical evaluation of existing sensing techniques, Cyclostationary Detection (using the Spectral Correlation Function and the Spectral Coherence Function) is identified as a particularly promising method, with considerable potential to achieve efficient and reliable wideband detection of the used/idle spectral bands/channels in Cognitive Radio applications.

[^9]A set of three research objectives are subsequently defined to jointly address the main research question:

1. Develop an enhancement to the most widely used algorithm to compute the Spectral Correlation Function (SCF) and, therefore, the Spectral Coherence Function (SCoF) - the FAM (Fast Fourier Transform - FFT - Accumulation Method) - to allow efficient computation of the SCF or the SCoF in subregions of the support $(f ; \alpha)$ plane (frequency/cycle frequency plane);

## Justification

The spectrum sensing task aims to achieve spectrum awareness of several (possible large number) bands licensed to (possibly) different PUs, therefore leading to a wideband spectrum sensing scenario. The SU must scan the possible sub-bands and determine which are vacant for opportunistic use. Knowing only general PU signal characteristics (e.g., data rate, guard interval, cyclic prefix) will define a small ( $f ; \alpha$ ) region to compute the SCF or the SCoF for each sub-band, without interest in the entire $(f ; \alpha)$ plane.
2. Obtain the probability density of both the SCF and the SCoF estimates, using the FAM algorithm, when only background noise is present on the channel;

## Justification

To the author's best knowledge, this result was not obtained, and it was not published in the open literature, for the FAM algorithm.
Obtaining the exact or approximate probability density of the SCF estimates should provide the estimate of the variance of the background noise, using the SCF estimates in the $(f ; \alpha)$ plane.
3. Based on the results from Objective 2, to develop Constant False Alarm Rate (CFAR) and Constant Detection Rate (CDR) detectors for signal presence/absence in the sub-bands of interest;

## Justification

The characterization of the probability density of the estimates of the SCF and the SCoF, using the FAM algorithm, when only noise is present, is addressed, which will allow the definition of an adaptive threshold to develop a blind (with respect to the noise statistics) Constant False Alarm Rate (CFAR) detector (using the SCoF) and also a CFAR and a CDR detector when that characterization is used to obtain an estimate of the background noise variance (using the SCF ).

Thus, the major goal is to develop new spectrum sensing algorithms and detectors that can quickly and robustly detect the white spaces available in a (possible) wideband frequency range. This will allow a standalone mechanism for Dynamic Spectrum Access of licensed bands, with opportunistic
access by unlicensed users when channels are detected as idle, and leaving (with residual impact on the Licensed User system) when they become occupied.

### 1.4 Validation

Validation will be accomplished by simulation of the overall system, and comparison with other algorithms available and described, namely by comparing the probability of detection, for a fixed probability of false alarm, as a function of the signal-to-noise ratio, and for the same observation length. Then, validation will also comprise the acceptance of this thesis work by submitting it for publication in international journals and conferences indexed in the Web of Science.

### 1.5 Original Contributions

The main contributions of the work reported in this dissertation can be described as:

- A new method - the zFAM: Zoom FFT Accumulation Method - was derived, to efficiently compute the SCF in a local/zoomed region of the support $(f ; \alpha)$ plane (frequency/cycle frequency plane);
- The vectorization of the FAM (or zFAM) algorithm was undertaken, to reach a quadratic form to compute the SCF. Based on the eigenvalue analysis of this quadratic form matrix, exact expressions were obtained for the pdf of the SCF estimates when the input signal is a Gaussian r.v.. Approximations were given for the pdf of the SCF, obtained for low and medium number of averaging segments - avoiding the computational problems of the exact pdf. Also, a Gaussian approximation for the SCoF was derived when the number of averaging segments is high, which depends solely on the segment overlap and the number of averaging segments;
- Two new Signal-of-Interest CFAR detectors were obtained using the SCF and the SCoF, and a new CDR detector was obtained using the SCF.

The results from chapter three, namely the proposed zFAM algorithm, were already published in Elsevier's Digital Signal Processing Journal:

- Miguel Tavares, José Gerald, João Goes (2022) 'Zoom discrete spectral correlation function, with application to cyclostationary signal detection', Digital Signal Processing, vol. 121. doi: 10.1016/j.dsp.2021.103316.

Submitting chapters four and five results for publication in international journals and conferences indexed in the Web of Science is an utmost priority, namely the FAM probability density of the estimates of the SCF and SCoF when only noise is present and the proposed CFAR and CDR detectors.

### 1.6 Thesis Organization

To fulfill the developed work report, this thesis is organized in six chapters as follows:

- Chapter 1 presents the framing and problem formulation and the overview of the evolution in algorithms, frameworks, and regulation of spectrum sensing for dynamic spectrum access. This chapter also includes the research question, hypothesis, and approach of the work developed in this thesis.
- Chapter 2 addresses the dynamic spectrum access models already in use or in a study phase for possible deployment, giving the foundations of SDR and CR. This chapter includes the state-of-the-art in spectrum sensing, leading to the identification of cyclostationarity detection as the method with considerable potential to achieve efficient and reliable wideband detection of the used/idle spectral bands in Cognitive Radio applications.
- Chapter 3 addresses the Theory of Cyclic Spectral Analysis, where the most widely used algorithm to compute the Spectral Correlation Function (SCF) - the FAM algorithm - is enhanced, to enable the efficient computation of the SCF in a local/zoomed region of the $(f ; \alpha)$ plane (frequency/cycle frequency plane). This will provide the quick identification of spectral bands/channels in use by PUs or free, in a wideband sensing scenario.
- Chapter 4 presents the characterization of the probability density of the estimates of the SCF and SCoF when only noise is present, using the FAM (or zFAM) algorithm, which will allow the definition of an adaptive threshold to develop a blind (with respect to the noise statistics) Constant False Alarm Rate (CFAR) detector (using the SCoF) and also a CFAR detector when that characterization is used to obtain an estimate of the background noise variance (using the SCF).
- Chapter 5 describes the proposed CFAR and CDR detectors (considering blind and with noise variance estimator cases), and simulation results are presented to assess their performance, compared to other methods available in the open scientific literature.
- Chapter 6 summarizes the key findings presented in this thesis, with concluding remarks and an indication of some possible future research directions following the achieved results.


## Chapter 2

## 2 Dynamic Spectrum Access and Spectrum Sensing

This chapter addresses the Dynamic Spectrum Access (DSA) models already in use or in a study phase for possible deployment, giving the foundations of Software Defined Radio and Cognitive Radio. It also includes the literature review in Spectrum Sensing, leading to the identification of Cyclostationarity Detection as the method with considerable potential to achieve efficient and reliable wideband detection of the used/idle spectral bands in CR applications.

As shown in the previous chapter, nowadays, spectrum scarcity goes along with contradictory radio channels underutilization. The regulatory adoption of a Fixed Spectrum Allocation policy allocates different spectrum bands to specific services, with exclusive use by authorized (licensed) users. This policy forbids other users to access those frequency bands, even when they are not in use by licensed users. Moreover, although this policy guarantees some desired quality of service and interference avoidance between services, it also contributes to the full spectrum allocation, with an inherently reduced spectrum available for deploying new services.

This leads to a natural need to switch to other spectrum allocation strategies, namely to a DSA as opposed to the FSA. The strategy here is to allow users without a license, referred to as Secondary Users, to access the spectrum of authorized users - referred to as Primary Users - if it is not in use. Another strategy is to allow the use of the PUs spectrum by SUs, even when in use, as long as the interference caused by the SU can be confined to an acceptable level defined by the PU. These strategies correspond to two known models: the former as the Opportunistic Spectrum Access (OSA) model and the latter as the Concurrent Spectrum Access (CSA) model. Yet another strategy is to allow any PU to lease its spectrum, when not in use, to any SU that needs some frequency band(s). This lease is settled for geographic and time slots for one or several frequency bands. Known as Leased Shared Access (LSA), this model is under test and regulation in the EU, to be used in the $2.3-2.4 \mathrm{GHz}$ band ${ }^{18}$. Further, in the EU, the terms on how the lease is processed must be agreed upon and approved by the National Regulatory Authority (NRA), thus giving a temporary individual license to the SU for frequency bands on location and time available in the LSA repository (geo-database) (Buckwitz, Engelberg and Rausch, 2014).

[^10]Figure 2.1 shows the taxonomy of several spectrum access models. OSA is also known as Overlay Spectrum Access and CSA as Underlay Spectrum Access. The Commons model addresses the case when the spectrum is not licensed to any PU. This is the typical situation for unlicensed bands, such as ISM bands, for example, the IEEE $802.11 \mathrm{~b} / \mathrm{g} / \mathrm{n}$ networks operating in the $2,4 \mathrm{GHz}$ band.


Figure 2.1 - Spectrum Access Models.

As referenced in section 1.1.3, the LSA model provides access to bands not in use by the PU, on a time and geographic basis, through the LSA Controller and Repository (geo-database). This concept is still under development to incorporate a spectrum sensing mechanism, thus providing a more dynamic spectrum usage.

With the SU's simultaneous usage of the PU band, the CSA model needs an interference-based sensing mechanism (see section 2.3) to guarantee that the interference stays below the maximum accepted by the PU. The FCC introduced the Interference Temperature metric to assess interference between the PU and SU in 2003, but later, in 2007, abandoned it after concluding on its impossible implementation.

A typical scenario of licensed spectrum usage by several PU is illustrated in Figure 2.2. A SU using the OSA model will sense the idle/used channels and switch communication to an idle channel whenever the present channel in use becomes active by the PU. A possible channel switching for the spectrum occupation depicted in Figure 2.2 - by an SU using the OSA model - could be the one illustrated in Figure 2.3. If more than one idle channel is detected, the SU must decide which channel to use next. This leads to some form of knowledge (prior or acquired) by the SU, to switch to the channel with the best communication characteristics to guarantee some desired quality of service.


Figure 2.2 - Spectrum usage from several PUs.


Figure 2.3 - SU operation using the OSA model.

### 2.1 Software Defined Radio (SDR)

In collaboration with the IEEE P1900 group, the Wireless Innovation Forum ${ }^{19}$ established a definition of SDR as a "radio in which some or all of the physical layer functions are softwaredefined". Physical-layer functions include, for instance, tuning, antenna diversity, up/down conversion, synchronization, modulation and demodulation. These physical-layer functions are implemented in software, which can be easily modified by software/firmware update or upgrade, thus allowing new wireless features or technologies to be added to existing radio systems without needing additional hardware.

SDR has its genesis in the military industry, with funded research by the U.S. Department of Defense. The first mention of such a radio system appeared in 1970 as a "digital receiver" and later, in 1984, as a "software radio". This was used to refer to a prototype developed by E-Systems, Inc. (now Raytheon Technologies), providing programmable interference cancellation using

[^11]adaptive digital filters.
In 1991, J. Mitola seems to independently reinvent the same "software radio" term (Mitola, 1993). The final "software-defined radio" term is due to Stephen Blust, who coined it in 1995 ${ }^{20}$ (Mitola et al., 2015) and has since then been the adopted reference by the academic and industrial community.

In a radio with some (or all) of the physical layer functions performed in software, as in SDR, the ultimate goal is to push the Analog-to-Digital and Digital-to-Analog converters (ADC and DAC) as close as possible to the antenna so that all the signal processing tasks can be done in software. The functional diagram of an ideal SDR is depicted in Figure 2.4, assuming the ADC has an antialiasing filter incorporated, and the DAC has an incorporated reconstruction filter. The only analog elements would be the TX/RX switch (or duplexor) and the Power Amplifier (PA) for the transmitted signal and a Low-Noise Amplifier (LNA) for the received signal.


Figure 2.4 - Functional diagram of an ideal SDR.

Of course, a realizable ADC converter does not have infinite bandwidth or infinite dynamic range and also, typically, has the largest noise figure of all components in the RF front end. Therefore, a feasible functional diagram of an SDR (as illustrated in Figure 2.5) will also have software controllable superheterodyne down converter with a Local Oscillator (LO), tunable Bandpass and Lowpass filters (BPF and LPF), and an Automatic Gain Control (AGC) for the best adaptation to the ADC voltage range, in the receiving branch. Similarly, on the transmitting branch, there will be a tunable LPF for signal reconstruction after the DAC, followed by superheterodyne upconversion and possibly tunable BPF prior to the PA.

There has been a wide development of SDR systems, available commercially, flexible and fullfledged, and targeting a low-cost end-product. As of March 2021, almost two hundred SDR systems

[^12]could be found online (Wikipedia, 2022). This Wikipedia page (updated on June 2022) lists 121 SDR systems commercially available.

The flexibility and high performance of these SDR products have allowed a quick and low-cost development and testing of complex systems, e.g., frequency spectrum monitoring (Yagoub et al., 2018), implementation of an Automatic Information System (AIS) transponder (Marques et al., 2019) an RF exposure assessment (Bechet et al., 2019) or radar for Unmanned Aerial Vehicle (UAV) localization (Santos and Sebasti, 2019).


Figure 2.5 - Functional diagram of a feasible SDR.

### 2.2 Cognitive Radio and Dynamic Spectrum Management

Cognitive Radio (CR) was coined by J. Mitola III in 1999 (Mitola and Maguire, 1999), extending the Software Defined Radio paradigm, which would enable the overall improvement of radio communications performance in interaction with spectrum usage. By integrating sensors, intelligence and adaptability, CR systems can monitor radio and network environment and change parameters such as power levels, carrier frequency, protocols, etc., adapting as necessary.

Two general definitions for Cognitive Radio are commonly used. The first from (Mitola, 2000):
A goal-driven framework in which the radio autonomously observes the radio environment, infers contexts, assesses alternatives, generates plans, supervises multimedia services, and learns from its mistakes.
and the second by (Haykin, 2005):
An intelligent wireless communication system that is aware of its surrounding environment and that uses the methodology of understanding by building to learn from the environment and adapts its internal states to statistical variations in the incoming RF stimuli by making corresponding changes in certain operating parameters in real-time.

Therefore, the overall framing of CR requires multidisciplinary contributions, namely from wireless communication and networking, signal processing, artificial intelligence, decision and optimization
theory. A typical cognitive cycle is illustrated in Figure 2.6, adapted from (Haykin, 2005) and (Liang, 2020). An SU with these CR capabilities would continuously sense the radio spectrum to obtain information on spectrum holes ${ }^{21}$, channel state and interference/noise information. With this information, it would then determine and adjust operating parameters (e.g., transmission power, carrier frequency, modulation). The radio environment would be analyzed with the information collected throughout its operational life (such as channel and traffic statistics and modeling or interference characterization) to be stored (learned) to help future decisions.


Figure 2.6 - Typical Cognitive Cycle for CR.

The first standards for the practical implementation of the cognitive radio technology were targeted to provide wireless broadband communication using unused VHF/UHF TV bands (TV White Spaces - TVWS) ranging from 54 to 862 MHz (Renfors et al., 2017).

As described in section 1.1.1, the first standard, ECMA-392 (ECMA International, 2012), now in the second edition, was created in 2009, aiming at a Wireless Local Area Network (WLAN). Then, the IEEE 802.22 (IEEE International, 2019) (C. R. Stevenson, G. Chouinard, Z. Lei, W. Hu, 2009) in 2011, and later IEEE 802.11af (IEEE International, 2013) (A. B. Flores, R. E. Guerra, E. W. Knightly, 2013) in 2013, issued standards for Wireless Regional Area Network (WRAN) and Wireless Local Area Network (WLAN), respectively.

Also, the IEEE DySPAN-SC ${ }^{22}$ (2010) (Dynamic Spectrum Access Networks - Standards Committee) committee develops standards in the areas of dynamic spectrum access, cognitive radio, interference management, coordination of wireless systems, advanced spectrum management, and next-generation radio systems. This committee was initially known as IEEE 1900 (2005) and

[^13]later as IEEE SCC 41 (2007) (IEEE Standards Coordinating Committee 41).
Since 2005, the IEEE Communications Society has organized the annual IEEE International Symposium on Dynamic Spectrum Access Networks (IEEE DySPAN), which is considered to have a major influence on policy and technology research and development, in this scientific area, in the United States, Europe and Asia.

Although present CR research focuses on exploring more potential applications in CR technology by using artificial intelligence (AI) (Chen et al., 2020), most existing CR test pilots focus on exploring and realizing the cognitive capability to facilitate the DSA.

The practical implementation of CR systems presents challenges in both software and hardware subsystems. Robust and highly efficient communication protocols (mainly developed in software) must be used in the RF devices to take advantage of the shared spectrum. In the hardware subsystem, the biggest challenge is to perform the spectrum sensing efficiently. The spectrum swath to be sensed can be very large, and the decision on the spectrum holes available must be made quickly and accurately. Also, the detection of the presence of the PU on a licensed band previously detected as a spectrum hole should be made quickly, for the SU to stop transmitting, thus avoiding significant interference on the PU service.

After detecting spectrum holes, which may cover a large frequency range, including licensed or unlicensed bands, a CR system must decide which band(s) to use to guarantee the QoS requirements of the service(s) used by the SU . Channel state information estimation should also be performed on detected spectrum holes to allow possible band switching to optimize spectrum utilization and service quality maintenance. Thus, the CR must perform the spectrum sensing to permit the DSA and also spectrum management, by selecting the optimal channels to be used, based on the spectrum sensing results.

### 2.3 Spectrum sensing

The primary objective of spectrum sensing is to detect, within a (probably) large frequency band, whether or not PUs are using their specific licensed frequency channel, as well as detecting if there is no activity on unlicensed bands. As indicated in Figure 2.7, there are three established types of spectrum sensing: Non-cooperative sensing, Cooperative sensing, and Interference-based sensing (Hossain, E., Niyato, D., \& Han, 2009), (Ariananda, Lakshmanan and Nikoo, 2009), (Amrutha and Karthikeyan, 2017), (Kakalou et al., 2018).

Selecting the method for spectrum sensing and PU detection must address three issues in order to have an accurate and fast system to identify spectrum holes. Firstly, the method must present low spectral leakage, or low smearing of the original signal spectrum, to increase the spectrum holes detection probability. Secondly, the sensing period must be low, meaning that the number of acquired samples needed to make a decision must be small to allow a fast sensing and quick decision of vacant channels. Finally, the method should present low computational complexity, allowing its implementation on (possible) low-performance mobile devices and low power dissipation.


Figure 2.7 - Types of spectrum sensing.

## Non-cooperative Spectrum Sensing

The non-cooperative spectrum sensing is performed by an SU to detect if an active PU is using the frequency band being sensed. The decision is made based on time-limited observation/samples of the received signal. The commonly known methods to carry out this type of spectrum sensing are Energy Detection, Matched Filter Detection, Eigenvalue Based Detection, and Cyclostationary feature detection (Figure 2.7).

The Cooperative Spectrum Sensing method will need information obtained by non-cooperative sensing performed by participant nodes in the network. However, this method is, by itself, a significant scientific study area. The non-cooperative method is the spectrum sensing method addressed in this thesis, which will be the focus of the following subsections.

## Cooperative Spectrum Sensing

Using a non-cooperative spectrum sensing strategy, an SU may not always detect the signal from a PU, whether from location separation or channel fading (Hossain, E., Niyato, D., \& Han, 2009). The typical hidden node problem (collision) is illustrated in Figure 2.8. Consider that nodes A and $B$ are licensed users (PUs), and nodes C and D are unlicensed users (SUs). When node A is actively transmitting to node B , node C cannot detect the signal. Therefore, after sensing the channel, node C will decide on the absence of any PU and might start transmitting to node D , causing a collision in nodes A-B communication, thus interfering with the PUs channel usage.

Cooperative spectrum sensing can be used to solve this problem, where multiple SUs will sense the channel using some method from the non-cooperative model and exchange information between active SUs. This way, the hidden node problem can be solved, and detection probability improved. In the example of Figure 2.8, although node C cannot detect de signal from node A , node D can sense it; in a cooperative environment, node C would receive this information and restrain itself from using the channel.


Figure 2.8 - The hidden node problem.

Several methods for cooperative sensing have been proposed, using either a centralized or distributed architecture, e.g. (Ganesan and Li, 2005) and (Zhao, Zheng and Yang, 2005), and more recent surveys in (Akyildiz, Lo and Balakrishnan, 2011), (Amrutha and Karthikeyan, 2017) or (Kakalou et al., 2018).

## Interference-based Sensing

In a CSA scheme (see Figure 2.1), the interference caused by a transmitting SU to a receiving PU must be confined to an acceptable level. To quantify this interference, the FCC (FCC, 2003) proposed a metric called the Interference Temperature. However, as noticed by (Akyildiz et al., 2006), measuring the Interference temperature would require the SU to know the location of PU receivers, which is impractical in most cases. In 2007, the FCC terminated the proceeding of the "Interference Temperature Model for Quantifying and Managing Interference" (FCC, 2003) ${ }^{23}$, concluding there was no technical procedure to implement it. Nevertheless, modifications and generalizations of the interference temperature concept are still being developed by the scientific community (Ghanekar, Dhole and Patil, 2014), (Liang, 2020).

### 2.3.1 Detection Problem

In a non-cooperative spectrum sensing model, the detection of an active PU by one SU can be stated as a binary hypothesis testing (Kay, 1998). Considering the case when there is only noise, the null hypothesis, $\mathcal{H}_{0}$, and the alternative hypothesis, the case when the PU is transmitting, $\mathcal{H}_{1}$, the sampled received signal, $r(n)$, is given by (under each hypothesis) as:

[^14]\[

$$
\begin{align*}
& \mathcal{H}_{o}: r(n)=w(n) \\
& \mathcal{H}_{1}: r(n)=s(n)+w(n) \tag{2.1}
\end{align*}
$$
\]

where $s(n)$ and $w(n)$ are the transmitted signal by the PU observed at the SU , and additive white Gaussian noise (AWGN), respectively. The noise signal is assumed to have zero mean and variance $\sigma^{2}$, so $w \sim \mathcal{N}\left(0, \sigma^{2}\right)$ and samples are independent and identically distributed (i.i.d.). Also, in this model, it is assumed that no interference is present from other SU or PU , or that it can be incorporated into the given signal model. We also assume that all signals $s(n), w(n)$, and $r(n)$ are real.

The four cases of the detection problem lead to the corresponding associated probabilities:

1. Declare $\mathcal{H}_{1}$ under $\mathcal{H}_{1}$ hypothesis: associated with the Probability of Detection $\left(P_{D}\right)$
2. Declare $\mathcal{H}_{0}$ under $\mathcal{H}_{1}$ hypothesis: associated with the Probability of Missing $\left(P_{M}\right)$
3. Declare $\mathcal{H}_{1}$ under $\mathcal{H}_{0}$ hypothesis: associated with the Probability of False Alarm $\left(P_{F A}\right)$
4. Declare $\mathcal{H}_{0}$ under $\mathcal{H}_{0}$ hypothesis: associated with the Probability of Correct Rejection.

The performance of the spectrum sensing detector is assessed by the two probabilities: Probability of Detection $\left(P_{D}\right)$ and the Probability of False Alarm $\left(P_{F A}\right)$. The $P_{D}$ defines the probability of the detector to correctly detect the presence of the PU , and $P_{F A}$ the probability of mistakenly declaring that the PU is present. The Probability of Missing $\left(P_{M}\right)$ is the probability of declaring that no PU is present when it is, in fact, using the channel. Finally, the Probability of Correct Rejection is the probability of correctly declaring that the PU is not present.

The detector should operate within given error levels, which directly relate to $P_{D}$ and $P_{F A}$. A detector should have a high detection probability (or low miss detection probability) to not interfere with the PU activity. However, having a high false alarm probability makes the detection process inefficient as more possible time slots could be used by the SU if not mistakenly declaring the presence of the PU. Unfortunately, making $P_{D}$ close to unity alongside making $P_{F A}$ close to zero are incompatible objectives.

Considering the simple case to detect the presence of a Signal-of-Interest (SoI), with some decision statistic modeled as a constant when present, i.e., $s(n)=A$ in (2.1), the resulting decision process for each hypothesis is illustrated in Figure 2.9. The probability density function (pdf) for the hypothesis $\mathcal{H}_{0}$ is of a $\mathcal{N}\left(0, \sigma^{2}\right)$ r.v., whereas for hypothesis $\mathcal{H}_{1}$ is of a $\mathcal{N}\left(A, \sigma^{2}\right)$ r.v.. Given a decision threshold, $\gamma$, a simple decision rule is to declare that the PU signal is present if the received signal is bigger than the defined threshold. The probability of detection and false alarm are then given by:

$$
\begin{gathered}
P_{D}=\operatorname{Pr}\left\{r>\gamma \mid \mathcal{H}_{1}\right\}=\frac{1}{\sqrt{2 \pi} \sigma} \int_{\gamma}^{+\infty} e^{-\frac{(r-A)^{2}}{2 \sigma^{2}}} d r=\frac{1}{2} \operatorname{erfc}\left(\frac{\gamma-A}{\sqrt{2} \sigma}\right) \\
P_{F A}=\operatorname{Pr}\left\{r>\gamma \mid \mathcal{H}_{0}\right\}=\frac{1}{\sqrt{2 \pi} \sigma} \int_{\gamma}^{+\infty} e^{-\frac{r^{2}}{2 \sigma^{2}}} d r=\frac{1}{2} \operatorname{erfc}\left(\frac{\gamma}{\sqrt{2} \sigma}\right)
\end{gathered}
$$



Figure 2.9 - Decision process for a constant decision statistic obtained from the Signal-of-Interest.

This way, one can obtain $\frac{\gamma}{\sqrt{2} \sigma}=\operatorname{erfc} c^{-1}\left(2 P_{F A}\right)$, and using the linear signal-to-noise ratio (SNR) $\mathrm{snr}=A^{2} / \sigma^{2}\left(\right.$ or in logarithmic units $\left.\mathrm{SNR}=10 \log _{10}\left(\frac{A^{2}}{\sigma^{2}}\right) \mathrm{dB}\right)$, the relation between $P_{D}$ and $P_{F A}$ is:

$$
\begin{equation*}
P_{D}=\frac{1}{2} \operatorname{erfc}\left(e r f c^{-1}\left(2 P_{F A}\right)-\sqrt{\frac{\mathrm{snr}}{2}}\right) \tag{2.2}
\end{equation*}
$$

Equation (2.2) allows depicting the detector performance by usual plots of $P_{D}$ vs. $P_{F A}$ for a given SNR (Receiver Operating Characteristic - ROC), and $P_{D}$ vs. SNR for a given $P_{F A}$.

Figure 2.10 plots of $P_{D}$ vs. $P_{F A}$ for varying SNR from -10 dB to +10 dB , with 2 dB spacing. In Figure 2.11 the $P_{D}$ vs. SNR is plotted for varying $P_{F A}\left(P_{F A}=10^{-1} ; 10^{-2} ; \ldots ; 10^{-6} ; 10^{-7}\right)$.

As can be inferred from Figure 2.9 or seen in the ROC (Figure 2.10), when we raise the threshold, $\gamma$, the $P_{F A}$ diminishes, but so does the $P_{D}$. Lowering the threshold will raise the $P_{D}$ but will also raise $P_{F A}$. From Figure 2.11, it can easily be seen that the detector performance (in terms of the $P_{D}$, for constant $P_{F A}$ ) will always be improved as the SNR is increased.

Two methods can be defined for the detector in a spectrum sensing scenario: a Constant Detection Rate (CDR) or a Constant False Alarm Rate (CFAR). When the objective is to guarantee the PU service, a CDR method should be used, using a high $P_{D}$. When the objective is to have a high SU throughput, the CFAR method is better, using a low $P_{F A}$.


Figure 2.10 - Receiver performance $P_{D}$ vs. $P_{F A}$ - Receiver Operating Characteristic (ROC).

The optimum criterium, known as the Neyman-Pearson test, will attempt to maximize the $P_{D}$, under the constraint of a fixed probability of false alarm, $P_{F A}=\alpha$ (Kay, 1998). Given an $N$ sample observation signal, $\boldsymbol{r}=[r(1), r(2), \ldots, r(N)]^{\prime}$ (where ' denotes the transpose operation) the Neyman-Pearson test will decide $\mathcal{H}_{1}$ if the likelihood ratio, $\Lambda(\boldsymbol{r})$, is greater or equal to a threshold, $\gamma$ :

$$
\begin{equation*}
\Lambda(\boldsymbol{r})=\frac{p_{\boldsymbol{r} \mid \mathcal{H}_{1}}\left(\boldsymbol{r} \mid \mathcal{H}_{1}\right)}{p_{\boldsymbol{r} \mid \mathcal{H}_{0}}\left(\boldsymbol{r} \mid \mathcal{H}_{0}\right)} \geq \gamma, \tag{2.3}
\end{equation*}
$$

where $p_{\boldsymbol{r} \mid \mathcal{H}_{1}}\left(\boldsymbol{r} \mid \mathcal{H}_{1}\right)$ denotes de joint distribution of the vector $\boldsymbol{r}$ under hypothesis $\mathcal{H}_{1}$, and the threshold is obtained from the constant $P_{F A}$ constraint:

$$
\begin{equation*}
\mathrm{P}_{\mathrm{FA}}=\int_{\{\boldsymbol{r}: \Lambda(\boldsymbol{r}) \geq \gamma\}} p_{\boldsymbol{r} \mid \mathcal{H}_{0}}\left(\boldsymbol{r} \mid \mathcal{H}_{0}\right) d \boldsymbol{r}=\alpha \tag{2.4}
\end{equation*}
$$

Or, in terms of the pdf of $\Lambda(\boldsymbol{r})$, or some monotonic function $T(\boldsymbol{r})$ of $\Lambda(\boldsymbol{r})$ :

$$
\begin{align*}
\mathrm{P}_{\mathrm{FA}} & =\int_{\lambda}^{+\infty} p_{\Lambda \mid \mathcal{H}_{0}}\left(l \mid \mathcal{H}_{0}\right) d l=\alpha \\
& =\int_{\gamma}^{+\infty} p_{T \mid \mathcal{H}_{0}}\left(t \mid \mathcal{H}_{0}\right) d t=\alpha \tag{2.5}
\end{align*}
$$



Figure 2.11 - Receiver performance $P_{D}$ vs. SNR.

Considering the simple case again, when the Signal-of-Interest in (2.1) is a constant, $A$, when the PU is active, or 0 when the PU is idle, the joint distributions are given by:

$$
\begin{aligned}
& p_{\boldsymbol{r} \mid \mathcal{H}_{1}}\left(\boldsymbol{r} \mid \mathcal{H}_{1}\right)=\frac{1}{\left(\sqrt{2 \pi \sigma^{2}}\right)^{N}} e^{-\frac{1}{2 \sigma^{2}} \sum_{n=1}^{N}(r(n)-A)^{2}} \\
& p_{\boldsymbol{r} \mid \mathcal{H}_{0}}\left(\boldsymbol{r} \mid \mathcal{H}_{0}\right)=\frac{1}{\left(\sqrt{2 \pi \sigma^{2}}\right)^{N}} e^{-\frac{1}{2 \sigma^{2}} \sum_{n=1}^{N} r(n)^{2}}
\end{aligned}
$$

So, the log-likelihood ratio is

$$
\log (\Lambda(\boldsymbol{r}))=-\frac{1}{2 \sigma^{2}} \sum_{n=1}^{N}\left(-2 r(n) A+A^{2}\right)=\frac{A}{\sigma^{2}} \sum_{n=1}^{N} r(n)-\frac{N A^{2}}{2 \sigma^{2}},
$$

and applying the Neyman-Pearson criterion leads to (where $T(\boldsymbol{r})$ is a monotonic function of $\Lambda(\boldsymbol{r})$, with adapted threshold, $\gamma$ )

$$
\begin{equation*}
T(\boldsymbol{r})=\frac{1}{N} \sum_{n=1}^{N} r(n) \geq \gamma, \tag{2.6}
\end{equation*}
$$

so that $T(\boldsymbol{r}) \left\lvert\, \mathcal{H}_{1} \sim \mathcal{N}\left(A, \frac{\sigma^{2}}{N}\right)\right.$ and $T(\boldsymbol{r}) \left\lvert\, \mathcal{H}_{0} \sim \mathcal{N}\left(0, \frac{\sigma^{2}}{N}\right)\right.$.

For a constant $P_{F A}$, as $P_{F A}=\operatorname{Pr}\left\{T(\boldsymbol{r}) \mid \mathcal{H}_{0} \geq \gamma\right\}=\frac{1}{2} \operatorname{erfc}\left(\frac{\gamma}{\sqrt{\frac{2 \sigma^{2}}{N}}}\right)$, the threshold is given by $\gamma=$ $\sqrt{\frac{2 \sigma^{2}}{N}} \operatorname{erfc} c^{-1}\left(2 P_{F A}\right)$.

For the probability of detection, one gets $P_{D}=\frac{1}{2} \operatorname{erfc}\left(\frac{\gamma-A}{\sqrt{\frac{2 \sigma^{2}}{N}}}\right)$, so finally (as snr $\left.=A^{2} / \sigma^{2}\right)$ :

$$
\begin{equation*}
P_{D}=\frac{1}{2} \operatorname{erfc}\left(e r f c^{-1}\left(2 P_{F A}\right)-\sqrt{\frac{N \times \mathrm{snr}}{2}}\right) \tag{2.7}
\end{equation*}
$$

It is easily seen that (2.2) is equal to the result obtained in (2.7) for $N=1$. In Figure 2.12 the $P_{D}$ vs. SNR is plotted for a fixed $P_{F A}=10^{-3}$, with varying sample length $(N=1 ; 10 ; 20 ; 30 ; 40 ; 50)$ along with the simulation results (dot makers) obtained from $2^{16}$ Monte Carlo runs.

Naturally, for a fixed SNR, the probability of detection increases as the sample size increases. To achieve a desired $P_{D}$, for a constant $P_{F A}$, the sample size can be computed for a given SNR of operation using (2.7).


Figure 2.12 - Receiver performance $P_{D}$ vs. SNR, for various $N$.

In the following section, the Neyman-Pearson criterion for making the trade-off between $P_{D}$ and $P_{F A}$ will be used for several detectors. For a fixed $P_{F A}$, the decision threshold is found and $P_{D}$ is obtained for varying SNR operation, thus obtaining a Constant False Alarm Rate (CFAR) detector.

Performance comparison of the detectors presented will show that cyclostationary feature detection is the best candidate for efficient PU detection, as data communication signals are cyclostationary. When known, exact or approximate expressions will be given for the probability of detection and false alarm, and also for the sample size needed to fulfill probability requirements, for a given SNR operation.

### 2.3.2 Energy Detection

Spectrum sensing using energy detection adopts a noncoherent approach to the problem, by just measuring the energy of the received signal and comparing it with some threshold, dependent on the noise variance. This method assumes no knowledge of the PU signal, so it is suitable for wideband spectrum sensing (Ma, Li and Juang, 2009). This is the lowest complexity method for spectrum sensing, as it does not involve complex signal processing tasks, as can be seen by its block diagram depicted in Figure 2.13. An enhanced energy detector is illustrated in Figure 2.14, where a wideband signal may be sampled, but energy detection can be done on specific frequency bands, by selecting the appropriate frequency bins from the FFT block output.


Figure 2.13 - Block diagram of the energy detector.


Figure 2.14 - Block diagram of the enhanced energy detector.

The e
nergy detector uses the normalized energy of the received signal as the test statistic, given by $T(\boldsymbol{r})=\boldsymbol{r}^{\prime} \boldsymbol{r} / \sigma^{2}$. When no PU is present $T(\boldsymbol{r})$ is the sum of $N$ squared standard normal r.v., so its probability density function (pdf) is a (central) chi-square distribution with $N$ degrees of freedom. When the PU is present, $T(\boldsymbol{r})$ is the sum $N$ squared normal r.v. with unit variance and mean equal to $s(n)^{2} / \sigma^{2}$, so its pdf is a non-central chi-square distribution with $N$ degrees of freedom
and noncentrality parameter $\lambda=\sum_{n=1}^{N} s(n)^{2} / \sigma^{2}=E_{s} / \sigma^{2}$ :

$$
\begin{align*}
& \mathcal{H}_{o}: T(\boldsymbol{r}) \sim \chi_{N}^{2} \\
& \mathcal{H}_{1}: T(\boldsymbol{r}) \sim \chi_{N}^{2}\left(E_{s} / \sigma^{2}\right)=\chi_{N}^{2}(\lambda) \tag{2.8}
\end{align*}
$$

Considering $F_{\chi_{N}^{2}}($.$) the cumulative distribution function (\mathrm{CDF})$ of $\chi_{N}^{2}$, and for a given $P_{F A}$, the detection threshold, $\gamma$, is given by

$$
\gamma=F_{\chi_{N}^{2}}^{-1}\left(1-P_{F A}\right)
$$

As shown in Appendix A, Demonstration 2.1, the probability of detection is given by:

$$
P_{D}=Q_{N / 2}\left(\sqrt{E_{s} / \sigma^{2}}, \sqrt{\gamma}\right)=Q_{N / 2}(\sqrt{N \times s n r}, \sqrt{\gamma})
$$

where $Q_{m}(a, b)$ is the generalized Marcum Q-function, and the signal-to-noise ratio is $s n r=$ $P_{s} / \sigma^{2}$ (the mean signal power is $\left.P_{s}=\lim _{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} s(n)^{2}\right)$

In Figure 2.15, the $P_{D}$ vs. SNR is plotted for varying $P_{F A}\left(10^{-1} ; 10^{-2} ; \ldots ; 10^{-6} ; 10^{-7}\right)$ considering a sample size of $N=20$. In Figure 2.16, the $P_{D}$ vs. SNR is also plotted, for a fixed $P_{F A}=10^{-3}$, but varying sample size, $N=10 ; 20 ; 30 ; 40 ; 50$. In both figures, simulation results are also plotted (dots markers) obtained from $2^{16}$ Monte Carlo runs.


Figure 2.15 - Receiver performance $P_{D}$ vs. SNR (Energy Detector, N=20).


Figure 2.16 - Receiver performance $P_{D}$ vs. SNR for fixed $P_{F A}$ (Energy Detector).

For a given $P_{F A}$, the detector performance will be better for a higher SNR; for a fixed SNR, the performance will be enhanced when increasing the sample size, $N$.

For energy detection, the noise variance is assumed to be known and constant. There will always be some fluctuation of the noise variance in any practical communication system, leading to a deteriorated detection performance (Yücek and Arslan, 2009). This detector must always have a feasible algorithm to estimate the noise variance, whether using a non-cooperative or cooperative sensing scheme. Some examples of these algorithms can be found, for instance, in (Zhu et al., 2008; Joshi, Popescu and Dobre, 2010; Teo, Zhong and Ng, 2010) or, more recently, in (Soofi, Potnis and Diwivedy, 2019).

### 2.3.2.1 Noise Uncertainty (SNR wall)

To assess the effect of noise variance uncertainty, and following the work from (Tandra and Sahai, 2008), we will consider the sample size large enough to invoke the central limit theorem (Durrett, 2019). Considering now the test statistic as the unnormalized $T(\boldsymbol{r})=\boldsymbol{r}^{\prime} \boldsymbol{r}=\sum_{n=1}^{N} r(n)^{2}$, when there is no uncertainty in the noise variance, we get:

$$
\begin{aligned}
& T(\boldsymbol{r}) \mid \mathcal{H}_{0} \sim \mathcal{N}\left(N \sigma^{2}, 2 N \sigma^{4}\right) \\
& T(\boldsymbol{r}) \mid \mathcal{H}_{1} \sim \mathcal{N}\left(N\left(P_{s}+\sigma^{2}\right), 2 N\left(P_{s}+\sigma^{2}\right)^{2}\right)
\end{aligned}
$$

where $P_{s}$ is the average signal power $\left(P_{s}=\lim _{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} s(n)^{2}\right)$ and the signal-to-noise-ratio is $s n r=P_{s} / \sigma^{2}$.

Using this approximation, and as in the previous subsection, the false alarm and detection probability are given by:

$$
\begin{gather*}
P_{F A}=\operatorname{Pr}\left\{T(\boldsymbol{r}) \mid \mathcal{H}_{0} \geq \gamma\right\}=\frac{1}{2} \operatorname{erfc}\left(\frac{\gamma-N \sigma^{2}}{2 \sigma^{2} \sqrt{N}}\right)  \tag{2.9}\\
P_{D}=\operatorname{Pr}\left\{T(\boldsymbol{r}) \mid \mathcal{H}_{1} \geq \gamma\right\}=\frac{1}{2} \operatorname{erfc}\left(\frac{\gamma-N\left(P_{s}+\sigma^{2}\right)}{2\left(P_{s}+\sigma^{2}\right) \sqrt{N}}\right) \tag{2.10}
\end{gather*}
$$

Using (2.9) and (2.10), and eliminating $\gamma$, gives the sample size needed for the detector, given the system parameters:

$$
\begin{equation*}
N=\frac{4\left(e r f c^{-1}\left(2 P_{F A}\right)-(1+s n r) e r f c^{-1}\left(2 P_{D}\right)\right)^{2}}{s n r^{2}} \tag{2.11}
\end{equation*}
$$

For a low signal-to-noise ratio, $1+s n r \approx 1$, so the sample size is $N=O\left(s n r^{-2}\right)$. This means that if the noise statistics are known, then, any signal, at arbitrarily low SNR, can be detected by increasing the sample size (or, equivalently, the sensing time).

When there is some uncertainty in noise variance, it can be modeled as being in an interval $\sigma_{n}^{2} \in$ $\left[\frac{1}{\rho} \sigma^{2} ; \rho \sigma^{2}\right]$, where $\sigma^{2}$ is the true noise variance, and $\rho>1$ is the parameter quantifying the uncertainty. The choice for this model is easily explained, as it gives an equal plus/minus variation of the true variance in decibel units. To obtain the sample size needed for some given system parameters, equations (2.9) and (2.10) must be adapted so that the detector can achieve the desired $P_{F A}$ and $P_{D}$ probabilities robustly. Therefore, we must have:

$$
\begin{gather*}
P_{F A}=\max _{\sigma_{n}^{2} \in\left[\frac{1}{\rho} \sigma^{2} ; \rho \sigma^{2}\right]} \frac{1}{2} \operatorname{erfc}\left(\frac{\gamma-N \sigma_{n}^{2}}{2 \sigma_{n}^{2} \sqrt{N}}\right)=\frac{1}{2} \operatorname{erfc}\left(\frac{\gamma-N \rho \sigma^{2}}{2 \rho \sigma^{2} \sqrt{N}}\right)  \tag{2.12}\\
P_{D}=\min _{\sigma_{n}^{2} \in\left[\frac{1}{\rho} \sigma^{2} ; \rho \sigma^{2}\right]} \frac{1}{2} \operatorname{erfc}\left(\frac{\gamma-N\left(P_{s}+\sigma_{n}^{2}\right)}{2\left(P_{s}+\sigma_{n}^{2}\right) \sqrt{N}}\right) \\
=\frac{1}{2} \operatorname{erfc}\left(\frac{\gamma-N\left(P_{s}+\frac{1}{\rho} \sigma^{2}\right)}{2\left(P_{s}+\frac{1}{\rho} \sigma^{2}\right) \sqrt{N}}\right) \tag{2.13}
\end{gather*}
$$

Now, using (2.12) and (2.13), and eliminating $\gamma$, gives the sample size needed for the detector, given the system parameters (valid for $\operatorname{snr}-\frac{\rho^{2}-1}{\rho}>0$ ):

$$
\begin{equation*}
N=\frac{4 \rho^{2}\left(e r f c^{-1}\left(2 P_{F A}\right)-\frac{1}{\rho}\left(\frac{1}{\rho}+\mathrm{snr}\right) e r f c^{-1}\left(2 P_{D}\right)\right)^{2}}{\left(\operatorname{snr}-\frac{\rho^{2}-1}{\rho}\right)^{2}} \tag{2.14}
\end{equation*}
$$

Again, for a low signal-to-noise ratio, $1+\mathrm{snr} \approx 1$, and considering a small noise variance uncertainty ( $\rho \approx 1$ ), the sample size can be approximated by:

$$
\begin{equation*}
N=\frac{4\left(\operatorname{erfc}^{-1}\left(2 P_{F A}\right)-\operatorname{erfc} c^{-1}\left(2 P_{D}\right)\right)^{2}}{\left(\mathrm{snr}-\frac{\rho^{2}-1}{\rho}\right)^{2}} \tag{2.15}
\end{equation*}
$$

From the result obtained (equation (2.15)), it can be seen that as the $s n r$ tends (from above) to $\frac{\rho^{2}-1}{\rho}$, the sample size needed tends to infinity. This $s n r$ value is known as the $s n r$ wall, for which (and below which) the estimator cannot detect the signal however large the sample size may be. In Figure 2.17, the sample size (as given by equation (2.14)) is plotted as a function of the signal-to-noise ratio when $P_{F A}=10^{-3}$ and $P_{D}=0.9$, for several values of the noise variance uncertainty $(\rho=1(0 \mathrm{~dB}), 0.001 \mathrm{~dB}, 0.01 \mathrm{~dB}, 0.1 \mathrm{~dB}, 1 \mathrm{~dB})$. Naturally, and as can be seen, the greater the noise uncertainty, the greater the SNR wall.


Figure 2.17-SNR wall - Sample size vs. SNR, for constant $P_{F A}$ with noise variance uncertainty.

For typical values of $P_{F A}$ and $P_{D}$, when the SNR is high, the sample size is approximately given by $N=4\left(\operatorname{erfc}^{-1}\left(2 P_{D}\right)\right)^{2}$.

Using (2.12) to obtain the needed threshold for a maximum $P_{F A}$, given the noise variance
uncertainty, $\rho$, and substituting in (2.13), we get the probability of detection, given the system parameters:

$$
\begin{equation*}
P_{D}=\frac{1}{2} \operatorname{erfc}\left(\frac{\rho}{\operatorname{snr}+\frac{1}{\rho}}\left(\operatorname{erfc} c^{-1}\left(2 P_{F A}\right)-\frac{\sqrt{N}}{2}\left(\frac{\operatorname{snr}-\frac{\rho^{2}-1}{\rho}}{\rho}\right)\right)\right. \tag{2.16}
\end{equation*}
$$

In Figure 2.18, the detector performance is illustrated by plotting $P_{D}$ vs. SNR (equation (2.16)) for several sample sizes $\left(N=10^{2}, 10^{3}, 10^{4}, 10^{5}, 10^{6}, 10^{7}\right)$ with $P_{F A}=10^{-3}$ and noise variance uncertainty, $\rho=0.1 \mathrm{~dB}=1.0233$, which clearly shows the SNR wall phenomenon. As can be seen, the probability of detection $\left(P_{D}\right)$ cannot be increased by increasing the sample size, when the operating SNR is below the $\mathrm{SNR}_{\text {wall }}=-13.37 \mathrm{~dB}$.


Figure 2.18 - SNR wall - $P_{D}$ vs. SNR for constant $P_{F A}$ with noise variance uncertainty of 0.1 dB .

### 2.3.3 Matched Filter Detection

Unlike the Energy Detector, the Matched Filter Detector (MFD) or Correlator Detector (CD) is a coherent detector that maximizes the SNR of the received signal in an additive white Gaussian noise scenario (Kay, 1998). Also, unlike the ED, the MFD requires the knowledge of the PU signal or specific PU signal components or structure to enable the distinction between the interference and noise. Most wireless communication technologies include some sort of pilot sequence or training sequences to allow channel estimation, presence beacon or synchronization reference. Knowing some $N$ point pilot from the $\mathrm{PU}, s(n)$, the filter impulse response of the matched filter is just $h(n)=s^{*}(-n)$.

However, MFD requires timing synchronization, and performance heavily depends on synchronizing the SU to the PU. Also, when doing wideband spectrum sensing, with PUs using different communication technologies, the SU must have different dedicated algorithms, thus increasing the complexity of the detector. Figure 2.19 shows the matched filter detector block diagram, whereas Figure 2.20 illustrates the correlator architecture.


Figure 2.19 - Block diagram of the matched filter detector.


Figure 2.20 - Block diagram of the correlator detector.

In the MFD, the PU signal is assumed to be known at the SU, and the test statistic is just $T(\boldsymbol{r})=$ $\boldsymbol{r}^{\prime} \boldsymbol{s}^{*}$ (Kay, 1998). The distribution of $T(\boldsymbol{r})$ under both hypotheses is given by (Kay, 1998):

$$
\begin{align*}
& \mathcal{H}_{o}: T(\boldsymbol{r}) \sim \mathcal{N}\left(0, N P_{s} \sigma^{2}\right) \\
& \mathcal{H}_{1}: T(\boldsymbol{r}) \sim \mathcal{N}\left(N P_{s}, N P_{s} \sigma^{2}\right), \tag{2.17}
\end{align*}
$$

where the mean signal power is $P_{s}=\lim _{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} s(n)^{2}$ and the signal-to-noise ratio is $s n r=$ $P_{s} / \sigma^{2}$. Therefore, the false alarm and detection probability are given by:

$$
\begin{gather*}
P_{F A}=\operatorname{Pr}\left\{T(\boldsymbol{r}) \mid \mathcal{H}_{0} \geq \gamma\right\}=\frac{1}{2} \operatorname{erfc}\left(\frac{\gamma}{\sigma \sqrt{2 N P_{s}}}\right)  \tag{2.18}\\
P_{D}=\operatorname{Pr}\left\{T(\boldsymbol{r}) \mid \mathcal{H}_{1} \geq \gamma\right\}=\frac{1}{2} \operatorname{erfc}\left(\frac{\gamma-N P_{s}}{\sigma \sqrt{2 N P_{s}}}\right) \tag{2.19}
\end{gather*}
$$

Using (2.18) and (2.19), and eliminating $\gamma$, gives the sample size needed for the detector, given the
system parameters:

$$
\begin{equation*}
N=\frac{2\left(e r f c^{-1}\left(2 P_{F A}\right)-e r f c^{-1}\left(2 P_{D}\right)\right)^{2}}{\mathrm{snr}} \tag{2.20}
\end{equation*}
$$

The sample size is $N=O\left(s n r^{-1}\right)$. Therefore, as in the ED case, if the noise statistics are known, then, any signal, at arbitrarily low SNR, can be detected by increasing the sample size (or, equivalently, the sensing time).

In Figure 2.21, the $P_{D}$ vs. SNR is plotted for varying $P_{F A}\left(10^{-1} ; 10^{-2} ; \ldots ; 10^{-6} ; 10^{-7}\right)$ considering a sample size of $N=20$. In Figure 2.22, the $P_{D}$ vs. SNR is also plotted, for a fixed $P_{F A}=10^{-3}$, but varying sample size, $N=10 ; 20 ; 30 ; 40 ; 50$. In both figures, simulation results are also plotted (dots markers) obtained from $2^{16}$ Monte Carlo runs.

For a given $P_{F A}$, the detector performance will be better for a higher SNR; for a fixed SNR, the performance will be enhanced when increasing the sample size, $N$.


Figure 2.21 - Receiver performance $P_{D}$ vs. SNR (Matched Filter, $\mathrm{N}=20$ ).


Figure 2.22 - Receiver performance $P_{D}$ vs. SNR for fixed $P_{F A}$ (Matched Filter).

### 2.3.4 Eigenvalue Based Detection

Consider the acquisition of $N$ samples of the received signal, as given by equation (2.1), and splitting them into $L$ blocks (called the "smoothing factor") of $N_{s}$ samples to form a $L \times N_{s}$ matrix $\boldsymbol{Y}$, such that:

$$
\boldsymbol{Y}=\left[\begin{array}{llll}
\boldsymbol{r}_{0} & \boldsymbol{r}_{1} & \ldots & \boldsymbol{r}_{N_{s}-1}
\end{array}\right]=\left[\begin{array}{cccc}
r_{0}(0) & r_{1}(0) & & r_{N_{s}-1}(0) \\
r_{0}(1) & r_{1}(1) & \cdots & r_{N_{s}-1}(1) \\
\vdots & \ddots & \vdots \\
r_{0}(L-1) & r_{1}(L-1) & \cdots & r_{N_{s}-1}(L-1)
\end{array}\right]
$$

and $\boldsymbol{r}_{k}=\left[\begin{array}{llll}r(k \times L) & r(k \times L+1) & \ldots & r((k+1) \times L-1)\end{array}\right]^{T}$.
When the PU is idle (the null hypothesis, $\mathcal{H}_{0}$ ) the received samples are samples from the noise process, so the non-diagonal elements of the covariance matrix of the received signal should theoretically be zero; the diagonal elements should contain the noise variance. Therefore, for some fixed $L$, when $N_{s} \rightarrow \infty$ the covariance matrix $\frac{1}{N_{s}} \boldsymbol{Y} \boldsymbol{Y}^{H}$ converges to $\sigma^{2} \boldsymbol{I}_{L}$. The $(L \times L)$ sample covariance matrix obtained from $\boldsymbol{Y}$, is given, alternatively by:

$$
\begin{equation*}
\boldsymbol{R}_{r}\left(N_{s}\right)=\frac{1}{N_{s}} \boldsymbol{Y} \boldsymbol{Y}^{H}=\frac{1}{N_{s}} \sum_{k=0}^{N_{s}-1} \boldsymbol{r}_{k} \boldsymbol{r}_{k}^{H} . \tag{2.21}
\end{equation*}
$$

This same model can be used when the SU has access to $L$ sampled signals (each with $N_{s}$ samples) obtained from $L$ antennas, or when the input signal is over-sampled by a factor of $L$ (Zeng and Liang, 2009).

The eigenvalues of the sample covariance matrix are obtained from the characteristic equation:

$$
\operatorname{det}\left(\boldsymbol{R}_{r}\left(N_{s}\right)-\lambda \boldsymbol{I}_{L}\right)=0
$$

where det is the determinant, $\lambda$ are the eigenvalues and $\boldsymbol{I}_{L}$ is the ( $L \times L$ ) identity matrix.
The Tracy-Widom distribution was found by Craig Tracy and Harold Widom (Tracy and Widom, $1996,2000)$ as the limiting law of the largest eigenvalue of an $(n \times n)$ Gaussian matrix. Few years later, Iain M. Johnstone (Johnstone, 2001) showed that the same limiting distribution applies to covariance matrixes, $\boldsymbol{X} \boldsymbol{X}^{H}$, derived from random rectangular $(n \times p)$ matrices $\boldsymbol{X}$, where $n$ and $p$ are large. The result from (Johnstone, 2001) can be expressed in the following theorem:

## Theorem 1

Assuming the noise is real, Gaussian with zero mean and variance, $\sigma^{2}$, and considering $\boldsymbol{A}_{n}\left(N_{s}\right)=\frac{N_{s}}{\sigma^{2}} \boldsymbol{R}_{n}\left(N_{s}\right)$; a centering parameter given by $\mu=\left(\sqrt{N_{s}-1}+\sqrt{L}\right)^{2}$ and a scaling parameter given by $\nu=\left(\sqrt{N_{s}-1}+\sqrt{L}\right)\left(\frac{1}{\sqrt{N_{s}-1}}+\frac{1}{\sqrt{L}}\right)^{1 / 3}$, and also that the dimension of $\boldsymbol{R}_{n}\left(N_{s}\right)$ is such that $\lim _{N_{s} \rightarrow \infty} \frac{L}{N_{s}}=y(0<y<1)$, then $\frac{\lambda_{\max }\left(\boldsymbol{A}_{r}\left(N_{s}\right)\right)-\mu}{\nu}$ converges to a TracyWidom distribution of order 1 :

$$
\begin{gather*}
\frac{\lambda_{\max }\left(\boldsymbol{A}_{n}\left(N_{s}\right)\right)-\mu}{\nu} \rightarrow \mathcal{T} \mathcal{W}_{1} \\
\mu=\left(\sqrt{N_{s}-1}+\sqrt{L}\right)^{2} ; \nu=\left(\sqrt{N_{s}-1}+\sqrt{L}\right)\left(\frac{1}{\sqrt{N_{s}-1}}+\frac{1}{\sqrt{L}}\right)^{1 / 3} \tag{2.22}
\end{gather*}
$$

Also, in the same conditions as in Theorem 1, in (Bai, 1999) (Theorem 2.16), it is shown that the limit of the highest and lowest eigenvalue are given by $\lim _{N_{s} \rightarrow \infty} \lambda_{\max }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)=\sigma^{2}(1+\sqrt{y})^{2}$ and $\lim _{N_{s} \rightarrow \infty} \lambda_{\min }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)=\sigma^{2}(1-\sqrt{y})^{2}$, respectively. Therefore one gets:

$$
\begin{align*}
& \lim _{N_{s} \rightarrow \infty} \lambda_{\max }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)=\frac{\sigma^{2}}{N_{s}}\left(\sqrt{N_{s}}+\sqrt{L}\right)^{2} \\
& \lim _{N_{s} \rightarrow \infty} \lambda_{\min }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)=\frac{\sigma^{2}}{N_{s}}\left(\sqrt{N_{s}}-\sqrt{L}\right)^{2} \tag{2.23}
\end{align*}
$$

The algorithms for eigenvalue detection rely on the properties given in equations (2.22) and (2.23), as shown in the following.

The centering and scaling parameters given in Equation (2.22) can be fine-tuned (given by Equation (2.24)), as indicated in (Nadler, 2011; Ma, 2012) to give a close match to the TracyWidom distribution. As can be seen in Figure 2.23, where the theoretical Tracy-Widom distribution is plotted for $L=20$ and $N_{s}=500$ (using the approximation given in (Chiani, 2014)), the simulation result for $2^{20}$ Monte Carlo runs with parameters corresponding to Equation (2.24) (Low $N_{s}, L$ approximation) are much closer to the distribution than with parameters corresponding to Equation (2.22) (High $N_{s}, L$ approximation).

$$
\begin{align*}
\mu & =\left(\sqrt{N_{s}-1 / 2}+\sqrt{L-1 / 2}\right)^{2} \\
\nu & =\left(\sqrt{N_{s}-1 / 2}+\sqrt{L-1 / 2}\right)\left(\frac{1}{\sqrt{N_{s}-1 / 2}}+\frac{1}{\sqrt{L-1 / 2}}\right)^{1 / 3}  \tag{2.24}\\
& =\mu \frac{(\sqrt{\mu})^{-\frac{2}{3}}}{\left(\left(N_{s}-1 / 2\right)(L-1 / 2)\right)^{\frac{1}{6}}}
\end{align*}
$$

Considering this new definition for $\mu$ and $\nu$, equations given in (2.23) are modified to:

$$
\begin{align*}
& \lim _{N_{s} \rightarrow \infty} \lambda_{\max }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)=\frac{\sigma^{2}}{N_{s}}\left(\sqrt{N_{s}-1 / 2}+\sqrt{L-1 / 2}\right)^{2}=\frac{\sigma^{2}}{N_{s}} \mu \\
& \lim _{N_{s} \rightarrow \infty} \lambda_{\min }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)=\frac{\sigma^{2}}{N_{s}}\left(\sqrt{N_{s}-1 / 2}-\sqrt{L-1 / 2}\right)^{2} \tag{2.25}
\end{align*}
$$

Figure 2.23 - Tracy-Widom density with simulation results for high and low $N_{s}, L$ approximations.

Considering the statistical covariance matrices of the signal and noise:

$$
\begin{aligned}
& \boldsymbol{R}_{s}=\mathrm{E}\left[\boldsymbol{s} \boldsymbol{s}^{H}\right] \\
& \boldsymbol{R}_{r}=\mathrm{E}\left[\boldsymbol{r} \boldsymbol{r}^{H}\right] \\
& \boldsymbol{R}_{n}=\mathrm{E}\left[\boldsymbol{n} \boldsymbol{n}^{H}\right]=\sigma^{2} \boldsymbol{I},
\end{aligned}
$$

then

$$
\begin{equation*}
\boldsymbol{R}_{r}=\boldsymbol{R}_{s}+\sigma^{2} \boldsymbol{I} \tag{2.26}
\end{equation*}
$$

When there is no PU signal $\boldsymbol{s}, \boldsymbol{R}_{s}=0$. Therefore, as $\lambda_{\max }\left(\boldsymbol{R}_{r}\right)=\lambda_{\text {max }}\left(\boldsymbol{R}_{s}\right)+\lambda_{\max }\left(\boldsymbol{R}_{n}\right)$, when there is no signal from the PU, $\lambda_{\max }\left(\boldsymbol{R}_{r}\right)=\lambda_{\text {max }}\left(\boldsymbol{R}_{n}\right)=\sigma^{2}$, and, when the PU is present, $\lambda_{\max }\left(\boldsymbol{R}_{r}\right)>\lambda_{\max }\left(\boldsymbol{R}_{n}\right)$. The same holds for the minimum eigenvalues.

Eigenvalue spectrum sensing algorithms may be proposed to allow signal presence detection by combining these eigenvalues properties (although not with the statistical covariance matrix but using the sample covariance matrix) and using their distribution and asymptotic values.

The two most used algorithms for eigenvalue spectrum sensing are addressed in the following subsections: the Maximum Eigenvalue Detection (MED) and the Maximum-Minimum eigenvalue Detection (MMED). Again, these are CFAR algorithms so that the detection threshold will be obtained for a given probability of false alarm (Kay, 1998).

### 2.3.4.1 Maximum Eigenvalue Detection (MED)

As seen, when there is no signal from the $\mathrm{PU}, \lambda_{\max }\left(\boldsymbol{R}_{r}\right)=\sigma^{2}$; when the PU is using the channel, $\lambda_{\text {max }}\left(\boldsymbol{R}_{r}\right)>\sigma^{2}$. The Maximum Eigenvalue Detection (Zeng, Choo and Liang, 2008) algorithm uses the maximum eigenvalue of the sample covariance matrix and a given $P_{F A}$ to obtain the CFAR detection threshold, $\gamma$, as (from the null hypothesis, $\mathcal{H}_{0}$ ):

$$
\begin{aligned}
P_{F A} & =P_{r}\left\{\lambda_{\max }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)>\gamma \sigma^{2}\right\} \\
& =P_{r}\left\{\frac{\sigma^{2}}{N_{s}} \lambda_{\max }\left(\boldsymbol{A}_{n}\left(N_{s}\right)\right)>\gamma \sigma^{2}\right\} \\
& =P_{r}\left\{\frac{\lambda_{\max }\left(\boldsymbol{A}_{n}\left(N_{s}\right)\right)-\mu}{\nu}>\frac{\gamma N_{s}-\mu}{\nu}\right\} .
\end{aligned}
$$

From (2.22), $\frac{\lambda_{\max }\left(\boldsymbol{A}_{n}\left(N_{s}\right)\right)-\mu}{\nu}$ follows a Tracy-Widom distribution, and considering $F_{1}($.$) the$ cumulative distribution function (cdf) of the $\mathcal{T} \mathcal{W}_{1}$, then

$$
\begin{equation*}
P_{F A}=1-F_{1}\left(\frac{\gamma N_{s}-\mu}{\nu}\right) . \tag{2.27}
\end{equation*}
$$

Using the centering and scaling parameters from Equation (2.24), one finally gets:

$$
\begin{gather*}
\gamma=\frac{\left(\sqrt{N_{s}-1 / 2}+\sqrt{L-1 / 2}\right)^{2}}{N_{s}}(1 \\
\left.+\frac{\left(\sqrt{N_{s}-1 / 2}+\sqrt{L-1 / 2}\right)^{-\frac{2}{3}}}{\left(\left(N_{s}-1 / 2\right)(L-1 / 2)\right)^{\frac{1}{6}}} \times F_{1}^{-1}\left(1-P_{F A}\right)\right) . \tag{2.28}
\end{gather*}
$$

To obtain the probability of detection, we will use the following approximation, which holds when $N_{s}$ is very large (Zeng and Liang, 2009):

$$
\begin{equation*}
\boldsymbol{R}_{r}\left(N_{s}\right) \approx \boldsymbol{R}_{s}\left(N_{s}\right)+\boldsymbol{R}_{n}\left(N_{s}\right) . \tag{2.29}
\end{equation*}
$$

Although not analyzed by the authors who proposed the MED algorithm, the probability of detection can be obtained, considering $\rho_{\max }$ the maximum eigenvalue of $\boldsymbol{R}_{s}\left(N_{s}\right)$. Under the approximation in (2.29), the probability of detection is:

$$
\begin{align*}
P_{D} & =\operatorname{Pr}\left\{\lambda_{\max }\left(\boldsymbol{R}_{r}\left(N_{s}\right)\right)>\gamma \sigma^{2}\right\} \\
& =\operatorname{Pr}\left\{\lambda_{\max }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)+\rho_{\max }>\gamma \sigma^{2}\right\} \\
& =\operatorname{Pr}\left\{\frac{N_{s}}{\sigma^{2}} \lambda_{\max }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)>N_{s}\left(\gamma-\frac{\rho_{\max }}{\sigma^{2}}\right)\right\}  \tag{2.30}\\
& =1-F_{1}\left(\frac{N_{s}\left(\gamma-\frac{\rho_{\max }}{\sigma^{2}}\right)-\mu}{\nu}\right)
\end{align*}
$$

Modeling the signal $s(n)$ as a Gaussian, zero mean and variance $\sigma_{s}^{2}$ r.v., and using the asymptotic value of the maximum eigenvalue, as given in equation (2.25), one gets:

$$
\begin{align*}
& P_{D}=1-F_{1}\left(\frac{N_{s}\left(\gamma-\frac{\sigma_{s}^{2}}{\sigma^{2}} \frac{\left(\sqrt{N_{s}-1 / 2}+\sqrt{L-1 / 2}\right)^{2}}{N_{s}}\right)-\mu}{\nu}\right)  \tag{2.31}\\
& =1-F_{1}\left(\frac{\left(N_{s} \gamma-\operatorname{snr} \times\left(\sqrt{N_{s}-1 / 2}+\sqrt{L-1 / 2}\right)^{2}\right)-\mu}{\nu}\right) .
\end{align*}
$$

Using (2.27) and (2.31), and eliminating $\gamma$, the sample size needed for the detector can now be obtained, given the system parameters, by (although not in a closed-form):

$$
\begin{equation*}
\frac{\left(\sqrt{N_{s}-1 / 2}+\sqrt{L-1 / 2}\right)^{-\frac{2}{3}}}{\left(\left(N_{s}-1 / 2\right)(L-1 / 2)\right)^{\frac{1}{6}}}\left(F_{1}^{-1}\left(1-P_{F A}\right)-F_{1}^{-1}\left(1-P_{D}\right)\right)=s n r . \tag{2.32}
\end{equation*}
$$

In Figure 2.24, the theoretical $P_{D}$ vs. SNR is plotted for varying $P_{F A}\left(10^{-1} ; 10^{-2} ; 10^{-3} ; 10^{-4}\right)$ considering $L=20$ and $N_{s}=500$. Simulation results are also plotted (dots markers) obtained from $2^{16}$ Monte Carlo runs. Naturally, as can be seen in the figure, for a given $P_{F A}$, the detector performance will be better for a higher SNR.

It is worth mentioning that the approximation made to obtain the $P_{D}$ is only valid when $N_{s} \rightarrow$
$\infty$, which clearly is not the case. Nevertheless, the probability of detection given by equation (2.31) has a close matching to the simulation results, even for a small $N_{s}$ value, especially at low SNR.


Figure 2.24 - Receiver performance $P_{D}$ vs. SNR (MED, $L=20, N_{s}=500$ ).

As in the case of Energy Detection (ED), the MED also assumes that the noise variance is known and constant. In (Zeng, Choo and Liang, 2008), it is shown that the MED algorithm performs similarly to the ED when the input Signal-of-Interest is modeled as Gaussian i.i.d.. For a highly correlated input signal, it is shown to perform much better than the ED.

### 2.3.4.2 Maximum-Minimum Eigenvalue Detection (MMED)

The Maximum-Minimum Eigenvalue Detector (MMED) proposed in (Zeng and Liang, 2007, 2009) attempts to remove the necessity of knowing the noise statistics, by using the asymptotic values of the maximum and also the minimum eigenvalues of the sample covariance matrix.

Specifically, the MMED algorithm uses the test statistic given by $\lambda_{\max }\left(\boldsymbol{R}_{r}\left(N_{s}\right)\right) / \lambda_{\min }\left(\boldsymbol{R}_{r}\left(N_{s}\right)\right)>\gamma$, where $\gamma>1$. As seen in section 2.3.4, when there is no signal from the PU, $\lambda_{\max }\left(\boldsymbol{R}_{r}\right)=\lambda_{\min }\left(\boldsymbol{R}_{r}\right)=\sigma^{2}$. The proposed statistic uses the distribution of the maximum eigenvalues and the asymptotic value of the minimum eigenvalue of the sample
covariance matrix and a given $P_{F A}$ to obtain the CFAR detection threshold, $\gamma$, as (from the null hypothesis, $\mathcal{H}_{0}$ ):

$$
P_{F A}=P_{r}\left\{\lambda_{\max }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right) / \lambda_{\text {min }}\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)>\gamma\right\} .
$$

Using the asymptotic value of the minimum eigenvalue, as given in equation (2.25), one gets:

$$
\begin{aligned}
& P_{F A}=P_{r}\left\{\lambda_{\max }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)>\gamma \frac{\sigma^{2}}{N_{s}}\left(\sqrt{N_{s}-1 / 2}-\sqrt{L-1 / 2}\right)^{2}\right\} \\
& =P_{r}\left\{\frac{\sigma^{2}}{N_{s}} \lambda_{\max }\left(\boldsymbol{A}_{n}\left(N_{s}\right)\right)>\gamma \frac{\sigma^{2}}{N_{s}}\left(\sqrt{N_{s}-1 / 2}-\sqrt{L-1 / 2}\right)^{2}\right\} \\
& =P_{r}\left\{\frac{\lambda_{\max }\left(\boldsymbol{A}_{n}\left(N_{s}\right)\right)-\mu}{\nu}>\frac{\gamma\left(\sqrt{N_{s}-1 / 2}-\sqrt{L-1 / 2}\right)^{2}-\mu}{\nu}\right\}
\end{aligned}
$$

From (2.22), $\frac{\lambda_{\max }\left(\boldsymbol{A}_{r}\left(N_{s}\right)\right)-\mu}{\nu}$ follows a Tracy-Widom distribution, and considering $F_{1}($.$) the CDF$ of the $\mathcal{T} \mathcal{W}_{1}$,

$$
\begin{equation*}
P_{F A}=1-F_{1}\left(\frac{\gamma\left(\sqrt{N_{s}-1 / 2}-\sqrt{L-1 / 2}\right)^{2}-\mu}{\nu}\right) \tag{2.33}
\end{equation*}
$$

Using the centering and scaling parameters from Equation (2.24), one finally gets:

$$
\begin{gather*}
\gamma=\frac{\left(\sqrt{N_{s}-1 / 2}+\sqrt{L-1 / 2}\right)^{2}}{\left(\sqrt{N_{s}-1 / 2}-\sqrt{L-1 / 2}\right)^{2}}(1 \\
\left.+\frac{\left(\sqrt{N_{s}-1 / 2}+\sqrt{L-1 / 2}\right)^{-\frac{2}{3}}}{\left(\left(N_{s}-1 / 2\right)(L-1 / 2)\right)^{\frac{1}{6}}} F_{1}^{-1}\left(1-P_{F A}\right)\right) . \tag{2.34}
\end{gather*}
$$

As in the case of the MED detector, using the approximation for very large $N_{s}$, given in (2.29) and considering $\rho_{\max }$ and $\rho_{\min }$, respectively, the maximum and minimum eigenvalue of $\boldsymbol{R}_{s}\left(N_{s}\right)$, the probability of detection is

$$
\begin{align*}
P_{D} & =\operatorname{Pr}\left\{\lambda _ { \operatorname { m a x } } \left(\boldsymbol{R}_{r}\left(N_{s}\right)>\gamma \lambda_{\min }\left(\boldsymbol{R}_{r}\left(N_{s}\right)\right\}\right.\right. \\
& =\operatorname{Pr}\left\{\frac { N _ { s } } { \sigma ^ { 2 } } \lambda _ { \operatorname { m a x } } \left(\boldsymbol{R}_{n}\left(N_{s}\right)>N_{s}\left(\gamma+\left(\gamma \rho_{\min }-\rho_{\max }\right) / \lambda_{\min }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)\right\}\right.\right.  \tag{2.35}\\
& =1-F_{1}\left(\frac{N_{s}\left(\gamma+\left(\gamma \rho_{\min }-\rho_{\max }\right) / \lambda_{\min }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)-\mu\right.}{\nu}\right)
\end{align*}
$$

For very large $N_{s}$, and as given in $(2.25), \lim _{N_{s} \rightarrow \infty} \lambda_{\min }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)=\frac{\sigma^{2}}{N_{s}}\left(\sqrt{N_{s}-1 / 2}-\sqrt{L-1 / 2}\right)^{2}$. However, as suggested in (Zeng and Liang, 2009) and confirmed in our simulations, using $\lim _{N_{s} \rightarrow \infty} \lambda_{\min }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)=\frac{\sigma^{2}}{\sqrt{N_{s}}}\left(\sqrt{N_{s}-1 / 2}-\sqrt{L-1 / 2}\right)$ yields slightly better results for $P_{D}$. In this case, $P_{D}$ is given by:

$$
\begin{align*}
& P_{D} \\
& =1-F_{1}\left(\frac{N_{s}\left(\gamma+\left(\gamma \rho_{\min }-\rho_{\max }\right) /\left(\frac{\sigma^{2}}{\sqrt{N_{s}}}\left(\sqrt{N_{s}-1 / 2}-\sqrt{L-1 / 2}\right)\right)\right)-\mu}{\nu}\right) . \tag{2.36}
\end{align*}
$$

Using (2.33) and (2.36) and eliminating $\gamma$, the sample size needed for the detector, given the system parameters, can be obtained, although not in a closed-form. However, this will not be done in this case because, as will be seen in the following, the equations obtained for both $P_{F A}$ and $P_{D}$ are too conservative, comparing to simulation results.

In Figure 2.25 , the $P_{D}$ vs. SNR is plotted for varying $P_{F A}\left(10^{-1} ; 10^{-2} ; 10^{-3} ; 10^{-4}\right)$ considering $L=20$ and $N_{s}=500$. This figure contains only the simulation results obtained from $2^{16}$ Monte Carlo runs. The input is a random QPSK signal ${ }^{24}$ (as in (Zeng and Liang, 2009), (Kortun et al., 2010)) with binary rate and carrier frequency equal to one-fourth of the sampling rate, with raisedcosine pulse shape with roll-off equal to 1 . Naturally, as can be seen in the figure, for a given $P_{F A}$, the detector performance will be better for a higher SNR.

However, these results are obtained by adjusting the threshold such that the correct $P_{F A}$ is achieved, similarly as done in (Zeng and Liang, 2009)(Zeng, Choo and Liang, 2008). In fact, using the threshold given by equation (2.34) will give a higher $P_{F A}$ than the desired value. The threshold given by (2.34) is satisfactory only for very large $L$ and $N_{s}$. In the case of the simulations presented in Figure 2.25, the probability of false alarm obtained without adjusting the threshold would be $P_{F A}=\left[0.59 \times 10^{-1} ; 1.4 \times 10^{-2} ; 3.5 \times 10^{-3} ; 1.0 \times 10^{-3}\right]$.

As to the probability of detection, as given by equation (2.36), even with the aforementioned adjusted value for $\lim _{N_{s} \rightarrow \infty} \lambda_{\min }\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right)$ (Zeng and Liang, 2009), the result obtained is still too conservative, with simulation results showing a much better performance than theoretically predicted. In Figure 2.26, the theoretical $P_{D}$ vs. SNR is plotted for $P_{F A}=10^{-4}$ along with the simulation results obtained. As can be seen, the theoretical curve is way too conservative, and the results obtained by simulation are clearly much better. More recently, in (Çiflikli and Ilgin, 2018), the authors obtained a new approximation to the probability of detection, but the results are still conservative. To the author's best knowledge, obtaining the optimal threshold and corresponding $P_{D}$ for this algorithm (presenting a close match to simulations results) is still an open question.

[^15]

Figure 2.25 - Receiver performance $P_{D}$ vs. SNR (MMED, $L=20, N_{s}=500$ ).


Figure 2.26 - Receiver performance $P_{D}$ vs. SNR (MMED, $L=20, N_{s}=500$ ) for $P_{F A}=10^{-4}$.

The advantage of the MMED algorithm is to avoid the necessity of knowing the noise variance, as is the case of the ED or MED detectors. As shown in (Zeng and Liang, 2009; Liu et al., 2010), the MMED will perform worse than the Energy Detector when there is no noise uncertainty, but better when the uncertainty is higher than approximately $0.5 d B$. This uncertainty level can be considered acceptable, as typical noise uncertainty on receiving devices is 1 to 2 dB (Sahai and Cabric, 2005).

The MMED algorithm was analyzed for other input modulation schemes and with real-world signals, e.g. (Kortun et al., 2010; Hamid et al., 2012; Pan, Wang and Shen, 2012). Compared to algorithms that need at least an estimate of the noise variance, the MMED will perform better when the noise uncertainty is moderate to high (typically above 1.5 dB ).

### 2.3.4.3 Other Eigenvalue Based Detectors

The MMED algorithm has also been proposed to be used in a multistage detector (Hamid, Bjorsell and Slimane, 2016), where the first stage uses the ED and the second stage uses the MMED detector (called the two-stage energy-maximum-minimum eigenvalue combined detector - 2EMC). When the ED declares the absence of the PU, the second stage, using the MMED, is now used to test for its presence. In the proposed method, this second stage is also responsible for estimating the noise variance to be used in the first stage, making the proposed multistage detector completely blind. Although presenting a better performance than MMED in the cases illustrated by the authors, the problem with this detector comes from the noise variance estimator. It is shown to perform well only when the used bandwidth of the PU signal is much smaller than the sensing bandwidth.

Several works have focused on eigenvalue detection, considering real word signals, namely captured Digital TV (DTV) signals in New York and Washington, DC (Tawil, 2006). In a recent paper, an eigenvalue detector was proposed and compared to other recent ones (Jin et al., 2019). Unfortunately, these detectors assume the knowledge (or estimation) of the noise variance, which uncertainty can degrade the performance on moderate to high noise uncertainty. Also, it is assumed that the PU signal has a sample covariance matrix with clear $p$ large eigenvalues compared to the complete eigenvalues set. The proposed method (called Sum of multiple large eigenvalues with estimated noise power - SMLE-ENP) was shown to perform better than several others already proposed, as indicated below. For the test statistics presented, $\lambda_{i}$ are the decreasing $\lambda_{\max }=\lambda_{1}>$ $\lambda_{2}>\cdots>\lambda_{L}=\lambda_{\min }(i=1 \ldots L)$ eigenvalues of the sample covariance matrix of the received signal. The variance of the noise is estimated assuming there are $N_{y}$ samples available of noise only, $y(n), n=0 \ldots N_{y}-1$, possibly obtained from a vacant channel or training phase. Therefore, the noise variance is estimated by $\hat{\sigma}^{2}=\frac{1}{N_{y}} \sum_{n=0}^{N_{y}-1} y(n) y(n)^{H}$. For the Roy's and Wilks's detector, it is assumed that the number of samples of noise only is equal to the number of received signal samples, so that the sample covariance matrix of the noise only, $\boldsymbol{R}_{y}\left(N_{s}\right)$, has the same dimension ( $L \times L$ ) of the corresponding to the received signal, $\boldsymbol{R}_{r}\left(N_{s}\right)$.

The algorithms compared are:

- Sum of Multiple Large Eigenvalue (SMLE-ENP) detector with estimated noise power, with test statistic (Jin et al., 2019):

$$
T_{S M L E-E N P}=\frac{\sum_{i=1}^{p} \lambda_{i}}{\hat{\sigma}^{2}}
$$

- Signal-Subspace Eigenvalues (SSE-ENP) detector with estimated noise power, with test statistic (Zhang et al., 2010):

$$
T_{S S E-E N P}=\frac{N_{s} p}{2}\left(\frac{\frac{1}{p} \sum_{i=1}^{p} \lambda_{i}}{\hat{\sigma}^{2}}-\ln \left(\frac{\left(\prod_{i=1}^{p} \lambda_{i}\right)^{1 / p}}{\hat{\sigma}^{2}}\right)-1\right)
$$

- Roy's detector with estimated noise power, with test statistic (Roy, 1953; Wei, Tirkkonen and Liang, 2014):

$$
T_{R-E N P}=\lambda_{\max }\left(\boldsymbol{R}_{y}^{-1}\left(N_{s}\right) \boldsymbol{R}_{r}\left(N_{s}\right)\right)
$$

- Wilks' detector with estimated noise power, with test statistic (Wilks, 1932; Wei, Tirkkonen and Liang, 2014):

$$
T_{W-E N P}=\frac{\operatorname{det}\left(\boldsymbol{R}_{y}\left(N_{s}\right)+\boldsymbol{R}_{r}\left(N_{s}\right)\right)}{\operatorname{det}\left(\boldsymbol{R}_{y}\left(N_{s}\right)\right)}
$$

- Maximum Eigenvalue Detector (ME-ENP), as given in section 2.3.4.1, with estimated noise power, with test statistic (Zeng, Choo and Liang, 2008):

$$
T_{M E D-E N P}=\frac{\lambda_{1}}{\hat{\sigma}^{2}}
$$

- Arithmetic to Geometric Mean (AGM) detector, with the test statistic (Zhang et al., 2010; Wei and Tirkkonen, 2012):

$$
T_{A G M}=\frac{\frac{1}{L} \sum_{i=1}^{L} \lambda_{i}}{\left(\prod_{i=1}^{L} \lambda_{i}\right)^{1 / L}}
$$

- Maximum-Minimum Eigenvalue Detector (MMED), as given in section 2.3.4.2, with test statistic (Zeng and Liang, 2007, 2009):

$$
T_{M M E D}=\frac{\lambda_{1}}{\lambda_{L}}
$$

- Maximum Eigenvalue to Arithmetic Mean (MEAM) detector, with the test statistic (Wang et al., 2010):

$$
T_{M E A M}=\frac{\lambda_{1}}{\sum_{i=1}^{L} \lambda_{i}}
$$

The ROC curves obtained for $S N R=-6 d B$, with $L=10, N_{s}=100, N_{y}=100$ and $p=4$, with the real-world sampled DTV signal, are reprinted (with IEEE permission) in Figure 2.27. As can be seen, algorithms that do not need noise statistic information have poor performance (MMED, MEAM and AGM, in increasing performance order) compared to estimated noise power counterparts (Roy's, SSE-ENP, Wilks', ME-ENP, and SMLE-ENP, also in increasing performance
order). However, noise uncertainty was not fully analyzed, which always results in highperformance degradation for medium to high uncertainty.


Figure 2.27 - ROC curves for various eigenvalue-based detectors by using the real-world DTV signals (from (Jin et al., 2019)/ 2327-4662 © 2018 IEEE - reused with IEEE permission)

### 2.3.5 Cyclostationary Feature Detection

Cyclostationary feature detection approach to PU detection is based on the fact that most humanmade communications signals are not accurately modeled as stationary, but rather more appropriately modeled as cyclostationary. Stationary signals have statistics that remain constant over time, while the ones of cyclostationary signals are periodic, due to the process of sampling, modulation, multiplexing or coding (Roberts, Brown and Loomis Jr., 1991), (Gardner, 1991).

For a wide-sense cyclostationary signal, $x(t)$, that exhibits cyclostationarity on second-order statistics, e.g., the autocorrelation function, a two-dimensional autocorrelation function can be computed, where the second dimension is referred to as the cyclic frequency. As it will be addressed in Chapter 3, the Cyclic Autocorrelation Function (CAF) is given by (Gardner, 1987)(Napolitano, 2019):

$$
\begin{equation*}
R_{x}^{\alpha}(\tau) \triangleq \lim _{T \rightarrow \infty} \frac{1}{T} \int_{T} x\left(t+\frac{\tau}{2}\right) x^{*}\left(t-\frac{\tau}{2}\right) e^{-j 2 \pi \alpha t} d t \tag{2.37}
\end{equation*}
$$

The Spectral Correlation Function (SCF) is, by definition, the Fourier transform of the cyclic autocorrelation function, given by:

$$
\begin{equation*}
S_{x}^{\alpha}(f) \triangleq \int_{-\infty}^{+\infty} R_{x}^{\alpha}(\tau) e^{-j 2 \pi f \tau} d \tau \tag{2.38}
\end{equation*}
$$

The SCF can also be interpreted as the spectral density of the correlation of the input signal $x(t)$ at frequencies $f+\frac{\alpha}{2}$ and $f-\frac{\alpha}{2}$, as will be seen in the next chapter. If the input signal is cyclostationary, the SCF will be different from zero for some cycle frequencies $\alpha \neq 0$.

Since the spectral components of white noise are uncorrelated, it does not contribute to the SCF of the received signal (when modeled as the sum of the Signal-of-Interest plus noise) at any $\alpha \neq 0$. For $\alpha=0$, where the SCF is now not insensitive to the noise, the SCF reduces to the known Power Spectral Density (PSD).

A normalization of the SCF, known as the Spectral Coherence Function (SCoF), is given as

$$
\begin{equation*}
C_{x}^{\alpha}(f) \triangleq \frac{S_{x}^{\alpha}(f)}{\left[\left(S_{x}^{0}(f+\alpha / 2)\right)^{*} \times S_{x}^{0}(f-\alpha / 2)\right]^{1 / 2}} \tag{2.39}
\end{equation*}
$$

The absolute value of the Spectral Coherence is in the range [0,1], and can be used for the detection or classification of the input signal, $x(t)$. Using the SCoF for signal detection or classification has the benefit of insensitivity to the channel effect. Considering the channel response, $h(t)$, with Fourier Transform $H(f)$, then, for the input signal, $x(t)$, and received signal $y(t)$ the following holds (Gardner, 1994):

$$
\begin{align*}
y(t) & =x(t) * h(t) \\
S_{y}^{\alpha}(f) & =H\left(f+\frac{\alpha}{2}\right) H^{*}\left(f-\frac{\alpha}{2}\right) S_{x}^{\alpha}(f)  \tag{2.40}\\
C_{y}^{\alpha}(f) & =C_{x}^{\alpha}(f)
\end{align*}
$$

As can be seen from (2.40), although the SCF can be distorted by the channel, the corresponding SCoF is the same as the obtained by the original (undistorted) input signal, as long as $H(f \pm \alpha / 2) \neq 0$.

As will be seen in the next chapter, the SCF can be obtained by a spectral correlation analyzer (see Figure 3.2), which will converge to the true SCF when the frequency resolution and time averaging tend to infinity $(\Delta f \rightarrow 0, \Delta t \rightarrow \infty)$. For the simulations presented in this section, we will consider that the SCF is obtained using $N p S e g$ points to estimate the spectral components (sliding FFT) and $P$ averaging segments with no decimation $(L=1)$. As will be seen, accurate estimation of the SCF will require a large $P$, which will be the case of the simulation results presented.

As also shown in (Gardner, 1994), a lowpass signal $([0, B] \mathrm{Hz})$ and a bandpass signal $([b, B] \mathrm{Hz})$ will have their corresponding regions of support of the SCF (or SCoF) as shown in Figure 2.28.


Figure 2.28 -Region of support of the SCF: a) bandpass signal; b) lowpass signal.

Therefore, knowing the signal's bandwidth to detect, one can define the region of the SCF to search for cyclostationary features. In the considered signal model (Equation (2.1)), the additive white Gaussian noise is purely stationary, so its SCF will be zero, except for $\alpha=0$ (Gardner, 1987).

Figure 2.29 illustrates the theoretical SCF of a random QPSK signal with symbol duration equal to $T_{0}$. As can be seen, cycle features (meaning that the SCF $\neq 0$ for some $\alpha \neq 0$ ) are obtained for $\alpha=k / T_{0}$.

When the received Signal-of-Interest is corrupted with additive noise, the resulting SCF will show leakage on the corresponding region of support. As an example (also used in the following), Figure 2.30a) shows the modulus of the SCF obtained for a two-channel QPSK modulation with a corresponding carrier frequency of $f_{c_{1}}=1 \mathrm{MHz}$ and $f_{c_{2}}=5 \mathrm{MHz}$, and symbol rate $f_{\text {sym }_{1}}=$ 1 Msps and $f_{\text {sym }}=2.25 \mathrm{Msps}$ (the sampling frequency is $f_{s}=18 \mathrm{MHz}$, with raised-cosine pulse shape with $100 \%$ excess bandwidth (roll-off, $\beta=1$ ). In this example, $N p S e g=128$ samples, with decimation factor, $L=N p S e g / 4=32$, and $P=2000$, so that the total number of samples is 64096, corresponding to a sensing period of approximately 3.56 ms .


Figure 2.29 - Theoretical SCF of a QPSK signal with $T_{0}$ symbol duration and rectangular pulse shape.

The cyclic features of each channel are clearly identified at $\alpha$ values corresponding to each channel symbol rate. However, as we are evaluating the SCF on a finite time duration interval, the SCF will not be zero on all the plane, except for $\alpha=0$ and $\alpha=f_{\text {sym }}$. Adding noise will result in further leakage on the resulting SCF, as can be seen in Figure 2.30b) for an SNR $=-5 \mathrm{~dB}$.

We now will assess the performance of a cyclostationary detector, considering the Signal-of-Interest as a QPSK modulated signal corrupted by additive Gaussian noise. Again, and as in (Zeng and Liang, 2009) (Kortun et al., 2010), this input is a random QPSK signal with binary rate and carrier frequency equal to one-fourth of the sampling rate, with raised-cosine pulse shape with roll-off equal to 1 .

For the first CFAR detector, we will use the SCF evaluated at $f=f_{s} / 4$ and $\alpha=f_{s} / 4$, where the cyclic feature of the QPSK signal is maximum. The detection threshold is adapted for each SNR simulated so that the $P_{F A}$ is the desired one. Note that this implies that the detector to be developed must have an estimate of the noise variance and must know the symbol rate.


Figure 2.30 - SCF for two-channel QPSK modulation with $f_{s}=18 \mathrm{MHz}$ and $f_{s y m_{1}}=1 \mathrm{Msps}, f_{c_{1}}=$ $1 \mathrm{MHz}, f_{\text {sym }_{2}}=2,25 \mathrm{Msps}, f_{c_{2}}=5 \mathrm{MHz}$. a) no noise; b) $\mathrm{SNR}=-5 \mathrm{~dB}$.

For the second CFAR detector, we will use the Spectral Coherence Function (SCoF). We will show in this thesis that for high $P$, with $L<N p S e g$, and when the input is just the noise component, the SCoF can be modeled as a normal distribution with zero mean and standard deviation $\sigma_{\text {SCoF }}=\frac{1}{\sqrt{2}} \sqrt{\frac{1+c^{2}}{1+(2 P-1) c}}$ with $c=L / N p S e g$, for its real and imaginary parts. This allows us to have a blind detector with no need for a noise variance estimate (nevertheless, one needs to know the cyclic feature location). The absolute value of the SCoF, used as the decision statistic, will have a Rayleigh distribution with a scale parameter equal to $\sigma_{\text {SCoF }}$. So given $F_{\text {Rayleigh }\left(\sigma_{S C o F}\right)}($.$) the$ cumulative distribution function (CDF) of Rayleigh $\left(\sigma_{S C o F}\right)$, and for a given $P_{F A}$, the detection threshold, $\gamma$, is given by

$$
\begin{equation*}
\gamma=F_{\text {Rayleigh }\left(\sigma_{S C o F)}\right)}^{-1}\left(1-P_{F A}\right) . \tag{2.41}
\end{equation*}
$$

In Figure 2.31, the $P_{D}$ vs. SNR is plotted for varying $P_{F A}\left(10^{-1} ; 10^{-2} ; 10^{-3} ; 10^{-4}\right)$ considering $L=1, N=10000$ and $N p S e g=16$ samples, for the SCF and the SCoF detector $(N=N p S e g+$ $(P-1) \times L)$. The results for the SCF detector are obtained by adjusting the threshold such that
the correct $P_{F A}$ is achieved. Naturally, as can be seen in the figure, for a given $P_{F A}$, the detectors performance will be better for a higher SNR.


Figure 2.31 - Receiver performance $P_{D}$ vs. SNR (SCF and SCoF, $N=10000, N p S e g=16, L=1$ ).

An important conclusion that can be obtained from the simulation results, is that there is only a slight performance degradation of the SCoF detector, when compared to the SCF detector. This is remarkable, considering that the SCoF is a blind detector (with respect to the noise statistics) and the SCF is not. However, as can be seen from (2.39), to compute the SCoF for some ( $f_{i} ; \alpha_{i}$ ) point, one needs the SCF of that point and also the SCF of two other points: $\left(f_{i}-\alpha_{i} / 2 ; 0\right)$ and ( $f_{i}+\alpha_{i} / 2 ; 0$ ), therefore increasing the computational cost.

### 2.3.6 Detectors Performance Comparison

We will now use the results present in the previous sections to make a comparison between the detectors. In the cases where the noise variance needs to be known, we will consider a reasonable uncertainty on its estimate, namely $\rho=0 \mathrm{~dB}$ (no uncertainty), $\rho=0.1 \mathrm{~dB}$ (low uncertainty), and $\rho=1 \mathrm{~dB}$ (moderate/high uncertainty). Further, as the matched filter detector needs the exact knowledge of the Signal-of-Interest, including timing synchronization, it will not be considered here, because, when wideband spectrum sensing is performed to detect spectrum holes, possible different communication technologies may be used on the wide bandwidth. For all simulation results presented, the probability of false alarm is $P_{F A}=0.1$, and the number of samples is $N=$
10000. As in previous sections, the input Signal-of-Interest is a random QPSK signal with a binary rate and carrier frequency equal to one-fourth of the sampling rate, with a raised-cosine pulse shape with roll-off equal to 1 .

In Figure 2.32, the $P_{D}$ vs. SNR is plotted for the Energy Detector (ED), Spectral Correlation Function Detector (SCF), and the Spectral Coherence Function Detector (SCoF).

Clearly, with no noise uncertainty, the Energy Detector outperforms all other detectors. Nevertheless, the ED performance degrades severely even for low uncertainty, which does not occur with the SCF detector (the performance is only slightly worse for $\rho=0.1 d B$ when compared to $\rho=0 d B)$. When the noise uncertainty increases, the SCF will clearly perform much better than the ED. As can also be seen in Figure 2.32, the SCoF detector has only marginal worse performance than the SCF for $\rho=0 d B$, but, as it is a blind detector with respect to the noise statistics, it does not degrade performance as noise uncertainty increases.

Figure 2.33 shows the $P_{D}$ vs. SNR obtained for the two blind detectors referenced before: the SCoF and the MMED. As can be seen, the SCoF detector performs much better, and, as detailed in section 2.3.4.2, the MMED will have a worse performance when using the threshold given by (2.34), and not by adjusting the threshold such that the correct $P_{F A}$ is achieved, as it was done to obtain the results illustrated in Figure 2.33.


Figure 2.32 - Receiver performance $P_{D}$ vs. SNR with noise uncertainty (SCoF, SCF, ED, $P_{F A}=$ $0.1 ; N=10000, N p S e g=16, L=1$ ).

Finally, in Figure 2.34, the $P_{D}$ vs. SNR is plotted for the SCF and the MED detectors, as both need an estimate of the noise variance.


Figure 2.33 - Receiver performance $P_{D}$ vs. SNR (SCoF, MMED, $P_{F A}=0.1 ; N=10000, L=20, N_{s}=$ 500).


Figure 2.34 - Receiver performance $P_{D}$ vs. SNR with noise uncertainty (SCF, MED, $P_{F A}=0.1 ; N=$ $\left.10000, L=20, N_{s}=500\right)$.

As can be seen from Figure 2.34, the MED detector will perform just slightly better than the SCF when there is no noise uncertainty and the SNR is low. When there is moderate to high noise uncertainty, the MED performance degrades quickly, with much worse performance than the SCF.

### 2.4 Chapter Summary

This chapter presented the Dynamic Spectrum Access (DSA) paradigm, where DSA networks provide the possibility of a highly efficient use or reuse of the radio spectrum. Specifically, the Opportunistic Spectrum Access (OSA) model was addressed, leading to the concept of Spectrum holes, meaning the idle channels not in use by some PU. Some SU may use this idle channel, provided it will stop transmitting as soon as the PU is detected again.

The need for flexible physical-layer functions of the radio to operate in this scenario led to the development of the Software Defined Radio (SDR), which is the enabling technology needed for the deployment of Cognitive Radio (CR) systems or Cognitive Radio Networks (CRN). A radio with CR capabilities will continuously sense the (probably wideband) radio spectrum and detect the spectrum holes, channel state and interference/noise information. With this information, it adjusts operating parameters, namely the transmission power, modulation, carrier frequency, etc., to (re)use the detected spectrum holes. Although a CRN will probably need a Cooperative Spectrum Sensing strategy to solve network issues like the hidden node problem, each node must be capable of sensing the channel in its surrounding neighborhood.

Four major actual digital signal processing techniques for spectrum sensing have been examined: Energy Detection, Matched Filter Detection, Eigenvalue Based Detection and Cyclostationary Feature Detection. The spectrum sensing techniques to use depend on the knowledge available about the PU signal and noise statistics.

For having low computational complexity, the Energy Detector is widely used for spectrum sensing. Nevertheless, the severe performance degradation when there is noise uncertainty, leading to the SNR wall, renders it useless in a low SNR environment. The Matched Filter Detector is optimal in the detection of a known PU signal in AWGN. However, different spectrum bands may have PUs with different technologies, leading to excessive computational complexity when sensing a wideband spectrum. Further, it is very sensitive to synchronization errors.

With higher computational complexity, the Eigenvalue Based Detectors can have good performance, but the sample size needs to be very large to obtain a not-so-too-conservative estimate of the threshold level for a given probability of false alarm. Nevertheless, it is in this class of detectors that arises the first blind detector with respect to the noise statistics.

Finally, although with high computational complexity, the Cyclostationary Feature Detection emerges as the best detector since human-made signals are cyclostationary. Specifically, the Spectral Correlation Function (SCF) Detector is much less sensitive to the noise uncertainty than the Energy detector but needs some information about the PU signal that generates the cyclostationary features (e.g., data rate, cyclic prefix, etc.). The Spectral Coherence Function (SCoF) Detector is the best blind detector (with similar performance as the SCF when there is no
noise uncertainty) with respect to the noise statistics, but it also needs some information about the PU signal that generates the cyclostationary features.

In the following, we will address the efficient algorithms to compute the SCF and SCoF and provide a modified algorithm to improve efficiency when detecting cyclostationary features on specific spectrum bands. This is especially useful when cyclostationary features' location is not exactly known and must be searched within the frequency and cycle frequency plane (Yeung and Gardner, 1996). Also, and for the algorithm used in this thesis (FAM - FFT Accumulation Method), the probability density of the spectral estimates of the SCF and SCoF is obtained, along with useful approximations. This will allow the development of a CFAR and a CDR detector when noise variance is known or estimated (using the SCF) and a CFAR blind detector (using the SCoF).


## Chapter 3

## 3 Cyclic Spectral Analysis

Efficient Cyclic Spectral Analysis (CSA) algorithms are seeing a growing relevance in late years, due to the convergence in the scientific community to consider this field an important and significant tool for signal analysis in various fields of science and engineering ((Roberts, Brown and Loomis Jr., 1991), (Gardner, Napolitano and Paura, 2006), (Gardner, 2018), (Napolitano, 2019)). Generally, communications signals are typically assumed to be stationary, i.e., signals whose statistical parameters, such as the mean and variance, do not vary with time. For this class of signals, a one-dimensional autocorrelation function can be obtained, and also its Power Spectrum Density (PSD) by computing the Fourier Transform (FT) of the autocorrelation function.

However, most human-made modulated signals used in communication systems are, in fact, not stationary but cyclostationary (and specifically wide-sense cyclostationary, for exhibiting cyclostationarity on second-order statistics, e.g., the autocorrelation function) due to the process of sampling, modulation, multiplexing or coding (Roberts, Brown and Loomis Jr., 1991), (Gardner, 1991). A signal is cyclostationary if it has statistical properties, namely its autocorrelation, with periodic changing features. For this class of cyclostationary signals, a two-dimensional autocorrelation function can be computed, where the second dimension is referred to as the cyclic frequency. As already referenced in section 2.3.5 (equation (2.37)), the Cyclic Autocorrelation Function (CAF) is given by (Gardner, 1987)(Napolitano, 2019):

$$
\begin{equation*}
R_{x}^{\alpha}(\tau) \triangleq \lim _{T \rightarrow \infty} \frac{1}{T} \int_{T} x\left(t+\frac{\tau}{2}\right) x^{*}\left(t-\frac{\tau}{2}\right) e^{-j 2 \pi \alpha t} d t \tag{3.1}
\end{equation*}
$$

which can be interpreted as the Fourier coefficient of a sinusoidal component with frequency $\alpha$ contained in a quadratic transformation of the input signal $x(t)$. Therefore, the signal $x(t)$ is cyclostationary if $R_{x}^{\alpha}(\tau)$ is different from zero at some time delay $\tau$ and cycle frequency $\alpha \neq 0$.

Considering a signal $x(t)$, the symmetric conventional autocorrelation function is given by

$$
\begin{equation*}
R_{x}(t, \tau)=E\left[x\left(t+\frac{\tau}{2}\right) x^{*}\left(t-\frac{\tau}{2}\right)\right] \tag{3.2}
\end{equation*}
$$

For a (wide sense) stationary signal $x(t)$, the autocorrelation (and mean) does not depend on the central time but only on the time difference (lag) hence $R_{x}\left(t_{1}, \tau\right)=R_{x}\left(t_{2}, \tau\right)=R_{x}(\tau)$. For the case of nonstationary but cyclostationary signals, the autocorrelation is a periodic function with period $T_{0}\left(\right.$ so $\left.R_{x}\left(t+T_{0}, \tau\right)=R_{x}(t, \tau)\right)$. As a periodic function, it can be expressed as a Fourier
series, given by $R_{x}(t, \tau)=\sum_{\alpha} R_{x}^{\alpha}(\tau) e^{j 2 \pi \alpha t}$, where $\alpha$ are the cycle frequencies, $\alpha=k / T_{0}$, and $R_{x}^{\alpha}(\tau)$ is the CAF (the Fourier coefficients at frequency $\alpha$ ). Therefore, The CAF is obtained by the coefficients of the Fourier series, given by

$$
R_{x}^{\alpha}(\tau)=\frac{1}{T_{0}} \int_{-T_{0} / 2}^{+T_{0} / 2} R_{x}(t, \tau) e^{-j 2 \pi \alpha t} d t
$$

Considering the signal as cyclo-ergodic and using the synchronized averaging technique (Gardner, 1987)(p. 362), the autocorrelation function, as given in (3.2), can be obtained by

$$
R_{x}(t, \tau)=\lim _{N \rightarrow \infty} \frac{1}{2 N+1} \sum_{n=-N}^{N} x\left(t+n T_{0}+\frac{\tau}{2}\right) x^{*}\left(t+n T_{0}-\frac{\tau}{2}\right)
$$

so that

$$
R_{x}^{\alpha}(\tau)=\lim _{N \rightarrow \infty} \frac{1}{(2 N+1) T_{0}} \sum_{n=-N}^{N} \int_{-T_{0} / 2}^{+T_{0} / 2} x\left(t+n T_{0}+\frac{\tau}{2}\right) x^{*}\left(t+n T_{0}-\frac{\tau}{2}\right) e^{-j 2 \pi \alpha t} d t
$$

Considering $T=(2 N+1) T_{0}$ and making the substitution $l=t+n T_{0}$, we get

$$
\begin{aligned}
& R_{x}^{\alpha}(\tau)=\lim _{T \rightarrow \infty} \frac{1}{T} \sum_{n=-N}^{N} \int_{-T_{0} / 2+n T_{0}}^{+T_{0} / 2+n T_{0}} x\left(l+\frac{\tau}{2}\right) x^{*}\left(l-\frac{\tau}{2}\right) e^{-j 2 \pi \alpha\left(l-n T_{0}\right)} d l \\
& =\lim _{T \rightarrow \infty} \frac{1}{T} \sum_{n=-N}^{N} \underbrace{e^{j 2 \pi \alpha n T_{0}}}_{=1} \int_{-T_{0} / 2+n T_{0}}^{+T_{0} / 2+n T_{0}} x\left(l+\frac{\tau}{2}\right) x^{*}\left(l-\frac{\tau}{2}\right) e^{-j 2 \pi \alpha l} d l \\
& =\lim _{T \rightarrow \infty} \frac{1}{T} \int_{-T_{0} / 2-N T_{0}}^{+T_{0} / 2+N T_{0}} x\left(l+\frac{\tau}{2}\right) x^{*}\left(l-\frac{\tau}{2}\right) e^{-j 2 \pi \alpha l} d l \\
& =\lim _{T \rightarrow \infty} \frac{1}{T} \int_{-T / 2}^{+T / 2} x\left(l+\frac{\tau}{2}\right) x^{*}\left(l-\frac{\tau}{2}\right) e^{-j 2 \pi \alpha l} d l
\end{aligned}
$$

which is the CAF presented in Equation (3.1).

The Spectral Correlation Function (SCF) is, by definition, the Fourier transform of the cyclic autocorrelation function, given by:

$$
\begin{equation*}
S_{x}^{\alpha}(f) \triangleq \int_{-\infty}^{+\infty} R_{x}^{\alpha}(\tau) e^{-j 2 \pi f \tau} d \tau \tag{3.3}
\end{equation*}
$$

The SCF can also be interpreted as the cross-spectrum of the input signal $x(t)$ at frequencies $f+$ $\frac{\alpha}{2}$ and $f-\frac{\alpha}{2}$, as will be seen in the following.

When using discrete-time input, the time and lag variables take integer values, and so $\tau / 2$ does not correspond to a direct signal input. In this case, the SCF can be estimated, and Inverse Fourier Transformed to obtain the CAF. Another way is to obtain the relation between the symmetric and asymmetric CAF. Considering the asymmetric CAF defined by

$$
\begin{equation*}
\widetilde{R}_{x}^{\alpha}(\tau)=\lim _{T \rightarrow \infty} \frac{1}{T} \int_{-T / 2}^{+T / 2} x(t+\tau) x^{*}(t) e^{-j 2 \pi \alpha t} d t \tag{3.4}
\end{equation*}
$$

and making the change of variable $\xi=t+\tau / 2$, we see that the symmetric CAF can be obtained from the asymmetric counterpart by simply scaling by a complex exponential ${ }^{25}$ :

$$
\begin{align*}
\widetilde{R}_{x}^{\alpha}(\tau) & =\lim _{T \rightarrow \infty} \frac{1}{T} \int_{-T / 2+\tau / 2}^{+T / 2+\tau / 2} x\left(\xi+\frac{\tau}{2}\right) x^{*}\left(\xi-\frac{\tau}{2}\right) e^{-j 2 \pi \alpha(\xi-\tau / 2)} d \xi \\
& =e^{j \pi \alpha \tau} \lim _{T \rightarrow \infty} \frac{1}{T} \int_{-T / 2}^{+T / 2} x\left(\xi+\frac{\tau}{2}\right) x^{*}\left(\xi-\frac{\tau}{2}\right) e^{-j 2 \pi \alpha \xi} d \xi  \tag{3.5}\\
& =e^{j \pi \alpha \tau} R_{x}^{\alpha}(\tau)
\end{align*}
$$

We now address the structure to obtain the SCF, starting by analyzing the cross-correlation and cross-spectrum of two input signals, $x(t)$ and $y(t)$.

Given the cross-correlation between two signals $x(t)$ and $y(t)$ :

$$
\begin{equation*}
R_{x y}(\tau)=E\left[x\left(t+\frac{\tau}{2}\right) y^{*}\left(t-\frac{\tau}{2}\right)\right] \tag{3.6}
\end{equation*}
$$

the corresponding cross-spectrum is given by

$$
\begin{equation*}
S_{x y}(f)=\int_{-\infty}^{+\infty} R_{x y}(\tau) e^{-j 2 \pi f \tau} d \tau \tag{3.7}
\end{equation*}
$$

Consider the block diagram of the cross-spectrum analyzer, as illustrated in Figure 3.1, where the input signals $x(t)$ and $y(t)$ are first filtered by a band-pass filter centered at $f_{i}$, with bandwidth $\Delta f \approx 1 / T$, following multiplication (with a conjugate branch), and finally averaging for $\Delta t$ seconds. The output signal, $S_{x y}\left(t, f_{i}\right)_{\Delta t \Delta f}$, is given by:

$$
\begin{equation*}
S_{x y}\left(t, f_{i}\right)_{\Delta t \Delta f}=\frac{1}{\Delta t} \int_{t-\Delta t / 2}^{t+\Delta t / 2} z(u) d u \tag{3.8}
\end{equation*}
$$

[^16]

Figure 3.1 - Block Diagram of a cross-spectrum analyzer.

The band-passed signals are given by (considering a sinc bandpass filter shape, with $\Delta f \approx 1 / T$, so that $H(f)=\operatorname{sinc}\left(T\left(f-f_{i}\right)\right)$ and corresponding impulse response $\left.h(t)=\frac{1}{T} u_{T}(t) e^{j 2 \pi f_{i} t}\right)$ :

$$
\begin{align*}
x_{f_{i}}(t) & =\frac{1}{T} \int x(t-v) u_{T}(v) e^{j 2 \pi f_{i} v} d v \\
y_{f_{i}}(t) & =\frac{1}{T} \int y(t-v) u_{T}(v) e^{j 2 \pi f_{i} v} d v \tag{3.9}
\end{align*}
$$

where

$$
u_{T}(t)=\operatorname{rect}\left(\frac{t}{T}\right)= \begin{cases}1 & |t| \leq T / 2  \tag{3.10}\\ 0 & |t|>T / 2\end{cases}
$$

To obtain a reliable result of the cross-spectrum using the given cross-spectrum analyzer, it is mandatory that $\Delta t \gg T$, giving the time-frequency product condition $\Delta f \Delta t \gg 1$.

Considering the averaging time, $\Delta t$, going to infinite, the output of the cross-spectrum analyzer becomes the mean of the normalized cross product, so:

$$
\begin{aligned}
\lim _{\Delta t \rightarrow \infty} S_{x y}\left(t, f_{i}\right)_{\Delta t \Delta f} & =E_{t}[z(t)] \\
& =E_{t}\left[\frac{1}{T^{2}} \iint x(t-u) u_{T}(u) e^{j 2 \pi f_{i} u} y^{*}(t-v) u_{T}(v) e^{-j 2 \pi f_{i} v} d u d v\right] \\
& =\frac{1}{\mathrm{~T}^{2}} \iint E_{t}\left[x(t-u) y^{*}(t-v)\right] u_{T}(u) u_{T}(v) e^{j 2 \pi f_{i}(u-v)} d u d v
\end{aligned}
$$

$$
=\frac{1}{\mathrm{~T}^{2}} \iint R_{x y}(v-u) u_{T}(u) u_{T}(v) e^{-j 2 \pi f_{i}(v-u)} d u d v
$$

Making the substitution $\beta=v-u$ and $\gamma=(v+u) / 2$, one gets

$$
\begin{aligned}
\lim _{\Delta t \rightarrow \infty} S_{x y}\left(t, f_{i}\right)_{\Delta t \Delta f} & =\frac{1}{\mathrm{~T}^{2}} \iint u_{T-|\beta|}(\gamma) u_{2 T}(\beta) R_{x y}(\beta) e^{-j 2 \pi f_{i} \beta} d \beta d \gamma \\
& =\frac{1}{\mathrm{~T}^{2}} \int(T-|\beta|) u_{2 T}(\beta) R_{x y}(\beta) e^{-j 2 \pi f_{i} \beta} d \beta \\
& =\int\left[\frac{1}{T}\left(1-\frac{|\beta|}{T}\right) u_{2 T}(\beta) e^{+j 2 \pi f_{i} \beta}\right]^{*} R_{x y}(\beta) d \beta
\end{aligned}
$$

Now, considering that the Fourier Transform $\operatorname{FT}\left\{\frac{1}{T}\left(1-\frac{|\beta|}{T}\right) u_{2 T}(\beta)\right\}=\operatorname{sinc}^{2}(T f)=W(f)$, so that $\operatorname{FT}\left\{\frac{1}{T}\left(1-\frac{|\beta|}{T}\right) u_{2 T}(\beta) e^{+j 2 \pi f_{i} \beta}\right\}=W\left(f-f_{i}\right)$, and $\operatorname{FT}\left\{R_{x y}(\beta)\right\}=S_{x y}(f)$, and using the Parseval Identity ${ }^{26}$ :

$$
\begin{equation*}
\lim _{\Delta t \rightarrow \infty} S_{x y}\left(t, f_{i}\right)_{\Delta t \Delta f}=\int W\left(f-f_{i}\right) S_{x y}(f) d f \tag{3.11}
\end{equation*}
$$

If the bandpass filters bandwidth approaches zero, to give a true cross-spectrum, then $W\left(f-f_{i}\right) \rightarrow W(0) \delta\left(f-f_{i}\right)=\delta\left(f-f_{i}\right), \mathrm{so}^{27}:$

$$
\begin{equation*}
\lim _{\Delta t \rightarrow \infty, \Delta f \rightarrow 0} S_{x y}\left(t, f_{i}\right)_{\Delta t \Delta f}=S_{x y}\left(f_{i}\right) \tag{3.12}
\end{equation*}
$$

The same result is obtained if the outputs of the bandpass filters are substituted by the complex envelope of the signal at a frequency $f_{i}$ with bandwidth $\Delta f$. In this case, the complex envelope is given by a Short-Time Fourier Transform:

$$
\begin{align*}
X_{T}\left(t, f_{i}\right) & =\frac{1}{T} \int_{t-T / 2}^{t+T / 2} x(v) e^{-j 2 \pi f_{i} v} d v  \tag{3.13}\\
& =\frac{1}{T} \int x(v) u_{T}(t-v) e^{-j 2 \pi f_{i} v} d v
\end{align*}
$$

Making the change of variable $u=t-v$, this becomes

$$
X_{T}\left(t, f_{i}\right)=\frac{1}{T} \int x(t-u) u_{T}(u) e^{-j 2 \pi f_{i}(t-u)} d u
$$

[^17]\[

$$
\begin{aligned}
& =\frac{1}{T} e^{-j 2 \pi f_{i} t} \int x(t-u) u_{T}(u) e^{+j 2 \pi f_{i} u} d u \\
& =\frac{1}{T} e^{-j 2 \pi f_{i} t} \times T \times x_{f_{i}}(t) \\
& =x_{f_{i}}(t) e^{-j 2 \pi f_{i} t}
\end{aligned}
$$
\]

Therefore, using the complex envelopes, and after conjugate and product (see Figure 3.1), one gets:

$$
X_{T}\left(t, f_{i}\right)\left(Y_{T}\left(t, f_{i}\right)\right)^{*}=x_{f_{i}}(t) e^{-j 2 \pi f_{i} t} y_{f_{i}}^{*}(t) e^{+j 2 \pi f_{i} t}=x_{f_{i}}(t) y_{f_{i}}^{*}(t)=z(t)
$$

Consider now the signals $u(t)$ and $v(t)$ obtained from $x(t)$ by a frequency shift:

$$
\begin{aligned}
& u(t)=x(t) e^{-j \pi \alpha_{i} t} \\
& v(t)=x(t) e^{+j \pi \alpha_{i} t}
\end{aligned}
$$

The cross-correlation is given by (Equation (3.6)):

$$
\begin{align*}
R_{u v}(\tau) & =E\left[u\left(t+\frac{\tau}{2}\right) v^{*}\left(t-\frac{\tau}{2}\right)\right] \\
& =E\left[x\left(t+\frac{\tau}{2}\right) e^{-j \pi \alpha_{i} t} x^{*}\left(t-\frac{\tau}{2}\right) e^{-j \pi \alpha_{i} t}\right]  \tag{3.14}\\
& =E\left[x\left(t+\frac{\tau}{2}\right) x^{*}\left(t-\frac{\tau}{2}\right) e^{-j 2 \pi \alpha_{i} t}\right]
\end{align*}
$$

which is the definition of the $\mathrm{CAF}, R_{x y}^{\alpha}(\tau)$. The cross-spectrum analyzer of Figure 3.1 can obtain the corresponding cross-spectrum if the input signal $x(t)$ is fed on the two input branches and frequency-shifted before bandpass filtering. This leads to the architecture of the spectral correlation analyzer depicted in Figure 3.2. Again, the output of the bandpass filters may be substituted by the corresponding complex envelope, giving:

$$
\begin{equation*}
S_{x}^{\alpha_{i}}\left(t, f_{i}\right)_{\Delta t \Delta f}=\left\langle X_{T}\left(t, f_{i}+\frac{\alpha_{i}}{2}\right) X_{T}^{*}\left(t, f_{i}-\frac{\alpha_{i}}{2}\right)\right\rangle_{\Delta t} \tag{3.15}
\end{equation*}
$$

Also, from Equation (3.12), we now have

$$
\begin{equation*}
\lim _{\Delta t \rightarrow \infty, \Delta f \rightarrow 0} S_{x}^{\alpha_{i}}\left(t, f_{i}\right)_{\Delta t \Delta f}=S_{x}^{\alpha_{i}}\left(f_{i}\right) \tag{3.16}
\end{equation*}
$$



Figure 3.2 - Block Diagram of a spectral correlation analyzer.

### 3.1 Symmetries of the SCF

As referenced in section 2.3.5, the region of support of the SCF for an input bandpass signal ( $[b, B]$ Hz ) sampled with a sampling rate $f_{s}$ is of a diamond shape as shown in Figure 3.3 (similar to Figure 2.28, and reprinted here for convenience). Naturally, for a general input signal, the full region of support is a diamond shape, as illustrated in Figure $3.7\left(-f_{s} / 2 \leq f<f_{s} / 2\right.$ and $-f_{s} \leq$ $\alpha<f_{s}$ ), which is equal to Figure 3.3 when $b=0$ and $B=f_{s} / 2$.


Figure 3.3 - Region of support of the SCF for a bandpass signal.

The sampling process generates a Spectral Correlation Function which has replicas centered at
multiples of $\left( \pm f_{s} / 2 ; \pm f_{s}\right)$ in the $(f ; \alpha)$ plane, so that the complete $\mathrm{SCF}, \hat{S}_{x}^{\alpha}$, is given by the aliasing formula (W.A. Gardner, 1986):

$$
\begin{equation*}
\hat{S}_{x}^{\alpha}(f)=\sum_{n, m=-\infty}^{+\infty} S_{x}^{\alpha+m f_{s}}\left(f-m \frac{f_{s}}{2}-n f_{s}\right) \tag{3.17}
\end{equation*}
$$

When the input signal $x(t)$ is bandlimited and Nyquist sampling is guaranteed $\left(f_{s}>2 B\right)$, each term in (3.17) corresponding to indexes $n$ and $m$ is disjoint from all other terms, and the complete SCF is as illustrated in Figure 3.4 (considering only the first nine terms around the origin).


Figure 3.4 - Region of support for the first terms around the origin for the SCF of a sampled signal.

Considering the definition of the CAF given by (3.1) and the SCF given by (3.3), when the input signal $x(t)$ is real, the following properties of the CAF can be obtained:

1. $R_{x}^{\alpha}(-\tau)=R_{x}^{\alpha}(\tau)$
2. $\quad R_{x}^{-\alpha}(\tau)=\left(R_{x}^{\alpha}(\tau)\right)^{*}$

So, for the SCF, similarly, one can obtain the following symmetry properties:

1. $S_{x}^{\alpha}(-f)=S_{x}^{\alpha}(f)$
2. $S_{x}^{-\alpha}(f)=\left(S_{x}^{\alpha}(f)\right)^{*}$

Therefore, to obtain the entire region of the SCF, only the first quadrant in Figure 3.7 needs to be
computed; the other quadrants can be obtained directly using the above symmetries.
For the complete SCF of a sampled signal, $\hat{S}_{x}^{\alpha}$, as given in (3.17), provided Nyquist sampling is used, the following periodicity properties apply, in addition to the above symmetry properties:

1. $\hat{S}_{x}^{\alpha}\left(f+f_{s}\right)=\hat{S}_{x}^{\alpha}(f)$
2. $\hat{S}_{x}^{\alpha+2 f_{s}}(f)=\hat{S}_{x}^{\alpha}(f)$
3. $\hat{S}_{x}^{\alpha+f_{s}}\left(f-f_{s} / 2\right)=\hat{S}_{x}^{\alpha}(f)$

### 3.2 The Spectral Coherence Function

The standard correlation coefficient, defined for two random variables, $X$ and $Y$ is given by:

$$
\begin{equation*}
\rho=\frac{E\left[\left(X-m_{X}\right)\left(Y-m_{Y}\right]\right)}{\sigma_{X} \sigma_{Y}} \tag{3.18}
\end{equation*}
$$

where $m_{X}, m_{Y}, \sigma_{X}$ and $\sigma_{Y}$ are, respectively, the mean values and standard deviation of $X$ and $Y\left(m_{X}=E[X] ; m_{Y}=E[Y] ; \sigma_{X}^{2}=E\left[\left(X-m_{X}\right)^{2}\right]\right.$ and $\left.\sigma_{Y}^{2}=E\left[\left(Y-m_{Y}\right)^{2}\right]\right)$.

The SCF can be viewed as

$$
\begin{equation*}
S_{x}^{\alpha}(f)=\lim _{T \rightarrow \infty} E\left[X_{T}(t, f+\alpha / 2) X_{T}^{*}(t, f-\alpha / 2)\right] \tag{3.19}
\end{equation*}
$$

Now, $E\left[X_{T}(t, f+\alpha / 2) X_{T}^{*}(t, f-\alpha / 2)\right]$ is just the correlation between two random variables given by $X_{T}(t, f+\alpha / 2)$ and $X_{T}(t, f-\alpha / 2)$ whose variance is, respectively, $E\left[\left|X_{T}(t, f+\alpha / 2)\right|^{2}\right]$ and $E\left[\left|X_{T}(t, f-\alpha / 2)\right|^{2}\right]^{28}$.

For finite $T$, the correlation coefficient becomes:

$$
\begin{equation*}
\rho=\frac{E\left[X_{T}(t, f+\alpha / 2) X_{T}^{*}(t, f-\alpha / 2)\right]}{\left[E\left[\left|X_{T}(t, f+\alpha / 2)\right|^{2}\right] E\left[\left|X_{T}(t, f-\alpha / 2)\right|^{2}\right]\right]^{1 / 2}} \tag{3.20}
\end{equation*}
$$

As $T \rightarrow \infty$, the numerator converges to $S_{x}^{\alpha}(f), \lim _{T \rightarrow \infty} E\left[\left|X_{T}(t, f+\alpha / 2)\right|^{2}\right]=S_{x}^{0}(f+\alpha / 2)$ and $\lim _{T \rightarrow \infty} E\left[\left|X_{T}(t, f-\alpha / 2)\right|^{2}\right]=S_{x}^{0}(f-\alpha / 2)$. Therefore, the correlation coefficient, called the Spectral Coherence Function (SCoF), is given by:

$$
\begin{equation*}
C_{x}^{\alpha}(f)=\frac{S_{x}^{\alpha}(f)}{\left[S_{x}^{0}(f+\alpha / 2) S_{x}^{0}(f-\alpha / 2)\right]^{1 / 2}} \tag{3.21}
\end{equation*}
$$

[^18]The SCoF is a normalization of the Spectral Correlation Function, and, since it is a valid correlation coefficient, its magnitude will be less than or equal to one. Also, since the SCF can, in general, be complex-valued, the SCoF lies in the closed unit disk in the complex plane.

As seen in the previous chapter (and will be used in the following to obtain a blind CFAR detector), the SCoF is a useful detection statistic for blindly determining significant cycle frequencies of the Signal-of-Interest when performing Spectrum Sensing.

### 3.3 Discrete-Time SCF

To allow a digital implementation for the computation of the SCF, the coss-spectrum analyzer and the spectral correlation analyzer (shown in Figure 3.1 and Figure 3.2, respectively) can be easily adapted from the analog counterparts for discrete-time signals.

Similarly, and from equation (3.15), the discrete Smoothed Cyclic Periodogram (SCP) is given by ${ }^{29}$ :

$$
\begin{equation*}
S_{x}^{\alpha_{i}}\left(n, f_{i}\right)_{\Delta t \Delta f}=\left\langle X_{T}\left(n, f_{i}+\frac{\alpha_{i}}{2}\right) X_{T}^{*}\left(n, f_{i}-\frac{\alpha_{i}}{2}\right)\right\rangle_{\Delta t} \tag{3.22}
\end{equation*}
$$

where $X_{T}\left(n, f_{i} \pm \alpha_{i} / 2\right)$ are the complex spectral components (complex envelope, or complex demodulates) obtained by narrowband, bandpass filtering of the input signal, $x(n)$, at frequencies $f_{i} \pm \alpha_{i} / 2$ followed by baseband down converting, and $\alpha$ is the cyclic frequency; $\langle.\rangle_{\Delta t}$ stands for the averaging over a $\Delta t$ period.

We can thus see that the computation of the smoothed cyclic cross periodogram involves the calculation of $X_{T}\left(n, f_{i}+\alpha_{i} / 2\right)$ for a sliding window of $T$ seconds (input channelization), followed by cross-product and averaging of the complex demodulates obtained at $f_{i} \pm \alpha_{i} / 2$ in a $\Delta t$ period. Given the sampling frequency, $f_{s}$, we will define $T=N_{p} / f_{s}$ and $\Delta t=N_{t} / f_{s}$.

The complex demodulates can be obtained by a short-time DFT (similarly as in (3.13)) as:

$$
\begin{equation*}
X_{T}(n, f)=\frac{1}{N_{p}} \sum_{k} a(k) x(k+n) e^{-j 2 \pi k f / f_{s}} \tag{3.23}
\end{equation*}
$$

where $a(k)$ is a data tapering window of length $T=N_{p} / f_{s}$.
Figure 3.5 illustrates the process of the sliding window over the input signal (channelization), whereas Figure 3.6 shows the complex demodulates (absolute values) for all data universe and a specific calculation of the cross product, $X_{T}\left(n, f_{i}+\alpha_{i} / 2\right) X_{T}^{*}\left(n, f_{i}-\alpha_{i} / 2\right)$, to obtain one point of the smoothed cyclic cross periodogram, $S_{x_{T}}^{\alpha_{i}}\left(n, f_{i}\right)_{\Delta t}$. In this example, the sliding window was applied without overlapping just for simplicity and ease of perception; this procedure will be

[^19]addressed in the following to obtain the best overlap between successive window samples.


Figure 3.5 - Input signal $x(t)$ and the corresponding channelization with a sliding window of $T$ seconds.


Figure 3.6 - Absolute value of complex demodulates $X_{T}(., f)$ and single $\left(f_{i} ; \alpha_{i}\right)$ points to calculate the smoothed cyclic cross periodogram.

As the input signal is sampled with a sampling rate $f_{s}$, the region of support of the spectral correlation is confined to the diamond shape illustrated in Figure 3.7 because the distinct possible values for $f \pm \alpha / 2$ must satisfy:

$$
\left\{\begin{array}{l}
-\frac{f_{s}}{2} \leq f+\frac{\alpha}{2} \leq+\frac{f_{s}}{2}  \tag{3.24}\\
-\frac{f_{s}}{2} \leq f-\frac{\alpha}{2} \leq+\frac{f_{s}}{2}
\end{array}\right.
$$



Figure 3.7 - Region of support of the spectral correlation.

For a given $\left(f_{i} ; \alpha_{i}\right)$, the spectral correlation can be calculated as

$$
\begin{equation*}
S_{x}^{\alpha_{i}}\left(n, f_{i}\right)_{\Delta f \Delta t}=\sum_{r} X_{T}\left(r, f_{k}\right) X_{T}^{*}\left(r, f_{l}\right) g(n-r), \tag{3.25}
\end{equation*}
$$

where $g(n-r)$ is a data tapering window of length $\Delta t=N_{t} / f_{s}$, provided that

$$
\left\{\begin{array} { l } 
{ f _ { l } = f _ { i } - \alpha _ { i } / 2 }  \tag{3.26}\\
{ f _ { k } = f _ { i } + \alpha _ { i } / 2 }
\end{array} \text { or, equivalently } \left\{\begin{array}{c}
f_{i}=\frac{f_{k}+f_{l}}{2} \\
\alpha_{i}=f_{k}-f_{l}
\end{array}\right.\right.
$$

To obtain the final definition of the spectral correlation, one can substitute the complex demodulates expression in $(3.25)^{30}$ to get:

$$
\begin{align*}
& S_{x}^{\alpha_{i}}\left(n, f_{i}\right)_{\Delta f \Delta t}=e^{-j 2 \pi \alpha_{i} n} \sum_{q} \sum_{r} m(q, r) x(n-q) x(n-r) \\
& m(q, r)=\sum_{p} a(q-p) a(r-p) g(p) e^{j 2 \pi f_{i}(q-r)} e^{j \pi \alpha_{i}(q+r)} \tag{3.27}
\end{align*}
$$

Equation (3.27) shows that the spectral correlation can be obtained by a quadratic transformation

[^20]of the input signal with a kernel $m(q, r)$ which contains the dependence on all the system parameters, namely, $f_{i}, \alpha_{i}, \Delta f$ and $\Delta \alpha$.

Again, similarly to equation (3.16), the SCF is given by the limits of the spectral correlation:

$$
\begin{equation*}
\lim _{\Delta t \rightarrow \infty, \Delta f \rightarrow 0} S_{x}^{\alpha_{i}}\left(n, f_{i}\right)_{\Delta t \Delta f}=S_{x}^{\alpha_{i}}\left(f_{i}\right) \tag{3.28}
\end{equation*}
$$

provided the data tapering windows are properly scaled, $\sum_{n} a^{2}(n)=\sum_{n} g(n)=1$ (Brown and Loomis, 1993), and, as previously, the time-frequency product $\Delta t \Delta f \gg 1$.

The SCF is, by definition,

$$
\begin{equation*}
S_{x}^{\alpha}(f) \triangleq \sum_{k=-\infty}^{k=+\infty} R_{x}^{\alpha}(k) e^{-j 2 \pi f k} \tag{3.29}
\end{equation*}
$$

where $R_{x}^{\alpha}(k)$ is the Cyclic Autocorrelation Function, defined by

$$
\begin{align*}
R_{x}^{\alpha}(k) & =\lim _{\Delta t \rightarrow+\infty}\left\langle x(n+k) e^{-j \pi \alpha(n+k)}\left(x(n) e^{j \pi \alpha n}\right)^{*}\right\rangle_{\Delta t} \\
& =\lim _{\Delta t \rightarrow+\infty}\left\langle x(n+k) x^{*}(n) e^{-j 2 \pi \alpha n}\right\rangle_{\Delta t} e^{j \pi \alpha k}  \tag{3.30}\\
& =\lim _{N \rightarrow+\infty} \frac{1}{2 N+1} e^{-j \pi \alpha k} \sum_{n=-N}^{N} x(n+k) x^{*}(n) e^{-j 2 \pi \alpha n}
\end{align*}
$$

Note that the discrete CAF, defined by Equation (3.30), is just the discretization of Equation (3.1), with the scaling needed to convert from the asymmetric to symmetric autocorrelation (Equations (3.4) and (3.5)).

### 3.3.1 Coverage of the SCF with the spectrum correlation analyzer

With proper choice of $f_{k}$ and $f_{l}$, full coverage of the region of support of the SCF can be obtained, as shown in Figure 3.8, for a small dimension example ( $N_{p}=8$ for each array). As can be seen, for regular (equally spaced) $f_{l}$ and $f_{k}\left(f_{k} ; f_{l}=-f_{s} / 2: f_{s} / N_{p}: f_{s} / 2-f_{s} / N_{p}\right)$, the coverage of the region of support of the SCF is not regular (meaning that $f_{i}$ and $\alpha_{i}$ are not equally spaced), which may be an undesired feature. This issue will be addressed later in section 3.5.


Figure 3.8 - Coverage of the region of support of the SCF.

### 3.3.2 The Cyclic Spectrum Analyzer (CSA) Cell

As shown in, e.g. (Roberts, Brown and Loomis Jr., 1991; Brown and Loomis, 1993), the kernel $m(q, r)$ can be expressed in terms of its Fourier Transform as

$$
\begin{equation*}
M(f, \alpha)=\sum_{q} \sum_{r} m(q, r) e^{-j 2 \pi(f+\alpha / 2) q} e^{j 2 \pi(f-\alpha / 2) r}, \tag{3.31}
\end{equation*}
$$

or, using the Fourier Transform of the data tapering windows $G(f)=\mathrm{FT}(g(n))$ and $A(f)=$ FT( $a(n)$ ):

$$
\begin{equation*}
M(f, \alpha)=G\left(\alpha-\alpha_{i}\right) A\left(f-f_{i}+\frac{\alpha-\alpha_{i}}{2}\right) A^{*}\left(f-f_{i}-\frac{\alpha-\alpha_{i}}{2}\right) . \tag{3.32}
\end{equation*}
$$

Using the definition of $f_{k}$ and $f_{l}$ (Equation (3.26)) one finally gets:

$$
\begin{equation*}
M(f, \alpha)=G\left(\alpha-\alpha_{i}\right) A\left(f-f_{k}+\frac{\alpha}{2}\right) A^{*}\left(f-f_{l}-\frac{\alpha}{2}\right) . \tag{3.33}
\end{equation*}
$$

The cyclic features of the input signal, present in its SCF, will be transmitted to the output of the spectral correlation analyzer if they belong to the kernel $M(f, \alpha)$ domain.

The bandwidth of $a(n)$ if approximately $\Delta f=1 / T$, and of $g(n)$ is in the order of $1 / \Delta t$. As can be inferred from equation (3.33), the kernel $M(f, \alpha)$ will have an approximate frequency resolution given by $\Delta f$, and a cycle resolution of $\Delta \alpha=1 / \Delta t$. Combining with the time frequency product needed for a reliable estimate of the spectral correlation $(\Delta f \Delta t \gg 1)$ the relation between frequency and cycle frequency resolution is $\Delta f \gg \Delta \alpha$.

Figure 3.9 illustrates the region of support of the kernel $M(f, \alpha)$, where the diamond shape is called the Channel-Pair Region, defined by the intersection of $A\left(f-f_{k}+\frac{\alpha}{2}\right)$ and $A^{*}\left(f-f_{l}-\frac{\alpha}{2}\right)$. As can be seen, the data tapering window $a(n)$ must be properly designed so that skirts and sidelobes are confined in the Channel-Pair Region, not allowing adjacent region features to be transmitted to the output, thus originating the cycle slip occurrence. Also, the same consideration must be taken to $g(n)$, although this is not so stringent as $\Delta f \gg \Delta \alpha$. The major part of the energy of $M(f, \alpha)$ is confined to the shaded region in Figure 3.9, defining the so-called Cyclic Spectrum Analyzer (CSA) cell.


Figure 3.9 - Region of support of $M(f, \alpha)$.
In Figure 3.10 and Figure 3.11, the components $A\left(f-f_{l}-\alpha / 2\right), A\left(f-f_{k}+\alpha / 2\right)$ and $G(\alpha-$ $\alpha_{i}$ ) of the kernel $M(f, \alpha)$ are illustrated for the case when both $a(n)$ and $g(n)$ are rectangular and Hamming windows, respectively. On both images $T=1, \Delta t=10$ and $\alpha_{i}$ and $f_{i}$ where chosen as $\alpha_{i}=4$ and $f_{i}=3$ (so $f_{l}=f_{i}-\alpha_{i} / 2=1$ and $f_{k}=f_{i}+\alpha_{i} / 2=5$ ). Figure 3.12 and Figure 3.13 show the channelizer $\left|A\left(f-f_{l}-\alpha / 2\right) \times A\left(f-f_{k}+\alpha / 2\right)\right|$ for a rectangular and Hamming window, respectively. In these figures, the diamond shape of the Channel-Pair Region, as conceptually depicted in Figure 3.9, can be clearly identified.

For the complete characterization of the kernel $M(f, \alpha)$, Figure 3.14, Figure 3.15 and Figure 3.16 show its absolute mesh and contour plots (for $[-0.1 ;-5 ;-10 ;-15 ;-20 ;-25] \mathrm{dB}$ contours), for different windows $a(n)$ and $g(n)$. Figure 3.14 is obtained when both $a(n)$ and $g(n)$ are a rectangular window, whereas Figure 3.15 is obtained for both defined as a Hamming window. Clearly, for the Hamming window case, the kernel is more confined with fewer skirts. Figure 3.16 shows the case when $a(n)$ is chosen as Hamming window, but $g(n)$ as rectangular. Clearly, in this case the kernel shows several skirts, which is not a desired characteristic, since cyclic features on those skirts will be propagated to the output for the specific $\left(f_{i} ; \alpha_{i}\right)$ region in analysis. Nevertheless, and as seen, for a reliable estimation of the SCF, it is necessary to satisfy the condition $\Delta t \gg T$. The result presented in Figure 3.16 is for $T=1$ and $\Delta t=10$; if $\Delta t=100$, the kernel mesh and contour plot obtained is depicted in Figure 3.17. As can be seen in this case, the kernel is much more confined,
defining a clear Cyclic Spectrum Analyzer cell, as conceptually illustrated in Figure 3.9. Therefore, typically $a(n)$ is chosen as a Hamming window, but $g(n)$ as a simpler rectangular window.


Figure 3.10 - Kernel $M(f, \alpha)$ components (absolute values) with rectangular windows; $T=1, \Delta t=10$, $f_{i}=3, \alpha_{i}=4\left(f_{k}=5 ; f_{l}=1\right)$.

Kernel $M(f, \alpha)$ components


Figure 3.11 - Kernel $M(f, \alpha)$ components (absolute values) with Hamming windows; $T=1, \Delta t=10$, $f_{i}=3, \alpha_{i}=4\left(f_{k}=5 ; f_{l}=1\right)$.


Figure 3.12-| $A^{*}\left(f-f_{l}-\alpha / 2\right) \times A\left(f-f_{k}+\alpha / 2\right) \mid$ with rectangular windows; $T=1, f_{i}=3, \alpha_{i}=4$.


Figure $3.13-\left|A^{*}\left(f-f_{l}-\alpha / 2\right) \times A\left(f-f_{k}+\alpha / 2\right)\right|$ with Hamming windows; $T=1, f_{i}=3, \alpha_{i}=4$.


Figure 3.14 - Kernel $|M(f, \alpha)|$ with rectangular windows; $T=1, \Delta t=10, f_{i}=3, \alpha_{i}=4$.


Figure 3.15 - Kernel $|M(f, \alpha)|$ with Hamming windows; $T=1, \Delta t=10, f_{i}=3, \alpha_{i}=4$.


Figure 3.16 - Kernel $|M(f, \alpha)|$ with $a(n)$ Hamming and $g(n)$ rectangular; $T=1, \Delta t=10, f_{i}=3, \alpha_{i}=$ 4.


Figure 3.17 - Kernel $|M(f, \alpha)|$ with $a(n)$ Hamming and $g(n)$ rectangular; $T=1, \Delta t=100, f_{i}=3$,

$$
\alpha_{i}=4
$$

### 3.3.3 The Cyclic Spectrum Analyzer Cell with decimation

To enhance efficiency, the outputs of the filters to calculate the complex demodulates can be decimated by a factor $L\left(L<N_{p}\right)$. In this case, and for a given $\left(f_{i} ; \alpha_{i}\right)$, the SCF can now be calculated as (Roberts, Brown and Loomis Jr., 1991; Brown and Loomis, 1993):

$$
\begin{equation*}
S_{x}^{\alpha_{i}}\left(n L, f_{i}\right)_{\Delta f \Delta t}=\sum_{r} X_{T}\left(r L, f_{k}\right) X_{T}^{*}\left(r L, f_{l}\right) g_{c}(n-r), \tag{3.34}
\end{equation*}
$$

where $g_{c}(n)$ now has length $P=N_{t} / L$, and $f_{k}=f_{i}+\alpha_{i} / 2$ and $f_{l}=f_{i}-\alpha_{i} / 2$. Although reducing the complexity, now the kernel $M(f, \alpha)$ shows replicas for $\alpha=n f_{s} / L$ (see Figure 3.18). The decimation factor must be chosen to minimize the adjacent Channel-Pair region features to be propagated to the output of the current CSA cell, originating the phenomenon referred to as cyclic aliasing.


Figure 3.18 - Region of support of $M(f, \alpha)$ with a decimation factor of $L$.

As shown in (Roberts, Brown and Loomis Jr., 1991), the optimal value can be obtained for $L=$ $N_{p} / 4$, minimizing the cycle leakage and cycle aliasing, provided the use of suitable data tapering windows. Also, as shown for the case without decimation, the data tapering $a(n)$ can be implemented by a Hamming window avoiding cycle leakage, but the data tapering $g(n)$ can be just a rectangular window with minimum effect on performance.

### 3.4 Digital implementation - The FFT accumulation Method (FAM)

The complex demodulates, as expressed by (3.23), can be discretized for a finite set of frequencies as

$$
\begin{equation*}
X_{T}\left(n, f_{m}\right)=\frac{1}{N_{p}} \sum_{k=-\frac{N_{p}}{2}}^{\frac{N_{p}}{2}-1} a(k) x(k+n) e^{-j 2 \pi k f_{m} / f_{s}} \tag{3.35}
\end{equation*}
$$

where $f_{m}=m \frac{f_{s}}{N_{p}}$.
Considering the SCF estimated by (3.34) and now taking small (positive and negative) $\Delta \alpha$ steps from the ( $f_{i} ; \alpha_{i}$ ) point, the SCF can be expressed as (Roberts, Brown and Loomis Jr., 1991):

$$
\begin{equation*}
S_{x}^{\alpha_{i}+q \Delta \alpha}\left(n L, f_{i}\right)_{\Delta f \Delta t}=\sum_{r} X_{T}\left(r L, f_{k}\right) X_{T}^{*}\left(r L, f_{l}\right) g(n-r) e^{-j 2 \pi r q / P} \tag{3.36}
\end{equation*}
$$

where the summation can be evaluated with a $P$ point FFT $\left(P=N_{t} / L\right)$. Also, the summation in (3.35) can also be evaluated with a $N_{p}$ point FFT, which must be post multiplied by $e^{-j 2 \pi \frac{k}{N_{p}} n L}$, $k=-N_{p} / 2, \ldots,+N_{p} / 2$, to account for the time delay of each time window of the input signal channelization (phase compensation).

Note that using (3.36) will make the region of support of the kernel $M(f, \alpha)$ slide along the $\alpha$ axis by $q \Delta \alpha$, as illustrated in Figure 3.19. In this case, the CSA cell (grey shaded region) may move toward the edge of the Channel-Pair region where the time-frequency resolution product tends to zero, thus giving unreliable results. As suggested in (Roberts, Brown and Loomis Jr., 1991), only half of the points in (3.36) should be retained for $q \Delta \alpha$ around the centre ( $q=0$ ) to avoid unreliable points estimates.


Figure 3.19 - Region of support of $M(f, \alpha)$ for the FAM algorithm.

Equations (3.35) and (3.36), with summations evaluated using FFTs, resume one of the most widely used algorithms to obtain the estimate of the SCF, known as the FFT Accumulation Method (FAM), which is illustrated in Figure 3.20. Here, the method is illustrated using input channelization with NpSeg samples per Segment/Channel, and first FFT with $N_{p}$ points. Although other efficient algorithms for estimation of the SCF have been proposed in the nineties (Brown and Loomis, 1988)(Brown and Loomis, 1993)31, FAM is still recognized as the most computationally efficient algorithm in the specialized literature (Antoni, Xin and Hamzaoui, 2017).

The following signal processing flow describes the FAM algorithm (with architecture illustrated in Figure 3.20)

1. Given the frequency resolution required, $\Delta f$, obtain $N_{p}=f_{s} / \Delta f$ and $L=N_{p} / 4$ or $L=$ $N p S e g / 4$ if $N_{p} \neq N p S e g$ ( $f_{k}$ and $f_{l}$ will both be on the range $-f_{s} / 2: \Delta f:+f_{s} / 2-\Delta f$ );
2. The total number of samples is $N_{t}=N p S e g+(P-1) \times L \approx P \times L$. Given the cycle frequency resolution required, $\Delta \alpha$, obtain $P=f_{s} /(\Delta \alpha L)^{32}$. Note that one must have $\Delta \alpha \ll \Delta f ;$
3. If the input signal length is less than $N_{t}$, zero fill, otherwise truncate to a length $N_{t}$;
4. Channelize the input signal with $P$ channels of length $N p S e g$, with a sliding hamming window shifted by $L$ samples for each channel;
5. Compute the FFT for each channel;
6. Phase-compensate each channel to account for the time shift of each one;
7. Obtain the cross product of the complex demodulates for all channel combinations;
8. Compute the FFT for all cross products, and keep only $P / 2$ around the center;
9. Fill the SCF matrix, finding the corresponding $\left(f_{i} ; \alpha_{i}\right)$ pair, from $f_{k}, f_{l}$ and $q \Delta \alpha$.


Figure 3.20 - FAM algorithm architecture.

Following all FAM steps will compute the SCF with $N_{p}$ points in the frequency axis, and $N_{p} \frac{P}{2}$

[^21]points in the cycle frequency axis.
If step 8 is suppressed, and just a sum of all $P$ channels is taken (equivalent to the middle point obtained with the FFT of step 8), the SCF is obtained in an $N_{p} \times N_{p}$ grid (although non-regular) in the $(f ; \alpha)$ plane. We will use this case in the following for having a manageable computational complexity. Note that using step 8 requires an additional computation of $N_{p}^{2}$ " $P$-point" FFTs.

Further, considering only the SCF obtained in that $N_{p} \times N_{p}$ grid, all points have the same variance (Roberts, Brown and Loomis Jr., 1991) as the time-frequency product remains constant for each point (from Figure 3.19, only with $q=0$, the CSA cell is equal for all $\left(f_{i} ; \alpha_{i}\right)$ points, as in Figure 3.18).

### 3.4.1 FAM computation of the SCF

As shown in Figure 3.8, the SCF region of support coverage when using the FAM algorithm is not regular, which is usually a desired feature when computing the SCF. Of course, a regular $N_{p} \times N_{p}$ points in the $(f ; \alpha)$ plane resolution can be obtained with FAM, when the frequency spacing is $f_{s} / N_{p}$ and the cyclic frequency spacing is $2 f_{s} / N_{p}$.

It is even more important to note that with a regular FFT (as used in the FAM algorithm) and depending on the region of interest of the SCF, complex demodulates for specific frequencies ( $f_{k} ; f_{l}$ ) may not be possible to compute for the entire region, given the FFT frequency spacing of $f_{s} / N_{p}$, starting at $-f_{s} / 2\left(f_{k} ; f_{l}=\left[-f_{s} / 2: f_{s} / N_{p}: f_{s} / 2-f_{s} / N_{p}\right]\right)$. This can be especially important when searching for some unknown cyclic signature, where fine resolution is needed. Also, when the signal-to-noise ratio (SNR) is low, cyclostationary detection involves a large integration time, which further stresses the requirement of fine cycle frequency resolution (William A. Gardner, 1986; Yeung and Gardner, 1996).

As an example, we will consider a two-channel digital QPSK communication, with $100 \%$ excess bandwidth raised-cosine pulse shape; sampling frequency $f_{s}=18 \mathrm{MHz}$ and baud/symbol-rate equal to $f_{\text {sym }}=1 \mathrm{Msps}$ with a carrier frequency $f_{c_{1}}=1 \mathrm{MHz}$ for the first channel, and $f_{\text {sym }_{2}}=$ 2.25 Msps with $f_{c_{2}}=5 \mathrm{MHz}$ for the second channel. This example is similar to the one presented in section 2.3.5, with each channel having a strong cycle feature at $\alpha=f_{\text {sym }}$. Figure 3.21 shows the SCF of the received signal obtained using the FAM algorithm with $N_{p}=256$ samples and $P=2000\left(N p S e g=128\right.$, so that $L=N p S e g / 4=32$ and $N_{t}=N p S e g+(P-1) L=64,096$ samples), corresponding to a sensing period of approximately 3.56 ms .

As can be seen, the cyclic signature of the second channel is detected, but not the one of the first channel. The cycle frequency spacing is $2 f_{s} / N_{p}$ and the symbol-rate of the second channel is an integer multiple of this spacing $(2.25 /(2 \times 18 / 256)=16)$, but not the symbol-rate of the first channel $(1 /(2 \times 18 / 256)=7$. (1) $)$. The only solution using the FAM algorithm is to increase $N_{p}$ to enhance coverage. However, as can be seen in Figure 3.22, even increasing to $N_{p}=2048$, no cyclic signature for the first channel is present.


Figure 3.21 - SCF for two-channel QPSK communication ( $N_{p}=256, P=2000, f_{s}=18 \mathrm{MHz}$ ) with $f_{s_{s m_{1}}}=1 \mathrm{Msps}, f_{c_{1}}=1 \mathrm{MHz}, f_{s y m_{2}}=2.25 \mathrm{Msps}, f_{c_{2}}=5 \mathrm{MHz}$.


Figure 3.22 - SCF for two-channel QPSK communication ( $N_{p}=2048, P=2000, f_{s}=18 \mathrm{MHz}$ ) with

$$
f_{s y m_{1}}=1 \mathrm{Msps}, f_{c_{1}}=1 \mathrm{MHz}, f_{s y m_{2}}=2.25 \mathrm{Msps}, f_{c_{2}}=5 \mathrm{MHz}
$$

As illustrated in Figure 3.23, where the SCF is plotted for fixed $f=1 \mathrm{MHz}$ and $\alpha=0 \ldots f_{s}$ for various values of $N_{p}$, only for $N_{p}=2^{16}=65536$ we get a clearly distinguishable cycle feature for the first channel $(\alpha=1 \mathrm{MHz})$.


Figure 3.23 - SCF for the 1 MHz channel QPSK communication ( $P=2000, f_{s}=18 \mathrm{MHz}$ ) with $f_{\text {sym }}=1 \mathrm{Msps}, f_{c_{1}}=1 \mathrm{MHz}$, for fixed $f=1 \mathrm{MHz}$ and varying $N_{p}$.

The FAM algorithm could be adapted to use a non-power-of-two number of points, dropping the efficiency gain on the FFT computation for that case. We could then use $N_{p}=252$ (a multiple of 36 and close to the original $N_{p}=256$ ) obtaining the SCF depicted in Figure 3.24. As now $f_{\text {sym }_{1}} /\left(2 f_{s}\right) \times N_{p}=1 /(2 \times 18) \times 252=7$ is an integer, the cycle signature is obtained for channel 1. However, for channel 2, we have $f_{\text {sym }_{2}} /\left(2 f_{s}\right) \times N_{p}=2,25 /(2 \times 18) \times 252=15.75$, which is not an integer, so the cycle signature for channel two is lost.


Figure 3.24 - SCF for two-channel QPSK communication ( $N_{p}=252, P=2000, f_{s}=18 \mathrm{MHz}$ ) with $f_{\text {sym }_{1}}=1 \mathrm{Msps}, f_{c_{1}}=1 \mathrm{MHz}, f_{\text {sym }_{2}}=2.25 \mathrm{Msps}, f_{c_{2}}=5 \mathrm{MHz}$.

We will now consider a similar two-channel digital QPSK communication, with $100 \%$ excess bandwidth raised-cosine pulse shape; sampling frequency $f_{s}=18 \mathrm{MHz}$ and baud/symbol-rate equal to $f_{\text {sym }_{1}}=562.5 \mathrm{ksps}$, with a carrier frequency $f_{c_{1}}=1 \mathrm{MHz}$ for the first channel, and $f_{\text {sym }_{2}}=2.25 \mathrm{Msps}$ with $f_{c_{2}}=5 \mathrm{MHz}$ for the second channel (similar to the previous example, but different first channel symbol rate, guaranteeing that $f_{s y m_{1,2}} /\left(2 f_{s}\right) \times N_{p}$ are integers). Figure 3.26 shows the SCF obtained using the FAM algorithm with $N_{p}=256$ samples and $P=2000$. Clearly, the cycle signatures of both channels can be detected.


Figure 3.25 - SCF for two-channel QPSK communication ( $N_{p}=288, P=2000, f_{s}=18 \mathrm{MHz}$ ) with

$$
f_{s y m_{1}}=1 \mathrm{Msps}, f_{c_{1}}=1 \mathrm{MHz}, f_{s y m_{2}}=2.25 \mathrm{Msps}, f_{c_{2}}=5 \mathrm{MHz}
$$



Figure 3.26 - SCF for two-channel QPSK communication ( $N_{p}=256, P=2000, f_{s}=18 \mathrm{MHz}$ ) with $f_{s y m_{1}}=562.5 \mathrm{ksps}, f_{c_{1}}=1 \mathrm{MHz}, f_{s y m_{2}}=2.25 \mathrm{Msps}, f_{c_{2}}=5 \mathrm{MHz}$.

Considering the same transmitted signal, but now with a $0.1 \%$ deviation in the receiver sampling frequency, the resulting SCF obtained is shown in Figure 3.27 (in this case, the deviation of the receiver sampling frequency is $-0.1 \%$ ). As can be seen, the cycle signatures vanish completely, showing the high cycle frequency sensitivity when performing cyclostationary analysis.

In Figure 3.28, this same scenario of sampling frequency deviation is simulated for the previous example where $f_{\text {sym }_{1}}=1 \mathrm{Msps}$ and $f_{\text {sym }_{2}}=2.25 \mathrm{Msps}$, with $N_{p}=288$ (original SCF is shown in Figure 3.25). Again, as can be seen, the cycle signature also vanishes for both channels.


Figure 3.27 - SCF for two-channel QPSK communication $\left(N_{p}=256, P=2000, f_{s}=(1-0.1 \%) \times\right.$ 18 MHz ) with $f_{s_{s y m_{1}}}=562.5 \mathrm{ksps}, f_{c_{1}}=1 \mathrm{MHz}, f_{\text {sym }_{2}}=2.25 \mathrm{Msps}, f_{c_{2}}=5 \mathrm{MHz}$.

The need for fine resolution on the cycle frequency, due to impairments on system parameters, or the long integration time needed when the signal-to-noise ratio is low, motivated the search for a modified FAM algorithm, able to efficiently obtain the SCF in specific subregions of the $(f ; \alpha)$ plane. In the following section, a zoom-FAM (zFAM) algorithm (which can address this zoom/local computing of the SCF ) is presented, and its computational cost is also given and compared to the FAM.

The zFAM algorithm can easily be used to calculate the SCF in a local region of the ( $f ; \alpha$ ) plane, with similar complexity as the FAM algorithm, for the same $N_{p}$. When the cycle features of the signal are known, within some uncertainty, the SCF in a zoomed local region can be efficiently obtained by zFAM. As an example, and for the scenario illustrated in Figure 3.21, the use of zFAM to obtain the SCF in a zoomed region around $\left(f_{i} ; \alpha_{i}\right)=\left(f_{c_{1}} ; f_{\text {sym }}\right)$ and $\left(f_{i} ; \alpha_{i}\right)=\left(f_{c_{2}} ; f_{\text {sym }_{2}}\right)$ with the same $N_{p}=256$ points is shown in Figure 3.29 and Figure 3.30, respectively. Clearly, now
the cycle signatures for both channels are well-defined and identifiable.


Figure 3.28 - SCF for two-channel QPSK communication $\left(N_{p}=288, P=2000, f_{s}=(1-0.1 \%) \times\right.$ 18 MHz ) with $f_{\text {sym }_{1}}=1 \mathrm{Msps}, f_{c_{1}}=1 \mathrm{MHz}, f_{\text {sym }_{2}}=2.25 \mathrm{Msps}, f_{c_{2}}=5 \mathrm{MHz}$.


Figure 3.29 - Zoom SCF for two-channel QPSK communication ( $N_{p}=256, P=2000, f_{s}=18 \mathrm{MHz}$ ) with $f_{s y m_{1}}=1 \mathrm{Msps}, f_{c_{1}}=1 \mathrm{MHz}, f_{\text {sym }_{2}}=2.25 \mathrm{Msps}, f_{c_{2}}=5 \mathrm{MHz}$ (obtained by zFAM, around

$$
\left.\left(f_{i} ; \alpha_{i}\right)=\left(f_{c_{1}} ; f_{\text {sym }_{1}}\right)\right)
$$



Figure 3.30 - Zoom SCF for two-channel QPSK communication ( $N_{p}=256, P=2000, f_{s}=18 \mathrm{MHz}$ ) with $f_{\text {sym }_{1}}=1 \mathrm{Msps}, f_{c_{1}}=1 \mathrm{MHz}, f_{\text {sym }}^{2}=2.25 \mathrm{Msps}, f_{c_{2}}=5 \mathrm{MHz}$ (obtained by zFAM, around $\left.\left(f_{i} ; \alpha_{i}\right)=\left(f_{c_{2}} ; f_{\text {sym }}\right)\right)$.

### 3.5 The Zoom FAM (zFAM) algorithm

The proposed algorithm, named zFAM (zoom FAM), will solve the issues pointed out in the previous section regarding the FAM, namely computing the SCF in specific regions of the $\left(f_{i} ; \alpha_{i}\right)$ plane. This will allow the extraction of cyclic features on small identified regions with high frequency or cyclic frequency resolution. Also, the zFAM point coverage region will have a regular spacing.

Considering the region $\Delta f=f_{i_{2}}-f_{i_{1}}$ and $\Delta \alpha=\alpha_{i_{2}}-\alpha_{i_{1}}$, we are interested in computing the SCF in a regular grid of that region, as shown in Figure 3.313. As seen, the ( $f_{i} ; \alpha_{i}$ ) are obtained by $\left(f_{i} ; \alpha_{i}\right)=\left(\frac{f_{k}+f_{l}}{2} ; f_{k}-f_{l}\right)$, so that the corresponding points and path of Figure 3.31 correspond to the ones shown in Figure 3.32, for $f_{k}$ and $f_{l}$.

[^22]

Figure 3.31 - Region and point path to compute the $\operatorname{SCF}\left(f_{i} ; \alpha_{i}\right)$.

$$
\begin{aligned}
& f_{i 1}+\frac{\alpha_{i 1}}{2} \quad f_{i 2}+\frac{\alpha_{i 1}}{2}
\end{aligned}
$$

Figure 3.32 - Region and point path to compute the $\operatorname{SCF}\left(f_{k} ; f_{l}\right)$.

Usually, one will consider a regular grid with $N_{p}$ points, where $f_{i}=f_{i_{1}}+\frac{\Delta f}{N_{p}} n$ and $\alpha_{i}=\alpha_{i_{1}}+$ $\frac{\Delta \alpha}{N_{p}} m$ with $n, m=0 \ldots N_{p}-1$. In the following, we will consider the more general case when $n, m=0 \ldots N_{p}$ to account for the whole region (with $\left(N_{p}+1\right) \times\left(N_{p}+1\right)$ points).

The basis of the proposed algorithm is to allow the computation of the SCF, following the FAM algorithm closely, but in ( $\Delta f ; \Delta \alpha$ ) subregions of the $\left(f_{i} ; \alpha_{i}\right)$ plane, starting with arbitrary frequencies $f_{i_{1}}$ and $\alpha_{i_{1}}$, such that the corresponding $\left(f_{k} ; f_{l}\right)$ points can be obtained from an equally spaced array. If such regions can be found, then the complex demodulates given in Equation (3.23) for all $f_{k}$ and $f_{l}$ frequencies can be computed resorting to the Chirp Transform Algorithm (CTA).

The FAM algorithm uses a regular FFT, so, depending on the region of interest, complex demodulates on frequencies $\left(f_{k} ; f_{l}\right)$ may not be possible to compute, given the FFT frequency spacing of $f_{s} / N_{p}$, starting at $-f_{s} / 2$. Differently, the CTA can compute the Discrete Fourier Transform of an input signal on any set of equally spaced frequencies. In Appendix B, and for completeness, the CTA algorithm is reviewed, and its computational cost is obtained, which will be used to obtain the computational cost of both FAM and zFAM algorithms.

In the following sections, the conditions needed to be met (for using the zFAM algorithm) when choosing the $(\Delta f ; \Delta \alpha)$ subregions of the $(f ; \alpha)$ plane of the SCF are derived. As will be shown, a
very high zoom of the SCF can be obtained on a relatively small part of the plane, keeping a low computational cost. This is the main objective of the proposed zFAM algorithm. This high-zoom SCF region can also be obtained by the FAM algorithm, with an increase in computational cost, if the $f_{k}$ and $f_{l}$ frequencies of that band are all an integer multiple of $f_{s} / N_{p}\left(N_{p}\right.$ could be arbitrarily increased, with a corresponding increase in computation cost). But, if $f_{k}$ and $f_{l}$ frequencies of that high-zoom SCF region are not an integer multiple of $f_{s} / N_{p}$, the FAM algorithm is unable to compute the SCF in that region. Conversely, as will be shown, the zFAM algorithm can compute the SCF for any subregion, provided that $\Delta \alpha=L \Delta f$ or $\Delta f=L \Delta \alpha$ ( $L$ integer).

In general, we will need to compute one CTA of the input signal for the $f_{k}$ frequencies and a second CTA for the $f_{l}$ ones. This will be referred to as the Dual CTA method in the following. We will also find conditions when $f_{k}$ and $f_{l}$ can be joined or merged in a single frequency array, in which case only one CTA needs to be computed. This case will be referred to as the Single CTA method.

### 3.5.1 Obtaining $f_{k}$ and $f_{l}$ independently (Dual CTA method)

In general, although we may have a regular grid for $\left(f_{i} ; \alpha_{i}\right)=\left(f_{i_{1}}+n \frac{\Delta f}{N_{p}} ; \alpha_{i_{1}}+m \frac{\Delta \alpha}{N_{p}}\right)(n, m=$ $\left.0 \ldots N_{p}\right)$ the corresponding values for $\left(f_{k} ; f_{l}\right)$ may not have a regular spacing. We now analyze two special cases and derive the conditions needed so that the $\left(f_{k} ; f_{l}\right)$ values can be obtained from an equally spaced frequency array.

Given the set of points $\left(f_{i} ; \alpha_{i}\right)=\left(f_{i_{1}}+n \frac{\Delta f}{N_{p}} ; \alpha_{i_{1}}+m \frac{\Delta \alpha}{N_{p}}\right)$ and the correspondence $\left\{\begin{array}{c}f_{l}=f_{i}-\alpha_{i} / 2 \\ f_{k}=f_{i}+\alpha_{i} / 2\end{array}\right.$, we get

$$
\begin{equation*}
\left(f_{k} ; f_{l}\right)=\left(f_{i_{1}}+\frac{\Delta f}{N_{p}} n+\frac{\alpha_{i_{1}}}{2}+\frac{\Delta \alpha}{2 N_{p}} m ; f_{i_{1}}+\frac{\Delta f}{N_{p}} n-\frac{\alpha_{i_{1}}}{2}-\frac{\Delta \alpha}{2 N_{p}} m\right) \tag{3.37}
\end{equation*}
$$

We then must guarantee that a constant spacing on $f_{k}, \Delta f_{k}$, and $f_{l}, \Delta f_{l}$, covers all points defined in Equation (3.37) and also that $\Delta f_{k}$ and $\Delta f_{l}$ are the maximum possible to minimize the number of points needed. Both conditions are met if:

$$
\frac{f_{k, l}-\min \left(f_{k, l}\right)}{\Delta f_{k, l}} \text { is integer and minimum }
$$

From Equation (3.37), we get:

$$
\frac{f_{k}-\min \left(f_{k}\right)}{\Delta f_{k}}=\frac{\frac{\Delta f}{N_{p}} n+\frac{\Delta \alpha}{2 N_{p}} m}{\Delta f_{k}}
$$

$$
\frac{f_{l}-\min \left(f_{l}\right)}{\Delta f_{l}}=\frac{\frac{\Delta f}{N_{p}} n+\frac{\Delta \alpha}{2 N_{p}}\left(m+N_{p}\right)}{\Delta f_{l}}
$$

We now address two possible cases where both conditions can be satisfied, obtaining the corresponding $f_{k}$ and $f_{l}$ spacing, and the number of points needed ( $N_{p_{1}}=\frac{\max \left(f_{k, l}\right)-\min \left(f_{k, l}\right)}{\Delta f_{k, l}}+1$ ). Also, the correspondence between each $\left(f_{i} ; \alpha_{i}\right)$ and $\left(f_{k} ; f_{l}\right)$ points is also given.

## - Case $\Delta \alpha=L \Delta f$

In this case

$$
\begin{gathered}
\frac{f_{k}-\min \left(f_{k}\right)}{\Delta f_{k}}=\frac{\frac{\Delta f}{N_{p}} n+\frac{\Delta \alpha}{2 N_{p}} m}{\Delta f_{k}}=\frac{\frac{\Delta f}{N_{p}}\left(n+\frac{\mathrm{L}}{2} m\right)}{\Delta f_{k}} \\
\frac{f_{l}-\min \left(f_{l}\right)}{\Delta f_{l}}=\frac{\frac{\Delta f}{N_{p}} n-\frac{\Delta \alpha}{2 N_{p}}\left(m+N_{p}\right)}{\Delta f_{l}}=\frac{\frac{\Delta f}{N_{p}}\left(n+\frac{\mathrm{L}}{2}\left(m+N_{p}\right)\right)}{\Delta f_{l}}
\end{gathered}
$$

To guarantee both are integer and minimum, for the possible values of $L$, we have two cases:

- $\boldsymbol{L}$ even $(j=L / 2$ is an integer $)$

$$
\begin{gathered}
\frac{f_{k}-\min \left(f_{k}\right)}{\Delta f_{k}}=\frac{\frac{\Delta f}{N_{p}}\left(n+\frac{\mathrm{L}}{2} m\right)}{\Delta f_{k}}=\frac{\frac{\Delta f}{N_{p}}(n+j \times m)}{\Delta f_{k}} \\
\frac{f_{l}-\min \left(f_{l}\right)}{\Delta f_{l}}=\frac{\frac{\Delta f}{N_{p}}\left(n+j \times\left(m+N_{p}\right)\right)}{\Delta f_{l}}
\end{gathered}
$$

So, to guarantee that both are integer and minimum, one must have $\Delta f_{k}=\frac{\Delta f}{N_{p}}$ and $\Delta f_{l}=\frac{\Delta f}{N_{p}}$.
The number of points is $N_{p_{1}}=\frac{\max \left(f_{k}\right)-\min \left(f_{k}\right)}{\Delta f_{k}}+1=\frac{\Delta f+\frac{\Delta \alpha}{2}}{\frac{\Delta f}{N_{p}}}+1=N_{p}\left(\frac{L}{2}+1\right)+1$.
In this case, the available points are

$$
\left(f_{k} ; f_{l}\right)=\left(f_{i_{1}}+\frac{\alpha_{i_{1}}}{2}+\frac{\Delta f}{N_{p}} k ; f_{i_{1}}-\frac{\alpha_{i_{1}}}{2}+\frac{\Delta f}{N_{p}} l\right) .
$$

Therefore, using (3.37) gives:

$$
\left\{\begin{array}{l}
f_{i_{1}}+\frac{\Delta f}{N_{p}} n+\frac{\alpha_{i_{1}}}{2}+\frac{\Delta \alpha}{2 N_{p}} m=f_{i_{1}}+\frac{\alpha_{i_{1}}}{2}+\frac{\Delta f}{N_{p}} k \\
f_{i_{1}}+\frac{\Delta f}{N_{p}} n-\frac{\alpha_{i_{1}}}{2}-\frac{\Delta \alpha}{2 N_{p}} m=f_{i_{1}}-\frac{\alpha_{i_{1}}}{2}+\frac{\Delta f}{N_{p}} l
\end{array}\right.
$$

which leads to (as $\Delta \alpha=L \Delta f$ )

$$
\left\{\begin{array}{l}
k=n+\frac{L}{2} m \\
l=n+\frac{L}{2}\left(N_{p}-m\right)
\end{array}\right.
$$

- 

$\boldsymbol{L}$ odd $(j=(L-1) / 2$ is an integer $)$

$$
\begin{aligned}
\frac{f_{k}-\min \left(f_{k}\right)}{\Delta f_{k}} & =\frac{\frac{\Delta f}{N_{p}}\left(n+\frac{\mathrm{L}}{2} m\right)}{\Delta f_{k}}=\frac{\frac{\Delta f}{N_{p}}\left(n+j \times m+\frac{m}{2}\right)}{\Delta f_{k}} \\
\frac{f_{l}-\min \left(f_{l}\right)}{\Delta f_{l}} & =\frac{\frac{\Delta f}{N_{p}}\left(n+j \times\left(m+N_{p}\right)+\frac{\left(m+N_{p}\right)}{2}\right)}{\Delta f_{l}}
\end{aligned}
$$

So, to guarantee that both are integer and minimum, one must have $\Delta f_{k}=\frac{\Delta f}{2 N_{p}}$ and $\Delta f_{l}=\frac{\Delta f}{2 N_{p}}$.

The number of points is $N_{p_{1}}=\frac{\max \left(f_{k}\right)-\min \left(f_{k}\right)}{\Delta f_{k}}+1=\frac{\Delta f+\frac{\Delta \alpha}{2}}{\frac{\Delta f}{2 N_{p}}}+1=N_{p}(L+2)+1$.
In this case, the available points are

$$
\left(f_{k} ; f_{l}\right)=\left(f_{i_{1}}+\frac{\alpha_{i_{1}}}{2}+\frac{\Delta f}{2 N_{p}} k ; f_{i_{1}}-\frac{\alpha_{i_{1}}}{2}+\frac{\Delta f}{2 N_{p}} l\right) .
$$

Therefore, using (3.37), we get

$$
\left\{\begin{array}{l}
f_{i_{1}}+\frac{\Delta f}{N_{p}} n+\frac{\alpha_{i_{1}}}{2}+\frac{\Delta \alpha}{2 N_{p}} m=f_{i_{1}}+\frac{\alpha_{i_{1}}}{2}+\frac{\Delta f}{2 N_{p}} k \\
f_{i_{1}}+\frac{\Delta f}{N_{p}} n-\frac{\alpha_{i_{1}}}{2}-\frac{\Delta \alpha}{2 N_{p}} m=f_{i_{1}}-\frac{\alpha_{i_{1}}}{2}+\frac{\Delta f}{2 N_{p}} l
\end{array}\right.
$$

which leads to (as $\Delta \alpha=L \Delta f)$

$$
\left\{\begin{array}{l}
k=2 n+L m \\
l=2 n+L\left(N_{p}-m\right)
\end{array}\right.
$$

## - Case $\Delta f=L \Delta \alpha$

$$
\begin{gathered}
\frac{f_{k}-\min \left(f_{k}\right)}{\Delta f_{k}}=\frac{\frac{\Delta f}{N_{p}} n+\frac{\Delta \alpha}{2 N_{p}} m}{\Delta f_{k}}=\frac{\frac{\Delta \alpha}{N_{p}}\left(L n+\frac{m}{2}\right)}{\Delta f_{k}} \\
\frac{f_{l}-\min \left(f_{l}\right)}{\Delta f_{l}}=\frac{\frac{\Delta f}{N_{p}} n-\frac{\Delta \alpha}{2 N_{p}}\left(m+N_{p}\right)}{\Delta f_{l}}=\frac{\frac{\Delta \alpha}{N_{p}}\left(L n+\frac{\left(m+N_{p}\right)}{2}\right)}{\Delta f_{l}}
\end{gathered}
$$

To guarantee that both are integer and minimum, we must have $\Delta f_{k}=\frac{\Delta \alpha}{2 N_{p}}$ and $\Delta f_{l}=\frac{\Delta \alpha}{2 N_{p}}$. The number of points is $N_{p_{1}}=\frac{\max \left(f_{k}\right)-\min \left(f_{k}\right)}{\Delta f_{k}}+1=\frac{\Delta f+\frac{\Delta \alpha}{2}}{\frac{\Delta_{\alpha}}{2 N_{p}}}+1=N_{p}(2 L+1)+1$.

The available points are:

$$
\left(f_{k} ; f_{l}\right)=\left(f_{i_{1}}+\frac{\alpha_{i_{1}}}{2}+\frac{\Delta \alpha}{2 N_{p}} k ; f_{i_{1}}-\frac{\alpha_{i_{1}}}{2}+\frac{\Delta \alpha}{2 N_{p}} l\right)
$$

Therefore, using (3.37), we get

$$
\left\{\begin{array}{l}
f_{i_{1}}+\frac{\Delta f}{N_{p}} n+\frac{\alpha_{i_{1}}}{2}+\frac{\Delta \alpha}{2 N_{p}} m=f_{i_{1}}+\frac{\alpha_{i_{1}}}{2}+\frac{\Delta \alpha}{2 N_{p}} k \\
f_{i_{1}}+\frac{\Delta f}{N_{p}} n-\frac{\alpha_{i_{1}}}{2}-\frac{\Delta \alpha}{2 N_{p}} m=f_{i_{1}}-\frac{\alpha_{i_{1}}}{2}+\frac{\Delta \alpha}{2 N_{p}} l
\end{array}\right.
$$

which leads to (as $\Delta f=L \Delta \alpha$ )

$$
\left\{\begin{array}{l}
k=2 L n+m \\
l=2 L n+\left(N_{p}-m\right)
\end{array}\right.
$$

Table 3.1 summarizes the above results for all cases. As can be seen, the frequency spacing needed for $f_{k}$ and $f_{l}$ is always equal $\left(\Delta f_{k}=\Delta f_{l}\right)$. Further, looking for the number of points needed for the CTA, the computational cost will be the lowest when $\Delta \alpha=L \Delta f$ with $L$ even, followed by the same case when $L$ is odd. The worst case, with the highest computational cost, is the case when $\Delta f=L \Delta \alpha$. This issue should be given attention when choosing the region to calculate the highresolution SCF.

| Case | L | $\begin{aligned} & \Delta f_{k, l}=\Delta f_{k} \\ & =\Delta f_{l} \end{aligned}$ | $N_{p_{1}}$ | Point assignment |
| :---: | :---: | :---: | :---: | :---: |
| $\Delta \alpha=L \Delta f$ | L even | $\frac{\Delta f}{N_{p}}$ | $N_{p}\left(\frac{L}{2}+1\right)+1$ | $\left\{\begin{array}{l} k=n+\frac{L}{2} m \\ l=n+\frac{L}{2}\left(N_{p}-m\right) \end{array}\right.$ |
|  | L odd | $\frac{\Delta f}{2 N_{p}}$ | $N_{p}(L+2)+1$ | $\left\{\begin{array}{l} k=2 n+L m \\ l=2 n+L\left(N_{p}-m\right) \end{array}\right.$ |
| $\Delta f=L \Delta \alpha$ | any | $\frac{\Delta \alpha}{2 N_{p}}$ | $N_{p}(2 L+1)+1$ | $\left\{\begin{array}{l}k=2 L n+m \\ l=2 L n+\left(N_{p}-m\right)\end{array}\right.$ |

Table 3.1 - Summary table for separate $f_{k}$ and $f_{l}$.

### 3.5.2 Joining $f_{k}$ and $f_{l}$ (Single CTA method)

We can now check if there are any cases when merging the values needed for $f_{k}$ and $f_{l}$ in a single frequency array of equally spaced points is possible.

Figure 3.33 shows the range of $f_{k}$ and $f_{l}$, where it can easily be seen that the first condition for the merge to be possible is $f_{i_{2}}-\frac{\alpha_{i_{1}}}{2}>f_{i_{1}}+\frac{\alpha_{i_{1}}}{2}$ which is equivalent to $\Delta f>\alpha_{i_{1}}$.


Figure $3.33-f_{k}$ and $f_{l}$ range.

However, we must also guarantee that frequencies coincide. The new frequency array is given by the points $f=\left[f_{i_{1}}-\frac{\alpha_{i_{1}}}{2}: \Delta f_{k, l}: f_{i_{2}}+\frac{\alpha_{i_{2}}}{2}\right]$, and the frequency points will coincide if, starting from the lowest value of $f_{l}$, we can reach the lower value of $f_{k}$ with an integer increment, $q$, of $\Delta f_{k, l}$, that is:

$$
f_{i_{1}}-\frac{\alpha_{i_{2}}}{2}+q \times \Delta f_{k, l}=f_{i_{1}}+\frac{\alpha_{i_{1}}}{2}
$$

Hence, we must guarantee that there is an integer $q$, such that:

$$
q=\frac{\Delta \alpha}{2 \Delta f_{k, l}}+\frac{\alpha_{i_{1}}}{\Delta f_{k, l}} .
$$

If the merging is possible, the number of points needed is given by:

$$
N_{p_{2}}=\frac{\max \left(f_{k}\right)-\min \left(f_{l}\right)}{\Delta f_{k, l}}+1=\frac{f_{i_{2}}+\frac{\alpha_{i_{2}}}{2}\left(f_{i_{1}}-\frac{\alpha_{i_{2}}}{2}\right)}{\Delta f_{k, l}}+1=\frac{\Delta f+\Delta \alpha+\alpha_{i_{1}}}{\Delta f_{k, l}}+1
$$

From this new frequency vector, $f_{l}$ and $f_{k}$ are obtained as a simple subset, starting at index 0 and $q$, respectively, with length equal to $N_{p_{1}}$ (as given in Table 3.1).

Again, for the several cases considered, we will obtain the condition needed to be met.

- Case $\Delta \alpha=L \Delta f$

○ $L$ even

In this case $\Delta f_{k, l}=\frac{\Delta f}{N_{p}}$, so

$$
q=\frac{L N_{p}}{2}+\frac{\alpha_{i_{1}}}{\Delta f_{k, l}}
$$

and $q$ will be an integer if $\alpha_{i_{1}}=p \times \Delta f_{k, l}$ for any $p$ integer.

The number of points is $N_{p_{2}}=N_{p}(L+1)+p+1$.

## - $L$ odd

In this case $\Delta f_{k, l}=\frac{\Delta f}{2 N_{p}}$, so

$$
q=L N_{p}+\frac{\alpha_{i_{1}}}{\Delta f_{k, l}}
$$

and $q$ will be an integer if $\alpha_{i_{1}}=p \times \Delta f_{k, l}$ for any $p$ integer.

The number of points is $N_{p_{2}}=2 N_{p}(L+1)+p+1$.

- Case $\Delta f=L \Delta \alpha$

In this case $\Delta f_{k, l}=\frac{\Delta \alpha}{2 N_{p}}$, so

$$
q=N_{p}+\frac{\alpha_{i_{1}}}{\Delta f_{k, l}}
$$

and $q$ will be an integer if $\alpha_{i_{1}}=p \times \Delta f_{k, l}$ for any $p$ integer.
The number of points is $N_{p_{2}}=2 N_{p}(L+1)+p+1$.

Therefore, it is possible to use a single array of frequencies if the following conditions are both met:

$$
\begin{gathered}
\Delta f>\alpha_{i_{1}} \\
p=\frac{\alpha_{i_{1}}}{\Delta f_{k, l}} \text { is integer }
\end{gathered}
$$

Table 3.2 summarizes the above results when the joining of $f_{k}$ and $f_{l}$ in a single frequency array is possible.

| Case | $\mathbf{L}$ | $\boldsymbol{\Delta} \boldsymbol{f}_{\boldsymbol{k}, \boldsymbol{l}}=\boldsymbol{\Delta} \boldsymbol{f}_{\boldsymbol{k}}$ <br> $=\boldsymbol{\Delta} \boldsymbol{f}_{\boldsymbol{l}}$ | $\boldsymbol{N}_{\boldsymbol{p}_{\mathbf{2}}}$ | Indexes for $\boldsymbol{f}_{l}$ and <br> $\boldsymbol{f}_{\boldsymbol{k}}\left(u=0 \ldots N_{p_{1}}-1\right)$ |
| :---: | :--- | :---: | :---: | :---: |
| $\boldsymbol{\Delta} \boldsymbol{\alpha}=\boldsymbol{L} \boldsymbol{\Delta} \boldsymbol{f}$ | L even | $\frac{\Delta f}{N_{p}}$ | $N_{p}(L+1)+p$ <br> +1 | $\left\{\begin{array}{l}f_{l}: u \\ f_{k}: \frac{L N_{p}}{2}+p+u\end{array}\right.$ |
|  | L odd | $\frac{\Delta f}{2 N_{p}}$ | $2 N_{p}(L+1)$ <br> $+p+1$ | $\left\{\begin{array}{l}f_{l}: u \\ f_{k}: L N_{p}+p+u\end{array}\right.$ |
|  | any | $\frac{\Delta \alpha}{2 N_{p}}$ | $2 N_{p}(L+1)$ <br> $+p+1$ | $\left\{\begin{array}{l}f_{l}: u \\ f_{k}: N_{p}+p+u\end{array}\right.$ |

Table 3.2 - Summary table when joining $f_{k}$ and $f_{l}$.

In the following section, we will use the complexity analysis of the CTA algorithm to find, in the case when $f_{k}$ and $f_{l}$ can be merged, which method (Single CTA or Dual CTA) has a lower computational cost.

### 3.5.3 Computational Cost of the Single and Dual CTA methods

We will start by analyzing the computational cost of the CTA and address the optimal decision when it is possible to join $f_{k}$ and $f_{l}$ in a single array. In this case, we will see when it is preferably to calculate two CTA for frequencies $f_{k}$ and $f_{l}$ independently, or a single CTA for the joint frequency array.

As given in Appendix B (Equation (B.3)), the computational cost of the CTA algorithm is

$$
\begin{equation*}
F_{C T A}=6 N_{p}+10 L_{C T A} \log _{2}\left(L_{C T A}\right)+6 L_{C T A}+6 N_{p_{C T A}}, \tag{3.38}
\end{equation*}
$$

where $N_{p}$ is the input signal sample length, $N_{p_{\text {CTA }}}$ is the number of points to calculate (output sample size) and $L_{C T A}=$ nextfftlength $\left(N_{p}+N_{p_{C T A}}-1\right)^{34}$.

In the following, we will consider $L_{C T A}=2^{l}{ }^{l}{ }^{\text {CTA }}$, which is the FFT/IFFT size needed to compute in both methods (Single/Dual CTA). We will show that when $L_{C T A}$ is the same for both methods,
${ }^{34} y=$ nextfftlength $(x)$ returns a power-of-two, $y=2^{n}$, such that $2^{n} \geq x$.
the Single CTA has less FLoating-point OPeration (FLOP) count than the Dual CTA (of course that if $L_{C T A}$ for Dual CTA is bigger than for Single CTA, this last one will always have a lower FLOP count). Finally, when $L_{C T A}$ for Dual CTA is lower than for Single CTA, then Dual CTA will have a lower FLOP count.

For all cases under consideration $(\Delta \alpha=L \Delta f, L$ even; $\Delta \alpha=L \Delta f, L$ odd; $\Delta f=L \Delta \alpha)$ the total FLOPs count for the Single/Double CTA method is (from Equation (3.38)):

$$
\begin{aligned}
& F_{\text {Single } C T A}=6 N_{p}+10 l_{C T A} \times 2^{l_{C T A}}+6 \times 2^{l_{C T A}}+6 N_{p_{\text {Single } C T A}} \\
& F_{\text {Dual } C T A}=12 N_{p}+20 l_{C T A} \times 2^{l_{C T A}}+12 \times 2^{l_{C T A}}+12 N_{p_{\text {Dual } C T A}}
\end{aligned}
$$

For both methods to have the same $L_{C T A}$, it implies, for the Single CTA, to have

$$
2^{l_{C T A}-1}<N_{p}+N_{p_{\text {Single } C T A}}-1 \leq 2^{l_{C T A}}
$$

and, for the Dual CTA, to have

$$
2^{l_{C T A}-1}<N_{p}+N_{p_{\text {Dual } C T A}}-1 \leq 2^{l_{C T A}}
$$

From (3.38), one can obtain:

$$
N_{p}+N_{p_{C T A}}=\left(F_{C T A}-2^{l_{C T A}}\left(10 l_{C T A}+6\right)\right) / 6
$$

Combining with the above inequalities, it leads to:

$$
\begin{gathered}
\left(10 l_{C T A}+9\right) 2^{l_{C T A}}+6<F_{\text {Single } C T A} \leq\left(10 l_{C T A}+12\right) 2^{l_{C T A}}+6 \\
\left(20 l_{C T A}+18\right) 2^{l_{C T A}}+12<F_{\text {Dual } C T A} \leq\left(20 l_{C T A}+24\right) 2^{l_{C T A}}+12
\end{gathered}
$$

For the Single CTA method to present a lower FLOP count, it suffices that its upper limit is less than the Dual CTA lower limit:

$$
\begin{aligned}
& \left(10 l_{C T A}+12\right) 2^{l_{C T A}}+6<\left(20 l_{C T A}+18\right) 2^{l_{C T A}}+12 \\
& \quad\left(10 l_{C T A}+6\right) 2^{l_{C T A}}>-6
\end{aligned}
$$

which is always true.
One could argue that if $L$ for the Single CTA method is on the next power-of- 2 slot, the FLOP count is still less. It is easy to show that, in that case, the lower limit for Single CTA for $l_{C T A}=$ $l_{C T A}+1$ is bigger than the Dual CTA upper limit, thus proving that this hypothesis is false:

$$
\begin{aligned}
\left(10\left(l_{C T A}+1\right)+9\right) 2^{l_{C T A}+1}+6 & >\left(20 l_{C T A}+24\right) 2^{l_{C T A}+12} \\
2^{l_{C T A}} & >\frac{6}{14}
\end{aligned}
$$

which is always true.

This means that the analysis on which conditions one method is better than the other can be assessed by analyzing not all the FLOP count, but only the number of points needed for the FFT/IFFT operation $\left(L_{C T A}\right)$. The method with a lower $L_{C T A}$ will have a lower FLOP count. If $L_{C T A}$ is equal, then the Single CTA will always have a lower FLOP count.

Now, and when a Single CTA is possible, we will analyze which of the two methods is preferable, for having lower computational complexity.

We will consider two conditions:
Condition 1: $2^{u-1}<N_{p}+N_{p_{\text {Single } C T A}}-1 \leq 2^{u}$, for some integer $u$
Condition 2: $2^{u-1}<N_{p}+N_{p_{\text {Dual } C T A}}-1$, for some integer $u$
Guaranteeing that Condition 1 is met, the Single CTA method is better if Condition $\mathbf{2}$ is also met; if Condition 2 is not met, then Dual CTA is better. In fact, guaranteeing that Condition 1 is met means that $L_{\text {Single CTA }}$ is $2^{u}$; now, if Condition 2 is also met, then $L_{D u a l ~ C T A ~}$ is $2^{u}$ or other greater power-of-two, so the Single CTA method is better. Also, guaranteeing that Condition 1 is met but not Condition 2, means that $L_{D u a l ~ C T A ~}$ is a lower power-of-two than $L_{\text {Single CTA }}$, so the Dual CTA method is better.

We will now study the ranges of $L$ and $p=\frac{\alpha_{i_{1}}}{\Delta f_{k, l}}$, combined with Conditions 1 and 2 , to assess when each method is better.

- Case $\Delta \alpha=L \Delta f, L$ even

In this case, we have:

$$
\begin{array}{ll}
N_{p_{\text {Single CTA }}}=(L+1) 2^{n_{p}}+p+1 & L_{\text {Single } C T A}=\text { nextfftlength }\left((L+2) 2^{n_{p}}+p\right) \\
N_{p_{\text {Dual CTA }}}=\frac{(L+2)}{2} 2^{n_{p}+1} & L_{\text {Dual } C T A}=\text { nextfftlength }\left(\frac{(L+4)}{2} 2^{n_{p}}\right)
\end{array}
$$

and $p=\alpha_{i_{1}} /\left(\frac{\Delta f}{N_{p}}\right)$ is an integer, to allow the Single CTA method.

The conditions that must be met so that Single CTA has lower computational cost are:
Condition 1: $2^{u-1}<(L+2) 2^{n_{p}}+p \leq 2^{u}$
Condition 2: $2^{u-1}<\frac{(L+4)}{2} 2^{n_{p}}$

We will take the possible values for $L$ in the form $L=2^{x-1}+2 n, n \leq 2^{x-2}-1$, so that $L=$ $2^{x-1} \ldots 2^{x}-2$.

From Condition 1, we get:

$$
2^{u-1}<\left(2^{x-1}+2 n+2\right) 2^{n_{p}}+p \leq 2^{u}
$$

The worst-case for the first inequality is for $\beta=0$, leading to

$$
2^{u-1}<2^{x+n_{p}-1}+2(n+1) 2^{n_{p}} \leq 2^{u}
$$

Both inequalities are always verified for $u=x+n_{p}$, if $2(n+1) \leq 2^{x-1}$ or $n \leq 2^{x-2}-1$. We note that in this case, the first inequality of Condition 1 is always verified, but we must address the possible values of $p$ for the second inequality.

We now turn to Condition 2 to verify its validity:

$$
\begin{aligned}
\frac{\left(2^{x-1}+2 n+4\right)}{2} 2^{n_{p}} & >2^{x+n_{p}-1} \\
2^{x-2}+(n+2) & >2^{x-1}
\end{aligned}
$$

This leads to $n>2^{x-2}-2$, or $n \geq 2^{x-2}-1$.
To satisfy Condition 1, we must have $n \leq 2^{x-2}-1$, and for Condition 2, $n \geq 2^{x-2}-1$. So, the only possible value to satisfy both conditions is $n=2^{x-2}-1$, where we get $L=$ $2^{x}-2$.

As mentioned, we must now check the possible values of $p$ that satisfy the second inequality of Condition 1.

$$
\begin{aligned}
\left(2^{x}-2+2\right) 2^{n_{p}}+p & \leq 2^{x+n_{p}} \\
p & \leq 0
\end{aligned}
$$

Therefore, for the Case $\Delta \alpha=L \Delta f, K$ even, the Single CTA method is only better for $L=$ $2^{x}-2$ and $p=\alpha_{i_{1}} /\left(\frac{\Delta f}{N_{p}}\right)=0$, so $\alpha_{i_{1}}=0$.

- Case $\Delta \alpha=L \Delta f, L$ odd

In this case, we have:

$$
\begin{array}{ll}
N_{p_{\text {Single CTA }}}=2(L+1) 2^{n_{p}}+p+1 & L_{\text {Single CTA }}=\operatorname{nextfftlength}\left((2 L+3) 2^{n_{p}}+p\right) \\
N_{p_{\text {Dual CTA }}}=(L+2) 2^{n_{p}}+1 & L_{\text {Dual CTA }}=\operatorname{nextfftlength}\left((L+3) 2^{n_{p}}\right)
\end{array}
$$

and $p=\alpha_{i_{1}} /\left(\frac{\Delta f}{2 N_{p}}\right)$ is an integer, to allow the Single CTA method.

The conditions that must be met so that Single CTA has lower computational cost are:

Condition 1: $2^{u-1}<(2 L+3) 2^{n_{p}}+p \leq 2^{u}$
Condition 2: $2^{u-1}<(L+3) 2^{n_{p}}$

We will take the possible values for $L$ in the form $L=2^{x-1}-1+2 n, n \leq 2^{x-2}-1$, so that $L=2^{x-1}-1 \ldots 2^{x}-3$.

From Condition 1, we get:

$$
2^{u-1}<\left(2^{x}-2+4 n+3\right) 2^{n_{p}}+p \leq 2^{u}
$$

The worst-case for the first inequality is for $p=0$, leading to

$$
2^{u-1}<2^{x+n_{p}}+(4 n+1) 2^{n_{p}} \leq 2^{u}
$$

Both inequalities are always verified for $u=x+n_{p}+1$, if $(4 n+1) \leq 2^{x}$ or $n \leq x^{x-2}-1$. We note that in this case, the first inequality of Condition $\mathbf{1}$ is always verified, but we must address the possible values of $p$ for the second inequality.

We now turn to Condition 2 to verify its validity:

$$
\begin{gathered}
\left(2^{x-1}-1+2 n+3\right) 2^{n_{p}}>2^{x+n_{p}} \\
2^{x-1}+2(n+1)>2^{x}
\end{gathered}
$$

This leads to $n>2^{x-2}-1$, or $n \geq 2^{x-2}$.
To satisfy Condition 1, we must have $n \leq 2^{x-2}-1$, and for Condition $2, n \geq 2^{x-2}$. So, there is no value of $L$ that satisfies both conditions.

Therefore, for the Case $\Delta \alpha=L \Delta f, L$ odd, the Dual CTA method is always better.

## - Case $\Delta f=L \Delta \alpha$

In this case, we have:

$$
\begin{array}{ll}
N_{p_{\text {Single CTA }}}=2(L+1) 2^{n_{p}}+p+1 & L_{\text {Single } C T A}=\operatorname{nextfftlength}\left((2 L+3) 2^{n_{p}}+p\right) \\
N_{p_{\text {Dual } C T A}}=(2 L+1) 2^{n_{p}}+1 & L_{\text {Dual CTA }}=\operatorname{nextfftlength}\left(2(L+1) 2^{n_{p}}\right) \\
\text { and } p=\alpha_{i_{1}} /\left(\frac{\Delta \alpha}{2 N_{p}}\right) \text { is an integer, to allow the Single CTA method. }
\end{array}
$$

The conditions that must be met so that Single CTA has lower computational cost are:
Condition 1: $2^{u-1}<(2 L+3) 2^{n_{p}}+p \leq 2^{u}$
Condition 2: $2^{u-1}<2(L+1) 2^{n_{p}}$

We will take the possible values for $L$ in the form $L=2^{x-1}+n, n \leq 2^{x-1}-1$, so that $L=$ $2^{x-1} \ldots 2^{x}-1$.

From Condition 1, we get:

$$
2^{u-1}<\left(2^{x}+2 n+3\right) 2^{n_{p}}+p \leq 2^{u}
$$

The worst-case for the first inequality is for $p=0$, leading to

$$
2^{u-1}<2^{x+n_{p}}+(2 n+3) 2^{n_{p}} \leq 2^{u}
$$

Both inequalities are always verified for $u=x+n_{p}+1$, if $(2 n+3) \leq 2^{x}$ or $n \leq 2^{x-1}-2$. Comparing with the previous range obtained for $n\left(n \leq 2^{x-1}-1\right)$, we note that the case $n=2^{x-1}-1$, which corresponds to $L=2^{x}-1$, must be addressed separately, which is done in the following.

We note that in the condition $n \leq 2^{x-1}-2$, the first inequality of Condition 1 is always verified, but we must address the possible values of $p$ for the second inequality.

We now turn to Condition 2 to verify its validity:

$$
\begin{aligned}
2\left(2^{x-1}+n+1\right) 2^{n_{p}} & >2^{x+n_{p}} \\
2^{x}+2(n+1) & >2^{x}
\end{aligned}
$$

This leads to $n>-1$, or $n \geq 0$.
To satisfy Condition 1, we must have $n \leq 2^{x-1}-2$, and for Condition 2, $n \geq 0$. So, the possible values to satisfy both conditions are $n \leq 2^{x-1}-2$, where we get $L \leq 2^{x}-2$.

As mentioned, we must now check the possible values of $p$ that satisfy the second inequality of Condition 1.

$$
\begin{aligned}
\left(2\left(2^{x-1}+n\right)+3\right) 2^{n_{p}}+p & \leq 2^{x+n_{p}+1} \\
2^{x}+(2 n+3)+\frac{p}{N_{p}} & \leq 2^{x+1} \\
p & \leq N_{p}\left(2^{x}-2 n-3\right)
\end{aligned}
$$

For the outstanding case $L=2^{x}-1$ we get from Condition 1:

$$
\begin{array}{r}
2^{u-1}<\left(2^{x+1}-2+3\right) 2^{n_{p}}+p \leq 2^{u} \\
2^{u-1}<2^{x+n_{p}+1}+2^{n_{p}}+p \leq 2^{u}
\end{array}
$$

For $p=0$, both inequalities are always true for $u=x+n_{p}+2$.
From Condition 2, we have:

$$
2\left(2^{x}-1+1\right) 2^{n_{p}}>2^{x+n_{p}+1}
$$

$$
2^{x+n_{p}+1}>2^{x+n_{p}+1}
$$

This inequality is always false.

Therefore, for the Case $\boldsymbol{\Delta} f=\boldsymbol{K} \boldsymbol{\Delta} \alpha$ :

- If $L=2^{x}-1$, the Dual CTA is always better.
- For $L=2^{x-1}+n, \quad n \leq 2^{x-1}-2$, the Single CTA is better only if $p \leq$ $N_{p}\left(2^{x}-2 n-3\right)$

In conclusion, for the most common case, when $N_{p}$ is a power-of-two, the conditions defining which method is better are:

- Case $\Delta \alpha=L \Delta f, L$ even
- If $L=2^{x}-2$ and $\alpha_{i_{1}}=0$, use Single CTA
- otherwise, use Dual CTA
- Case $\Delta \alpha=L \Delta f, L$ odd
- Always use Dual CTA
- Case $\Delta f=L \Delta \alpha$, and $p=\alpha_{i_{1}} / \Delta f_{k, l}=\alpha_{i_{1}} /\left(\frac{\Delta \alpha}{2 N_{p}}\right)$ is integer
- If $L=2^{x}-1$, use Dual CTA
- If $L=2^{x-1}+n, \quad n \leq 2^{x-1}-2$ and $p=\alpha_{i_{1}} /\left(\frac{\Delta \alpha}{2 N_{p}}\right)=0 \ldots N_{p}\left(2^{x}-2 n-3\right)$, use Single CTA, otherwise use Dual CTA


### 3.6 Computational Cost of the FAM and zFAM algorithms

In this section, the computational cost of the zFAM algorithm is given and compared with the original FAM. Although the primary use of the zFAM is to obtain a local (zoomed) part of the SCF, we will compare both algorithms when only the first quadrant of the SCF is computed, which is the only part needed when the input signal is real, due to the SCF symmetries.

Table 3.3 indicates the computational cost for both FAM and zFAM algorithms, detailing the cost for each step for a single input channel. Therefore, the total cost of each algorithm will be the indicated, multiplied by the number of $P$ channels on the channelization process. Also, all operations are assumed to have complex operators, and the number of real FLOPs is obtained as indicated in Table B. 1 of Appendix B.

As explained in 3.4, we will consider the FAM algorithm implementation when the second FFT is not used (step 8), and only the sum of all $P$ channel is computed.

| Step | FAM | zFAM |
| :---: | :---: | :---: |
| Channelization | - | - |
| Windowing | $6 N_{p}$ | $6 N_{p}$ |
| FFT | $10\left(\frac{N_{p}}{2}\right) \log _{2}\left(N_{p}\right)$ | - |
|  | [80 | Case $\Delta \alpha=L \Delta f, L$ even $\begin{gathered} N_{p_{\text {Dual CTA }}}=N_{p}\left(\frac{L}{2}+1\right)+1 \\ N_{p_{\text {Single CTA }}}=N_{p}(L+1)+\frac{\alpha_{i 1}}{\Delta f} N_{p}+1 \end{gathered}$ <br> Case $\Delta \alpha=L \Delta f, L$ odd $\begin{aligned} N_{p_{\text {Dual CTA }}} & =N_{p}(L+2)+1 \\ N_{p_{\text {Single CTA }}} & =2 N_{p}(L+1)+2 \frac{\alpha_{i 1}}{\Delta f} N_{p}+1 \end{aligned}$ <br> Case $\Delta f=L \Delta \alpha$ $\begin{aligned} N_{p_{\text {Dual CTA }}} & =N_{p}(2 L+1)+1 \\ N_{p_{\text {Single CTA }}} & =2 N_{p}(L+1)+2 \frac{\alpha_{i 1}}{\Delta \alpha} N_{p}+1 \end{aligned}$ <br> For all cases, $L_{C T A}=\operatorname{nextfftlength}\left(N_{p}+N_{p \frac{\text { Single }}{\text { Dual }} C T A}-1\right)$ |
| Dual CTA | - | $\begin{gathered} 2\left(6 N_{p}+10 L_{C T A} \log _{2}\left(L_{C T A}\right)+6 L_{C T A}\right. \\ \left.+6 N_{p_{\text {Dual CTA }}}\right) \\ \hline \end{gathered}$ |
| Single CTA | - | $\begin{gathered} 6 N_{p}+10 L_{C T A} \log _{2}\left(L_{C T A}\right)+6 L_{C T A} \\ +6 N_{p_{\text {Single }} \text { CTA }} \\ \hline \end{gathered}$ |
| Phase-compensation | $6 N_{p}$ | Single CTA  <br>  $6\left(N p_{\text {Single CTA }}\right)$ <br> Dual CTA  <br>  $6(2 N p)$ |
| Complex demodulates multiply | $6 N_{p}^{2}$ | $6 N_{p}^{2}$ |
| Sum | $2 N_{p}$ | $2 N_{p}$ |

Table 3.3 - Computational cost for FAM and zFAM.

As can be seen from Table 3.3, the FAM algorithm has a fixed number of FLOPs (per input channel) given by

$$
\begin{equation*}
F_{F A M}=14 N_{p}+5 N_{p} \log _{2}\left(N_{p}\right)+6 N_{p}^{2} \tag{3.39}
\end{equation*}
$$

For the zFAM algorithm, the total number of FLOPs depends on the region being calculated. For this performance comparison, we will get the number of operations needed by FAM and zFAM to obtain the SCF for the first quadrant of the $(f ; \alpha)$ plane (the only part needed when the input signal is real) in a $N_{p} \times N_{p}$ grid.

For a given $N_{p}$, the FAM will produce a $N_{p} \times N_{p}$ matrix for all four quadrants of the $(f ; \alpha)$ plane. Therefore, we will use the values from Table 3.3, but, for the FAM algorithm, the FFT and phasecompensation steps must now be computed with $2 N_{p}$ points. In the case of the zFAM algorithm, as $\Delta \alpha=2 \Delta f$ and $\alpha_{i_{1}}=0$ for the first quadrant, we will use the Single CTA method with $N_{p_{\text {Single } C T A}}=N_{p}(L+1)+1=3 N_{p}+1 ; L_{C T A}=\operatorname{nextfftlength}\left(N_{p}+3 N_{p}\right)$, and, if $N_{p}$ is a power-of-2, $L_{C T A}=4 N_{p}$.

So, the total number of FLOPs for the FAM algorithm (per input channel and for the first quadrant, 1 Q ) is

$$
\begin{equation*}
F_{F A M_{1 Q}}=30 N_{p}+10 N_{p} \log _{2}\left(N_{p}\right)+6 N_{p}^{2} \tag{3.40}
\end{equation*}
$$

For zFAM (in the same conditions), one gets:

$$
\begin{equation*}
F_{z F A M_{1 Q}}=154 N_{p}+40 N_{p} \log _{2}\left(N_{p}\right)+6 N_{p}^{2}+12 \tag{3.41}
\end{equation*}
$$

Figure 3.34 shows the FLOP ratio $F_{z F A M_{1 Q}} / F_{F A M_{1 Q}}$ as a function of $N_{p}$ and for $N p \geq 64$ (in a practical situation, $N_{p}$ will be high). As expected, the zFAM has a higher computational cost than the FAM algorithm due to the high computational cost required by the CTA transform. Nevertheless, for high $N_{p}$, both algorithms show similar cost due to the dominance of the $N_{p}^{2}$ term in the total number of operations.

However, the zFAM algorithm is not tailored to address the computation of the entire region of support of the SCF but to a zoom/local subregion. Let us assume we wish to compute the SCF in a subregion defined in the spacing interval of the frequency and cycle frequency $\Delta f \times \Delta \alpha$ with $N_{p}^{\prime} \times N_{p}^{\prime}$ points grid, as shown in Figure 3.35, and also let us assume an equal dimension subregion, i.e., $\Delta f=\Delta \alpha$.


Figure 3.34 - Comparison of the FAM and zFAM algorithms (FLOP ratio) for the first-quadrant computation of the SCF (no zoom/local region).


Figure 3.35 - Zoom region to compute the SCF.

For zFAM, and using the results from Table 3.3, we have $L=1$, and, considering the worst case of a Dual CTA, $N_{p_{\text {Dual CTA }}}=3 N^{\prime}{ }_{p}+1$ and $L_{C T A}=\operatorname{nextfftlength}\left(4 N^{\prime}{ }_{p}\right)=4 N^{\prime}{ }_{p}$ assuming $N^{\prime}{ }_{p}$ is a power-of-two. The total number of FLOPs is:

$$
\begin{equation*}
F_{z F A M}=300 N_{p}^{\prime}+80 N_{p}^{\prime} \log _{2}\left(N_{p}^{\prime}\right)+6 N_{p}^{\prime 2}+24 \tag{3.42}
\end{equation*}
$$

For the FAM, we will need the computation of the first FFT and phase-compensation of the entire frequency range array. Assuming the initial spacing of $f_{s} / N_{p}$, the total number of points in the FFT is now $N_{p} N_{p}^{\prime}$. Using the results from Table 3.3, one gets:

$$
\begin{equation*}
F_{F A M}=6 N_{p}+\left(5 N_{p} N_{p}^{\prime} \log _{2}\left(N_{p} N_{p}^{\prime}\right)+6 N_{p} N_{p}^{\prime}\right)+6 N_{p}^{\prime 2}+2 N_{p}^{\prime} \tag{3.43}
\end{equation*}
$$

In this case, the correspondence of the needed complex demodulates obtained after the first FFT and phase-compensation would have to be obtained for the specific points $\left(f_{i} ; \alpha_{i}\right)$ of the SCF in the subregion. Further, the coverage of this subregion will not be regular, as in the case of complete first-quadrant computation of the SCF using the FAM algorithm.

Although there is no general relation between $N_{p}$ and $N_{p}^{\prime}$, we will usually be interested in a high zoom SCF so that $N_{p} N_{p}^{\prime}$ is high. This way, we will compare the computational cost when using FAM or zFAM for high $N_{p}$ and $N_{p}^{\prime}$.

Figure 3.36 shows the FLOP ratio $F_{z F A M} / F_{F A M}$ as a function of $N_{p}$ for several local region dimensions ( $N_{p}^{\prime}$ ). As can be seen, the zFAM has a much lower computation cost than FAM, being lower than $5 \%$ (compared to FAM) for $N_{p} \geq 2^{10}$.


Figure 3.36 - Comparison of the FAM and zFAM algorithms (FLOP ratio) for a zoom/local region of the SCF with varying $N_{p}^{\prime}$.

For the case presented in section 3.4.1, with receiver sampling frequency deviation, when using the FAM algorithm, the cycle signatures vanished with $N_{p}=256$ (Figure 3.27). Now, using zFAM for the zoom region around $\left(f_{i} ; \alpha_{i}\right)=\left(f_{c_{1}} ; f_{\text {sym }}\right)$ delimited by the corresponding square as in Figure 3.35 , with $N^{\prime}{ }_{p}=256$, one obtains the SCF shown in Figure 3.37, where the cycle feature is now visible. For this case, and from Figure 3.36, with $N_{p}=N_{p}^{\prime}=256$, zFAM takes only about $10 \%$ the computational time of FAM.


Figure 3.37 - Zoom SCF obtained by zFAM in the square region of the cycle feature with receiver sampling frequency deviation.

### 3.7 Chapter Summary

This chapter presented the general theory of Cyclic Spectral Analysis (CSA), introducing the twodimensional Cyclic Autocorrelation Function (CAF) for cyclostationary signals. The corresponding Spectral Correlation Function - SCF (defined as the Fourier Transform of the CAF) and the Spectral Coherence Function (SCoF) are then obtained to enable cyclostationary signal detection or CSA.

The continuous-time analysis was given, which led to the Spectral Correlation Analyzer structure to obtain the SCF. The discretization of the Spectral Correlation Analyser was then pursued, including the decimation analysis to increase performance. For the digital implementation
algorithm to compute the SCF (and, therefore, the SCoF), the FFT Accumulation Method (FAM) was detailed, and its characteristics were analyzed.

Performing a (possible) wideband spectrum sensing and detection involves analyzing a large frequency swath to perform cyclic feature detection. But, generally, only a small region of the $(f ; \alpha)$ plane contains these features. This led to the search for an innovative adaptation or modification of the FAM algorithm to efficiently obtain the SCF in a zoomed/local sub-band of the entire plane. The zoom FAM (zFAM) algorithm has been proposed and detailed, and its computational cost was compared to the original FAM. When the entire ( $f ; \alpha$ ) plane needs to be computed, the FAM is naturally more efficient than the zFAM. However, for practical cases, when a small fraction of the ( $f ; \alpha$ ) plane is needed, the zFAM can perform significantly better than the FAM algorithm, reaching less than $5 \%$ of the FAM computational cost in many cases.

## Chapter 4

## 4 Probability Density of the Spectral Estimates of the FAM and zFAM Algorithms

Since published in 1988 (Brown and Loomis, 1988), the FAM algorithm has been used to compute the SCF or SCoF, and, although considered the most computationally efficient (Antoni, Xin and Hamzaoui, 2017), there has been no thorough statistical analysis of the algorithm. To the author's best knowledge, no published scientific work in the open literature addressed this topic.

This chapter analyzes the statistics of spectral estimates using the FAM, or the proposed zFAM algorithms, by deriving the pdf obtained for the points covering the ( $f ; \alpha$ ) plane. Obtaining the pdf will allow the development of a CFAR Primary User detection using the SCF (with noise floor estimation) or the SCoF (blind detector with no statistical information on the noise).

In Section 4.1, we analyze the FAM algorithm and obtain its vector version and the Quadratic Form equivalent of a single ( $f ; \alpha$ ) point on the frequency/cyclic frequency plane. Throughout this chapter - as the frequency range is $\left[0 ; f_{s} / 2\right]$ and the cyclic frequency range is $\left[0 ; f_{s}\right]$ - the plane point will be given as $(f ; \alpha)=\left(K \times \frac{f_{s} / 2}{N p S e g} ; \Delta \times \frac{f_{s}}{N p S e g}\right)$ with $K, \Delta=0 \ldots N p S e g-1$.

Section 4.2 analyzes the estimates obtained by the FAM algorithm, using its Quadratic Form equivalent when the input is a Gaussian $\mathcal{N}\left(0, \sigma^{2}\right)$ r.v.. Exact expressions for the pdf of the estimates along the ( $f ; \alpha$ ) plane are derived and reduced complexity approximations are obtained, with justification and comparison to the former when the number of averaging segments, $P$ is low to medium.

Finally, in Section 4.3, we obtain the approximate pdf of the SCF and SCoF, when the number of averaging segments is high. Except for the $\Delta=0$ case, these functions will be approximated by a Gaussian density, as justified in Appendix E. The approximation given for the SCoF is a main result of this thesis, which will enable the development of the blind estimator referenced.

### 4.1 Vectorized FAM algorithm and Quadratic Form

The FAM algorithm was analyzed in section 3.4, where Equations (3.35) and (3.36) summarize the signal processing tasks needed to evaluate the SCF, with the implementation shown in Figure 3.20.

Also, in section 3.4, the processing steps were systematized, and, for some given $N_{p}, N p S e g$ and $P$ parameters, the following steps are performed ( $L=N p S e g / 4$ ):

1. Channelize the input signal with $P$ channels of length $N p S e g$, with samples shifted by $L$ samples for each channel;
2. Apply a Hamming window on each channel signal;
3. Compute the $N_{p}$-point FFT for each channel;
4. Phase-compensate each channel to account for the time shift of each one;
5. Obtain the cross product of the complex demodulates for all channel combinations;
6. Sum all points obtained for each cross product;
7. Fill the SCF matrix, finding the corresponding $\left(f_{i} ; \alpha_{i}\right)$ pair, from $f_{k}, f_{l}$.

These signal processing steps can be vectorized, as illustrated in Figure 4.1, where the operation denoted by $\odot$ stands for the Hadamard product, and $a(n), n=0 \ldots N p S e g-1$ is the Hamming window of size NpSeg.


a) Channelization


FFT( $x w$ )

c) FFT
d) Phase-compensation


Figure 4.1 - Vectorized FAM algorithm signal processing steps.

As seen, the input signal, $s(n)$, is segmented in $P$ overlapping segments, each with $N p S e g$ samples, so that the $i$ th segment is $x_{i}(n)=s(n+L i)$, where $L=N p S e g / 4$ and the indexes are $i=0 \ldots P$ and $n=0 \ldots N p S e g-1$.

In the following, the statistical properties of the FAM algorithm are analyzed for the case when the input signal $s(n)=w(n)$, and $w(n)$ is a wide-sense, Gaussian sequence with $\mathcal{N}\left(0, \sigma^{2}\right)$. All the FAM processing for a single $\left(f_{i} ; \alpha_{i}\right)$ point is given in matrix-vector notation, and the resulting estimate is shown to be a quadratic form of the input vector. The pdf corresponding to this quadratic form is obtained and clustered for several points in the $(f ; \alpha)$ plane. Then, several approximations are obtained for the pdf, and several simulation results are performed to assess their validity.

Considering the FAM algorithm processing steps, as illustrated in Figure 4.1, but only for a single $\left(f_{i} ; \alpha_{i}\right)$ point, and for a single segment, $P$, then, given ${ }^{35}$

$$
\begin{array}{ll}
f_{i}=K \frac{f_{s}}{2 N p S e g}, & K=0 \ldots N p S e g-1, \\
\alpha_{i}=\Delta \frac{f_{s}}{N p S e g}, & \Delta=0 \ldots N p S e g-1, \tag{4.1}
\end{array}
$$

the FFT points after phase compensation (steps a), b), c), and d)) are computed by

$$
X_{f_{i} \pm \frac{\alpha_{i}}{2}}=\frac{1}{N p S e g} \sum_{n} a(n) x(n) e^{-j \frac{2 \pi n}{f_{s}}\left(f_{i} \pm \frac{\alpha_{i}}{2}\right)} e^{-j \frac{2 \pi P L}{f_{s}}\left(f_{i} \pm \frac{\alpha_{i}}{2}\right)}
$$

giving:

[^23]\[

$$
\begin{align*}
X_{f_{i}+\frac{\alpha_{i}}{2}} & =\frac{1}{N p S e g} \sum_{n} a(n) x(n) e^{-j 2 \pi n\left(\frac{K}{2 N p S e g}+\frac{\Delta}{2 N p S e g}\right)} e^{-j 2 \pi P L\left(\frac{K}{2 N p S e g}+\frac{\Delta}{2 N p S e g}\right)} \\
X_{f_{i}-\frac{\alpha_{i}}{2}} & =\frac{1}{N p S e g} \sum_{m} a(m) x(m) e^{-j 2 \pi m\left(\frac{K}{2 N p S e g}-\frac{\Delta}{2 N p S e g}\right)} e^{-j 2 \pi P L\left(\frac{K}{2 N p S e g}-\frac{\Delta}{2 N p S e g}\right)} . \tag{4.2}
\end{align*}
$$
\]

The contribution to the SCF of this segment is, then:

$$
\begin{align*}
& X_{f_{i}+\frac{\alpha_{i}}{2}} \times X_{f_{i}-\frac{\alpha_{i}}{2}}^{*} \\
& =\frac{1}{N p S e g^{2}} \sum_{n} \sum_{m} a(n) x(n) a(m) x(m) e^{-j \frac{\pi K}{N p S e g}(n-m)} e^{-j \frac{\pi \Delta}{N p S e g}(n+m)} e^{-j \frac{2 \pi P L \Delta}{N p S e g}} . \tag{4.3}
\end{align*}
$$

For the first segment $(P=0)$, equation (4.3) can be written in matrix form as:

$$
\begin{equation*}
X_{f_{i}+\frac{\alpha_{i}}{2}} \times X_{f_{i}-\frac{\alpha_{i}}{2}}^{*}=\frac{1}{N p S e g^{2}} \boldsymbol{x}^{T} \underbrace{\left(\operatorname{diag}(a) \times \boldsymbol{C}_{K, \Delta} \times \operatorname{diag}(a)\right.}_{=\boldsymbol{G}_{\boldsymbol{K}, \boldsymbol{\Delta}}^{0}}) \boldsymbol{x} \tag{4.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(\boldsymbol{C}_{K, \Delta}\right)_{m, n}=e^{-j_{N p S S g}^{\pi K}(n-m)} e^{-j \frac{\pi \Delta}{N_{p S e g}}(n+m)} \tag{4.5}
\end{equation*}
$$

with $\boldsymbol{G}_{\boldsymbol{K}, \boldsymbol{\Delta}}^{0}=\operatorname{diag}(a) \times \boldsymbol{C}_{K, \Delta} \times \operatorname{diag}(a)$ and with the input signal samples arranged in a column vector $\boldsymbol{x}$.

For another segment $P,\left(P=0, \ldots, P_{\max }\right)$, equation (4.4) is modified as:

$$
X_{f_{i}+\frac{\alpha_{i}}{2}} \times X_{f_{i}-\frac{\alpha_{i}}{2}}^{*}=\frac{1}{N p S e g^{2}} \boldsymbol{x}^{T}\left[\begin{array}{ccc}
\mathbf{0} & \mathbf{0} & \mathbf{0}  \tag{4.6}\\
\mathbf{0} & \boldsymbol{G}_{K, \Delta}^{P}=\boldsymbol{G}_{K, \Delta}^{0} e^{-j \frac{2 \pi P L \Delta}{N p S e g}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \boldsymbol{0}, ~
\end{array}\right] \boldsymbol{x}
$$

where the $\mathbf{0}$ submatrices have the dimension adapted to the adequate $P$ segment and overlap considered. For example, to the $P$-th segment and with no overlap, the top submatrix is $\mathbf{0}_{P \times N p S e g, P \times N p S e g}$; with $50 \%$ overlap, it is $\mathbf{0}_{P \times \frac{N p S e g}{2}, P \times \frac{N p S e g}{2}}$.

The SCF estimate for a specific $\left(f_{i} ; \alpha_{i}\right)$ point can thus be obtained as follows:

1. Obtain $\boldsymbol{G}_{K, \Delta}^{0}$ as given in (4.4);
2. For the number of averaging segments, obtain the corresponding matrix as given in (4.6), $\boldsymbol{G}_{K, \Delta}^{P}$ with $\mathbf{0}$ submatrices;
3. Add all the matrices obtained in previous steps (zero padding all to the appropriate final size) to obtain $\boldsymbol{G}_{K, \Delta}$ (see Figure 4.2 for visualization of the procedure);
4. Compute the SCF as:

$$
\begin{equation*}
S_{x}^{\alpha_{i}}\left(f_{i}\right)=\frac{1}{P \times N p S e g^{2}} \boldsymbol{x}^{T} \boldsymbol{G}_{K, \Delta} \boldsymbol{x} \tag{4.7}
\end{equation*}
$$



Figure 4.2 - Quadratic form matrix to obtain the SCF for a specific point.
Therefore, the SCF can be obtained by the quadratic form expressed in step 4) (equation(4.7)).
The $\boldsymbol{G}_{K, \Delta}$ matrix in equation (4.47) can be substituted by a symmetric matrix obtained as (Demonstration 4.2):

$$
\begin{equation*}
\boldsymbol{G}_{K, \Delta}^{\prime}=\frac{\boldsymbol{G}_{K, \Delta}+\left(\boldsymbol{G}_{K, \Delta}\right)^{T}}{2} . \tag{4.8}
\end{equation*}
$$

This is especially important because the eigenvalues of the real and the imaginary part of this modified matrix are all real (see Demonstration 4.3). The whole procedure can be maintained only by simply substituting the $\boldsymbol{G}_{K, \Delta}^{0}$ matrix in equation (4.4) by a symmetric matrix obtained as in (4.8). This is equivalent to using the following matrix in (4.5):

$$
\begin{gather*}
\left.\begin{array}{l}
\frac{\left(\boldsymbol{C}_{K, \Delta}\right)_{m, n}+\left(\boldsymbol{C}_{K, \Delta}\right)^{T}}{m, n} \\
=\frac{1}{2}\left(e^{-j \frac{\pi K}{N p S e g}(n-m)} e^{-j \frac{\pi \Delta}{N p S e g}(n+m)}+e^{+j} \frac{\pi K}{N p \operatorname{Seg}}(n-m)\right. \\
\left.e^{-j \frac{\pi \Delta}{N p S e g}(n+m)}\right) \\
=e^{-j \frac{\pi \Delta}{N p S e g}(n+m)} \cos \left(\frac{\pi K}{N p S e g}(n-m)\right) \\
=\cos \left(\frac{\pi K}{N p S e g}(n-m)\right)
\end{array}\right]\left[\cos \left(\frac{\pi \Delta}{N p S e g}(n+m)\right)-j \sin \left(\frac{\pi \Delta}{N p S e g}(n+m)\right)\right]
\end{gather*}
$$

We note that the procedure described to obtain the SCF (steps leading to the matrix in Figure 4.2) when $\Delta=0$, corresponds to the well-known Welch's Method for Power Spectrum Estimation (Welch, 1967)(Proakis and Manolakis, 1996), where $\frac{\left(\boldsymbol{C}_{K, \Delta}\right)_{m, n}+\left(\boldsymbol{C}_{K, \Delta}\right)^{T}{ }_{m, n}}{2}=\cos \left(\frac{\pi K}{N p S e g}(n-m)\right)$ (Johnson and Long, 1999) ${ }^{36}$.

[^24]
### 4.2 FAM Spectral Estimates Characterization

The Moment Generating Function (MGF), $M(s)$, of a quadratic form $y=\boldsymbol{x}^{T} \boldsymbol{A} \boldsymbol{x}$ when $\boldsymbol{x}$ is a zero-mean Gaussian vector with variance $\sigma^{2}$, is given as (Sharf, 1991):

$$
\begin{equation*}
M(s)=\prod_{i=1}^{D}\left(\frac{1}{1-2 \lambda_{i} s}\right)^{\frac{\nu_{i}}{2}} \tag{4.10}
\end{equation*}
$$

where $\lambda_{i}$ are the $D$ distinct, nonzero eigenvalues of $\sigma^{2} \boldsymbol{A}$, and $\nu_{i}$ is the multiplicity of the $\lambda_{i}$ eigenvalue. The mean and variance of $y$ are given by (Sharf, 1991):

$$
\begin{align*}
\mu_{Y} & =\sum_{i=1}^{D} \lambda_{i} \nu_{i} \\
\sigma_{Y}^{2} & =\sum_{i=1}^{D} 2 \lambda_{i}^{2} \nu_{i} \tag{4.11}
\end{align*}
$$

The pdf corresponding to the MGF in (4.10) can be shown to be

$$
\begin{equation*}
f_{Y}(y)=\frac{g}{\Gamma\left(\frac{N}{2}\right)} y^{\frac{N}{2}-1} \Phi_{2}\left(\frac{\nu_{1}}{2}, \ldots, \frac{\nu_{D}}{2} ; \frac{N}{2} ;-\frac{1}{2 \lambda_{1}} y,-\frac{1}{2 \lambda_{2}} y, \ldots,-\frac{1}{2 \lambda_{D}} y\right) \tag{4.12}
\end{equation*}
$$

where $g=\prod_{i=1}^{D}\left(2 \lambda_{i}\right)^{-\nu_{i} / 2} ; \Phi_{2}$ is the generalized hypergeometric function (Roberts and Kaufman, 1966):

$$
\begin{align*}
\Phi_{2}\left(b_{1}, \ldots, b_{n} ; c ; x_{1}, \ldots, x_{n}\right) & =\sum_{m_{1} \ldots m_{n}} \frac{\left(b_{1}\right)_{m 1} \ldots\left(b_{n}\right)_{m_{n}}}{(c)_{m_{1}+\cdots+m_{n}} m_{1}!\ldots m_{n}!} x_{1}^{m_{1}} \ldots x_{n}^{m_{n}} \\
m_{1} \ldots m_{n} & =0, \ldots,+\infty \tag{4.13}
\end{align*}
$$

The Pochhammer symbol is:

$$
\begin{equation*}
(a)_{n}=a(a+1)(a+2) \ldots(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)} \tag{4.14}
\end{equation*}
$$

Although the pdf of the quadratic form is given by (4.12), the computation of the generalized hypergeometric function may render it difficult to use because of its computational burden.

In the following, we will analyze special cases of the $(f ; \alpha)$ plane when computing the SCF, aiming to obtain more practical solutions. For each of these cases, the theoretical and simulation results
will be shown for selected points in the plane, given by specific values of $(K ; \Delta)$.

### 4.2.1 The case when $\Delta=0$ (Welch Algorithm)

When computing the SCF for $\Delta=0$, the FAM algorithm is equal to Welch's algorithm, as noted in the previous section, where the quadratic form matrix is given by

$$
\begin{align*}
& \operatorname{diag}(\boldsymbol{a}) \boldsymbol{C}_{K, 0} \operatorname{diag}(\boldsymbol{a}), \text { where } \\
& \boldsymbol{C}_{K, 0}=\cos \left(\frac{\pi K}{N p S e g}(n-m)\right) \tag{4.15}
\end{align*}
$$

The analysis of the density estimates when there is only one averaging segment, and when there are multiple segments but no overlap of the input samples, was performed in (Johnson and Long, 1999). In this work - using a partial fraction expansion (Oppenheim, Willsky and Nawab, 1996) and the work developed in (Imhof, 1961) - it was shown that when the eigenvalues have even multiplicity, the pdf of the quadratic form, with the MGF as expressed in equation (4.10) is given by $\left(h_{i}=\nu_{i} / 2 ; d_{i}=1 / 2 \lambda_{i} ; g=\prod_{i=1}^{D}\left(d_{i}\right)^{h_{i}}=\prod_{i=1}^{D}\left(2 \lambda_{i}\right)^{-h_{i}} ; D\right.$ is the number of distinct eigenvalues):

$$
\begin{equation*}
f_{Y}(y)=g \sum_{i=1}^{D} \sum_{j=1}^{h_{i}} A_{i j} \frac{1}{(j-1)!} y^{j-1} e^{-d_{i} y} \tag{4.16}
\end{equation*}
$$

where

$$
A_{i j}=\frac{1}{\left(h_{i}-j\right)!} \sum_{n=0}^{h_{i}-j} \frac{D_{i}^{(n)}\left(d_{i}\right)}{n!} \sum_{k=1, k \neq i}^{D} \frac{h_{k}}{\left(d_{i}-d_{k}\right)^{h_{i}-j-n}}
$$

and the derivatives of the MGF are:

$$
D_{i}^{(m+1)}(s)=\sum_{n=0}^{m}\binom{m}{n} D_{i}^{(n)}(s) \sum_{k=1, k \neq i}^{D} h_{k} \frac{(m-n)!}{\left(-d_{k}-s\right)^{m-n+1}}
$$

The result obtained in (4.16) was then used in (Johnson and Long, 1999) with the premise that "Even when the eigenvalues are distinct, they are typically in near pairs for commonly used windows.".

In Appendix C , we show that the eigenvalues do come in pairs but only for the rectangular window, odd number of averaging segments, $P$, and for $K$ even. This way, the assumption that the eigenvalues, although distinct, come in near pairs with other standard windows is expected, as
applying a non-rectangular window will typically smooth the spectrum and, therefore, the obtained eigenvalues will depart slightly from the equal value paring.

Unfortunately, computing the pdf of the quadratic form using (4.16) will have numerical problems on the exponential component when the matrix has a high number of eigenvalues (high number of averaging segments). Typically, the $d_{i}=1 / 2 \lambda_{i}$ will have a highly increasing pattern with a large dynamic range, and numeric issues arise when computing $f_{Y}(y)$.

In the following, we will obtain an approximation of the pdf using the quadratic form MGF (equation (4.10)) or its corresponding Characteristic Function, avoiding the numerical problems of equation (4.16).

## A physics-based approach approximation

The MFG of the quadratic form was obtained in equation (4.10), which, when the eigenvalues come in pairs (using $h_{i}=\nu_{i} / 2$ ) is:

$$
\begin{equation*}
M(s)=\prod_{i=1}^{D}\left(\frac{1}{1-2 \lambda_{i} s}\right)^{h_{i}}=\prod_{i=1}^{D}\left(\frac{d_{i}}{d_{i}-s}\right)^{h_{i}} . \tag{4.17}
\end{equation*}
$$

Let us consider a general function, $f(s)$, given by:

$$
\begin{equation*}
f(s)=\prod_{i=1}^{D}\left(\frac{d_{i}}{d_{i}-s}\right)^{m_{i}} . \tag{4.18}
\end{equation*}
$$

The log-derivative of $f(s)$ is:

$$
\begin{equation*}
\frac{f^{\prime}(s)}{f(s)}=(\log (f(s)))^{\prime}=\sum_{i=1}^{D} \frac{m_{i}}{d_{i}-s} . \tag{4.19}
\end{equation*}
$$

Considering $r_{i}^{2}=d_{i}-s$, then (4.19) becomes:

$$
\begin{equation*}
\frac{f^{\prime}(s)}{f(s)}=\sum_{i=1}^{D} \frac{m_{i}}{r_{i}^{2}} . \tag{4.20}
\end{equation*}
$$

The result expressed in (4.20) can be interpreted as the resulting scaled gravitational (or electric) force on a unitary mass (charge) particle at $s$, from $D$ particles with mass (charge) $m_{i}$, at distances $r_{i}$ from the unitary particle. This force will be equal to the equivalent force of a single particle with mass (charge) $m$ at a distance $d$, so that one could write:

$$
\frac{f_{1}^{\prime}(s)}{f_{1}(s)}=\frac{m}{d-s} .
$$

Therefore, the original function $f_{1}(s)$ would be:

$$
\begin{equation*}
f_{1}(s)=\left(\frac{d}{d-s}\right)^{m} \tag{4.21}
\end{equation*}
$$

Applying this result to the MGF, as in equation (4.17), we would approximate it by:

$$
\begin{equation*}
M(s)=\prod_{i=1}^{D}\left(\frac{d_{i}}{d_{i}-s}\right)^{h_{i}} \approx\left(\frac{d}{d-s}\right)^{m} \tag{4.22}
\end{equation*}
$$

## An Electrical Engineering-based approach approximation

Another perspective can be obtained by using the quadratic form's characteristic function (CF). Using the definition from (Lukacs, 1970)(p.196), the relation between the MFG and the CF is $M(s=-j w)=\varphi(w)^{37}$, and as also referenced in (Sharf, 1991) (p.65):

$$
\begin{equation*}
\varphi(w)=\prod_{i=1}^{D}\left(\frac{d_{i}}{d_{i}+j w}\right)^{h_{i}} \tag{4.23}
\end{equation*}
$$

Applying the same approximation that led to (4.22), one now obtains:

$$
\begin{equation*}
\varphi(w)=\prod_{i=1}^{D}\left(\frac{d_{i}}{d_{i}+j w}\right)^{h_{i}} \approx\left(\frac{d}{d+j w}\right)^{m} \tag{4.24}
\end{equation*}
$$

Equation (4.24) can be interpreted as approximating a cascade of low-pass filters with poles at $w_{p_{i}}=d_{i}=1 / 2 \lambda_{i}$, by a low order cascade of equal low-pass filters, all with a common pole at $w_{p}=d$.

Figure 4.3 shows an example of the location of the poles for $P=65$ averaging segments. Typically, the eigenvalues follow a decreasing pattern so that the corresponding poles will be highly increasing, as can be best seen in the poles' location (in logarithmic scale). The concentration of poles near $w=0$ suggests a possible approximation by a single pole multiple filters as expressed by (4.24).

[^25]Filter pole location


Filter pole logarithmic location


Figure 4.3 - Pole location of the Characteristic Function interpreted as a cascade of low-pass filters.

The approximation expressed in equation (4.22) is the basis of the following work, where we will obtain the pdf of the quadratic form, avoiding the numerical problems of (4.16).

When the eigenvalues come in pair $(+\lambda ;+\lambda)$, the multiplicity is 2 , so $h_{i}=\nu_{i} / 2=1$. Therefore (4.22) becomes, after changing the variables' names:

$$
\begin{equation*}
M(s)=\prod_{i=1}^{D} \frac{d_{i}}{d_{i}-s} \approx\left(\frac{\beta}{\beta-s}\right)^{\alpha} \tag{4.25}
\end{equation*}
$$

The MGF of this approximation, $M_{1}(s)$ :

$$
\begin{equation*}
M_{1}(s)=\left(\frac{\beta}{\beta-s}\right)^{\alpha} \tag{4.26}
\end{equation*}
$$

is the MGF of a gamma density with shape parameter equal to $\alpha$ and rate parameter equal to $\beta$, with pdf given as (Papoulis and Pillai, 2002):

$$
\begin{equation*}
f_{Y}(y)=\frac{\beta^{\alpha}}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \tag{4.27}
\end{equation*}
$$

The moments of a r.v. Y, can be obtained by its MGF or CF, as (Papoulis and Pillai, 2002) (Lukacs, 1970):

$$
\begin{align*}
& E\left[Y^{k}\right]=\left.\frac{\partial^{k} M_{Y}(s)}{\partial s^{k}}\right|_{s=0} \\
& E\left[Y^{k}\right]=\left.\frac{1}{j^{k}} \frac{\partial^{k} \varphi(w)}{\partial w^{k}}\right|_{w=0} \cdot{ }^{38} \tag{4.28}
\end{align*}
$$

We then consider obtaining the parameters of the proposed approximation by calculating and equating the moments obtained using both MGFs, $M(s)$ and $M_{1}(s)$.

The normal derivative of any function can be expressed as a function of its log-derivative:

$$
f^{\prime}(x)=\frac{\partial}{\partial x} f(x)=f(x)\left[\frac{f^{\prime}(x)}{f(x)}\right]=f(x)[\log (f(x))]^{\prime} .
$$

The log-derivative of $M(s)$ is

$$
\frac{M^{\prime}(s)}{M(s)}=\left[\log (M(s)]^{\prime}=\sum_{i=1}^{D}\left[\log \left(d_{i}\right)-\log \left(d_{i}-s\right)\right]^{\prime}=\sum_{i=1}^{D} \frac{1}{d_{i}-s}\right.
$$

So one can obtain:

$$
\begin{aligned}
& \frac{\partial}{\partial s} M(s)=M(s) \sum_{i=1}^{D} \frac{1}{d_{i}-s} \\
& \frac{\partial^{2}}{\partial s^{2}} M(s)=\left(\frac{\partial}{\partial s} M(s)\right) \sum_{i=1}^{D} \frac{1}{d_{i}-s}+M(s) \sum_{i=1}^{D}\left(\frac{1}{d_{i}-s}\right)^{2} \\
& \frac{\partial^{3}}{\partial s^{3}} M(s)=\left(\frac{\partial^{2}}{\partial s^{2}} M(s)\right) \sum_{i=1}^{D} \frac{1}{d_{i}-s}+3\left(\frac{\partial}{\partial s} M(s)\right) \sum_{i=1}^{D}\left(\frac{1}{d_{i}-s}\right)^{2}+M(s) \sum_{i=1}^{D}\left(\frac{1}{d_{i}-s}\right)^{3} .
\end{aligned}
$$

And, in general:

$$
\begin{equation*}
\frac{\partial^{n}}{\partial s^{n}} M(s)=\sum_{k=1}^{n} \frac{(n-1)!}{(n+1-k)!}\left(\frac{\partial^{n-k}}{\partial s^{n-k}} M(s)\right) \sum_{i=1}^{D}\left(\frac{1}{d_{i}-s}\right)^{k} . \tag{4.29}
\end{equation*}
$$

As for $M_{1}(s)$, its log-derivative is

$$
\frac{M_{1}^{\prime}(s)}{M_{1}(s)}=\left[\log \left(M_{1}(s)\right]^{\prime}=\alpha[\log (\beta)-\log (\beta-s)]^{\prime}=\frac{\alpha}{\beta-s}\right.
$$

[^26]So one can obtain:

$$
\begin{aligned}
\frac{\partial}{\partial s} M_{1}(s) & =\frac{\alpha}{\beta-s} M_{1}(s) \\
\frac{\partial^{2}}{\partial s^{2}} M_{1}(s) & =\frac{\alpha+1}{\beta-s} \frac{\partial}{\partial s} M_{1}(s) \\
\frac{\partial^{3}}{\partial s^{3}} M_{1}(s) & =\frac{\alpha+2}{\beta-s} \frac{\partial^{2}}{\partial s^{2}} M_{1}(s)
\end{aligned}
$$

And, in general:

$$
\begin{equation*}
\frac{\partial^{n}}{\partial s^{n}} M_{1}(s)=\frac{\alpha+(n-1)}{\beta-s} \frac{\partial^{n-1}}{\partial s^{n-1}} M_{1}(s) \tag{4.30}
\end{equation*}
$$

As $M(0)=M_{1}(0)=1$, and the approximation only has two parameters ( $\alpha$ and $\beta$ ), we will force equality of the successive moments obtained by (4.28) applied to $M(s)$ and $M_{1}(s)$ until these parameters can be obtained. Table 4.1 shows the results obtained for the moments up to order three, alongside the variance.

| Moment | By $\boldsymbol{M}(\boldsymbol{s})$ | By $\boldsymbol{M}_{\mathbf{1}}(\boldsymbol{s})$ |
| :---: | :---: | :---: |
| $E[Y]$ | $\sum \frac{1}{d_{i}}$ | $\frac{\alpha}{\beta}$ |
| $E\left[Y^{2}\right]$ | $\sum\left(\frac{1}{d_{i}}\right)^{2}+\left(\sum \frac{1}{d_{i}}\right)^{2}$ | $\frac{\alpha(\alpha+1)}{\beta^{2}}$ |
| $E\left[Y^{3}\right]$ | $\sum\left(\frac{1}{d_{i}}\right)^{3}+\left(3 \sum \frac{1}{d_{i}}+\left(\sum \frac{1}{d_{i}}\right)^{2}\right) \sum\left(\frac{1}{d_{i}}\right)^{2}$ | $\frac{\alpha(\alpha+1)(\alpha+2)}{\beta^{2}}$ |
| $+\left(\sum \frac{1}{d_{i}}\right)^{4}$ | $\frac{\alpha}{\beta^{2}}$ |  |
| $\operatorname{var}(Y)$ <br> $=E\left[Y^{2}\right]-(E[Y])^{2}$ | $\sum\left(\frac{1}{d_{i}}\right)^{2}$ |  |

Table 4.1 - Moments obtained using $M(s)$ and $M_{1}(s)$.
Using just the moments of order one and two (and more straightforward with the variance), one gets:

$$
\begin{gathered}
\frac{E[Y]}{\operatorname{var}(Y)}=\frac{\sum \frac{1}{d_{i}}}{\sum\left(\frac{1}{d_{i}}\right)^{2}}=\frac{\frac{\alpha}{\beta}}{\frac{\alpha}{\beta^{2}}} \\
\frac{(E[Y])^{2}}{\operatorname{var}(Y)}=\frac{\left(\sum \frac{1}{d_{i}}\right)^{2}}{\sum\left(\frac{1}{d_{i}}\right)^{2}}=\frac{\left(\frac{\alpha}{\beta}\right)^{2}}{\frac{\alpha}{\beta^{2}}}
\end{gathered}
$$

and so:

$$
\begin{align*}
& \beta=\frac{\sum_{i=1}^{D} \frac{1}{d_{i}}}{\sum_{i=1}^{D}\left(\frac{1}{d_{i}}\right)^{2}}=\frac{\sum_{i=1}^{D} 2 \lambda_{i}}{\sum_{i=1}^{D}\left(2 \lambda_{i}\right)^{2}} \\
& \alpha=\frac{\left(\sum_{i=1}^{D} \frac{1}{d_{i}}\right)^{2}}{\sum_{i=1}^{D}\left(\frac{1}{d_{i}}\right)^{2}}=\frac{\left(\sum_{i=1}^{D} 2 \lambda_{i}\right)^{2}}{\sum_{i=1}^{D}\left(2 \lambda_{i}\right)^{2}} \tag{4.31}
\end{align*}
$$

Figure 4.4 shows the pdf obtained by simulation ( $2^{20}$ runs, $N p S e g=64, P=33$ and $(K ; \Delta)=$ $(4 ; 0)$ ) along with the theoretical result obtained by the MGF $M(s)$ and by using the proposed approximation for the MGF, $M_{1}(s)$. Figure 4.5 show the cdf (in logarithmic scale) obtained for the same cases. As can be seen, the approximation is quite good, with a slight difference around the maximum of the pdf, which naturally reflects the corresponding small variation of the cdf for low probability. It should be stressed that this approximation has no numerical computational problems, whereas when using (4.16), numerical issues arise even for low $P$.

Figure 4.6 shows the real and imaginary components of the Characteristic Functions $(\varphi(w)$ and $\varphi_{1}(w)$ ) corresponding to $M(s)$ and $M_{1}(s)$, which can be seen to be in close agreement.


Figure 4.4 - pdf obtained by simulation, by $M(s)$ and by $M_{1}(s)$ for $N p S e g=64, P=33$ and $(K, \Delta)=$ $(4,0)$.


Figure 4.5 - cdf (in logarithmic scale) obtained by simulation, by $M(s)$ and by $M_{1}(s)$ for $N p S e g=64$, $P=33$ and $(K, \Delta)=(4,0)$.



Figure 4.6 - Real and Imaginary parts of the Characteristic Functions $\left(\varphi(w)\right.$ and $\left.\varphi_{1}(w)\right)$ corresponding to $M(s)$ and $M_{1}(s)$ for $N p S e g=64, P=33$ and $(K, \Delta)=(4,0)$.

### 4.2.1.1 A Maximum-Likelihood Estimator

As will be seen in the following, the $\Delta=0$ case will always be a different case on the $(f ; \alpha)$ plane, with a very skewed pdf structure, not feasibly approximated by a Gaussian distribution even for a large number of averaging segments.

We will compare the maximum-likelihood (ML) estimate of the noise variance from the samples obtained for the SCF when using the exact pdf, and when using its approximation.

In the $\Delta=0$ case, when the eigenvalues come in $(+\lambda ;+\lambda)$ pairs, the pdf of the estimate is (from (4.16)):

$$
\begin{equation*}
f_{Y}(y)=g \sum_{i=1}^{D} A_{i 1} e^{-d_{i} y} \tag{4.32}
\end{equation*}
$$

Given the definition of $g$ and $A_{i 1}$ (as used in (4.16)), equation (4.32) can be expressed with the explicit dependence on the noise variance, $\sigma^{2}$ and the eigenvalues of the quadratic form matrix obtained for $\sigma^{2}=1, \lambda_{i}=\sigma^{2} \eta_{i}$. Equation (4.32) is now:

$$
\begin{equation*}
f_{Y}(y)=\frac{g^{\prime}}{\sigma^{2}} \sum_{i=1}^{D} A_{i 1}^{\prime} e^{-\frac{y}{2 \sigma^{2} \eta_{i}}} \tag{4.33}
\end{equation*}
$$

where $g^{\prime}$ and $A_{i 1}^{\prime}$ are obtained from $g$ and $A_{i 1}$ when $\sigma^{2}=1$.
The log-likelihood can be expressed as:

$$
\begin{aligned}
\mathcal{L}(y, \sigma) & =\sum_{n=0}^{N-1} \log f_{Y}\left(y_{n}\right) \\
& =N \times \log \left(\frac{g^{\prime}}{\sigma^{2}}\right)+\sum_{n=0}^{N-1} \log \left(\sum_{i=1}^{D} A_{i 1}^{\prime} e^{-\frac{y_{n}}{2 \sigma^{2} \eta_{i}}}\right)
\end{aligned}
$$

The ML estimate of $\sigma$ will be obtained for $\frac{\partial \mathcal{L}(y, \sigma)}{\partial \sigma}=0$, so:

$$
\frac{\partial \mathcal{L}(y, \sigma)}{\partial \sigma}=-\frac{2 N}{\sigma}+\sum_{n=0}^{N-1}\left(\frac{\sum_{i=1}^{D} \frac{y_{n} A_{i 1}^{\prime}}{\eta_{i} \sigma^{3}} e^{-\frac{y_{n}}{2 \sigma^{2} \eta_{i}}}}{\sum_{i=1}^{D} A_{i 1}^{\prime} e^{-\frac{y_{n}}{2 \sigma^{2} \eta_{i}}}}\right)=0
$$

which leads to:

$$
\begin{equation*}
\sum_{n=0}^{N-1}\left(\frac{\frac{y_{n}}{\sigma^{2}} \sum_{i=1}^{D} \frac{A_{i 1}^{\prime}}{\eta_{i}} e^{-\frac{y_{n}}{2 \sigma^{2} \eta_{i}}}}{\sum_{i=1}^{D} A_{i 1}^{\prime} e^{-\frac{y_{n}}{2 \sigma^{2} \eta_{i}}}}\right)=2 N . \tag{4.34}
\end{equation*}
$$

Therefore, finding the ML estimate of $\sigma$, implies finding the $\sigma$ that satisfies (4.34). Unfortunately, for a high number of averaging segments, the high number of eigenvalues and their dispersion will make this task difficult because of numerical computation problems.

Using the proposed approximation, the pdf is given by (4.27), and, considering the estimates of $\alpha$ and $\beta$ as given in equation (4.31), the explicit dependence on the noise variance can be obtained as:

$$
\begin{equation*}
f_{Y}(y)=\frac{1}{\sigma^{2 \alpha^{\prime}}} \frac{\beta^{\prime \alpha}}{\Gamma(\alpha)} y^{\alpha \prime-1} e^{-\frac{\beta^{\prime}}{\sigma^{2}} y} \tag{4.35}
\end{equation*}
$$

As before, $\alpha^{\prime}$ and $\beta^{\prime}$ are obtained by (4.31) when $\sigma^{2}=1$ (in this case $\alpha=\alpha^{\prime}$ and $\beta=\beta^{\prime} / \sigma$ ).
The log-likelihood can now be expressed as:

$$
\begin{aligned}
\mathcal{L}(y, \sigma) & =\sum_{n=0}^{N-1} \log f_{Y}\left(y_{n}\right) \\
= & -2 \alpha^{\prime} N \times \log (\sigma)+\sum_{n=0}^{N-1} \log \left(\frac{\beta^{\prime \alpha}}{\Gamma(\alpha)} y_{n}^{\alpha^{\prime}-1}\right)-\sum_{n=0}^{N-1} \frac{\beta^{\prime}}{\sigma^{2}} y_{n}
\end{aligned}
$$

The ML estimate of $\sigma$ will be obtained for $\frac{\partial \mathcal{L}(y, \sigma)}{\partial \sigma}=0$, so:

$$
\frac{\partial \mathcal{L}(y, \sigma)}{\partial \sigma}=-\frac{2 \alpha^{\prime} N}{\sigma}+\frac{2 \beta^{\prime}}{\sigma^{3}} \sum_{n=0}^{N-1} y_{n}=0
$$

which leads to:

$$
\begin{equation*}
\hat{\sigma}_{M L}=\sqrt{\frac{\beta^{\prime}}{\alpha^{\prime}} \times \frac{1}{N} \sum_{n=0}^{N-1} y_{n}} \tag{4.36}
\end{equation*}
$$

Equation (4.36) shows that the ML variance estimate can be obtained by a simple scaled version of the mean of the input samples, which is incommensurably simpler to obtain than solving equation (4.34).

We will compare the results obtained from both cases, plotting the relative error and the variance of the $\hat{\sigma}_{M L}$ for an example case with $P=33, N p S e g=64,(K, \Delta)=(10,0)$, and varying $N=$ $2^{9}, 2^{10}, 2^{11}, 2^{12}, 2^{13}$. Five hundred trials were simulated for each case to obtain the mean and variance.

Figure 4.7 and Figure 4.8 show, respectively, the mean relative error and the variance of the estimate obtained for the exact and approximation cases. As can be seen, the mean decreases as the sample size increases, being less than $0,05 \%$ for all simulated cases. We can also see a very close agreement of the variance obtained in both cases.


Figure 4.7 - Relative $\hat{\sigma}_{M L}$ error, obtained by $M(s)$ and by $M_{1}(s)$ for $P=33, N p S e g=64$, and $(K, \Delta)=(10,0)$.


Figure 4.8 - Variance of $\hat{\sigma}_{M L}$, obtained by $M(s)$ and by $M_{1}(s)$ for $P=33, N p S e g=64$, and $(K, \Delta)=$ $(10,0)$.

Although mathematically intractable for the exact case, for the approximation, one can obtain the variance of the estimate by computing the Fisher information, $I_{N}(\sigma)$, given as:

$$
\begin{align*}
I_{N}(\sigma) & =-E\left[\frac{\partial^{2} \mathcal{L}(y, \sigma)}{\partial \sigma^{2}}\right] \\
\operatorname{var}\left(\hat{\sigma}_{M L}\right) & \geq \frac{1}{I_{N}(\sigma)} \tag{4.37}
\end{align*}
$$

As previously obtained, $\frac{\partial \mathcal{L}(y, \sigma)}{\partial \sigma}=-\frac{2 \alpha^{\prime} N}{\sigma}+\frac{2 \beta^{\prime}}{\sigma^{3}} \sum_{n=0}^{N-1} y_{n}$, so:

$$
\frac{\partial^{2} \mathcal{L}(y, \sigma)}{\partial \sigma^{2}}=\frac{2 \alpha^{\prime} N}{\sigma^{2}}-\frac{6 \beta^{\prime}}{\sigma^{4}} \sum_{n=0}^{N-1} y_{n} .
$$

As $E\left[y_{n}\right]=\frac{\alpha^{\prime}}{\beta^{\prime}} \sigma^{2}{ }^{39}$ one gets:

$$
I_{N}(\sigma)=-E\left[\frac{\partial^{2} \mathcal{L}(y, \sigma)}{\partial \sigma^{2}}\right]=-\frac{2 \alpha^{\prime} N}{\sigma^{2}}+\frac{6 \beta^{\prime}}{\sigma^{4}} \frac{\alpha^{\prime} N}{\beta^{\prime}} \sigma^{2}=\frac{4 \alpha^{\prime} N}{\sigma^{2}} .
$$

Therefore, the variance of the estimate is:

$$
\begin{equation*}
\operatorname{var}\left(\hat{\sigma}_{M L}\right) \geq \frac{\sigma^{2}}{4 \alpha^{\prime} N} \tag{4.38}
\end{equation*}
$$

Although this result is an approximation, we can see the close match in Figure 4.9, where the variance obtained by simulation is plotted alongside equation (4.38). We note that this theoretical expression for the variance was obtained using the approximation case and not the exact one. If computed from this last case, it would always have to be less than the simulation results.

[^27]

Figure 4.9 - Variance of $\widehat{\sigma}_{M L}$ obtained by simulation and theoretical analysis.

### 4.2.2 The case when $K=0$ (Similar to Welch's Algorithm over the Cyclic Frequency axis)

When computing the SCF for $\mathrm{K}=0$, the FAM algorithm is similar to Welch's algorithm but now applied to the cycle frequency axis. In this case, the quadratic form matrix is given by (from (4.9)):

$$
\operatorname{diag}(\boldsymbol{a}) \boldsymbol{C}_{0, \Delta} \operatorname{diag}(\boldsymbol{a})
$$

where

$$
\begin{equation*}
C_{0, \Delta}=\cos \left(\frac{\pi \Delta}{N p S e g}(n+m)\right)-j \sin \left(\frac{\pi \Delta}{N p S e g}(n+m)\right) . \tag{4.39}
\end{equation*}
$$

Now, and differently from the $\Delta=0$ case, the quadratic form generates complex estimates. Resulting from a sin/cos pair, the eigenvalues of the real and imaginary components will be equal. Further, and following a similar process as developed in the previous section, for an odd number of averaging segments, rectangular window, and even $\Delta$, the eigenvalues of the quadratic form matrix, $\boldsymbol{G}_{0, \Delta}$, will come in $(+\lambda ;-\lambda)$ pairs. From equation (4.10), the MFG is:

$$
M(s)=\prod_{i=1}^{D}\left(\frac{1}{1-2 \lambda_{i} s}\right)^{\frac{\nu_{i}}{2}}
$$

For the eigenvalue pattern, with multiplicity equal to $\nu_{i}=1$, this becomes:

$$
\begin{align*}
M(s) & =\prod_{i=1}^{D / 2}\left(\frac{1}{1-2 \lambda_{i} s}\right)^{\frac{1}{2}}\left(\frac{1}{1+2 \lambda_{i} s}\right)^{\frac{1}{2}} \\
& =\prod_{i=1}^{D / 2}\left(\frac{1}{1-\left(2 \lambda_{i}\right)^{2} s^{2}}\right)^{\frac{1}{2}} \tag{4.40}
\end{align*}
$$

In Appendix D, we show that a general MGF of the form:

$$
M(s)=\prod_{i=1}^{D / 2}\left(\frac{d_{i}^{2}}{d_{i}^{2}-s^{2}}\right)^{h_{i}}
$$

can be expressed as

$$
M(s)=g_{1} \sum_{i=1}^{D / 2} \sum_{j=1}^{h_{i}} \frac{A_{i, j}}{\left(d_{i}^{2}-s^{2}\right)^{j}}
$$

where $h_{i}=\nu_{i} / 2 ; d_{i}=1 / 2 \lambda_{i} ; g_{1}=\prod_{i=1}^{D / 2}\left(d_{i}^{2}\right)^{h_{i}}=\prod_{i=1}^{D / 2}\left(\left(2 \lambda_{i}\right)^{2}\right)^{-h_{i}}$. The corresponding pdf is, as shown:

$$
\begin{equation*}
f_{Y}(y)=g_{1} \sum_{i=1}^{D / 2} \sum_{j=1}^{h_{i}} A_{i j} \frac{|y|^{j-\frac{1}{2}}}{\sqrt{\pi} \Gamma(j)\left(2 d_{i}\right)^{j-\frac{1}{2}}} K_{j-\frac{1}{2}}\left(d_{i}|y|\right) \tag{4.41}
\end{equation*}
$$

As with equation (4.16) in the previous section, equation (4.41) will also have numerical problems on the exponential component when the matrix has a high number of eigenvalues. Therefore, and following the same approach as in the previous section, we will now approximate the MGF by:

$$
\begin{align*}
M(s)= & \prod_{i=1}^{D / 2}\left(\frac{d_{i}^{2}}{d_{i}^{2}-s^{2}}\right)^{h_{i}} \approx\left(\frac{\alpha^{2}}{\alpha^{2}-s^{2}}\right)^{\lambda} \\
& M_{1}(s)=\left(\frac{\alpha^{2}}{\alpha^{2}-s^{2}}\right)^{\lambda} \tag{4.42}
\end{align*}
$$

Equation (4.42) is a particular case of the MGF of a variance-gamma distribution. The general variance-gamma MGF and corresponding pdf are given by (Gaunt, 2014):

$$
\begin{equation*}
f_{Y}(y)=\frac{\left(\alpha^{2}-\beta^{2}\right)^{\lambda}}{\sqrt{\pi} \Gamma(\lambda)(2 \alpha)^{\lambda-\frac{1}{2}}}|y-\mu|^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}(\alpha|y-\mu|) e^{\beta|y-\mu|} \tag{4.43}
\end{equation*}
$$

with MGF:

$$
\begin{equation*}
M_{Y}(s)=\left(\frac{\alpha^{2}-\beta^{2}}{\alpha^{2}-(\beta+s)^{2}}\right)^{\lambda} e^{\mu s} \tag{4.44}
\end{equation*}
$$

Therefore, the approximation given by equation (4.42) corresponds to a variance gamma distribution with $\beta=0$ and $\mu=0$.

As in the previous section, we will compute the moments using the MGF and its approximation as given by (4.42), forcing equality of successive moments until the parameters $\alpha$ and $\lambda$ can be obtained.

Using the log-derivative with $M(s)$, one can obtain:

$$
\begin{aligned}
& M(s)=\prod\left(\frac{d_{i}^{2}}{d_{i}^{2}-s^{2}}\right)^{\frac{1}{2}} \\
& \frac{\partial}{\partial s} M(s)= M(s) \sum \frac{s}{d_{i}^{2}-s^{2}} \\
& \frac{\partial^{2}}{\partial s^{2}} M(s)=\left(\frac{\partial}{\partial s} M(s)\right) \sum \frac{s}{d_{i}^{2}-s^{2}}+M(s) \sum \frac{d_{i}^{2}+s^{2}}{\left(d_{i}^{2}-s^{2}\right)^{2}} \\
& \frac{\partial^{3}}{\partial s^{3}} M(s)=\left(\frac{\partial^{2}}{\partial s^{2}} M(s)\right) \sum \frac{s}{d_{i}^{2}-s^{2}}+2\left(\frac{\partial}{\partial s} M(s)\right) \sum \frac{d_{i}^{2}+s^{2}}{\left(d_{i}^{2}-s^{2}\right)^{2}}+M(s) \sum \frac{2 s\left(3 d_{i}^{2}+s^{2}\right)}{\left(d_{i}^{2}-s^{2}\right)^{3}} \\
& \frac{\partial^{4}}{\partial s^{4}} M(s)=\left(\frac{\partial^{3}}{\partial s^{3}} M(s)\right) \sum \frac{s}{d_{i}^{2}-s^{2}}+3\left(\frac{\partial^{2}}{\partial s^{2}} M(s)\right) \sum \frac{d_{i}^{2}+s^{2}}{\left(d_{i}^{2}-s^{2}\right)^{2}} \\
&+3\left(\frac{\partial}{\partial s} M(s)\right) \sum \frac{2 s\left(3 d_{i}^{2}+s^{2}\right)}{\left(d_{i}^{2}-s^{2}\right)^{3}}+M(s) \sum \frac{6\left(d_{i}^{4}+6 d_{i}^{2} s^{2}+s^{4}\right)}{\left(d_{i}^{2}-s^{2}\right)^{4}}
\end{aligned}
$$

As for $M_{1}(s)$ one gets:

$$
\begin{aligned}
M_{1}(s)= & \left(\frac{\alpha^{2}}{\alpha^{2}-s^{2}}\right)^{\lambda} \\
\frac{\partial}{\partial s} M_{1}(s)= & M_{1}(s) \frac{2 \lambda s}{\alpha^{2}-s^{2}} \\
\frac{\partial^{2}}{\partial s^{2}} M_{1}(s)= & \left(\frac{\partial}{\partial s} M_{1}(s)\right) \frac{2 \lambda s}{\alpha^{2}-s^{2}}+M_{1}(s) \frac{2 \lambda\left(\alpha^{2}+s^{2}\right)}{\left(\alpha^{2}-s^{2}\right)^{2}} \\
\frac{\partial^{3}}{\partial s^{3}} M_{1}(s)= & \left(\frac{\partial^{2}}{\partial s^{2}} M_{1}(s)\right) \frac{2 \lambda s}{\alpha^{2}-s^{2}}+2\left(\frac{\partial}{\partial s} M_{1}(s)\right) \frac{2 \lambda\left(\alpha^{2}+s^{2}\right)}{\left(\alpha^{2}-s^{2}\right)^{2}}+M_{1}(s) \frac{4 \lambda s\left(3 \alpha^{2}+s^{2}\right)}{\left(\alpha^{2}-s^{2}\right)^{3}} \\
\frac{\partial^{4}}{\partial s^{4}} M_{1}(s)= & \left(\frac{\partial^{3}}{\partial s^{3}} M_{1}(s)\right) \frac{2 \lambda s}{\alpha^{2}-s^{2}}+3\left(\frac{\partial^{2}}{\partial s^{2}} M_{1}(s)\right) \frac{2 \lambda\left(\alpha^{2}+s^{2}\right)}{\left(\alpha^{2}-s^{2}\right)^{2}} \\
& +3\left(\frac{\partial}{\partial s} M_{1}(s)\right) \frac{4 \lambda s\left(3 \alpha^{2}+s^{2}\right)}{\left(\alpha^{2}-s^{2}\right)^{3}}+M_{1}(s) \frac{12 \lambda\left(\alpha^{4}+6 \alpha^{2} s^{2}+s^{4}\right)}{\left(\alpha^{2}-s^{2}\right)^{4}}
\end{aligned}
$$

Table 4.2 shows the results obtained for the moments up to order four, alongside the variance, skewness, and kurtosis, when using $M(s)$ and $M_{1}(s)$.

| Moment | By $\boldsymbol{M}(\boldsymbol{s})$ | By $\boldsymbol{M}_{\mathbf{1}}(\boldsymbol{s})$ |
| :---: | :---: | :---: |
| $E[Y]$ | 0 | 0 |
| $E\left[Y^{2}\right]$ | $\sum \frac{1}{d_{i}^{2}}$ | $\frac{2 \lambda}{\alpha^{2}}$ |
| $E\left[Y^{3}\right]$ | 0 | 0 |
| $E\left[Y^{4}\right]$ | $3\left(\sum \frac{1}{d_{i}^{2}}\right)^{2}+6 \sum \frac{1}{d_{i}^{4}}$ | $\frac{12 \lambda(\lambda+1)}{\alpha^{4}}$ |
| $\operatorname{var}(Y)$ <br> $=E\left[Y^{2}\right]-(E[Y])^{2}$ | $\sum \frac{1}{d_{i}^{2}}$ | $\frac{2 \lambda}{\alpha^{2}}$ |
| Skewness $(Y)$ <br> $=E\left[\left(\frac{Y-E[Y]}{\sqrt{\operatorname{var}(Y)}}\right)^{3}\right]$ | 0 | 0 |
| Kurtosis $(Y)$ <br> $=E\left[\left(\frac{Y-E[Y]}{\sqrt{\operatorname{var}(Y)}}\right)^{4}\right]$ | $\frac{3\left(\sum \frac{1}{d_{i}^{2}}\right)^{2}+6 \sum \frac{1}{d_{i}^{4}}}{\left(\sum \frac{1}{d_{i}^{2}}\right)^{2}}$ | $3\left(1+\frac{1}{\lambda}\right)$ |

Table 4.2 - Moments obtained using $M(s)$ and $M_{1}(s)$.
Using the moments of order two and four (and more straightforward with the variance and the kurtosis), one gets:

$$
\begin{aligned}
\operatorname{Kurtosis}(Y) & =\frac{3\left(\sum \frac{1}{d_{i}^{2}}\right)^{2}+6 \sum \frac{1}{d_{i}^{4}}}{\left(\sum \frac{1}{d_{i}^{2}}\right)^{2}}=3\left(1+\frac{1}{\lambda}\right) \\
\operatorname{var}(Y) & =\sum \frac{1}{d_{i}^{2}}=\frac{2 \lambda}{\alpha^{2}}
\end{aligned}
$$

and so:

$$
\begin{align*}
& \lambda=\frac{\left(\sum \frac{1}{d_{i}^{2}}\right)^{2}}{2 \sum \frac{1}{d_{i}^{4}}} \\
& \alpha=\sqrt{\frac{\sum \frac{1}{d_{i}^{2}}}{\sum \frac{1}{d_{i}^{4}}}} \tag{4.45}
\end{align*}
$$

Figure 4.10 shows the pdf obtained by simulation $\left(2^{20}\right.$ runs, $N p S e g=64, P=33$ and $(K ; \Delta)=$
$(0 ; 4))$ along with the theoretical result obtained by the MGF $M(s)$ and by using the proposed approximation for the MGF, $M_{1}(s)$. Figure 4.11 shows the cdf (in logarithmic scale) obtained for the same cases. As can be seen, the approximation is quite good, with a negligible difference around the maximum of the pdf and for all the cdf. We again note that this approximation has no numerical computational problems, whereas when using (4.41), numerical issues arise even for low $P$.

Figure 4.12 shows the Characteristic Functions $\left(\varphi(w)\right.$ and $\varphi_{1}(w)$, both real functions) corresponding to $M(s)$ and $M_{1}(s)$, which can be seen to be in close agreement.


Figure 4.10 - pdf obtained by simulation, by $M(s)$ and by $M_{1}(s)$ for $N p S e g=64, P=33$ and $(K, \Delta)=(0,4)$.


Figure 4.11 - cdf (in logarithmic scale) obtained by simulation, by $M(s)$ and by $M_{1}(s)$ for $N p S e g=64$, $P=33$ and $(K, \Delta)=(0,4)$.


Figure 4.12 - Characteristic Functions $\left(\varphi(w)\right.$ and $\left.\varphi_{1}(w)\right)$ corresponding to $M(s)$ and $M_{1}(s)$ for

$$
N p S e g=64, P=33 \text { and }(K, \Delta)=(0,4) .
$$

### 4.2.3 The case when $K=\Delta$

In this case, the quadratic form matrix is given by (from (4.9)):

$$
\operatorname{diag}(\boldsymbol{a}) \boldsymbol{C}_{K, K} \operatorname{diag}(\boldsymbol{a})
$$

where

$$
\begin{gather*}
C_{K, K}=\cos \left(\frac{\pi K}{N p S e g}(n-m)\right) \mathrm{e}^{-j \frac{\pi K}{N p S e g}(n+m)}  \tag{4.46}\\
=\frac{1}{2}\left[\cos \left(\frac{2 \pi K}{N p S e g} n\right)+\cos \left(\frac{2 \pi K}{N p S e g} m\right)\right]-\frac{j}{2}\left[\sin \left(\frac{2 \pi K}{N p S e g} n\right)+\sin \left(\frac{2 \pi K}{N p S e g} m\right)\right]
\end{gather*}
$$

Here, and as in the $K=0$ case, the quadratic form generates complex estimates. The real and imaginary matrix obtained from (4.46) has two eigenvalues in the form $(+\lambda ;-\lambda)$. Similarlly to the $K=0$ case, for an odd number of averaging segments, rectangular window, and even $K$ and even $\Delta$, the eigenvalues of the quadratic form matrix, $\boldsymbol{G}_{K, \Delta}$, will come in $(+\lambda ;-\lambda)$ pairs. Therefore, the resulting approximation of the MGF leads to a variance gamma distribution with the $\alpha$ and $\lambda$ parameters estimated by equation (4.45).

Figure 4.13 shows the pdf obtained by simulation ( $2^{20}$ runs, $N p S e g=64, P=33$ and $(K ; \Delta)=$ $(4 ; 4))$ along with the theoretical result obtained by the MGF $M(s)$ and by using the proposed approximation for the MGF, $M_{1}(s)$ as given by (4.42). Figure 4.14 shows the cdf (in logarithmic scale) obtained for the same cases. As in the case $K=0$, the approximation is quite good, with a negligible difference obtained for the pdf and cdf.


Figure 4.13 - pdf obtained by simulation, by $M(s)$ and by $M_{1}(s)$ for $N p S e g=64, P=33$ and

$$
(K, \Delta)=(4,4)
$$



Figure 4.14 - cdf (in logarithmic scale) obtained by simulation, by $M(s)$ and by $M_{1}(s)$ for $N p S e g=64$,

$$
P=33 \text { and }(K, \Delta)=(4,4) .
$$

### 4.2.3.1 The subcase when $\mathrm{K}=\Delta=\mathrm{NpSeg} / 2$

In this case, the quadratic form matrix is given by (from (4.46)):

$$
\begin{align*}
\boldsymbol{C}_{K, K} & =\frac{1}{2}[\cos (\pi n)+\cos (\pi m)]-\frac{j}{2}[\sin (\pi n)+\sin (\pi m)] \\
& =\frac{1}{2}[\cos (\pi n)+\cos (\pi m)] \tag{4.47}
\end{align*}
$$

This case is equal to $K=\Delta$, but the quadratic form generates only real estimates. The matrix obtained from (4.47) has two eigenvalues in the $(+\lambda ;-\lambda)$ form. As in the $K=\Delta$ case, for an odd number of averaging segments and rectangular window, the eigenvalues of the quadratic form matrix, $\boldsymbol{G}_{\frac{N p S e g}{2}, \frac{N p S_{e g}}{2}}$, will come in $(+\lambda ;-\lambda)$ pairs, and the resulting approximation of the MGF leads to a variance gamma distribution with the $\alpha$ and $\lambda$ parameters estimated by equation (4.45).

### 4.2.4 All other cases $(\mathrm{K} \neq 0, \Delta \neq 0, K \neq \Delta)$

In the general case that this section addresses, the quadratic form matrix is given by (from (4.9)):

$$
\operatorname{diag}(\boldsymbol{a}) \boldsymbol{C}_{K, \Delta} \operatorname{diag}(\boldsymbol{a})
$$

where

$$
\begin{align*}
& \boldsymbol{C}_{K, \Delta}=\cos \left(\frac{\pi K}{N p S e g}(n+m)\right) \boldsymbol{C}_{0, \Delta}  \tag{4.48}\\
& =\cos \left(\frac{\pi K}{N p S e g}(n+m)\right)\left[\cos \left(\frac{\pi \Delta}{N p S e g}(n+m)\right)-j \sin \left(\frac{\pi \Delta}{N p S e g}(n+m)\right)\right] .
\end{align*}
$$

For an odd number of averaging segments, rectangular window, even $K$, and even $\Delta$, the eigenvalues of the quadratic form matrix, $\boldsymbol{G}_{K, \Delta}$, will come in $(+\lambda ;+\lambda ;-\lambda ;-\lambda)$ pairs. From equation (4.10), the MFG is:

$$
M(s)=\prod_{i=1}^{D}\left(\frac{1}{1-2 \lambda_{i} s}\right)^{\frac{\nu_{i}}{2}}
$$

For the eigenvalue pattern, with multiplicity equal to $\nu_{i}=2$ (so $h_{i}=1$ ), this becomes:

$$
\begin{align*}
M(s) & =\prod_{i=1}^{D / 2}\left(\frac{1}{1-2 \lambda_{i} s}\right)\left(\frac{1}{1+2 \lambda_{i} s}\right) \\
& =\prod_{i=1}^{D / 2}\left(\frac{1}{1-\left(2 \lambda_{i}\right)^{2} s^{2}}\right) . \tag{4.49}
\end{align*}
$$

Comparing equation (4.49) with the MGF obtained for the case $K=0$ (equation (4.40)) we see that they are identical when squaring each term of (4.40).

As in the case of $K=0$, the pdf of the quadratic form is given by (4.41). Making a similar approximation of the MGF:

$$
M(s)=\prod_{i=1}^{D / 2}\left(\frac{d_{i}^{2}}{d_{i}^{2}-s^{2}}\right)^{h_{i}} \approx\left(\frac{\alpha^{2}}{\alpha^{2}-s^{2}}\right)^{\lambda}
$$

and

$$
\begin{equation*}
M_{1}(s)=\left(\frac{\alpha^{2}}{\alpha^{2}-s^{2}}\right)^{\lambda} \tag{4.50}
\end{equation*}
$$

will lead to the same result in estimating the $\alpha$ and $\lambda$ parameters of the variance-gamma distribution, only with $\lambda$ being twice the value obtained by equation (4.45) (in the previous case $h_{i}$ was $1 / 2$, and now $h_{i}=1$ ). Therefore, the parameters $\alpha$ and $\lambda$ are given by:

$$
\begin{align*}
& \lambda=\frac{\left(\sum \frac{1}{d_{i}^{2}}\right)^{2}}{\sum \frac{1}{d_{i}^{4}}} \\
& \alpha=\sqrt{\frac{\sum \frac{1}{d_{i}^{2}}}{\sum \frac{1}{d_{i}^{4}}}} \tag{4.51}
\end{align*}
$$

Figure 4.15 shows the pdf obtained by simulation ( $2^{20}$ runs, $N p S e g=64, P=33$ and $(K ; \Delta)=$ $(12 ; 4)$ ) along with the theoretical result obtained by the MGF $M(s)$ and by using the proposed approximation for the MGF, $M_{1}(s)$. Figure 4.16 shows the cdf (in logarithmic scale) obtained for the same cases. As can be seen, the approximation is quite good, with a negligible difference for the pdf and the cdf.

Figure 4.17 shows the Characteristic Functions $\left(\varphi(w)\right.$ and $\varphi_{1}(w)$, both real functions) corresponding to $M(s)$ and $M_{1}(s)$, which can be seen to be in close agreement.


Figure 4.15 - pdf obtained by simulation, by $M(s)$ and by $M_{1}(s)$ for $N p S e g=64, P=33$ and $(K, \Delta)=(12,4)$.


Figure 4.16 - cdf (in logarithmic scale) obtained by simulation, by $M(s)$ and by $M_{1}(s)$ for $N p S e g=64$, $P=33$ and $(K, \Delta)=(12,4)$.


Figure 4.17 - Characteristic Functions $\left(\varphi(w)\right.$ and $\left.\varphi_{1}(w)\right)$ corresponding to $M(s)$ and $M_{1}(s)$ for $N p S e g=64, P=33$ and $(K, \Delta)=(12,4)$.

### 4.3 Approximate pdf of the SCF and SCoF for a high number of averaging segments

It is a well-known result that when the Fisher Transformation (or Fisher z-transformation) (Fisher, 1928) is applied to the sample correlation coefficient between two random variables, the resulting variable is approximately normal (Nuttall and Carter, 1981; Bortel and Sovka, 2007; Miles, 2011). Considering $r$ the sample correlation coefficient, the Fisher z-transformation is given as:

$$
\begin{equation*}
z=\frac{1}{2} \ln \left(\frac{1+r}{1-r}\right)=\operatorname{atanh}(r) \tag{4.52}
\end{equation*}
$$

The same result is obtained when considering the coherence between two random variables, $x$ and $y$, given by

$$
\gamma_{x y}(f)=\frac{\left|G_{x y}(f)\right|}{G_{x x}(f) G_{y y}(f)},
$$

where $G_{x y}(f)$ is the cross-power spectrum function (Bortel and Sovka, 2007; Miles, 2011).
Although, to our best knowledge, there is no known reference to the application of the Fisher ztransformation to the Spectral Coherence Function (SCoF), we conjecture that the result obtained for the correlation coefficient, or the coherence, should hold for the SCoF real and imaginary components.

The SCoF, the Fisher z-transformation of the SCoF, and the SCF calculated herein will be compared to normal distributions using the standard statistical analysis tools: histograms, quantilequantile plots (Q-Q plots), and Probability-Probability plots (P-P plots) as referenced in f

For the histogram, the sample variance, $\sigma$, is computed from the input signal, and the simulation results are plotted alongside a normal distribution, $\mathcal{N}\left(0, \sigma^{2}\right)$. For the case of the SCoF, this normal distribution is computed only for the $[-1 ;+1]$ interval. Similarly, the Q-Q and P-P plots also use this variance estimate, so that, if the sample distribution is close to a normal distribution, the points in these plots should lay on a straight line (see Appendix E for details).

In Appendix E, thorough simulation results are presented to assess the validity of this conjecture. For convenience, a few results will be replicated in this section, illustrating the conclusions obtained. The figures presented show the results obtained for several points in the $(f ; \alpha)$ plane, as indicated in Figure E. 1 (Appendix E). When the number of segments $(P)$ used to compute the SCoF is low, the sample distribution does not fit a theoretical normal distribution. This is also the case when addressing the SCoF Fisher z-transformation. This is easily seen in Figure 4.18 and Figure 4.19, for low $P(P=17)$, for the SCoF and its Fisher z-transformation, respectively.

From the results obtained in Appendix E, the SCoF sample distribution would be approximately normal for $P \geq 129$ (or conservatively, $P \geq 257$ ). In the case of the SCoF Fisher z-transformation,
this assumption would be feasible for $P \geq 33$ (or conservatively, $P \geq 65$ ). This can be seen in Figure 4.20 and Figure 4.21, for high $P(P=129)$, for the SCoF and its Fisher z-transformation, respectively.

Finally, for the SCF, and for low $P$, the sample distribution does not fit a theoretical normal distribution, as can be seen in Figure $4.22(P=33)$. Nevertheless, this would possibly be accepted for $P \geq 129$ (or conservatively, $P \geq 257$ or even, $P \geq 513$ ), as shown in Figure 4.23 for $P=257$.




Figure 4.18 - Histogram, Q-Q plot, and P-P plot of the real part of the $\mathrm{SCoF}, P=17, N p S e g=128$.



Figure 4.19 - Histogram, Q-Q plot, and P-P plot of the real part of the SCoF Fisher z-transformation,

$$
P=17, N p S e g=128
$$





Figure 4.20 - Histogram, Q-Q plot, and P-P plot of the real part of the $\mathrm{SCoF}, P=129, N p S e g=128$.



Figure 4.21 - Histogram, Q-Q plot, and P-P plot of the real part of the SCoF Fisher z-transformation, $P=129, N p S e g=128$.




Figure 4.22 - Histogram, Q-Q plot, and P-P plot of the real part of the $\mathrm{SCF}, P=33, N p S e g=128$.




Figure 4.23 - Histogram, Q-Q plot, and P-P plot of the real part of the SCF, $P=257, N p S e g=128$.

Given the SCoF estimates, $C$, its Fisher z-transformation, $Z$, will have an approximately normal distribution, $\mathcal{N}\left(0, \sigma^{2}\right)$, so that:

$$
f_{Z}(z)=\frac{1}{\sqrt{2 \pi} \sigma} e^{-\frac{z^{2}}{2 \sigma^{2}}}
$$

As $Z=$ atanh $(C)$, and using the normal approximation for $Z$, then the SCoF will have a pdf given by:

$$
\begin{equation*}
f_{C}(c)=\frac{1}{\left(1-c^{2}\right) \sqrt{2 \pi} \sigma} e^{-\frac{\operatorname{atanh}^{2}(c)}{2 \sigma^{2}}} \tag{4.53}
\end{equation*}
$$

As the number of averaging segments increases, the variance of $Z$ (and $C$ ) will decrease so that the pdf of $Z$ will progressively be in a narrow neighborhood around $z=0$. In this case, $\operatorname{atanh}(C) \approx C$, and so the pdf of $C$ will be approximately given by the normal distribution $\mathcal{N}\left(0, \sigma^{2}\right)$. Figure 4.24 illustrates this behavior for $N p S e g=128$ in one $(f ; \alpha) \operatorname{point}(K ; \Delta)=$ (112,16): for low $P,(P=17)$ the histograms have distinct behavior, especially on the tails; for high $P,(P=129)$ the histograms are almost identical.


Figure 4.24 - Histograms for the SCoF and its Fisher z-transformation, $P=17$ and $P=129, N p S e g=$ $128,(K, \Delta)=(112,16)$.

### 4.3.1 SCF and SCoF Variance estimation

When approximating the SCF estimates (real and imaginary components) by a Gaussian density, which is feasible for high $P$ and $\Delta \neq 0$, the mean and variance can be estimated directly by equation (4.11), as the SCF is obtained directly from the quadratic form. Therefore, obtaining the eigenvalues of the quadratic form matrix allows direct computing of the SCF variance, provided the input noise variance is known or estimated.

For the SCoF, there is no direct estimate available as it is obtained by the ratio of estimates that are not independent (equation (3.21))

$$
C_{x}^{\alpha}(f)=\frac{S_{x}^{\alpha}(f)}{\left[S_{x}^{0}(f+\alpha / 2) S_{x}^{0}(f-\alpha / 2)\right]^{1 / 2}} .
$$

As seen in the previous section, for $\Delta \neq 0$, the $\operatorname{SCF} S_{x}^{\alpha}(f)$, can be approximated by a variance gamma density, and, for $\Delta=0$, the $\operatorname{SCF} S_{x}^{0}(f+\alpha / 2)$ or $S_{x}^{0}(f-\alpha / 2)$ by a gamma density.

When these three components are independent, the resulting $C_{x}^{\alpha}(f)$ can be obtained as shown in Appendix A, Demonstration 4.5. A possible estimate of the SCoF variance, when approximated by a Gaussian density, is to use the resulting density when the components are independent, at the origin, and equaling to the Gaussian pdf also at the origin. Using equation (A.9) leads to:

$$
\frac{\alpha_{x}}{2 \sqrt{\pi} \alpha_{y}}\left(\frac{\Gamma\left(\lambda_{y}+\frac{1}{2}\right)}{\Gamma\left(\lambda_{y}\right)}\right)^{2} \frac{\Gamma\left(\lambda_{x}-\frac{1}{2}\right)}{\Gamma\left(\lambda_{x}\right)}=\frac{1}{\sqrt{2 \pi} \sigma_{S C o F}}
$$

where $\lambda_{x}$ and $\alpha_{x}$ are the parameters of the variance gamma density of the $S_{x}^{\alpha}(f)$ component approximation, and the $\lambda_{y}$ and $\alpha_{y}$ are the parameters of the gamma density approximation of
both $S_{x}^{0}(f+\alpha / 2)$ or $S_{x}^{0}(f-\alpha / 2){ }^{40}$. Therefore, the standard deviation of the SCoF would be estimated as:

$$
\begin{equation*}
\sigma_{S C o F}=\frac{\sqrt{2} \alpha_{y}}{\beta}\left(\frac{\Gamma\left(\lambda_{y}\right)}{\Gamma\left(\lambda_{y}+\frac{1}{2}\right)}\right)^{2} \frac{\Gamma(\alpha)}{\Gamma\left(\alpha-\frac{1}{2}\right)} \tag{4.54}
\end{equation*}
$$

Figure 4.25 shows the results obtained for the standard deviation of the SCoF when using (4.54) and by simulation results, for varying $P$, and for $\operatorname{NpSeg}=128,(K, \Delta)=(50,25)$. In Figure 4.26 the corresponding relative error is drawn for the same scenario. As can be seen, the agreement is quite good, with a relative error of less than $1 \%$ for $P \geq 33$.


Figure 4.25 - Comparison of $\sigma_{S C o F}$ obtained by (4.54) and by simulation for varying $P(N p S e g=128$,

$$
(K, \Delta)=(50,25)) .
$$

[^28]

Figure 4.26 - Relative error of $\sigma_{\text {SCoF }}$ obtained by (4.54) and by simulation for varying $P(N p S e g=128$, $(K, \Delta)=(50,25))$.

Although equation (4.54) can be used in general, for large $P$ it will be computationally complex because the quadratic form matrix will present a large number of eigenvalues.

We will obtain another simpler approximation for the SCoF variance following another perspective.
When $L=0$, there will be no smoothing in the computation of the SCF, and all segments will be equal, so the Coherence Function ( SCoF ) will have its absolute value always equal to 1, i.e., $\left|C_{x}^{\alpha}(f)\right|=1$. Therefore $C=C_{x}^{\alpha}(f)=e^{j \theta}$, where $\theta$ is uniformly distributed: $\theta \sim \mathcal{U}(0,2 \pi)$.

In this case, the real and imaginary parts of the SCoF are given by:

$$
\begin{gather*}
R=\operatorname{Re}\{C\}=\cos (\theta) \\
I=\operatorname{Im}\{C\}=\sin (\theta) \tag{4.55}
\end{gather*}
$$

and the marginal densities can be obtained as (Appendix A, Demonstration 4.1):

$$
\begin{equation*}
f_{R}(r)=f_{I}(i)=\frac{1}{\pi \sqrt{1-r^{2}}}, \quad-1<r<1 . \tag{4.56}
\end{equation*}
$$

Using equation (4.56), one gets:

$$
\begin{align*}
& E[\operatorname{Re}\{C\}]=E[\operatorname{Im}\{C\}]=0 \\
& \operatorname{var}[\operatorname{Re}\{C\}]=\operatorname{var}[\operatorname{Im}\{C\}]=\frac{1}{2} . \tag{4.57}
\end{align*}
$$

When $L=N p S e g$, there is no overlap between segments. For large $P$, the SCoF will be approximated by a Gaussian density, as shown previously. Therefore, one could argue that the real and imaginary SCoF estimates could be obtained as a mean of $P$ independent variables with variance equal to when $L=0$, that is $1 / 2$. In this case, we would get:

$$
\begin{align*}
& E[\operatorname{Re}\{C\}]=E[\operatorname{Im}\{C\}]=0 \\
& \operatorname{var}[\operatorname{Re}\{C\}]=\operatorname{var}[\operatorname{Im}\{C\}]=\frac{1}{2 P} \tag{4.58}
\end{align*}
$$

Figure 4.27 shows the simulation results obtained for the real part of the SCoF for varying $P(P=$ $5,9,17,33,65,129,257,513,1025)$ and $N p S e g=128$, for all possible values of the overlap $L=$ $0 \ldots \mathrm{NpSeg}$. Here we opted to graph the standard deviation and not the variance for better image perception. As can be seen, the previous analysis made for $L=0$ and $L=N p S e g$ is confirmed (the mean relative error obtained for $L=0$ is $0.0001 \%$ and $0.03 \%$ for $L=N p S e g$ ).

Although no further analysis can be made for other overlap values, we will propose a polynomial function to approximate the variance as:

$$
\begin{equation*}
\operatorname{var}[\operatorname{Re}\{C\}] \approx \frac{1}{2} \frac{1+l^{2}}{1+(2 P-1) \times l} \tag{4.59}
\end{equation*}
$$

where $l=L / N p S e g$. Equation (4.59) will give the exact same value as obtained in (4.57) and (4.58) for the $l=0$ and $l=1$ cases, respectively.

Equation (4.59) is also plotted in Figure 4.27 alongside the simulation results obtained for $N p S e g=128,(K ; \Delta)=(50 ; 25)$, and varying $P(P=5,9,17,33,65,129,257,513,1025)$. Again, for large $P$, the proposed approximation is quite accurate. Figure 4.28 shows the relative error when estimating the SCoF standard deviation as a function of the overlap (as given by equation (4.59)) compared to the simulation results. As can be seen, the relative error is low, and, in the case of the FAM algorithm where $L / N p S e g=0.25$, this error is (in absolute value) less than $1 \%$.

The same relative error is also plotted in Figure 4.29 for $N p S e g=64,128,256$ and for $P=$ 5,9,33. Analysis of this figure shows that the results obtained are almost independent of NpSeg (higher values of $P$ where not drawn because they are similar to the $P=33$ case.


Figure 4.27 - Approximate SCoF standard deviation as a function of the overlap for $\mathrm{NpSeg}=128$ and varying $P((K ; \Delta)=(50 ; 25))$.


Figure 4.28 - Relative Error of the SCoF standard deviation as a function of the overlap for $\mathrm{NpSeg}=$ 128 and varying $P((K ; \Delta)=(50 ; 25))$.


Figure 4.29 - Relative Error of the SCoF standard deviation as a function of the overlap for several $P$ and $\operatorname{NpSeg}((K ; \Delta)=(50 ; 25))$.

Finally, in Figure 4.30 and Figure 4.31, we show the comparative results (absolute and relative error) obtained by equations (4.54) and (4.59), alongside simulation results for varying $P$, with $N p S e g=128$ and $(K ; \Delta)=(50 ; 25)$. In this case, when using equation (4.59), the value for $l$ is 0.25 as used in the FAM algorithm. As can be seen, both results are similar, with very close agreement with the simulation results, for high $P$.


Figure 4.30 - Comparison of $\sigma_{\text {SCoF }}$ obtained by (4.54), (4.59), and by simulation for varying $P$ $(N p$ Seg $=128,(K, \Delta)=(50,25))$.


Figure 4.31 - Relative error of $\sigma_{\text {SCoF }}$ obtained by (4.54), (4.59), and by simulation for varying $P$ $(N p$ Seg $=128,(K, \Delta)=(50,25))$.

Showing that modeling the real and imaginary parts of the SCoF as Gaussian with zero mean and variance given by (4.59) when the averaging segments, $P$, is high, is a major contribution of this work. This allows using the absolute value of the SCoF as a decision statistic to detect if some SoI is present, with a preset probability of false alarm $\left(P_{F A}\right)$.

### 4.4 Chapter Summary

This chapter presented the analysis of the FAM algorithm in a vectorized way, giving its quadratic form equivalent. Based on the eigenvalue analysis of the quadratic form matrix, exact expressions were obtained for the pdf of the SCF estimates produced by the FAM on the $(f ; \alpha)$ plane, when the input signal is a Gaussian $\mathcal{N}\left(0, \sigma^{2}\right)$ r.v.

Then, several approximations were given for the pdf obtained for low and medium number of averaging segments - avoiding the computational problems of the exact pdf - for the various ( $f ; \alpha$ ) plane regions.

Finally, extensive simulations were performed to show that the SCF and the SCoF could be modeled as a Gaussian r.v. when the number of averaging segments is high ( $P \geq 129$ or conservatively, $P \geq 257$ ) and for $\Delta \neq 0$ (for $\Delta=0$, the estimates should be approximately modeled as a gamma density). An estimate of the SCF variance was also given, using the quadratic form matrix eigenvalues.

In the case of the SCoF, two models were further given, to estimate its variance. One uses the parameters of the approximate densities of the SCF as variance gamma (for $K \neq 0, \Delta \neq 0$ ) and as gamma (for $\Delta=0$ ). The second model, a major contribution of this work, solely depends on the segment overlap and number of averaging segments and presents a similar agreement with the simulation results as the first one. The following chapter will use these models to develop and analyze cyclostationary signal detectors.

## Chapter 5

## 5 Cyclostationarity Detectors for Spectrum Sensing

Using the results obtained in the previous chapters, and specifically the statistic characterization of the estimates obtained by the FAM (or zFAM) algorithm for the SCF and SCoF, we will now propose two cyclostationarity detectors using the cumulative distribution function (cdf) of the estimates, with an adaptive threshold to guarantee a constant false alarm rate detection (CFAR detectors). Using the SCF, we will also propose a CDR detector with estimation of the SCF variance at the cycle feature location.

We will evaluate and compare the proposed detectors with other proposed cyclostationary and entropy detectors presented in the open literature. Also, several wireless channel models, namely AWGN with fading and shadowing channels, are considered.

### 5.1 Multipath Fading and Shadowing Channels

Multipath fading occurs when the transmitted signal reaches a receiver through several paths with different amplitudes and phases. The combined sum of these received signals, with a constructive or destructive interference effect, is the result of the multipath fading.

Several models represent the multipath fading behavior, depending on the wireless radio environment. We will consider the general Rayleigh (Sklar, 1997; Proakis, 2000), Rice (Rice, 1948; Proakis, 2000), and Nakagami-m (Nakagami, 1960; Proakis, 2000) models, and also the more recent $\kappa-\mu$ and $\eta-\mu$ fading models (Yacoub, 2007), and also the shadowing model, to characterize the random fluctuation of the received signal intensity. The first set of models considers the line-ofsight (LOS) and non-line-of-sight (NLOS) between emitter and receiver scenarios, in homogeneous outdoor and indoor environments. The second set considers also the LOS and NLOS scenarios, but also in a non-homogeneous and enclosed environments such as airplanes, buses, trains.

### 5.1.1 Rayleigh Fading Channel

The Rayleigh fading is typically used to model channels when there is no direct path between emitter and receiver (NLOS scenario) and when many objects scatter the emitter signal before it
arrives at the receiver. In this Rayleigh fading channel, the pdf of the signal envelope, $R$, can be given by:

$$
\begin{equation*}
f_{R}(r)=\frac{r}{\sigma^{2}} e^{-\frac{r^{2}}{2 \sigma^{2}}} \tag{5.1}
\end{equation*}
$$

where the mean power is $E\left[R^{2}\right]=2 \sigma^{2}$.

### 5.1.2 Rice Fading Channel

The Rice fading is commonly used to model channels when there is a direct path between emitter and receiver (LOS scenario) and when many objects scatter the emitter signal before it arrives at the receiver. In this Rice fading channel, the pdf of the signal envelope, $R$, is given by:

$$
\begin{equation*}
f_{R}(r)=\frac{r}{\sigma^{2}} e^{-\frac{r^{2}+\delta^{2}}{2 \sigma^{2}}} I_{o}\left(\frac{\delta r}{\sigma^{2}}\right) \tag{5.2}
\end{equation*}
$$

where $I_{0}($.$) is the modified Bessel function of the first kind with order zero, and the mean power$ is $E\left[R^{2}\right]=2 \sigma^{2}+\delta^{2}$. The $\delta$ and $\sigma$ are, respectively, the non-centrality and scale parameters. The Rician $K$-factor, is defined as the ratio of the power of the dominant component ( $\delta^{2}$ ) and the mean power of the scattered component ( $2 \sigma^{2}$ ), so that $K=\delta^{2} / 2 \sigma^{2}$.

When there is no dominant component (i.e., there is no LOS), the Rice distribution ((5.2) with $\delta=0$ ) becomes equal to the Rayleigh distribution, given by (5.1).

### 5.1.3 Nakagami- $m$ Fading Channel

The Nakagami- $m$ fading is a generalized model, typically associated with multipath fading in a LOS and NLOS environment, and has been found to be a good fitting model for the mobile radio channel (Dersch and Braun, 1991). The pdf of the signal envelope, $R$, for the Nakagami- $m$ channel can be obtained as the sum of $m$ i.i.d. random Rayleigh r.v., expressed as:

$$
\begin{equation*}
f_{R}(r)=\frac{2 m^{m} r^{2 m-1}}{\Gamma(m) \Omega^{m}} e^{-\frac{m r^{2}}{\Omega}}, m \geq \frac{1}{2} \tag{5.3}
\end{equation*}
$$

where $\Gamma($.$) is the gamma function, m$ is the fading severity parameter, and the mean power is $E\left[R^{2}\right]=\Omega$.

For $1 / 2 \leq m<1$ this distribution will model a more severe fading scenario than the Rayleigh one.

When $m=1$, the Nakagami- $m$ distribution is identical to the Rayleigh distribution, with $\Omega=$ $2 \sigma^{2}$. Also, when there is a direct component, the Nakagami- $m$ distribution approximates the Rice distribution for $m>1$ (Dersch and Braun, 1991).

### 5.1.4 $\kappa-\mu$ Fading Channel

The $\kappa-\mu$ fading is a recently introduced, more general fading model, providing an enhanced fit to the empirical data observed in different scenarios under the line-of-sight (LOS) channel case.

As referenced in (Yacoub, 2007), the fading characterization of the mobile radio channel can, generally, be well described by the Nakagami- $m$ distribution. Nevertheless, other distributions best describe the fading channel in several commonly found situations. Other situations can also be found for which none of the previous standard distributions seems to fit the collected data.

Departing from the homogenous diffuse scattering scenario, the $\kappa-\mu$ distribution was obtained considering a non-homogeneous environment due to the spatial correlation of the scattering surfaces. The received signal is assumed to be formed by separable multipath signal clusters, each propagating in a homogeneous environment. Further, the power of the scattered multipath clusters is assumed to be equal, and the power of the dominant component in each cluster is assumed to be arbitrary. The delay times within each scattered cluster are assumed to be similar, with large delay spreads between different clusters. The pdf of the signal envelope, $R$, for the $\kappa-\mu$ fading channel is given by (Yacoub, 2007):

$$
\begin{equation*}
f_{R}(r)=\frac{2 \mu(1+\kappa)^{\frac{\mu+1}{2}} r^{\mu}}{\kappa^{\frac{\mu-1}{2}} \mathrm{e}^{\mu \kappa} \Omega^{\frac{\mu+1}{2}}} e^{-\frac{\mu(\kappa+1) r^{2}}{\Omega}} I_{\mu-1}\left(2 \mu \sqrt{\frac{\kappa(1+\kappa)}{\Omega}} r\right) \tag{5.4}
\end{equation*}
$$

where the mean power is $E\left[R^{2}\right]=\Omega$.
The $\kappa$ and $\mu$ parameters have the following physical interpretation: $\kappa>0$ is the ratio of the total power of the dominant components to the total power of the scattered components; $\mu$ is related to the number of multipath clusters.

This highly generalized fading model includes the previous fading distributions: for $\kappa \rightarrow 0$, it coincides with the Nakagami- $m$ model with $m=\mu$; for $\kappa \rightarrow 0$ and $\mu=1$, it coincides with the Rayleigh model; for $\mu=1$, it coincides with the Rice model with Rician $K$-factor equal to $\kappa$ ( $K=$ $\kappa)$.

### 5.1.5 $\quad \eta-\mu$ Fading

Similarly to the $\kappa-\mu$ fading model described in the previous section, the $\eta-\mu$ is a recently introduced general fading model, providing an enhanced fit to the empirical data observed in
different scenarios under the non-line-of-sight (NLOS) channel case.
Here, the received signal is assumed to be formed by separable multipath signal clusters, each propagating in a homogeneous environment. The delay times within each scattered cluster are assumed to be similar, with large delay spreads between different clusters. The pdf of the signal envelope, $R$, for the $\eta-\mu$ fading channel is given by (Yacoub, 2007):

$$
\begin{equation*}
f_{R}(r)=\frac{4 \sqrt{\pi} \mu^{\mu+\frac{1}{2}} h^{\mu} r^{2 \mu}}{\Gamma(\mu) H^{\mu-\frac{1}{2}} \Omega^{\mu+\frac{1}{2}}} e^{-\frac{2 \mu h r^{2}}{\Omega}} I_{\mu-\frac{1}{2}}\left(\frac{2 \mu H r^{2}}{\Omega}\right) \tag{5.5}
\end{equation*}
$$

where the mean power is $E\left[R^{2}\right]=\Omega$.
Although expressed mathematically by the same model as given in equation (5.5), this type of fading can appear in two formats, with different physical interpretations. In Format 1, the In-phase and Quadrature components of the fading signal within each cluster are assumed to be independent and with possible different power; In Format 2, the In-phase and Quadrature components are assumed to be correlated with equal power. The interpretation of $\eta$ and how to obtain the parameters $h$ and $H$ is given in Table 5.1

|  | Format 1 | Format 2 |
| :---: | :---: | :--- |
| Physical model | The In-phase and Quadrature <br> components of the fading signal <br> within each cluster are assumed to <br> be independent and with possible <br> different power | The In-phase and Quadrature <br> components of the fading signal within <br> each cluster are assumed to be <br> correlated with equal power |
| $\eta$ | $0<\eta<\infty: \eta$ is the power ratio <br> between the In-phase and <br> Quadrature components of each <br> cluster scattered signal | $-1<\eta<1: \eta$ is the correlation <br> coefficient between In-phase and <br> Quadrature components of each cluster <br> scattered signal |
| $h$ | $h=\frac{2+\eta^{-1}+\eta}{4}$ | $h=\frac{1}{1-\eta^{2}}$ |
| $H$ | $H=\frac{\eta^{-1}-\eta}{4}$ | $H=\frac{\eta}{1-\eta^{2}}$ |

Table 5.1 - Parameters of the two formats for the $\eta-\mu$ distribution.

The $\eta$ parameter in one format can be obtained from the other by $\eta_{\text {Format } 1 ; 2}=\frac{1-\eta_{\text {Format } 2 ; 1}}{1+\eta_{\text {Format } 2 ; 1}}$.
This general fading model also includes several other known fading distributions. The Rayleigh distribution is obtained for $\mu=0.5$ and setting $\eta=1$ in Format 1 or $\eta=0$ in Format 2; the Nakagami- $m$ distribution can be obtained for $\mu=m$ and setting $\eta \rightarrow 0$ or $\eta \rightarrow \infty$ in Format 1 or
$\eta \rightarrow \pm 1$ in Format 2.

### 5.1.6 Shadowing Channels

Shadowing in a wireless channel makes the received signal power to fluctuate due to objects' obstruction of the propagation path between emitter and receiver. This phenomenon is modeled similarly to the fading effect, with the pdf of the received signal envelope, $R$, given by a lognormal distribution (Hashemi, 1993):

$$
\begin{equation*}
f_{R}(r)=\frac{1}{\sqrt{2 \pi} \sigma r} e^{-\frac{(\ln (r)-\mu)^{2}}{2 \sigma^{2}}} \tag{5.6}
\end{equation*}
$$

where $\mu$ and $\sigma^{2}$ are, respectively, the mean and variance of the distribution in $\mathrm{dB}: \mu=E[\ln (r)]$, $\sigma^{2}=\operatorname{var}(\ln (r))$.

As referenced in (Hashemi, 1993), the traditional Rayleigh or Rice multipath fading models are adequate for small area scenarios, where the channel is assumed to have spatial homogeneity. For large area scenarios, with space inhomogeneity of the channel, the lognormal distribution more adequately fits the experimental data.

### 5.1.7 Models for Experimental Results

The cdf of the fading distribution is a useful representation of the fading phenomenon. We can use the curve to obtain a particular fade depth probability. For a hypothetic fading cdf, as illustrated in Figure 5.1, the probability of a (maximum) 20 dB fade is $1 \%$. Therefore, one could assure a reliable link for $99 \%$ usage if a 20 dB margin is guaranteed in the link budget project. Any cdf curve lying above the one in Figure 5.1 corresponds to a more severe fading (worse zone), meaning that the same maximum fading intensity will have a higher occurrence probability; accordingly, a curve laying below corresponds to a less severe fading environment (better zone).

We will use the previous fading and shadowing channel models for simulation, with parameters obtained for the worst-case experimental results reported in the scientific literature. Therefore, we will use the model for which the corresponding cdf curve lies in the uppermost part of the graphic representation.


Figure 5.1 - Cumulative distribution functions (cdf) for some fading models.

We used experimental results obtained and modeled as Rayleigh or Nakagami- $m$, in (Sánchez, Urquiza-Aguiar and Paredes, 2021); Rayleigh, Rice, and Nakagami- $m$ reported in (Turbic, Ambroziak and Correia, 2017); Nakagami- $m, \kappa-\mu$ and $\eta-\mu$ reported in (Abouraddy and Elnoubi, 2000; F. C. Martins, H. B. Tercius, 2004; Yacoub, 2007); Log-normal shadowing reported in (Erceg et al., 2004; Hao et al., 2006).

As an example, in Figure 5.2 and Figure 5.3, the pdf and cdf are drawn, respectively, for the $\kappa-\mu$ fading model. As can be seen, the worst-case corresponds to $\kappa=1.5798, \mu=0.875$, and $\sigma=1$.

The worst-case distribution parameters for each model were obtained in a similar way and are shown in Table 5.2.

Figure 5.4 and Figure 5.5 show, respectively, the pdf and cdf of the fading, with parameters for each model (Rayleigh, Rice, Nakagami- $m$, $\kappa-\mu$, and $\eta-\mu$ Format 1 and 2). Figure 5.6 and Figure 5.7 show the pdf and cdf for the Log-normal distribution that models the shadowing channel.


Figure 5.2 - Probability distribution functions (pdf) for the $\kappa-\mu$ fading model.


Figure 5.3 - Cumulative distribution functions (cdf) for the $\kappa-\mu$ fading models.

| Fading/Shadowing Channel | Parameters |
| :--- | :--- |
| Rayleigh | $\sigma=0.8$ |
| Rice | $K=1.231$ |
|  | $\sigma=0.5$ |
| Nakagami-m | $m=0.7054$ |
|  | $\sigma=1$ |
| $\kappa-\mu$ | $\kappa=1.5798$ |
|  | $\mu=0.875$ |
|  | $\sigma=1$ |
| $\eta-\mu$, Format 1 | $\eta=0.2$ |
|  | $\mu=0.5094$ |
|  | $\sigma=1$ |
| $\eta-\mu$, Format 2 | $\eta=0.2$ |
|  | $\mu=0.5094$ |
|  | $\sigma=1$ |
| Shadowing | $\sigma=5 \mathrm{~dB}$ |

Table 5.2 - Parameters of worst cases for each fading and shadowing models.


Figure 5.4 - Probability distribution functions (pdf) for the fading models.


Figure 5.5 - Cumulative distribution functions (cdf) for the fading models.


Figure 5.6 - Probability distribution functions (pdf) for the shadowing model.


Figure 5.7 - Cumulative distribution functions (cdf) for the shadowing model.

### 5.2 Proposed Cyclostationarity Detector

In this section, we will present the proposed cyclostationarity detectors based on the SCoF and the SCF. Assuming no knowledge of the noise statistics and without any assumption on the Signal-of-Interest (apart from presenting cyclostationary features), we will propose a CFAR detector using the SCoF and a CFAR detector using the SCF. Modeling the input signal as a random Gaussian process, we will propose a CDR detector using the SCF.

### 5.2.1 CFAR Detector Using the SCoF

As shown in the previous chapter, the real and imaginary part of the SCoF can be modeled as $\mathcal{N}\left(0 ; \sigma_{\text {SCoF }}^{2}\right)$, when the received signal is noise only, so that its absolute value has a Rayleigh distribution with a scale parameter equal to $\sigma_{S C o F}$. So, given $F_{\text {Rayleigh }\left(\sigma_{S C o F}\right)}($.$) the cumulative$ distribution function (CDF) of Rayleigh $\left(\sigma_{S C o F}\right)$, and for a given False Alarm Probability, $P_{F A}$, the detection threshold, $\gamma$, is given by

$$
\begin{equation*}
\gamma=F_{\text {Rayleigh }\left(\sigma_{S C o F}\right)}^{-1}\left(1-P_{F A}\right) \tag{5.7}
\end{equation*}
$$

where, from equation (4.59),

$$
\begin{equation*}
\sigma_{S C o F}=\frac{1}{\sqrt{2}} \sqrt{\frac{1+l^{2}}{1+(2 P-1) l}}, \text { with } l=L / N p S e g . \tag{5.8}
\end{equation*}
$$

Therefore, given the intended $P_{F A}$, and with $P$ averaging segments, one obtains $\sigma_{S C o F}$ and the threshold $\gamma$. With the $P$ segments of $N p S e g$ samples each, one computes the $\operatorname{SCoF}$ for the $\left(f_{i} ; \alpha_{i}\right)$ point of the cyclostationary feature and decide if the PU is present (hypothesis $\mathcal{H}_{1}$ ) or not (hypothesis $\mathcal{H}_{0}$ ) according to:

$$
\left|C_{x}^{\alpha_{i}}\left(f_{i}\right)\right| \begin{gather*}
\mathcal{H}_{1}  \tag{5.9}\\
\underset{\mathcal{H}_{0}}{\gtrless} \\
\gtrless
\end{gather*} .
$$

### 5.2.2 CFAR and CDR Detectors Using the SCF

As shown in the previous chapter, the real and imaginary part of the SCF can also be modeled as $\mathcal{N}\left(0 ; \sigma_{S C F}^{2}\right)$, when the input signal is noise only. Therefore its absolute value has a Rayleigh distribution with a scale parameter equal to $\sigma_{S C F}$. Given $F_{\operatorname{Rayleigh}\left(\sigma_{S C F}\right)}($.$) the cumulative$ distribution function (CDF) of Rayleigh $\left(\sigma_{S C F}\right)$, and for a given False Alarm Probability, $P_{F A}$, the detection threshold, $\gamma_{S C F}$, is given by

$$
\begin{equation*}
\gamma_{S C F}=F_{\text {Rayleigh }\left(\sigma_{S C F}\right)}^{-1}\left(1-P_{F A}\right) \tag{5.10}
\end{equation*}
$$

## A CFAR Detector

But now, differently from the SCoF case, we do not have an approximation for $\sigma_{S C F}$. We will assume some knowledge about the cyclic features profile of the Signal-of-Interest, and obtain the SCF on the $\left(f_{i} ; \alpha_{i}\right)$ point of the cyclostationarity feature to detect, and another far away point on the cyclic frequency axis, for the same $f_{i}$. With finite input sample length, the SCF will show leakage on all the $(f ; \alpha)$ plane when the SoI is present. The choice of a far $\alpha$ point is justified for the lower leakage along the cyclic axis.

Assuming that for the far point, the SCF can be close to the case when the input signal is noise only, we will estimate $\sigma_{S C F}$ using a fixed number of SCF samples, and obtain the adaptive threshold given by $(5.10)$ to be used with the SCF obtained for the $\left(f_{i} ; \alpha_{i}\right)$ point.

The maximum likelihood estimator of the mean and standard deviation of a $\mathcal{N}\left(0 ; \sigma^{2}\right)$ r.v. $x$, using $N$ points is ${ }^{41}$ :

[^29]\[

$$
\begin{align*}
& \hat{\mu}_{N}=\frac{1}{N} \sum_{i=0}^{N-1} x_{i} \\
& \hat{\sigma}_{N}=\sqrt{\frac{1}{N} \sum_{i=0}^{N-1}\left(x_{i}-\hat{\mu}_{N}\right)^{2}} \tag{5.11}
\end{align*}
$$
\]

Therefore, for the proposed CFAR detector, and given the intended $P_{F A}$, with $P$ averaging segments, one obtains $\sigma_{S C F}$ using $N$ samples of the far point SCF (mean of the real and imaginary part) as given by equations (5.11). Then, using (5.10), one obtains the adaptive threshold $\gamma_{S C F}$. With the $P$ segments of $N p S e g$ samples each, one computes the SCF for the $\left(f_{i} ; \alpha_{i}\right)$ point of the cyclostationary feature and decide if the PU is present (hypothesis $\mathcal{H}_{1}$ ) or not (hypothesis $\mathcal{H}_{0}$ ) according to:

$$
\left|S_{x}^{\alpha_{i}}\left(f_{i}\right)\right| \stackrel{\mathcal{H}_{1}}{\underset{\mathcal{H}_{0}}{\gtrless}} \gamma_{S C F} .
$$

## A CDR Detector

As seen in section 4.1, the SCF computed by the FAM algorithm can be expressed in a Quadratic Form so that, for the received signal $\boldsymbol{r}=\boldsymbol{x}+\boldsymbol{n}$, the SCF for any $\left(f_{i} ; \alpha_{i}\right)$ point can obtained by:

$$
\begin{align*}
S_{r}^{\alpha_{i}}\left(f_{i}\right) \sim \boldsymbol{r}^{\boldsymbol{T}} \boldsymbol{G} \boldsymbol{r} & =(\boldsymbol{x}+\boldsymbol{n})^{T} \boldsymbol{G}(\boldsymbol{x}+\boldsymbol{n}) \\
& =\underbrace{\underbrace{\boldsymbol{G} \boldsymbol{x}}_{S_{x}^{T}}}_{S_{1}}+\underbrace{2 \boldsymbol{x}^{T} \boldsymbol{G} \boldsymbol{n}}_{S_{x n}} \tag{5.13}
\end{align*}+\underbrace{\boldsymbol{n}^{T} \boldsymbol{G} \boldsymbol{n}}_{S_{n}}, ~ l
$$

When the SoI signal, $\boldsymbol{x}$, is unknown, we can try to model it as a random Gaussian variable with $\mathcal{N}\left(0 ; \sigma_{x}^{2}\right)$ (as in (Zeng and Liang, 2007)), so that, for a high number of averaging segments, and as seen in the previous chapter, the real and imaginary components of $S_{x n}$ can be approximately modeled as $\mathcal{N}\left(0 ; \sigma_{S_{x n}}^{2}\right)$ and $S_{n}$ as $\mathcal{N}\left(0 ; \sigma_{S_{n}}^{2}\right)$. The $S_{x}$ term is the SCF estimate in the noiseless case.

As in the case of the CFAR detector, we will use a far estimate of the SCF in the cycle frequency axis (where, with low leakage, the SCF will be approximately equal to $S_{n}$ ) to estimate $\sigma_{S_{n}}^{2}$. Assuming independence from $S_{n}$ and $S_{1}=S_{x}+S_{x n}$, the variance of the real and imaginary components of $S_{1}$ can be obtained.

Both $S_{n}$ and $S_{x n}$ are modeled with zero mean; using the SCF computed for the cycle feature point $\left(f_{i} ; \alpha_{i}\right)$, the mean of the real and imaginary parts of $S_{x}$ can be estimated. Assuming that the fraction of the squared means of $S_{x}$ (real and imaginary components) is equal to the fraction of the variance of $S_{x}$ (real and imaginary components) - which is not proven, but we could see it in practical simulation examples - the variance of $S_{x}$ and $S_{x n}$ can be estimated.

The following signal processing flow describes the proposed algorithm:

1. Compute the SCF for the cycle feature point and for a far point in the cycle frequency axis, $\mathrm{SCF}_{1}$ and $\mathrm{SCF}_{2}$, respectively, in a continuous way from the input signal;
2. Use $N$ points from $\mathrm{SCF}_{2}$ to estimate $\sigma_{S_{n}}^{2}$;
3. Use $N$ points from $\mathrm{SCF}_{1}$ to estimate the variance of the real and imaginary components of $S_{1}$ :

$$
\begin{align*}
& \sigma_{S_{1_{R}}}^{2}=\operatorname{var}\left(\operatorname{Re}\left(\mathrm{SCF}_{1}\right)\right)-\sigma_{S_{n}}^{2} \\
& \sigma_{S_{1_{I}}}^{2}=\operatorname{var}\left(\operatorname{Im}\left(\mathrm{SCF}_{1}\right)\right)-\sigma_{S_{n}}^{2} \tag{5.14}
\end{align*}
$$

4. Use $N$ points from $\mathrm{SCF}_{1}$ to estimate the mean of the real and imaginary components of $S_{x}$ :

$$
\begin{align*}
& m_{S_{x_{R}}}=\operatorname{mean}\left(\operatorname{Re}\left(\mathrm{SCF}_{1}\right)\right) \\
& m_{S_{x_{I}}}=\operatorname{mean}\left(\operatorname{Im}\left(\mathrm{SCF}_{1}\right)\right) \tag{5.15}
\end{align*}
$$

5. Compute the fraction of the squared mean of $S_{x}$ (real and imaginary) and obtain the variance of $S_{x}$ and $S_{x n}$, assuming the fraction is equal to the fraction of the variance of $S_{x}$ (real and imaginary components):

$$
\begin{array}{ll}
\alpha_{R}^{2}=\frac{m_{S_{x_{R}}}^{2}}{m_{S_{x_{R}}}^{2}+m_{S_{x_{I}}}^{2}} & \alpha_{I}^{2}=\frac{m_{S_{x_{I}}}^{2}}{m_{S_{x_{R}}}^{2}+m_{S_{x_{I}}}^{2}}=1-\alpha_{R}^{2}  \tag{5.16}\\
\sigma_{S_{x_{R}}}^{2}=\alpha_{R}^{2} \sigma_{S_{x}}^{2} & \sigma_{S_{x_{I}}}^{2}=\alpha_{I}^{2} \sigma_{S_{x}}^{2}=\left(1-\alpha_{R}^{2}\right) \sigma_{S_{x}}^{2}
\end{array}
$$

So,

$$
\left\{\begin{array} { r l } 
{ \sigma _ { S _ { x _ { R } } } ^ { 2 } + \sigma _ { S _ { x n _ { R } } } ^ { 2 } } & { = \sigma _ { S _ { 1 _ { R } } } ^ { 2 } }  \tag{5.17}\\
{ \sigma _ { S _ { x _ { I } } } ^ { 2 } + \sigma _ { S _ { x n _ { I } } } ^ { 2 } } & { = \sigma _ { S _ { 1 _ { I } } } ^ { 2 } }
\end{array} \Leftrightarrow \left\{\begin{array}{c}
\sigma_{S_{x}}^{2}=\frac{\sigma_{S_{1_{R}}}^{2}-\sigma_{S_{1_{I}}}^{2}}{\alpha_{R}^{2}-\alpha_{I}^{2}} \\
\sigma_{S_{x n}}^{2}=\sigma_{S_{1_{R}}}^{2}-\alpha_{R}^{2} \sigma_{S_{x}}^{2}
\end{array}\right.\right.
$$

The previous algorithm will estimate the mean and variance of the real and imaginary components of $S_{x}$. Two situations can arise when using this algorithm: 1) the variance estimate is real and positive; 2) the variance estimate is zero or an imaginary number so that the real part is zero. These cases will be addressed in the following.

## Case When The Variance Estimate is Zero or Complex (Low SNR)

In this case, we can only take the real part of the estimate, which will be equal to zero. We will have $\sigma_{S_{x_{R}}}^{2}=\sigma_{S_{x_{I}}}^{2}=0$, so that the pdf of $S_{x_{R}}$ is $\delta\left(x-m_{S_{x_{R}}}\right)$ and the pdf of $S_{x_{I}}$ is $\delta\left(x-m_{S_{x_{I}}}\right)$.

The real and imaginary components of the $\mathrm{SCF}_{1}$ are, therefore, $\mathcal{N}\left(m_{S_{x_{R}}}, m_{S_{x_{I}}} ; \sigma_{S_{n}}^{2}+\sigma_{S_{x n}}^{2}\right)$. The absolute value of the $\mathrm{SCF}_{1}$ has a Rice distribution, $f_{S}(s)$, with $\lambda=m_{S_{x_{R}}}^{2}+m_{S_{x_{I}}}^{2}$ and $\sigma^{2}=\sigma_{S_{n}}^{2}+\sigma_{S_{x n}}^{2}$ (Papoulis and Pillai, 2002) (similar to (5.2) with $\lambda=\delta^{2}$ ):

$$
\begin{equation*}
f_{S}(s)=\frac{s}{\sigma^{2}} e^{-\frac{s^{2}+\lambda}{2 \sigma^{2}}} I_{o}\left(\frac{s \sqrt{\lambda}}{\sigma^{2}}\right) . \tag{5.18}
\end{equation*}
$$

The cdf of the Rice distribution is given as:

$$
\begin{equation*}
F_{S}(s)=1-Q_{m}\left(\frac{\sqrt{\lambda}}{\sigma} ; \frac{s}{\sigma}\right), \tag{5.19}
\end{equation*}
$$

where $Q_{m}(a, b)$ is the generalized Marcum Q -function.
For a given minimum Probability of Detection, $P_{D}$, the adaptive threshold, $\gamma_{S C F}$, is obtained as

$$
\begin{equation*}
\gamma_{S C F}=F_{S}^{-1}\left(1-P_{D}\right)=\left.s\right|_{Q_{m}\left(\frac{\sqrt{\lambda}}{\sigma} ; \frac{s}{\sigma}\right)=P_{D}} . \tag{5.20}
\end{equation*}
$$

The decision whether the PU is present (hypothesis $\mathcal{H}_{1}$ ) or not (hypothesis $\mathcal{H}_{0}$ ) is undertaken according to:

$$
\begin{equation*}
\left|S_{x}^{\alpha_{i}}\left(f_{i}\right)\right| \stackrel{\mathcal{H}_{1}}{\gtrless} \mathcal{H}_{0} \gamma_{S C F} . \tag{5.21}
\end{equation*}
$$

## Case When the Variance Estimate is Real and Different from Zero (Medium/High SNR)

In this case, the mean of $S_{x_{R}}$ and $S_{x_{I}}$ is not modeled as constant, but will have mean $m_{S_{x_{R}}}, m_{S_{x_{I}}}$ and variance $\sigma_{S_{x_{R}}}^{2}, \sigma_{S_{x_{I}}}^{2}$. With different variances, the square sum of $S_{x_{R}}$ and $S_{x_{I}}$ (equal to $\lambda$ ) can not be explicitly modeled. We will model it as a gamma distribution, because it will allow us to obtain the pdf of $\left|S_{x}^{\alpha_{i}}\left(f_{i}\right)\right|$, which we will validate by simulation examples.

As $\lambda=S_{x_{R}}^{2}+S_{x_{I}}^{2}$, then (Appendix A, Demonstration 5.1):

$$
\begin{align*}
E[\lambda] & =m_{S_{x_{R}}}^{2}+\sigma_{S_{x_{R}}}^{2}+m_{S_{x_{I}}}^{2}+\sigma_{S_{x_{I}}}^{2} \\
\operatorname{var}(\lambda) & =2 \sigma_{S_{x_{R}}}^{2}\left(\sigma_{S_{x_{R}}}^{2}+2 m_{S_{x_{R}}}^{2}\right)+2 \sigma_{S_{x_{I}}}^{2}\left(\sigma_{S_{x_{I}}}^{2}+2 m_{S_{x_{I}}}^{2}\right) \tag{5.22}
\end{align*}
$$

The gamma distribution $\Gamma(K, \theta)=\Gamma\left(K, \frac{\Omega}{K}\right)$, where $K$ is the shape parameter and $\theta$ the scale parameter (similar to eq. (4.27) with $\alpha=K$ and $\theta=1 / \beta$ ) with:

$$
\begin{align*}
& \theta=\frac{\operatorname{var}(\lambda)}{E[\lambda]} \\
& \mathrm{K}=\frac{E[\lambda]}{\theta}  \tag{5.23}\\
& \Omega=K \theta
\end{align*}
$$

is

$$
\begin{equation*}
f_{\Lambda}(\lambda)=\frac{\lambda^{K-1} e^{-\frac{\lambda}{\theta}}}{\Gamma(K) \theta^{K}} \tag{5.24}
\end{equation*}
$$

The pdf of the absolute value of the $\mathrm{SCF}_{1}$ can now be obtained from the previous case (the Rician distribution) as:

$$
\begin{equation*}
f_{S}(s)=E_{\lambda}\left[\frac{s}{\sigma^{2}} e^{-\frac{s^{2}+\lambda}{2 \sigma^{2}}} I_{o}\left(\frac{s \sqrt{\lambda}}{\sigma^{2}}\right)\right] \tag{5.25}
\end{equation*}
$$

which, in (Alfano and De Maio, 2007), is shown to be :

$$
\begin{equation*}
f_{S}(s)=\left(\frac{2 \sigma^{2} K}{2 \sigma^{2} K+\Omega}\right) \frac{s}{\sigma^{2}} e^{-\frac{s^{2}}{2 \sigma^{2}}}{ }_{1} F_{1}\left(K ; 1 ; \frac{\Omega s^{2}}{2 \sigma^{2}\left(2 \sigma^{2} K+\Omega\right)}\right) \tag{5.26}
\end{equation*}
$$

As shown in Appendix A, Demonstration 5.2, the cdf of the absolute value of the $\mathrm{SCF}_{1}$ is (eq. (A.16)):

$$
\begin{align*}
F_{S}(s)=\frac{1}{2 \sigma^{2}}( & \left.\frac{2 \sigma^{2} K}{2 \sigma^{2} K+\Omega}\right)^{K} s^{2} \Phi_{2}(1 \\
& \left.\quad-K, K, 2,-\frac{s^{2}}{2 \sigma^{2}} ;-\frac{K s^{2}}{2 \sigma^{2} K+\Omega}\right) \tag{5.27}
\end{align*}
$$

Again, for a given minimum Probability of Detection, $P_{D}$, the adaptive threshold, $\gamma_{S C F}$, is obtained as

$$
\begin{equation*}
\gamma_{S C F}=F_{S}^{-1}\left(1-P_{D}\right) \tag{5.28}
\end{equation*}
$$

The decision whether the PU is present (hypothesis $\mathcal{H}_{1}$ ) or not (hypothesis $\mathcal{H}_{0}$ ) is
undertaken according to:

$$
\begin{equation*}
\left|S_{x}^{\alpha_{i}}\left(f_{i}\right)\right| \stackrel{\mathcal{H}_{1}}{\underset{\mathcal{H}_{0}}{\gtrless}} \gamma_{S C F} . \tag{5.29}
\end{equation*}
$$

For a QPSK Signal-of-Interest with symbol rate and carrier frequency equal to one-fourth of the sampling rate, with raised-cosine pulse shape with roll-off equal to 1 , and for $N p S e g=128$ and $P=257$, the pdf of $\left|S_{x}^{\alpha_{i}}\left(f_{i}\right)\right|$ was obtained by simulation for varying $\operatorname{SNR}\left(\left(f_{i} ; \alpha_{i}\right)=\right.$ $\left.\left(f_{s} / 4 ; f_{s} / 4\right)\right)$. When the variance estimate is zero, only the Rice pdf can be obtained (equation (5.18)); when the variance is positive and real, the proposed pdf can also be obtained (equation (5.26)).

Figure 5.8 shows the histogram obtained for $\left|S_{x}^{\alpha_{i}}\left(f_{i}\right)\right|$ alongside the theoretical pdfs, when possible to obtain, for $S N R=-15,-10,-5,0,5 d B$. As can be seen, for low SNR (a) and b) cases), the Rice pdf fits the empirical pdf obtained, and the variance estimate was zero. When the variance estimate could be obtained (c)-e) cases) the proposed pdf fits the one obtained by simulation, and the Rice pdf gives unsatisfactory results.


e)

Figure 5.8 - Probability Density Function (pdf) of $\left|S_{x}^{\alpha_{i}}\left(f_{i}\right)\right|$ for varying SNR. a) $\mathrm{SNR}=-15 \mathrm{~dB}$; b)

$$
\mathrm{SNR}=-10 \mathrm{~dB} ; \text { c) } \mathrm{SNR}=-5 \mathrm{~dB} ; \mathrm{d}) \mathrm{SNR}=0 \mathrm{~dB} ; \text { e) } \mathrm{SNR}=5 \mathrm{~dB} .
$$

### 5.3 CREST and ENTROPY Detectors

We will assess the performance of the detectors proposed in the previous section and compare them with two recent CFAR detectors proposed in the open literature: the CREST detector and the ENTROPY detector. As shown in Chapter 2, the cyclostationary detector outperforms other standard detectors, especially when there is noise uncertainty. These two detectors were chosen for specifically claiming to perform better than standard cyclostationary detection.

### 5.3.1 CREST Detector

In (Kim et al., 2007), the Crest Factor (CF) was introduced for signal detection and feature extraction. After computing the Spectral Coherence Function (SCoF), $C_{x}^{\alpha_{i}}\left(f_{i}\right)$, the cycle frequency profile (also named $\alpha$-domain profile) is given as (for $N \alpha$ points, and $\alpha \neq 0$ ):

$$
\begin{equation*}
I\left(\alpha_{i}\right)=\max _{f_{i}}\left|C_{x}^{\alpha_{i}}\left(f_{i}\right)\right| \tag{5.30}
\end{equation*}
$$

The Crest Factor of $I\left(\alpha_{i}\right)$ is then obtained as:

$$
\begin{equation*}
\mathrm{CF}=\frac{\max \left(I\left(\alpha_{i}\right)\right)}{\left(\sum_{i=1}^{N} I^{2}\left(\alpha_{i}\right)\right) / N} . \tag{5.31}
\end{equation*}
$$

The proposed detection scheme was based on the CF's simulation (or real measures) when only AWGN was fed to the detector to obtain the CF's histogram. The empirical cdf of CF is estimated
from the histogram, and, for a given Probability of False Alarm, $P_{F A}$, the decision threshold is obtained as the minimum value for which the cdf is bigger than $1-P_{F A}$.

In (Kadjo, Yao and Mansour, 2017), the authors proposed to model the CF as a Generalized Extreme Value (GEV) distribution to obtain the decision threshold directly from its inverse cdf. More recently, the same method was revisited and more detailed in (Kadjo et al., 2021).

The GEV distribution is defined as $(\xi \neq 0)$ :

$$
\begin{equation*}
f_{X}(x)=\frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} \times \exp \left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right\} \tag{5.32}
\end{equation*}
$$

where $\xi$ is the shape parameter, $\sigma$ is the scale parameter, and $\mu$ is the location parameter.
The cdf of the GEV distribution is:

$$
\begin{equation*}
F_{X}(x)=\exp \left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right\} \tag{5.33}
\end{equation*}
$$

For a given Probability of False Alarm, $P_{F A}$, the adaptive threshold, $\gamma_{C R E S T}$, is obtained as

$$
\begin{equation*}
\gamma_{C R E S T}=F_{X}^{-1}\left(1-P_{F A}\right) \tag{5.34}
\end{equation*}
$$

The decision whether the PU is present (hypothesis $\mathcal{H}_{1}$ ) or not (hypothesis $\mathcal{H}_{0}$ ) is undertaken according to:

$$
C F \stackrel{\mathcal{H}_{1}}{\underset{\mathcal{H}_{0}}{\gtrless} \gamma_{\text {CREST }} .}
$$

Although no explicit details were given in the previous referenced papers about the simulation of the CF in an AWGN-only environment, the parameters were obtained by maximizing the quadratic error as $\xi=-0.027, \sigma=0.295$, and $\mu=8.20$.

We used the FAM algorithm, with $N p S e g=128, P=257$, and $N=128$, to obtain the SCoF in a $128 \times 128$ region of the first quadrant of the $(f ; \alpha)$ plane. The $N$-point cyclic-frequency profile was computed, and the CF obtained (equation (5.31)) for $2^{17}$ runs. Figure 5.9 shows the histogram obtained alongside the theoretical pdf of the GEV distribution, using the parameters estimated by the built-in function gevfit in MATLAB ${ }^{\circ}$. In our case, the parameters obtained are $\xi=0.022, \sigma=$ 0.0881 , and $\mu=1.4066$, and we can seen a very good fit of the GEV distribution to the simulation results.


Figure 5.9 - Probability Density Function (pdf) of the Crest Factor (CF)

### 5.3.2 ENTROPY Detector

In (Zhang, Zhang and Melodia, 2010), a frequency-domain entropy-based detector was proposed for spectrum sensing in cognitive radio, claiming a significant improvement compared to energy and cyclostationarity detectors. As in section 2.3, the received signal is given as in equation (2.1):

$$
\begin{aligned}
& \mathcal{H}_{o}: r(n)=w(n) \\
& \mathcal{H}_{1}: r(n)=s(n)+w(n),
\end{aligned}
$$

where $s(n)$ and $w(n)$ are the transmitted signal by the PU observed at the SU , and additive white Gaussian noise (AWGN), respectively. The noise signal is assumed to have zero mean and variance $\sigma^{2}$, so $w \sim \mathcal{N}\left(0, \sigma^{2}\right)$.

Applying the Discrete Fourier Transform to a $N$ point sample input, the general received signal is given as

$$
R(k)=S(k)+W(k), k=0 \ldots N,
$$

where $R, S$, and $W$ are the complex spectrum of the received signal, primary signal (if present), and noise, respectively.

Under the null hypothesis $\left(\mathcal{H}_{o}\right)$, the real and imaginary parts of $R=W$ have Gaussian
distribution for being a linear combination of Gaussian random variables. The complex spectrum $W(k)$ is a Gaussian variable with mean and variance given by (Zhang, Zhang and Melodia, 2010):

$$
\begin{aligned}
& E[W(k)]=\frac{1}{N} \sum_{n=0}^{N-1} E[w(n)] e^{-j \frac{2 \pi n k}{N}}=0 \\
& \operatorname{var}[W(k)]=\frac{1}{N^{2}} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} E\left[w^{2}(n)\right]=\frac{\sigma^{2}}{N} .
\end{aligned}
$$

Therefore, $Y=|W|=\sqrt{W_{R}^{2}+W_{I}^{2}}$ will have a Rayleigh distribution with $\sigma_{\text {Rayleigh }}=\sigma / \sqrt{2 N}$. With $N$ samples, the discrete entropy of the spectrum is obtained through the $L$ bin histogram of $Y$ :

$$
\begin{equation*}
T(Y)=-\sum_{i=0}^{L-1} p_{i} \log _{2}\left(p_{i}\right), \quad p_{i}=\frac{k_{i}}{N} \tag{5.36}
\end{equation*}
$$

where $k_{i}$ are the number of $Y$ samples within each bin $i$.
In the case of white noise, the entropy was shown to be approximately given as:

$$
\begin{equation*}
H_{L}(Y) \cong \ln \left(\frac{L}{\sqrt{2}}\right)+\frac{\gamma}{2}+1 \tag{5.37}
\end{equation*}
$$

where $\gamma$ is the Euler-Mascheroni constant ${ }^{42}$.
The proposed detector assumed that $H_{L}(Y)$ was not exactly constant, but could be modeled as $\mathcal{N}\left(H_{L}, \sigma_{\epsilon}^{2}\right)$, with $H_{L}$ obtained by (5.37), so the decision threshold, for a given Probability of False Alarm, is obtained as:

$$
\begin{equation*}
\gamma_{E N T R O P Y}=H_{L}+Q^{-1}\left(1-P_{F A}\right) \sigma_{\epsilon} \tag{5.38}
\end{equation*}
$$

The decision whether the PU is present (hypothesis $\mathcal{H}_{1}$ ) or not (hypothesis $\mathcal{H}_{0}$ ) is undertaken according to:

$$
\begin{equation*}
T(Y) \stackrel{\mathcal{H}_{1}}{\underset{ }{\gtrless}} \gamma_{\text {ENTROPY }} . \tag{5.39}
\end{equation*}
$$

Performance results were given for $L=15$ ( $L$ should be greater than 10 ) and for an unjustified $\sigma_{\epsilon}^{2} / H_{L}=10^{-3}$, claiming a significant improvement compared to energy and cyclostationarity detectors.

In (Zhao, 2013), a new two-stage entropy detector was proposed but using $Y=|W|^{2}=W_{R}^{2}+W_{I}^{2}$.

$$
{ }^{42} \gamma=\lim _{n \rightarrow \infty}\left(-\log (n)+\sum_{k=1}^{n} \frac{1}{k}\right)
$$

As in the previous case the first stage detection is performed as:

$$
T_{1}(Y)\left\{\begin{array}{c}
\leq \gamma_{E N T R O P Y}-\Delta_{0}: \text { Decide } \mathcal{H}_{0}  \tag{5.40}\\
\geq \gamma_{E N T R O P Y}+\Delta_{0}: \text { Decide } \mathcal{H}_{1} \\
\text { else: go to stage } 2
\end{array}\right.
$$

In the second stage, another $N$-point sample is used to compute a new entropy estimate, $T_{2}(Y)$, (equation (5.36)). The second stage detection is performed as:

$$
\frac{T_{1}(Y)+T_{2}(Y)}{2}\left\{\begin{array}{l}
\leq \gamma_{E N T R O P Y}: \text { Decide } \mathcal{H}_{0}  \tag{5.41}\\
\geq \gamma_{E N T R O P Y}: \text { Decide } \mathcal{H}_{1}
\end{array}\right.
$$

The threshold $\gamma_{E N T R O P Y}$ was obtained for a given $P_{F A}$ by simulating the AWGN-only environment, and using the computed empirical cdf of $T(Y)$.

Recently, in (Prieto et al., 2018), the power spectrum used to compute the entropy was averaged using $K$ periodograms of $Y=|W|^{2}=W_{R}^{2}+W_{I}^{2}$, and a two-stage detector, similar to the one in (Zhao, 2013), was applied for detection.

Another entropy detector was proposed in (Muñoz, Martínez and Hernandez, 2020), using the Rényi entropy:

$$
\begin{equation*}
T(Y)=\frac{1}{1-\alpha} \log _{2}\left(\sum_{i=0}^{L-1} p_{i}^{\alpha}\right)=\frac{1}{1-\alpha} \log _{2}\left(\sum_{i=0}^{L-1}\left(\frac{k_{i}}{N}\right)^{\alpha}\right) . \tag{5.42}
\end{equation*}
$$

This is a single-stage detector, using $\alpha=2.5$ and $L=15$, and the threshold was also obtained for a given $P_{F A}$ by simulating the AWGN-only environment, and using the computed empirical cdf of $T(Y)$.

A more recent entropy detector was proposed in (Hong and Xu, 2021) using the Welch method to obtain the power spectrum estimates and compute the entropy. This is a two-stage detector, similar to the one in (Zhao, 2013), but considered using $m$ averaging estimates of the entropy in the second stage, instead of only two. As claimed to have better performance than energy, cyclostationary, and entropy detection (compared to (Zhang, Zhang and Melodia, 2010) and (Muñoz, Martínez and Hernandez, 2020)) we will use this entropy detector to compare with other detector schemes.

### 5.4 Performance of Detection Schemes

In this section, simulation results are presented to assess the performance of the proposed detectors using the SCoF and the SCF and then compared to the CREST and ENTROPY detectors.

We considered a QPSK signal-of-interest with symbol rate and carrier frequency equal to onefourth of the sampling rate, with raised-cosine pulse shape with roll-off equal to 1 , and for $\mathrm{NpSeg}=$

128 and $P=257$. For the proposed detectors, the total number of samples for each detection is $N=(P-1) \times L+N p S e g$, with $L=N p S e g / 4$, so that $N=8320$ samples. For the CREST detector, the same values of NpSeg and $P$ were used, and the Crest Factor was computed in a $128 \times 128$ grid on the first quadrant of the $(f ; \alpha)$ plane. For the ENTROPY detector, the estimation of the power spectrum used 8320 samples to obtain $N=128$ frequency slots ( $\left[0 ; f_{s} / 2\right]$ ) with $L=15$ bins to obtain the entropy. In all simulation cases, $2^{15}$ Monte-Carlo runs were performed to obtain the results presented. As commonly used in the literature, the Probability of False Alarm in the CFAR detectors and the Probability of Detection in the CDR detectors are $P_{F A}=0.1$ and is $P_{D}=0.9$, respectively, according to the IEEE 802.22 WRAN standard (IEEE International, 2019).

In the following subsections, simulation results are given for the case of flat slow-fading and flat fast-fading. The slow-fading is assumed to be constant along the detection period; for the fastfading, it is constant for each symbol duration.

Finally, section 5.4.3 presents simulation results for a wideband IEEE 802.11 OFDM received signal.

### 5.4.1 Performance Analysis in an AWGN and Slow-Fading Environment

The detectors' performance will be assessed by obtaining the Probability of Detection for varying SNR, for a fixed Probability of False Alarm, in the CFAR case, or by obtaining the Probability of False Alarm for varying SNR, for a fixed Probability of Detection.

In Figure 5.10, the simulation results for the proposed CFAR SCoF detector are presented, for the AWGN-only case, alongside the seven fading models and shadowing fading model. Figure 5.11 shows similar results for the proposed CFAR SCF detector, where the far cycle frequency point was chosen as $\left(f_{c} ; 1.5 \times f_{\text {symbol }}\right)$; the cyclic feature to detect is at $\left(f_{c} ; f_{\text {symbol }}\right)$. Figure 5.12 also shows similar results for the proposed CDR SCF detector.

As can be seen from Figure 5.10, the proposed CFAR SCoF detector can maintain the prescribed $P_{F A}$ for all fading scenarios. The performance degradation is visible when we consider the fading phenomenon, and the more severe performance degradation occurs for the Nakagami model. The channel shadowing model, as expected, causes the worst performance degradation.

As for the proposed CFAR SCF detector, the same conclusion can be drawn, but the detector fails to maintain a constant $P_{F A}$ at high SNR (see Figure 5.11). This was expected, because, as the SNR increases, the leakage effect on the computation of the SCF will be increased, so the estimation of the noise variance will be less accurate. This detector shows better performance than the SCoF, and although the $P_{F A}$ can not be kept constant for high SNR, it is always less than the value prescribed.

From Figure 5.12, we can see that the proposed CDR SCF detector, works well in the case of the AWGN-only scenario. For any fading model used, it fails to maintain a constant $P_{D}$. As with the CFAR SCoF detector, the performance also is highly degraded when considering the shadowing
channel.


Figure 5.10 - Proposed SCoF CFAR detector $P_{D}$ and $P_{F A}$ vs. SNR.


Figure 5.11 - Proposed SCF CFAR detector $P_{D}$ and $P_{F A}$ vs. SNR.


Figure 5.12 - Proposed SCF CDR detector $P_{D}$ and $P_{F A}$ vs. SNR.

We now compare the performance of the proposed CFAR SCoF detector with the CREST and ENTROPY detectors for all the fading scenarios. Figure 5.13 till Figure 5.20 show the results of the three detectors in an AWGN only, Rayleigh, Rice, Nakagami, $\kappa-\mu, \eta-\mu$, Type 1 and 2 , and shadowing fading scenarios.

As can be seen, the CFAR SCoF detector outperforms the other detectors in all the cases considered. Again, the shadowing channel produces the worst performance degradation for all detectors. Further, the ENTROPY detector has the worse performance of all detectors in all fading scenarios. As with the SCoF detector, both the CREST and ENTROPY detectors show the worst performance degradation for the Nakagami and $\eta-\mu$ Type 1 fading scenarios.


Figure 5.13 - Comparison of proposed SCoF, CREST, and ENTROPY detectors $P_{D}$ and $P_{F A}$ vs. SNR AWGN only.


Figure 5.14 - Comparison of proposed SCoF, CREST, and ENTROPY detectors $P_{D}$ and $P_{F A}$ vs. SNR Rayleigh fading.


Figure 5.15 - Comparison of proposed SCoF, CREST, and ENTROPY detectors $P_{D}$ and $P_{F A}$ vs. SNR Rice fading.


Figure 5.16 - Comparison of proposed SCoF, CREST, and ENTROPY detectors $P_{D}$ and $P_{F A}$ vs. SNR Nakagami fading.


Figure 5.17 - Comparison of proposed SCoF, CREST, and ENTROPY detectors $P_{D}$ and $P_{F A}$ vs. SNR -$\kappa-\mu$ fading.


Figure 5.18 - Comparison of proposed SCoF, CREST, and ENTROPY detectors $P_{D}$ and $P_{F A}$ vs. SNR -$\eta-\mu$, Type 1 fading.


Figure 5.19 - Comparison of proposed SCoF, CREST, and ENTROPY detectors $P_{D}$ and $P_{F A}$ vs. SNR -$\eta-\mu$, Type 2 fading.


Figure 5.20 - Comparison of proposed SCoF, CREST, and ENTROPY detectors $P_{D}$ and $P_{F A}$ vs. SNR Shadowing fading.

### 5.4.2 Performance Analysis in an AWGN and Fast-Fading Environment

This section presents simulation results for the three detectors when a flat fast-fading scenario is used. In this case, the fading is constant for each symbol duration, not the detection interval.

Figure 5.21 shows the results obtained for the proposed CFAR SCoF detector. As can be seen, now the performance is not affected by the fast-fading effect, and can even be enhanced when considering the Rice fading model. The randomization due to the fading effect on the received signal samples when considering the entire detection period may explain this unanticipated effect. This randomization process does not occur with slow fading, as, in this case, the fading effect is constant during the detection period.

Figure 5.22 and Figure 5.23 show the results obtained for the proposed CFAR SCF and CDR SCF detector (with $N=100$ SCF samples for variance estimation), respectively. Again, the performance is now independent of the fading model. Due to the leakage effect in the computation of the SCF, the CFAR SCF detector is incapable of maintaining a constant $P_{F A}$ for high SNR as in the case of the slow-fading scenario. However, the CDR SCF detector can now maintain the constant $P_{D}$, which did not occur for the slow-fading scenario.

As can be seen in Figure 5.24, the CREST detector is also insensitive to the fast fading, but the ENTROPY detector shows a slight performance degradation, especially with Rayleigh fading (see Figure 5.25).


Figure 5.21 - Proposed SCoF CFAR detector $P_{D}$ and $P_{F A}$ vs. SNR with a fast-fading model.


Figure 5.22 - Proposed SCF CFAR detector $P_{D}$ and $P_{F A}$ vs. SNR with a fast-fading model.


Figure 5.23 - Proposed SCF CDR detector $P_{D}$ and $P_{F A}$ vs. SNR with a fast-fading model.


Figure 5.24 - CREST detector $P_{D}$ and $P_{F A}$ vs. SNR with a fast-fading model.


Figure 5.25 - ENTROPY detector $P_{D}$ and $P_{F A}$ vs. SNR with a fast-fading model.

### 5.4.3 Performance Analysis for an IEEE 802.11 SoI

This section will present results when the Signal-of-Interest (SoI) is a wideband OFDM signal generated according to the IEEE 802.11a/g WLAN standard.

The transmitted OFDM signal is constructed in a frame-by-frame scheme. A $N$ parallel binary subchannels streams are mapped to a specified signal constellation, $a_{n}, n=0 \ldots N-1$, according to the modulation used. These $N$ subchannels include the transmitted random binary user data and also pilot tones and guard bands. The resulting $N$ symbols are fed to an Inverse Fast Fourier Transform (IFFT) block, which, after parallel-to Serial conversion, results in the signal frame with duration $T_{S}$ seconds. For each IFFT bin, the corresponding carrier frequency is $f_{n}=n / T_{S}$. Afterward, a cyclic prefix is added, by copying the last portion of the frame with $T_{C P}$ seconds to the beginning of the frame. The complete OFDM frame will have a duration equal to $T=T_{S}+$ $T_{C P}$. After pulse-shaping using a $p(t)$ filter, the signal is up-converted to the RF carrier frequency used.

For the baseband OFDM signal, the SCF was shown to be (Adrat et al., 2009):

$$
\begin{align*}
& S_{x}^{\alpha}(f)= \\
& =\left\{\begin{array}{l}
\frac{1}{T} \sum_{n=0}^{N-1} P\left(f+\alpha / 2-f_{n}\right) P^{*}\left(f-\alpha / 2-f_{n}\right) \tilde{S}_{a_{n} a_{n}}\left(f+\alpha / 2-f_{n}\right) \\
\text { for } f_{n}=f_{m}, \alpha=z / T, z \in \mathbb{Z} \\
\frac{1}{T} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} P\left(f+\alpha / 2-f_{n}\right) P^{*}\left(f-\alpha / 2-f_{n}\right) \tilde{S}_{a_{n} a_{m}}\left(f+\alpha / 2-f_{n}\right) e^{-j 2 \pi\left(f_{n}-f_{m}\right) T_{G}} \\
\\
0, \\
\text { for } f_{n} \neq f_{m}, \alpha=f_{n}-f_{m}+z / T, z \in \mathbb{Z}
\end{array}\right. \\
& \text { otherwise }
\end{align*}
$$

Where $P(f)$ is the Fourier Transform of the shaping filter impulse response, $p(t)$ and $\tilde{S}_{a_{n} a_{m}}$ is the cross-spectral density of the user-transmitted symbols. Considering that the user transmits random symbols on each sub-channel, then $\tilde{S}_{a_{n} a_{m}}$ is zero, except for the pilot symbols.

As detailed in (Adrat et al., 2009), the cycle features of the SCF of the OFDM signal will be located at multiples of $1 / T$ and $1 / T_{S}$, as $z / T$ is a multiple of $1 / T$ and $f_{n}-f_{m}$ is a multiple of $1 / T_{S}$. The pilot symbols will contribute to the cycle features found in the SCF around $(f ; \alpha)=$ $\left(\left(f_{n}+f_{m}\right) / 2 ;\left(f_{n}-f_{m}\right)\right)$, for the $n$ and $m$ subchannels of the pilots. Further, for these cases, and due to the cycle prefix, whenever the relation $\left(f_{n}-f_{m}\right) \times T_{C P}$ is not an integer, then the feature at multiples of $1 / T_{S}$ are eliminated. This will be shown for a simulation of an IEEE WLAN 802.11 signal.

In the IEEE WLAN 802.11a/g standard, the IFFT length is $N=64$, with $\Delta f=312.5 \mathrm{kHz}$ separation, so that $T_{S}=1 / \Delta f=3.2 \mu \mathrm{~s}$. The cyclic prefix is $T_{C P}=0.8 \mu \mathrm{~s}$, corresponding to 16 symbols, so that the total frame is $T=4 \mu s$, transmitting $64+16=80$ symbols per frame $\left(f_{\text {symbol }}=80 / 4 \mu s=20 \mathrm{Msps}\right.$. On the carriers/sub-channels $f_{n}=i / T_{S}, i=\{ \pm 7, \pm 21\}+32$ the
same sequence of (BPSK modulated) pilot tones are sent, with sign inversion for $i=+21$. Further, eleven sub-channels $i=\{-32, \ldots,-27,+27, \ldots,+31\}+32$ are unused to guarantee guard bands for neighbor transmitting systems. The sub-channel $i=0$ is also excluded to avoid bias (DC null). Therefore, in a complete frame, there are 48 user data sub-channels, 12 zero sub-channels, and 4 pilot sub-channels, totaling 64 symbols. With 16 cyclic prefix symbols, the complete frame is 80 symbols long.

As detailed in (Adrat et al., 2009) and also in (Spooner and Nicholls, 2009), although only for the cycle frequency location, the cycle features are located at the $(f ; \alpha)$ points: $\left(f_{c} ; 0\right) \pm$ $\{(2.1875 ; 0),(6.5625 ; 0),(0 ; 4.25),(4.375 ; 4.25),(0 ; 4.5),(4.375 ; 4.5),(2.1875 ; 8.75),(0 ; 13)$, $(0 ; 13.25)\} \mathrm{MHz}$, with small features at $\left(f_{c} ; 0\right) \pm\{(2.1875 ; 0.25),(6.5625 ; 0.25),(2.1875 ; 8.5)$, $(2.1875 ; 9)\} \mathrm{MHz}$. Further, cycle features are also located at $\left(0 ; 2 f_{c}\right) \pm\{(2.1875 ; 0),(6.5625 ; 0)$, $(0 ; 4.25),(4.375 ; 4.25),(0 ; 4.5),(4.375 ; 4.5),(2.1875 ; 8.75),(0 ; 13),(0 ; 13.25)\} \quad \mathrm{MHz}$, with small features at $\left(0 ; f_{c}\right) \pm\{(2.1875 ; 0.25),(6.5625 ; 0.25),(2.1875 ; 8.5),(2.1875 ; 9)\} \mathrm{MHz}$.

The simulation results for the IEEE 802.11 signal, with a sampling frequency of $f_{s}=40 \mathrm{MHz}$ and carrier frequency $f_{c}=10 \mathrm{MHz}$ are shown in Figure 5.26 (mesh plot) and Figure 5.27 (contour plot) when only the pilot tones are sent. As can be seen, the cyclic features agree with the expected results. Figure 5.28 shows the simulation results for the SCF when the random user data is also considered. The SCF was computed with $\Delta \alpha=12.5 \mathrm{kHz}$, where cycle frequency features can exist. As referenced, the cyclic features can exist at multiples of $1 / T$ and $1 / T_{S}$, which can be combined by searching at $\Delta \alpha=1 / \mathrm{lcm}\left(T, T_{S}\right)=1 / \mathrm{lcm}(4 \mu s, 3.2 \mu s)=1 / 80 \mu s=12.5 \mathrm{kHz}$.


Figure 5.26 - SCF of an IEEE 802.11 signal - just pilots.

Relating to the effect of the cyclic prefix, and as an example, for $\alpha \in[4.25 ; 4.5] \mathrm{MHz}$, there could be a feature at $\alpha=f_{n}-f_{m}$. With $n=21$ and $m=7$, it leads to $\alpha=4.375 \mathrm{MHz}$. But, as $\alpha \times T_{C P}=4.375 \times 10^{6} \times 0.8 \times 10^{-6}=3.5$ is not an integer, there is no peak at $\alpha=4.375 \mathrm{MHz}$. However, for $n=21$ and $m=-7, \alpha=8.75 \mathrm{MHz}$. In this case $\alpha \times T_{C P}=8.75 \times 10^{6} \times 0.8 \times$ $10^{-6}=7$ is an integer, and the SCF shows a peak at $\alpha=8.75 \mathrm{MHz}$.


Figure 5.27 - Contour plot of the SCF of an IEEE 802.11 signal - just pilots.


Figure 5.28 - SCF of an IEEE 802.11 signal with random user data.

Simulation results to access the performance of the CFAR SCoF detector will use the stronger cyclic feature at $\left(f_{c}+2.1875 \times 10^{6} ; 8.75 \times 10^{6}\right) \mathrm{Hz}$. For the far SCF point, we will use $\left(f_{c}+\right.$ $\left.2.1875 \times 10^{6} ; 12.5 \times 10^{6}\right) \mathrm{Hz}$, showing no cycle feature and low leakage. Similarly to the narrowband modulation case, we will consider a sampling frequency four times de symbol rate, and a carrier frequency equal to one-fourth of the sampling rate $\left(f_{\text {symbol }}=20 \mathrm{Msps}, f_{c}=20 \mathrm{MHz}, f_{s}=\right.$ 80 MHz ).

As seen in the previous subsections, the Nakagami slow-fading model caused the worst performance degradation; therefore, it is the model we will use here.

Figure 5.29 shows the results of the proposed CFAR SCoF detector for AWGN-only and Nakagami slow fading cases. Figure 5.30 shows similar results for the proposed CFAR SCF detector. As can be seen, the CFAR SCoF detector can maintain the prescribed $P_{F A}=0.1$ in both cases, with a detection performance degradation when considering the fading effect. Compared to the narrowband case (Figure 5.10), the performance is now worse. This was anticipated as the SCoF at the cycle features of the IEEE 802.11 signal is smaller than in the QPSK case.

As for the proposed CFAR SCF detector, the same conclusion can be drawn, but the detector fails to maintain a constant $P_{F A}$ at high SNR. As in the narrowband case, this was expected because, as the SNR increases, the leakage effect on the computation of the SCF will be increased so that the variance estimation will be less accurate. Also, as in the narrowband case, this detector shows better performance than the SCoF, and although the $P_{F A}$ can not be kept constant for high SNR, it is always less than the value prescribed.


Figure 5.29 - Proposed SCoF CFAR detector $P_{D}$ and $P_{F A}$ vs. SNR (IEEE 802.11 SoI).

From Figure 5.31, we can see that the proposed CDR SCF detector, works well in the case of the AWGN-only scenario. When considering the fading model, it fails to maintain a constant $P_{D}$.


Figure 5.30 - Proposed SCF CFAR detector $P_{D}$ and $P_{F A}$ vs. SNR (IEEE 802.11 SoI).


Figure 5.31 - Proposed SCF CDR detector $P_{D}$ and $P_{F A}$ vs. SNR (IEEE 802.11 SoI).

Finally, Figure 5.32 and Figure 5.33 shows the results for the CREST and ENTROPY detectors, respectively. The CREST detector now has a performance degradation when compared to the narrowband case, and performs worse than the CFAR SCoF or the CFAR SCF detectors.


Figure 5.32 - CREST detector $P_{D}$ and $P_{F A}$ vs. SNR (IEEE 802.11 SoI).


Figure 5.33 - ENTROPY detector $P_{D}$ and $P_{F A}$ vs. SNR (IEEE 802.11 SoI ).

As to the ENTROPY detector, as can be seen in Figure 5.33, the detector fails to work. This was anticipated as the OFDM modulation presents an almost flat Power Spectral Density (usually called a picket-fence spectrum). Therefore, the ENTROPY detector can not discriminate the signal (SoI) from a white noise spectrum.

### 5.5 Chapter Summary

This chapter presented the proposed Constant False-Alarm Rate (CFAR) detectors using the Spectral Coherence Function (SCoF) and the Spectral Correlation Function (SCF), and also the Constant Detection Rate (CDR) SCF detectors.

The detectors' performance was assessed by simulation, considering a narrowband (QPSK) and a wideband (IEEE 802.11 OFDM) modulation for the SoI. The results include the fading channels with multipath fading model parameters obtained from real case scenarios (obtained from the open literature), namely Rayleigh, Rice, Nakagami, $\kappa-\mu, \eta-\mu$, and also the shadowing channel model. The proposed detectors performance was compared to other proposed detectors, namely the CREST and the ENTROPY detectors.

It was shown that the proposed detectors outperform any of the other detectors, for AWGN-only and when considering any fading model. The performance of the proposed detectors is almost unaffected when fast-fading is considered but shows performance degradation in the slow-fading case. Specifically, the CDR SCF detector shows unacceptable performance in the slow-fading case but is also almost unaffected in the fast-fading scenario.

The CFAR SCF detector, although needing the computation of two points of the SCF, performs better than the CFAR SCoF. Although it can not maintain the prescribed False-Alarm probability, it outperforms the CFAR SCoF detector presenting a lower $P_{F A}$.

## Chapter 6

## 6 Conclusions and Future Work

Cognitive Radio has been proposed as an enabling solution to achieve a feasible Dynamic Spectrum Access/Sharing in an opportunistic model. As secondary users must quickly and accurately detect the presence or absence of the primary user, spectrum sensing takes a crucial role in Cognitive Radio.

The primary objectives of this thesis were to analyze the best sensing techniques to achieve an efficient and reliable wideband detection of the used/idle spectral bands and propose adapted and new constant false-alarm rate and constant detection rate detectors.

### 6.1 Conclusions

A common comparison procedure was used to analyze the detection performance of four major digital signal processing techniques for spectrum sensing: Energy Detection, Matched Filter Detection, Eigenvalue Based Detection, and Cyclostationary Feature Detection.

Having low computational complexity, the energy eetector is widely used for spectrum sensing. However, it presents a severe performance degradation when there is noise uncertainty, rendering it useless in a low SNR environment. The matched filter detector is optimal for detecting a known PU signal in AWGN. However, different spectrum bands may have PUs with different technologies, leading to excessive computational complexity when sensing a wideband spectrum. Further, it is very sensitive to synchronization errors.

With higher computational complexity, the eigenvalue based detectors can perform well but leads to a high sensing time, as the adaptive threshold estimation for a given False-Alarm probability needs a very high sample size.

Finally, although with high computational complexity, the cyclostationary feature detection was shown to have the best detection performance, using the cycle features of (probably almost) all human-made communication signals. The spectral correlation function (SCF) detector is much less sensitive to noise uncertainty than the energy detector. However, it needs some information about the PU signal that generates the cyclostationary features (e.g., data rate, cyclic prefix, etc.). The spectral coherence function (SCoF) detector was shown to be the best blind detector (with similar
performance as the SCF when there is no noise uncertainty) with respect to the noise statistics, but it also needs some information about the PU signal that generates the cyclostationary features.

When cyclostationary features' location is not exactly known and must be searched within the frequency and cycle frequency plane ( $f ; \alpha$ ), cyclostationarity detection can be problematic for its computational complexity. After presenting the theory of cyclic spectral analysis, one of the most widely used algorithms to compute de spectral correlation function (and, therefore, the spectral coherence function) - the FFT Accumulation Method (FAM) - was detailed, to be adapted to compute the SCF in a small region of the ( $f ; \alpha$ ) plane. Generally, signal detection is accomplished by computing the SCF or the SCoF where the signal presents dominant cycle features, corresponding to a point or only a small region of the $(f ; \alpha)$ plane.

The zoom FAM (zFAM) algorithm was then proposed and detailed, and its computational cost was compared to the original FAM. When the entire ( $f ; \alpha$ ) plane needs to be computed, the FAM is naturally more efficient than the zFAM. However, for practical cases, when a small fraction of the plane is needed, the zFAM can perform significantly better than the FAM algorithm, reaching less than $5 \%$ of the FAM computational cost in many cases. Then, and for the FAM algorithm, the probability density of the spectral estimates of the SCF and SCoF were obtained, alongside useful approximations, when the input signal is a Gaussian $\mathcal{N}\left(0 ; \sigma^{2}\right)$ r.v. When a large number of averaging segments is used in the algorithm, it was shown that the estimates could be modeled as a Gaussian r.v., except for $\alpha=0$, when they are best modeled having a gamma density.

For the SCoF real and imaginary components, it was shown that a Gaussian density could be assumed (for a high number of averaging segments), with zero mean and variance only dependent on the segment overlap and number of averaging segments. This characterization of the SCF and SCoF obtained for the FAM algorithm allowed the development of two cyclostationary Constant False-Alarm Rate (CFAR) and one Constant Detection Rate (CDR) detector.

Using the statistical characterization of the SCoF, a new CFAR detector was proposed, with an adaptive threshold estimate obtained directly from the overlap and number of averaging segments, for a given Probability of False-Alarm, $P_{F A}$. A new CFAR detector, now using the SCF, was also proposed, with the estimation of the SCF variance using a point in the $(f ; \alpha)$ plane that does not present any cyclic feature. Similarly to estimating the noise variance, this also allowed obtaining an adaptive threshold for a given $P_{F A}$. Also, a new Constant Detection Rate (CDR) detector was proposed using these two points of the SCF.

The three detectors' performance was assessed by simulation for narrowband and wideband input signal cases (QPSK and OFDM). Further, the simulations included not only the AWGN-only case but also considered the multipath fading, with fading parameters obtained from real case scenarios. Fading included the Rayleigh, Rice, Nakagami, $\kappa-\mu, \eta-\mu$, and shadowing channel models. The detectors' performance was compared to other detectors, namely the CREST and the ENTROPY detectors.

It was shown that the proposed detectors outperform any of the other detectors for AWGN-only (including classic detectors) and when considering any fading model. The performance of the proposed detectors is almost unaffected when fast-fading is considered, but it shows degradation
in the slow-fading case. Further, the CDR SCF detector shows unacceptable performance in the slow-fading case but is also almost unaffected by fast-fading.

The CFAR SCF detector, although needing the computation of two points of the SCF, performs better than the CFAR SCoF. Although it can not maintain the prescribed False-Alarm probability at high SNR - as spectral leakage will be increased in the computation of the SCF, so that the variance estimation will be less accurate - it outperforms the CFAR SCoF detector presenting a lower $P_{F A}$.

### 6.2 Future work

In order to obtain a real-time Digital Signal Processing (DSP) prototype for spectrum sensing, based on the detectors proposed, a hardware implementation needs to be developed. We will aim to develop the proposed algorithms for an FPGA implementation, processing data from an RF front-end, implemented by SDR units. The sampled data from the RF front-end ADC will be processed in the DSP unit that will implement the detectors' algorithms. With a functional prototype, we envision the possibility of articulating with the work developed in 2020 by ANACOM's Leased Shared Access test pilots, to develop joint collaboration projects using both approaches.

Typically, when analyzing some detector performance, the Primary User (PU) is modeled as remaining stationary during the sensing period (as was done in this thesis). A possibly more realistic model is to consider that the PU is non-stationary, consisting of a noise-only segment, followed by a segment with a PU signal corrupted by noise. Analysis of the detectors' performance under a PU signal model with variable random duty-cycle (noise only and signal plus noise components) may reflect a more realistic scenario for spectrum sensing.

Although this research work focused on the FAM algorithm, we think that the methodology used to vectorize and obtain a quadratic form for the FAM algorithm can be extended to the other popular algorithm to compute the SCF: the Strip Spectral Correlation Analyzer (SSCA). This would allow to obtain its spectral estimates for noise-only input and develop similar CFAR and CDR detectors.

As a mid/long-term project, and not specifically addressing this research work, we will try to apply for a multi-institution project with European funding to develop a Cognitive Radio Testbed, similar to the FIT/CorteXlab ${ }^{43}$ located in Lyon, France. This infrastructure will comprise multiple SDR or FPGA DSP nodes, allowing programming from the physical layer up, and presenting a high level of RF flexibility. The Lab could be located outside Beja, using part of the former regional supply market (a near $5,370 \mathrm{~m}^{2}$ unused facility), isolated from external interference with shielding. Network connection from the nodes to a central processing unit guarantees access to program each node and also remote access to the system. A cooperation protocol between the project partners

[^30]and the French FIT Initiative/OneLab Consortium will allow the knowledge transfer necessary to the hardware/software implementation of the Lab, providing shared access to all the members.

## Appendix A-Demonstrations

## Demonstration 2.1 - Probability of Detection, normalized energy statistic

Considering the case when there is only noise, the null hypothesis, $\mathcal{H}_{0}$, and the case when the PU is transmitting the alternative hypothesis, $\mathcal{H}_{1}$, the sampled received signal, $r(n)$, is given by (under each hypothesis) as:

$$
\begin{align*}
& \mathcal{H}_{o}: r(n)=w(n) \\
& \mathcal{H}_{1}: r(n)=s(n)+w(n) \tag{A.1}
\end{align*}
$$

Using the normalized energy of the received signal as the test statistic, $T(\boldsymbol{r})=\boldsymbol{r}^{\prime} \boldsymbol{r} / \sigma^{2}$. When only noise is present, $T(\boldsymbol{r})$ is the sum of $N$ squared standard normal r.v., so its probability density function (pdf) is a (central) chi-square distribution with $N$ degrees of freedom. When the received signal has noise plus the Signal-of-Interest, $T(\boldsymbol{r})$ is the sum $N$ squared normal r.v. with unit variance and mean equal to $s(n)^{2} / \sigma^{2}$, so its pdf is a non-central chi-square distribution with $N$ degrees of freedom and noncentrality parameter $\lambda=\sum_{n=1}^{N} s(n)^{2} / \sigma^{2}=$ $E_{s} / \sigma^{2}:$

$$
\begin{align*}
& \mathcal{H}_{o}: T(\boldsymbol{r}) \sim \chi_{N}^{2} \\
& \mathcal{H}_{1}: T(\boldsymbol{r}) \sim \chi_{N}^{2}\left(E_{s} / \sigma^{2}\right)=\chi_{N}^{2}(\lambda) \tag{A.2}
\end{align*}
$$

Given $F_{\chi_{N}^{2}(.)}$ ) the cumulative distribution function (CDF) of $\chi_{N}^{2}$, and for a given $P_{F A}$, the detection threshold, $\gamma$, is given by

$$
\gamma=F_{\chi_{N}^{2}}^{-1}\left(1-P_{F A}\right) .
$$

The pdf of a r.v. $X$ with a (central) chi-square distribution with $N$ degrees of freedom is given by

$$
f_{X}(x)=\frac{1}{2^{N / 2} \Gamma(N / 2)} x^{\frac{N}{2}-1} e^{-\frac{x}{2}} .
$$

When a r.v. Y is a non-central chi-square distributed with $N$ degrees of freedom and noncentrality parameter $\lambda$, the pdf is given by:

$$
f_{Y}(y)=\frac{1}{2} e^{-\frac{y+\lambda}{2}}\left(\frac{y}{\lambda}\right)^{\frac{N}{4}-\frac{1}{2}} \mathrm{I}_{\frac{N}{2}-1}(\sqrt{\lambda y}) .
$$

Given $F_{X}(x)$ as the CDF of $f_{X}(x)$, the threshold needed to attain a probability of false alarm, $P_{F A}$, is given by $\gamma=F_{X}^{-1}\left(1-P_{F A}\right)$. Using this threshold, the probability of detection is given by:

$$
\begin{equation*}
P_{D}=\int_{\gamma}^{+\infty} f_{Y}(y) d y=\int_{\gamma}^{+\infty} \frac{1}{2} e^{-\frac{y+\lambda}{2}}\left(\frac{y}{\lambda}\right)^{\frac{N}{4}-\frac{1}{2}} \mathrm{I}_{\frac{N}{2}-1}(\sqrt{\lambda y}) d y \tag{A.3}
\end{equation*}
$$

The generalized Marcum Q-function is given by:

$$
Q_{m}(a, b)=\frac{1}{a^{m-1}} \int_{b}^{+\infty} x^{m} e^{-\frac{x^{2}-a^{2}}{2}} \mathrm{I}_{m-1}(a x)
$$

or, alternatively, changing variables $y=x^{2}$ :

$$
\begin{equation*}
Q_{m}(a, b)=\frac{1}{2} \frac{1}{a^{m-1}} \int_{b^{2}}^{+\infty} y^{\frac{m}{2}-\frac{1}{2}} e^{-\frac{y-a^{2}}{2}} \mathrm{I}_{m-1}(a \sqrt{y}) \tag{A.4}
\end{equation*}
$$

Using (A.3) and (A.4), it is easily obtained that:

$$
P_{D}=Q_{N / 2}(\sqrt{\lambda}, \sqrt{\gamma})
$$

Demonstration 4.1 - pdf of the real and imaginary components of an r.v. given by $\underline{C=e^{j \theta} \text {, where } \theta \text { is uniformly distributed: } \theta \sim \mathcal{U}(0,2 \pi) .}$

The real and imaginary parts of $C$ are given by:

$$
\begin{aligned}
& R=\operatorname{Re}\{C\}=\cos (\theta) \\
& I=\operatorname{Im}\{C\}=\sin (\theta)
\end{aligned}
$$

Considering the real component, $R=\cos (\theta)$, one has $r=g(\theta)=\cos (\theta)$.In the interval $[-\pi, \pi]$ there are two solutions: $\theta_{1}=\cos ^{-1}(r)$ and $\theta_{2}=2 \pi-\cos ^{-1}(r)$. Using (A.7), one gets:

$$
\begin{aligned}
& f_{R}(r)=\frac{f_{\Theta}\left(\theta_{1}\right)}{\left|g^{\prime}\left(\theta_{1}\right)\right|}+\frac{f_{\Theta}\left(\theta_{2}\right)}{\left|g^{\prime}\left(\theta_{2}\right)\right|} \\
& =\frac{1}{2 \pi} \frac{1}{\left|-\sin \left(\cos ^{-1}(r)\right)\right|}+\frac{1}{2 \pi} \frac{1}{\left|-\sin \left(2 \pi-\cos ^{-1}(r)\right)\right|} \\
& =\frac{1}{\pi} \frac{1}{\left|\sin \left(\cos ^{-1}(r)\right)\right|}
\end{aligned}
$$



Using the trigonometric circle (unit circle), it is simple to evaluate $\sin \left(\cos ^{-1}(r)\right)$ as $\sqrt{1-r^{2}}$, so one finally gets:

$$
\begin{equation*}
f_{R}(r)=\frac{1}{\pi \sqrt{1-r^{2}}}, \quad-1<r<1 \tag{A.5}
\end{equation*}
$$

Similar analysis on the imaginary part leads to the exact same result.

## Demonstration 4.2 - Quadratic form using a symmetric version of the matrix

Given the column vector $\boldsymbol{x}$, and an arbitrary matrix $\boldsymbol{A}$, the quadratic form $\boldsymbol{x}^{T} \boldsymbol{A} \boldsymbol{x}$ is equal to $\boldsymbol{x}^{T} \frac{\boldsymbol{A}+\boldsymbol{A}^{T}}{2} \boldsymbol{x}$.

$$
\begin{aligned}
\boldsymbol{x}^{T} \frac{\boldsymbol{A}+\boldsymbol{A}^{T}}{2} \boldsymbol{x} & =\frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{x}^{T} \boldsymbol{A} \boldsymbol{x}+\boldsymbol{x}^{T} \boldsymbol{A}^{T} \boldsymbol{x}\right)=\frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{x}^{T} \boldsymbol{A} \boldsymbol{x}+\left(\boldsymbol{x}^{T} \boldsymbol{A}^{T} \boldsymbol{x}\right)^{T}\right) \frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{x}^{T} \boldsymbol{A} \boldsymbol{x}+\left(\boldsymbol{x}^{T} \boldsymbol{A} \boldsymbol{x}\right)\right) \\
& =\boldsymbol{x}^{T} \boldsymbol{A} \boldsymbol{x}
\end{aligned}
$$

## Demonstration 4.3 - The eigenvalues of a hermitian matrix are all real

Consider a hermitian matrix $\boldsymbol{A}=\boldsymbol{A}^{H}$ And let $\boldsymbol{v}$ be a (possibly) complex eigenvector of a (possibly) complex eigenvalue, $\lambda$, then, for $\boldsymbol{v} \neq \mathbf{0}, \boldsymbol{A} \boldsymbol{v}=\lambda \boldsymbol{v}$. Therefore:

$$
\begin{equation*}
\boldsymbol{v}^{H} \boldsymbol{A} \boldsymbol{v}=\boldsymbol{v}^{H} \lambda \boldsymbol{v}=\lambda \boldsymbol{v}^{H} \boldsymbol{v} \tag{A.6}
\end{equation*}
$$

Applying the hermitian operator to equation (A.6), one gets:

$$
\begin{aligned}
& \left(\boldsymbol{v}^{H} \boldsymbol{A} \boldsymbol{v}\right)^{H}=\left(\lambda \boldsymbol{v}^{H} \boldsymbol{v}\right)^{H} \\
& \boldsymbol{v}^{H} \boldsymbol{A}^{H} \boldsymbol{v}=\lambda^{*} \boldsymbol{v}^{H} \boldsymbol{v} \\
& \boldsymbol{v}^{H} A \boldsymbol{v}=\lambda^{*} \boldsymbol{v}^{H} \boldsymbol{v}
\end{aligned}
$$

Using (A.6) implies that $\left(\lambda^{*}-\lambda\right) \boldsymbol{v}^{H} \boldsymbol{v}$, and, as $\boldsymbol{v} \neq \mathbf{0}$, one must have $\lambda^{*}=\lambda$, so the eigenvalues are real.

As a real symmetric matrix is hermitian, its eigenvalues are also real.

## Demonstration 4.4 - Probability Density Function of the square root of the product of gamma r.v.

Consider two independent gamma r.v. $X_{1,2} \sim \Gamma\left(\lambda_{1,2} ; \alpha\right)$ with shape parameters $\lambda_{1,2}$ and rate parameter $\alpha$. The pdf of the product $Y=X_{1} X_{2}$ is given by (Gaunt, 2016):

$$
p_{Y}(y)=\frac{2 \alpha^{\lambda_{1}+\lambda_{2}}}{\Gamma\left(\lambda_{1}\right) \Gamma\left(\lambda_{2}\right)} y^{\frac{\lambda_{1}+\lambda_{2}}{2}-1} K_{\lambda_{1}-\lambda_{2}}(2 \alpha \sqrt{y}), \quad y>0
$$

which, for $\lambda_{1}=\lambda_{2}=\lambda$ reduces to:

$$
p_{Y}(y)=\frac{2 \alpha^{2 \lambda}}{\Gamma^{2}(\lambda)} y^{\lambda-1} K_{o}(2 \alpha \sqrt{y}), \quad y>0 .
$$

Given a r.v. Y with pdf $f_{Y}(y)$ and a differentiable function $g(Y)$, the pdf of $z=g(Y)$ is given by

$$
\begin{equation*}
f_{Z}(z)=\sum_{i=1}^{n} \frac{f_{X}\left(x_{i}\right)}{\left|g^{\prime}\left(x_{i}\right)\right|} \tag{A.7}
\end{equation*}
$$

Where $x_{i}$ are the $n$ roots of $z=g(x)$ and $g^{\prime}\left(x_{i}\right) \neq 0$.
Considering $Z=g(Y)=\sqrt{Y}$, then $y_{1}=z^{2}$, so:

$$
\begin{aligned}
p_{Z}(z) & =\frac{1}{\left|g^{\prime}\left(y_{1}\right)\right|} p_{Y}\left(y_{1}\right) \\
& =2 z \times p_{Y}\left(z^{2}\right)
\end{aligned}
$$

Therefore,

$$
p_{Z}(z)=\frac{4 \alpha^{2 \lambda}}{\Gamma^{2}(\lambda)} z^{2 \lambda-1} K_{o}(2 \alpha z), \quad z>0
$$

## Demonstration 4.5 - Probability Density Function of the ratio of a variance gamma and square root of the product of gamma r.v.

Consider two independent gamma r.v. $G_{1,2} \sim \Gamma\left(\lambda_{y} ; \alpha_{y}\right)$. The pdf of the square root of the product of the r.v, $Y=\sqrt{G_{1} G_{2}}$ is given by (Demonstration 2.2):

$$
p_{Y}(y)=\frac{4 \alpha_{y}^{2 \lambda_{y}}}{\Gamma^{2}\left(\lambda_{y}\right)} y^{2 \lambda_{y}-1} K_{o}\left(2 \alpha_{y} y\right), \quad y>0
$$

Given a general variance gamma r.v., $W \sim V G\left(\lambda_{x} ; \alpha_{x} ; \beta_{x} ; \mu_{x}\right)$, with pdf:

$$
p_{W}(w)=\frac{\left(\alpha_{x}^{2}-\beta_{x}^{2}\right)^{\lambda_{x}}}{\sqrt{\pi} \Gamma\left(\lambda_{x}\right)}\left(\frac{\left|w-\mu_{x}\right|}{2 \alpha_{x}}\right)^{\lambda_{x}-\frac{1}{2}} e^{\beta_{x}\left(w-\mu_{x}\right)} K_{\lambda_{x}-\frac{1}{2}}\left(\alpha_{x}\left|w-\mu_{x}\right|\right)
$$

consider $X \sim V G\left(\lambda_{x} ; \alpha_{x} ; 0 ; 0\right)$ with pdf:

$$
p_{X}(x)=\frac{\alpha_{x}^{2 \lambda_{x}}}{\sqrt{\pi} \Gamma\left(\lambda_{x}\right)}\left(\frac{|x|}{2 \alpha_{x}}\right)^{\lambda_{x}-\frac{1}{2}} K_{\lambda_{x}-\frac{1}{2}}\left(\alpha_{x}|x|\right) .
$$

The pdf of the r.v. obtained as the ratio $Z=X / Y$ is given as

$$
p_{Z}(z)=\int_{-\infty}^{+\infty}|y| p_{X}(z y) p_{Y}(y) d y
$$

$$
\begin{aligned}
& =\int_{0}^{+\infty} y \frac{\alpha_{x}^{2 \lambda_{x}}}{\sqrt{\pi} \Gamma\left(\lambda_{x}\right)}\left(\frac{|z y|}{2 \alpha_{x}}\right)^{\lambda_{x}-\frac{1}{2}} K_{\lambda_{x}-\frac{1}{2}}\left(\alpha_{x}|z y|\right) \frac{4 \alpha_{y}^{2 \lambda_{y}}}{\Gamma^{2}\left(\lambda_{y}\right)} y^{2 \lambda_{y}-1} K_{o}\left(2 \alpha_{y} y\right) d y \\
& =\frac{4 \alpha_{x}^{2 \lambda_{x}} \alpha_{y}^{2 \lambda_{y}}}{\sqrt{\pi} \Gamma\left(\lambda_{x}\right) \Gamma^{2}\left(\lambda_{y}\right)} \frac{1}{\left(2 \alpha_{x}\right)^{\lambda_{x}-\frac{1}{2}}}|z|^{\lambda_{x}-\frac{1}{2}} \int_{0}^{+\infty} y^{\lambda_{x}+2 \lambda_{y}-\frac{1}{2}} K_{\lambda_{x}-\frac{1}{2}}\left(\alpha_{x}|z| y\right) K_{o}\left(2 \alpha_{y} y\right) d y .
\end{aligned}
$$

Using (6.576) from (Gradshteyn and Ryzhik, 2007) ${ }^{44}$, one finally gets:

$$
\begin{align*}
p_{Z}(z) & =\left(\frac{2 \alpha_{y}}{\alpha_{x}}\right)^{2 \lambda_{y}} \frac{\Gamma^{2}\left(\lambda_{x}+\lambda_{y}\right) \Gamma^{2}\left(\lambda_{y}+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma\left(\lambda_{x}\right) \Gamma^{2}\left(\lambda_{y}\right) \Gamma\left(\lambda_{x}+2 \lambda_{y}+\frac{1}{2}\right)}|z|^{-2 \lambda_{y}-1}  \tag{A.8}\\
& \times{ }_{2} F_{1}\left(\lambda_{x}+\lambda_{y}, \lambda_{y}+\frac{1}{2} ; \lambda_{x}+2 \lambda_{y}+\frac{1}{2} ; 1-\left(\frac{2 \alpha_{y}}{\alpha_{x}|z|}\right)^{2}\right)
\end{align*}
$$

## Approximation of $p_{Z}(z)$ as $|z| \rightarrow 0$

The asymptotic approximation of the Gauss hypergeometric function ${ }_{2} F_{1}(a, b ; c ; z)$ as $|z| \rightarrow \infty$ is:

$$
{ }_{2} F_{1}(a, b ; c ; z) \approx \frac{\Gamma(b-a) \Gamma(c)}{\Gamma(b) \Gamma(c-a)}(-z)^{-a}+\frac{\Gamma(a-b) \Gamma(c)}{\Gamma(a) \Gamma(c-b)}(-z)^{-b} \quad,|z| \rightarrow \infty
$$

Also, $1-\left(\frac{2 \alpha_{y}}{\alpha_{x}|z|}\right)^{2} \approx-\left(\frac{2 \alpha_{y}}{\alpha_{x}|z|}\right)^{2}$ as $|z| \rightarrow 0$, and using the above approximation, in (A.8), one gets:

$$
p_{Z}(z) \approx\left(\frac{2 \alpha_{y}}{\alpha_{x}}\right)^{2 \lambda_{y}} \frac{\Gamma^{2}\left(\lambda_{x}+\lambda_{y}\right) \Gamma^{2}\left(\lambda_{y}+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma\left(\lambda_{x}\right) \Gamma^{2}\left(\lambda_{y}\right) \Gamma\left(\lambda_{x}+2 \lambda_{y}+\frac{1}{2}\right)}|z|^{-2 \lambda_{y}-1}
$$

44

$$
\begin{aligned}
\int_{0}^{\infty} x^{-\lambda} K_{\mu}(a x) K_{\nu}(b x) d x & =\frac{2^{-2-\lambda} a^{-\nu+\lambda-1} b^{\nu}}{\Gamma(1-\lambda)} \Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \\
& \times \Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right) \\
& \times{ }_{2} F_{1}\left(\frac{1-\lambda+\mu+\nu}{2}, \frac{1-\lambda-\mu+\nu}{2} ; 1-\lambda ; 1-\frac{b^{2}}{a^{2}}\right)
\end{aligned}
$$

$[\operatorname{Re} a+b>0, \quad \operatorname{Re} \lambda<1-|\operatorname{Re} \mu|-|\operatorname{Re} \nu|]$

$$
\begin{aligned}
& \times\left[\frac{\Gamma\left(\frac{1}{2}-\lambda_{x}\right) \Gamma\left(\lambda_{x}+2 \lambda_{y}+\frac{1}{2}\right)}{\Gamma^{2}\left(\lambda_{y}+\frac{1}{2}\right)}\left(\frac{2 \alpha_{y}}{\alpha_{x}|z|}\right)^{-2\left(\lambda_{x}+\lambda_{y}\right)}\right. \\
& \left.\quad+\frac{\Gamma\left(\lambda_{x}-\frac{1}{2}\right) \Gamma\left(\lambda_{x}+2 \lambda_{y}+\frac{1}{2}\right)}{\Gamma^{2}\left(\lambda_{x}+\lambda_{y}\right)}\left(\frac{2 \alpha_{y}}{\alpha_{x}|z|}\right)^{-2\left(\lambda_{y}+\frac{1}{2}\right)}\right] \\
& =\left(\frac{\alpha_{x}}{2 \alpha_{y}}\right)^{2 \lambda_{x}} \frac{\Gamma^{2}\left(\lambda_{x}+\lambda_{y}\right) \Gamma\left(\frac{1}{2}-\lambda_{x}\right)}{\sqrt{\pi} \Gamma\left(\lambda_{x}\right) \Gamma^{2}\left(\lambda_{y}\right)}|z|^{2 \lambda_{x}-1}+\frac{\alpha_{x}}{2 \alpha_{y}} \frac{\Gamma^{2}\left(\lambda_{y}+\frac{1}{2}\right) \Gamma\left(\lambda_{x}-\frac{1}{2}\right)}{\sqrt{\pi} \Gamma\left(\lambda_{x}\right) \Gamma^{2}\left(\lambda_{y}\right)}, \quad|z| \rightarrow 0
\end{aligned}
$$

Therefore, $p_{Z}(0)$ can be approximated by:

$$
\begin{align*}
\lim _{|z| \rightarrow 0}\left\{p_{Z}(z)\right\} & =\frac{\alpha_{x}}{2 \alpha_{y}} \frac{\Gamma^{2}\left(\lambda_{y}+\frac{1}{2}\right) \Gamma\left(\lambda_{x}-\frac{1}{2}\right)}{\sqrt{\pi} \Gamma\left(\lambda_{x}\right) \Gamma^{2}\left(\lambda_{y}\right)} \\
& =\frac{\alpha_{x}}{2 \sqrt{\pi} \alpha_{y}}\left(\frac{\Gamma\left(\lambda_{y}+\frac{1}{2}\right)}{\Gamma\left(\lambda_{y}\right)}\right)^{2}\left(\frac{\Gamma\left(\lambda_{x}-\frac{1}{2}\right)}{\Gamma\left(\lambda_{x}\right)}\right) \tag{A.9}
\end{align*}
$$

When $x \rightarrow+\infty, \Gamma(x+\alpha) \sim \Gamma(x) x^{\alpha}, \alpha \in \mathbb{C}$. Therefore, for high $\lambda_{x}, \lambda_{y}$ (A.9) can be approximated by:

$$
\begin{equation*}
\lim _{|z| \rightarrow 0}\left\{p_{Z}(z)\right\}=\frac{\alpha_{x}}{2 \sqrt{\pi} \alpha_{y}} \frac{\lambda_{y}}{\sqrt{\lambda_{x}}} \tag{A.10}
\end{equation*}
$$

## Demonstration 5.1 - Mean and Variance of square $\mathcal{N}\left(\mu, \sigma^{2}\right)$ r.v

If $X$ is a r.v with $E[X]=\mu$ and $\operatorname{var}(X)=\sigma^{2}$, then, for $Y=X^{2}$ one has:

$$
\begin{align*}
E[Y] & =E\left[X^{2}\right]=(E[X])^{2}-\operatorname{var}(X)=\mu^{2}+\sigma^{2} \\
\operatorname{var}(\mathrm{Y}) & =\operatorname{var}\left(X^{2}\right)=E\left[X^{4}\right]-\left(E\left[X^{2}\right]\right)^{2}  \tag{A.11}\\
& =E\left[X^{4}\right]-\sigma^{4}-2 \sigma^{2} \mu^{2}-\mu^{4}
\end{align*}
$$

If $X \sim \mathcal{N}\left(\mu, \sigma^{2}\right)$, then $X=W+\mu$, where $W \sim \mathcal{N}\left(0, \sigma^{2}\right)$, for which

$$
E\left[W^{n}\right]=(n-1) \sigma^{2} E\left[W^{n-2}\right]
$$

and, for $n$ odd $(n=2 k+1) E\left[W^{2 k+1}\right]=0$.
We can now evaluate $E\left[X^{4}\right]$ by

$$
\begin{aligned}
E\left[X^{4}\right] & =E\left[(W+\mu)^{4}\right] \\
& =\underbrace{E\left[W^{4}\right]}_{3 \sigma^{2} E\left[W^{2}\right]}+4 \mu^{3} \underbrace{E\left[W^{3}\right]}_{=0}+6 \mu^{2} E\left[W^{2}\right]+4 \mu^{3} \underbrace{E[W]}_{=0}+E\left[\mu^{4}\right] \\
& =3 \sigma^{4}+6 \mu \sigma^{2}+\mu^{4} .
\end{aligned}
$$

From (A.11), we get:

$$
\begin{align*}
\operatorname{var}(\mathrm{Y}) & =E\left[X^{4}\right]-\sigma^{4}-2 \sigma^{2} \mu^{2}-\mu^{4} \\
& =3 \sigma^{4}+6 \mu \sigma^{2}+\mu^{4}-\sigma^{4}-2 \sigma^{2} \mu^{2}-\mu^{4}  \tag{A.12}\\
& =2 \sigma^{2}\left(\sigma^{2}+2 \mu^{2}\right) .
\end{align*}
$$

## Demonstration 5.2 - cdf from the pdf given by (5.26)

Given the pdf:

$$
f_{S}(s)=\left(\frac{2 \sigma^{2} K}{2 \sigma^{2} K+\Omega}\right) \frac{s}{\sigma^{2}} e^{-\frac{s^{2}}{2 \sigma^{2}}}{ }_{1} F_{1}\left(K ; 1 ; \frac{\Omega s^{2}}{2 \sigma^{2}\left(2 \sigma^{2} K+\Omega\right)}\right)
$$

and making the change of variable $y=s^{2}$, one gets:

$$
f_{Y}(y)=\frac{1}{2 \sigma^{2}}\left(\frac{2 \sigma^{2} K}{2 \sigma^{2} K+\Omega}\right) e^{-\frac{y}{2 \sigma^{2}}}{ }_{1} F_{1}\left(K ; 1 ; \frac{\Omega y}{2 \sigma^{2}\left(2 \sigma^{2} K+\Omega\right)}\right)
$$

The Moment Generating Function of $f_{Y}(y)$ is:

$$
\begin{gathered}
M G F_{Y}(s)=\mathcal{L}\left\{f_{Y}(y) ;-s\right\}=\mathcal{L}\left\{e^{-\frac{y}{2 \sigma^{2}}} f_{Y}^{1}(y) ;-s\right\} \\
=\left.\mathcal{L}\left\{f_{Y}^{1}(y) ;-s\right\}\right|_{s=s-\frac{1}{2 \sigma^{2}}}
\end{gathered}
$$

where

$$
\begin{equation*}
f_{Y}^{1}(y)=\frac{1}{2 \sigma^{2}}\left(\frac{2 \sigma^{2} K}{2 \sigma^{2} K+\Omega}\right) \quad{ }_{1} F_{1}\left(K ; 1 ; \frac{\Omega y}{2 \sigma^{2}\left(2 \sigma^{2} K+\Omega\right)}\right) . \tag{A.13}
\end{equation*}
$$

From (Magnus, Oberhettinger and Tricomi, 1954)[4.23.1], the following Laplace pair can be used to obtain $\mathcal{L}\left\{f_{Y}^{1}(y) ;-s\right\}$ :

$$
t^{\gamma-1} \times{ }_{1} F_{1}(\alpha ; \gamma ; \lambda t) \stackrel{\mathcal{L}}{\leftrightarrow} \Gamma(\gamma) s^{\alpha-\gamma}(s-\lambda)^{-\alpha}
$$

So that, with $\gamma=1, \alpha=K$ and $\lambda=\Omega / 2 \sigma^{2}\left(2 \sigma^{2} K+\Omega\right)$, one gets:

$$
\begin{equation*}
\mathcal{L}\left\{f_{Y}^{1}(y) ;-s\right\}=\frac{1}{2 \sigma^{2}}\left(\frac{2 \sigma^{2} K}{2 \sigma^{2} K+\Omega}\right)^{K} \underbrace{\Gamma(1)}_{=1} s^{K-1}\left(s-\frac{\Omega}{2 \sigma^{2}\left(2 \sigma^{2} K+\Omega\right)}\right)^{-K} . \tag{A.14}
\end{equation*}
$$

Therefore, after some manipulation, one gets:

$$
\begin{aligned}
& \operatorname{MGF}_{Y}(s)=\left.\mathcal{L}\left\{f_{Y}^{1}(y) ;-s\right\}\right|_{s=s-\frac{1}{2 \sigma^{2}}} \\
& =\frac{1}{2 \sigma^{2}}\left(\frac{2 \sigma^{2} K}{2 \sigma^{2} K+\Omega}\right)^{K} \frac{1}{(-s)}\left(1-\frac{1}{2 \sigma^{2} s}\right)^{K-1}\left(1-\frac{K}{\left(2 \sigma^{2} K+\Omega\right) s}\right)^{-K}
\end{aligned}
$$

Now, the cdf of $y$ can be obtained from the MGF as:

$$
\begin{aligned}
& F_{Y}(y)=\mathcal{L}\left\{\frac{M G F_{Y}(-s)}{s} ; y\right\} \\
& =\mathcal{L}\left\{\frac{1}{2 \sigma^{2}}\left(\frac{2 \sigma^{2} K}{2 \sigma^{2} K+\Omega}\right)^{K} s^{-2}\left(1+\frac{1}{2 \sigma^{2} s}\right)^{-(1-K)}\left(1+\frac{K}{\left(2 \sigma^{2} K+\Omega\right) s}\right)^{-K}\right\}
\end{aligned}
$$

From (Magnus, Oberhettinger and Tricomi, 1954)[4.24.3], the following Laplace pair can be used to obtain $F_{Y}(y)$ :

$$
t^{\gamma-1} \times \Phi_{2}\left(\beta, \beta^{\prime}, \gamma, x t ; \gamma t\right) \stackrel{\mathcal{L}}{\leftrightarrow} \Gamma(\gamma) s^{-\gamma}\left(1-\frac{x}{s}\right)^{-\beta}\left(1-\frac{\gamma}{s}\right)^{-\beta^{\prime}}
$$

where $\Phi_{2}$ is the confluent hypergeometric function given by (Gradshteyn and Ryzhik, 2007)[9.261.2] ${ }^{45}$

$$
\begin{aligned}
\Phi_{2}\left(\beta, \beta^{\prime}, \gamma, x, y\right) & =\sum_{m, n=0}^{\infty} \frac{(\beta)_{m}\left(\beta^{\prime}\right)_{n}}{m!n!(\gamma)_{m+n}} x^{m} y^{n} \\
& =\sum_{k=0}^{\infty} \frac{(\beta)_{k}}{k!(\gamma)_{k}} x^{k}{ }_{1} F_{1}\left(\beta^{\prime}, \gamma+k ; y\right) \\
\Phi_{2}\left(\beta, \beta^{\prime}, \gamma, x t, \gamma t\right) & =\sum_{k=0}^{\infty} \frac{(\beta)_{k}}{k!(\gamma)_{k}}(x t)^{k}{ }_{1} F_{1}\left(\beta^{\prime}, \gamma+k ; \gamma t\right)
\end{aligned}
$$

and $(\beta)_{K}$ is the pochhammer symbol, given by:

[^31]$$
(\beta)_{K}=\beta(\beta+1) \ldots(\beta+n-1)=\frac{\Gamma(\beta+K)}{\Gamma(\beta)} .
$$

So that, with $\gamma=2, \beta=1-K, \beta^{\prime}=K, x=-\frac{1}{2 \sigma^{2}}$ and $\gamma=-\frac{K}{2 \sigma^{2} K+\Omega}$, one gets:

$$
\begin{equation*}
F_{Y}(y)=\frac{1}{2 \sigma^{2}}\left(\frac{2 \sigma^{2} K}{2 \sigma^{2} K+\Omega}\right)^{K} y \Phi_{2}\left(1-K, K, 2,-\frac{y}{2 \sigma^{2}} ;-\frac{K y}{2 \sigma^{2} K+\Omega}\right) \tag{A.15}
\end{equation*}
$$

From (A.15), and remembering that $y=s^{2}$, one finally gets the $\operatorname{cdf} F_{S}(s)$ as:

$$
\begin{equation*}
F_{S}(s)=\frac{1}{2 \sigma^{2}}\left(\frac{2 \sigma^{2} K}{2 \sigma^{2} K+\Omega}\right)^{K} s^{2} \Phi_{2}\left(1-K, K, 2,-\frac{s^{2}}{2 \sigma^{2}} ;-\frac{K s^{2}}{2 \sigma^{2} K+\Omega}\right) \tag{A.16}
\end{equation*}
$$

## Appendix B - The Chirp Transform Algorithm (CTA)

The Chirp Transform Algorithm (CTA) allows efficient calculation of the Discrete Fourier Transform, $X\left(e^{j\left(w_{0}+k \Delta w\right)}\right)$ of a discrete signal $x(n)$, at uniformly-spaced samples along the unit circle, starting from a given frequency, $w_{0}$.

The CTA computes M uniformly spaced samples by:

$$
X\left(e^{j w_{k}}\right)=\sum_{n=0}^{N-1} x(n) e^{-j w_{k} n}, \quad k=0, \ldots, M-1
$$

where $w_{k}=w_{0}+k \Delta w$.
Defining $W=e^{-j \Delta w}$, one gets

$$
\begin{aligned}
& X\left(e^{j w_{k}}\right)=\sum_{n=0}^{N-1} x(n) e^{-j w_{0} n} W^{n k} \\
& =\sum_{n=0}^{N-1} x(n) e^{-j w_{0} n} W^{n^{2} / 2} W^{k^{2} / 2} W^{-(k-n)^{2} / 2}
\end{aligned}
$$

Letting $g(n)=e^{-j w_{0} n} W^{n^{2} / 2} x(n)$, the above equation can be written as

$$
X\left(e^{j w_{k}}\right)=W^{k^{2} / 2}\left(\sum_{n=0}^{N-1} g(n) W^{-(k-n)^{2} / 2}\right)
$$

Swapping $n$ by $k$ one gets

$$
\begin{aligned}
X\left(e^{j w_{n}}\right) & =W^{n^{2} / 2}\left(\sum_{k=0}^{N-1} g(k) W^{-(n-k)^{2} / 2}\right) \\
& =W^{n^{2} / 2}\left(g(n) * W^{-n^{2} / 2}\right)
\end{aligned}
$$

The system implemented directly from the above equation cannot be realized since it is neither stable nor causal. However, considering the application of the CTA to a finite-length and causal signal $x(n), W^{-n^{2} / 2}$ can be truncated from $-(N-1) \leq n \leq M-1$ and delayed by $(N-1)$ to obtain a causal impulse response, so that, considering

$$
h(n)=\left\{\begin{array}{c}
W^{-(n-N+1)^{2} / 2, \quad n=0, \ldots, M+N-2} \\
0, \quad \text { otherwise }
\end{array}\right.
$$

we get

$$
\begin{equation*}
y(n)=W^{(n-N+1)^{2} / 2}(g(n) * h(n)) \tag{B.1}
\end{equation*}
$$

and

$$
\begin{equation*}
X\left(e^{j w_{k}}\right)=y(k+N-1) . \tag{B.2}
\end{equation*}
$$

Figure B. 1 shows the block diagram of the CTA system obtained from (B.1). Finally, $X\left(e^{j w_{k}}\right)$ can be obtained directly from (B.2).


Figure B. 1 - Block diagram of the CTA for causal, finite-length impulse response.

## CTA using the Fast Fourier Transform

The CTA can be implemented by resorting to the Fast Fourier Transform to speed its computation. Specifically, the convolution from (B.1) can be evaluated as $g(n) * h(n)=\operatorname{IFFT}(\operatorname{FFT}(g(n)) *$ $\operatorname{FFT}(h(n))$. The FFT length, $L$, can be a power of $2\left(L=2^{m}\right)$, provided that $L \geq N+M-$ 1 , and only M samples are retained from $g(n) * h(n)$.

Considering, as before, that

$$
h(n)=\left\{\begin{array}{c}
W^{-(n-N+1)^{2} / 2, \quad} \quad n=0, \ldots, M+N-2 \\
0, \quad \text { otherwise }
\end{array}\right.
$$

and using the vector $\boldsymbol{n}=[0, \ldots, N+M-2]^{\mathrm{T}}$, the CTA can be efficiently implemented as illustrated in the following block diagram (Figure B.2).


Figure B. 2 - Block diagram of the CTA using the FFT.

## Computational Cost

We will detail the steps of the CTA algorithm and reach a final equation for the number of real FLOPs needed. Although several scientific papers have addressed this issue (Rajmic, Prusa and Wiesmeyr, 2014)(Bailey and Swarztrauber, 2005), no final equal result was presented and definitely established in the open literature.

Considering the computation of a decimation-in-time N -point FFT, with $N=n_{1} \times n_{2} \times \ldots \times n_{r}$, it can be implemented by a cascading of $\operatorname{FFT}\left(n_{i}\right)$ butterflies:

- $\frac{N}{n_{1}} \mathrm{FFT}\left(n_{1}\right)$
- $\frac{N}{n_{2}} \operatorname{FFT}\left(n_{2}\right)$
- $\frac{\dddot{N}}{n_{r}} \mathrm{FFT}\left(n_{r}\right)$

Thus, if N is a power-of- $2, N=2^{p}$, it can be implemented with a $\frac{N}{2} \log _{2}(N), \mathrm{FFT}(2)$ butterflies structure.

Although the number of floating-point multiplications used to be the measure for algorithm complexity, nowadays, this must be done considering both multiplications and addition (or subtractions) because most processors, either digital signal processors or arithmetic blocks in FPGAs, will take approximately the same time for both operations. Therefore, we will consider the number of real floating-point operations (FLOPs) needed for any algorithm presented in this thesis to assess its performance.

Following this methodology, the number of FLOPs for several operations is listed in Table B.1, where radix-n FFT structure and flop count are given in, e.g., (Löfgren and Nilsson, 2011).

| Operation | Number of real <br> FLOPs |
| :--- | :---: |
| Complex(a,b): $a+b$ | 2 |
| Complex(a,b): $a \times b$ | 6 |
| Real(a)/Complex(b): $a+b$ | 1 |
| Real(a)/Complex(b): $a \times b$ | 2 |
| FFT(2) | 10 |
| FFT(3) | 28 |
| FFT(5) | 68 |

Table B. 1 - Number of real Floating-point operations (FLOPs) for several operations.

For example, the general FFT(2) butterfly is implemented as shown in Figure B.3, where

$$
\begin{aligned}
& y_{0}=x_{0}+x_{1} w_{n}^{k} \\
& y_{1}=x_{0}-x_{1} w_{n}^{k}
\end{aligned}
$$

where the twiddle-factors are $w_{n}^{k}=e^{-j \frac{2 \pi k}{n}}$.
Considering the general case where the input is complex, we need one complex multiplication and two complex additions to compute the output, totaling 10 FLOPs, as indicated in Table B.1.


Figure B. 3 - FFT(2) butterfly structure.

The following signal processing flow resumes the steps for the CTA:

1. Input: signal $x(n)$ with length $N$; starting frequency, $w_{0}$; frequency step, $\Delta w$; number of frequencies to calculate, $M$;
2. Pre-calculation:

- $L=\operatorname{nextfftlength}(N+M-1)$
- Compute the $L$ length sequence, $h(n)=\left\{\begin{array}{c}W^{-(n-N+1)^{2} / 2}, n=0, \ldots, M+N-2 \\ 0, \quad \text { otherwise }\end{array}\right.$
- Compute the $L$ length sequence,

$$
c(n)=\left\{\begin{array}{r}
e^{-j w_{0} n} \times h^{*}(N-1+n), n=0, \\
0, \quad \text { otherwise }
\end{array} \quad . \quad, M+N-2\right.
$$

- Compute the $L$ length fast Fourier transform $H(k)=\operatorname{FFT}(h(n))$

3. Pre-multiply:

- Compute the $L$ length sequence, $y(n)=\left\{\begin{array}{c}x(n) \times c(n), n=0, \ldots, N \\ 0, \quad \text { otherwise }\end{array}\right.$

4. Convolution with fast Fourier transform:

- Compute the $L$ length fast Fourier transform $Y(k)=\operatorname{FFT}(y(n))$
- Compute the $L$ length sequence $G(k)=Y(k) \times H(k)$
- Compute the $L$ length inverse fast Fourier transform $Z(k)=\operatorname{IFFT}(G(k))$


## 5. Final Multiply:

- Compute output $X(k)=Z(N-1+k) \times h^{*}(N-1+k), k=0 \ldots M-1$

The bulk computation of the CTA algorithm is done on steps 3-5, assuming the calculations of steps 1-2 are precomputed. Therefore, for a complex input signal, we will need:

- $\quad N$ complex multiplications - step 3
- Two FFT with length $L$ (one FFT and one IFFT) - step 4
- $L$ complex multiplications - step 4
- $\quad M$ complex multiplications - step 5

Therefore, the total number of FLOPs for the CTA algorithm is:

$$
\begin{equation*}
F_{C T A}=6 N+10 \operatorname{Llog}_{2}(L)+6 L+6 M \tag{B.3}
\end{equation*}
$$

## Appendix C - Eigenvalues of the SCF Quadratic Form Matrix

The Quadratic form to obtain the SCF, when using the FAM algorithm, is given by equation (4.7), using the auxiliary matrices $\boldsymbol{G}_{K, \Delta}^{P}$, which in turn are obtained by $\boldsymbol{C}_{K, \Delta}$, or, preferably, its symmetric version as given by equation (4.9):

$$
\begin{aligned}
& \frac{\left(\boldsymbol{C}_{K, \Delta}\right)_{i, j}+\left(\boldsymbol{C}_{K, \Delta}\right)^{T}{ }_{i, j}}{2}= \\
& =\cos \left(\frac{\pi K}{N p S e g}(i-j)\right)\left[\cos \left(\frac{\pi \Delta}{N p S e g}(i+j)\right)-j \sin \left(\frac{\pi \Delta}{N p S e g}(i+j)\right)\right]
\end{aligned}
$$

For $\Delta=0$, the elements of the matrix are:

$$
\frac{\left(\boldsymbol{C}_{K, 0}\right)_{i, j}+\left(\boldsymbol{C}_{K, 0}\right)^{T}}{2}=\cos \left(\frac{\pi K}{N p S e g}(i-j)\right) .
$$

These are the elements of the matrix reported by (Johnson and Long, 1999), but considering the complete FFT frequency interval $\left(\left[0 ; f_{s}\right]\right)$ with $N p S e g$. As referenced in Section 4.1, that case is obtained by simply changing the argument in the cosine to:

$$
\begin{equation*}
\boldsymbol{C}_{K, 0}^{\prime}=\frac{\left(\boldsymbol{C}_{K, 0}\right)_{i, j}+\left(\boldsymbol{C}_{K, 0}\right)^{T}{ }_{i, j}}{2}=\cos \left(\frac{2 \pi K}{\text { NpSeg }}(i-j)\right) . \tag{C.1}
\end{equation*}
$$

All conclusions obtained in this appendix are valid for Section 4.2 by simply considering the cases when $K$ is even. Here, we will maintain the equation (C.1) version for direct comparison with the (Johnson and Long, 1999) work.

The base matrix $\boldsymbol{C}_{K, 0}^{\prime}$ is symmetric, and also persymmetric (generally named bisymmetric when both symmetries exist). A persymmetric $N \times N$ matrix, $A$, is symmetric with respect to its antidiagonal (northeast-to-southwest diagonal) so that its elements verify $a_{i, j}=a_{N-j+1, N-i+1}$ (Golub and Loan, 2013). From (C.1), the elements of $\boldsymbol{C}_{K, 0}^{\prime}$ verify the condition:

$$
\begin{aligned}
c_{N p S e g-j+1, N p S e g-i+1} & =\cos \left(\frac{2 \pi K}{N p S e g}(N p S e g-j+1-(N p S e g-i+1))\right) \\
& =\cos \left(\frac{2 \pi K}{N p S e g}(i-j)\right)=c_{i, j}
\end{aligned}
$$

The complete matrix for the quadratic form, $\boldsymbol{G}_{K, 0}$, as given in (4.7), is, by construction, also symmetric and persymmetric.

A persymmetric $N \times N$ matrix, $A$, must have the structure:

$$
A=\left[\begin{array}{cc}
J D^{T} J & B^{T} \\
B & D
\end{array}\right]
$$

where $J$ is the reverse identity matrix (exchange matrix), such that left-multiplying by $J$ reverses the row order of the matrix and right-multiplying reverses the column order. If the matrix is also symmetric (and thus bisymmetric), then $D=D^{T}$ and $B^{T}=J B J$.

Given a permutation matrix, $P$ :

$$
P=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
J & -I \\
J & I
\end{array}\right], \operatorname{inv}(P)=P^{T}
$$

the similarity transformation $P \times A \times P^{-1}=P \times A \times P^{T}$ - which we will name persymmetric transformation - leads to (Cantoni and Butler, 1976) ${ }^{46}$ :

$$
P \times A \times P^{T}=\left[\begin{array}{cc}
D-B \times J & \mathbf{0} \\
\mathbf{0} & D+B \times J
\end{array}\right]
$$

So the eigenvalues of $A$ are the eigenvalues of $L_{1}=D-B \times J$ and $L_{2}=D+B \times J$ (Reid, 1997).

The persymmetric transformation of the $\boldsymbol{G}_{K, 0}$ matrix leads to the following $L_{1}$ and $L_{2}$ matrices:

$$
\begin{align*}
& \left(L_{1}\right)_{i, j}=D_{i, j} \odot \cos \left((i-j) \frac{2 \pi K}{N p S e g}\right)-(B \times J)_{i, j} \odot \cos \left((i+j+1) \frac{2 \pi K}{N p S e g}\right) \\
& \left(L_{2}\right)_{i, j}=D_{i, j} \odot \cos \left((i-j) \frac{2 \pi K}{N p S e g}\right)+(B \times J)_{i, j} \odot \cos \left((i+j+1) \frac{2 \pi K}{N p S e g}\right) \tag{C.2}
\end{align*}
$$

The $D$ and $B \times J$ matrices are obtained from the FAM algorithm parameters, namely $N p S e g$ (and equivalently $L=N p S e g / 4$ ) and the number of averaging segments, $P$. For an odd number of averaging segments and considering $P=4 n+1, n=0,1, \ldots$ to obtain a $\boldsymbol{G}_{K, \Delta}$ matrix with dimension multiple of the base $\boldsymbol{C}_{K, 0}^{\prime}$ matrix dimension $(N p S e g \times N p S e g)$, the $D$ and $B \times J$ matrices can be obtained as follows.

For example, for $N p S e g=8, L=N p S e g / 4=2, P=5$, the $D$ matrix is given as:

[^32]\[

D=\left[$$
\begin{array}{llllllll}
4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 \\
4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 \\
3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 \\
3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$\right]
\]

And, in general, provided $P>3$, it can be obtained by concatenating a constant $L \times L$ matrix along the diagonal, starting from one to four $(N p S e g / L)$, and from the lower right position to the upper left. After reaching this value, the concatenating $L \times L$ matrix will maintain this value. Then, for each of these diagonal sub-matrices, concatenate horizontally and vertically with another constant $L \times L$ submatrices with decreasing values of the original sub-matrix to zero.

For $N p S e g=8, L=N p S e g / 4=2, P=9$, the $D$ matrix is given as:

$$
D=\left[\begin{array}{llllllllllll}
4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
3 & 3 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 0 & 0 \\
3 & 3 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 0 & 0 \\
2 & 2 & 3 & 3 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 \\
2 & 2 & 3 & 3 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 \\
1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 \\
1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 \\
0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right]
$$

For $N p S e g=8, L=N p S e g / 4=2, P=5$, the $B \times J$ matrix is given as:

$$
B \times J=\left[\begin{array}{cccccccc}
3 & 3 \\
3 & 3
\end{array}\right]\left[\begin{array}{lll}
2 & 2 \\
2 & 2
\end{array}\right]\left[\begin{array}{llll}
1 & 1 \\
1 & 1
\end{array}\right]
$$

And, in general, it can be obtained by concatenating a constant $L \times L$ matrix on the first block line with decreasing values from three $(N p S e g / L-1)$ to zero. Then, successive block matrices lines are obtained by shifting left the previous block matrix lines.

For $N p S e g=8, L=N p S e g / 4=2, P=9$, the $B \times J$ matrix is given as:

$$
\left.B \times J=\left[\begin{array}{cccccccccccc}
3 & 3 \\
3 & 3
\end{array}\right]\left[\begin{array}{llll}
2 & 2 \\
2 & 2
\end{array}\right]\left[\begin{array}{llllll}
1 & 1 \\
1 & 1
\end{array}\right]\left[\begin{array}{lllll}
0 & 0 \\
0 & 0
\end{array}\right] \begin{array}{lllll}
0 & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{cccccccccc}
0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 \\
2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0
\end{array}\right]
$$

We will show in the following that there is always a similar transformation such that $L_{2}=$ $P_{r} \times L_{1} \times P_{r}^{-1}$, and, specifically, a permutation transformation, so that $\operatorname{inv}\left(P_{r}\right)=P_{r}^{-1}=P_{r}^{T}$. This way, the eigenvalues of $L_{1}$ are equal to the ones of $L_{2}$, and so the eigenvalues of the $G_{K, 0}$ quadratic form matrix come in pairs.

The general form of the permutation matrix $P_{r}$ is

$$
P_{r}=\left(\left[\begin{array}{cc}
1 & 0  \tag{C.3}\\
0 & -1
\end{array}\right] \otimes J_{n p}\right) \otimes I_{n p 1}
$$

where $I_{n p 1}$ is the $n p 1 \times n p 1$ identity matrix, $J_{n p 1}$ is the $n p 1 \times n p 1$ exchange matrix, $\otimes$ stands for the Kronecker product, and $n p 1$ is given by:

$$
n p 1=\frac{N p S e g+(P-1) \times L / 2}{2 \times n p}
$$

The crucial parameter $n p$ is obtained by the following (Matlab) code ${ }^{47}$ :

```
if (K == 2*L)
np = 1;
else
    for n = 0:log2(L)
        if (mod(K, L/2^n) == 0)
        np = 2^(n);
        break;
        end
    end
end
```

That is: choose the minimum $n$ such that $K /\left(L / 2^{n}\right)$ is an integer. As $L$ is a power-of-two, there will always be a solution for $n$.

Specific cases can be obtained, namely:

- If $K$ is odd, then $n p=L$

[^33]- If $K=2 \times L$, then $n p=1$
- If $\bmod (K, 4)==2$, then $n p=L / 2$

All other possible values for $K$ belong to the condition the $\bmod (K, 4)=0$, and for those cases, there is no general solution to $n p$.

In the following, we will use the result expressed in Proposition 1:
Proposition 1: For the minimum $n$ that satisfies $K /\left(L / 2^{n}\right)=Q$ with $Q$ and integer, $Q$ is always odd.

## Proof:

As $L$ is a power-of-two, then $K /\left(L / 2^{n}\right)=Q \Longleftrightarrow K=Q \times 2^{l-n}$. If $Q$ is even, then $Q=$ $2 \times Q^{\prime}$, so $K=Q^{\prime} \times 2^{l-(n-1)}$, which means that $n-1$ satisfies the required relation. This is absurd since we required $n$ to be the minimum value. Therefore $Q$ must always be odd.

The condition expressed in Proposition 1 is always verified, except when $K=0$. In this case, there is no similar transformation between $L_{1}$ and $L_{2}$, and the eigenvalues do not come in pairs.

The inner permutation, as expressed in (C.3), will be performed on $n p \times n p$ submatrices within $L_{1}$ or $L_{2}$. As $n p \leq L$, the $d_{i, j}$ elements of matrix $D$ and the $b j_{i, j}$ elements of matrix $B \times J$ within those submatrices will all be equal, as the $L \times L$ submatrices of $D$ and $B \times J$ have equal elements. Further, this inner permutation will exchange elements in a $n p \times n p$ block as $c_{i, j} \leftrightarrow$ $\pm c_{n p-1-i, n p-1-j}$, where the sign is obtained in an alternating form for each neighbor block, as depicted in Figure 4.9.


Figure C. 1 - Sign change for neighbor block matrices.

Recalling that $L_{1}$ and $L_{2}$ (given by equation (C.2)) are a sum/difference of two terms, we will analyze each part separately. Considering:

$$
\begin{aligned}
& \left(L_{11}\right)_{i, j}=D_{i, j} \odot \cos \left((i-j) \frac{2 \pi K}{N p S e g}\right) \\
& \left(L_{12}\right)_{i, j}=(B \times J)_{i, j} \odot \cos \left((i+j+1) \frac{2 \pi K}{N p S e g}\right)
\end{aligned}
$$

then, $L_{1}=L_{11}-L_{12}$ and $L_{2}=L_{11}+L_{12}$. Therefore will we show that $L_{11}=P_{r} \times L_{11} \times P_{r}^{T}$ and that $L_{12}=-P_{r} \times L_{12} \times P_{r}^{T}$, leading to $L_{2}=P_{r} \times L_{1} \times P_{r}^{T}$.

Will will make use of the following identity:

$$
\begin{equation*}
\frac{K}{L / n p}=Q \Longleftrightarrow K=\frac{Q \times L}{n p}=\frac{Q \times N p S e g}{4 n p} \Leftrightarrow \frac{K}{N p S e g}=\frac{Q}{4 n p}, \tag{C.4}
\end{equation*}
$$

remembering that $Q$ is always odd.
As seen, in each of the $n p \times n p$ blocks, the corresponding blocks of $D$ and $B \times J$ are equal (and will remain equal after the permutation), so only the "cos" part of $L_{11}$ and $L_{12}$ needs to be addressed. We will analyze all the possible $n p \times n p$ blocks with the corresponding sign changes (see Figure C.1).

- Case $\left(i^{\prime}, j^{\prime}\right)=(2 m \times n p+i, 2 n \times n p+j)$

From Figure C.1, these correspond to the case when the sign for the permutation subblock is positive, so $c_{i, j} \leftrightarrow c_{n p-1-i, n p-1-j}$.

| ${ }_{n p}\{$ | + | - | + | - | $\ldots$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | - | + | - | + | $\ldots$ |
|  | + | - | + | - | $\ldots$ |
|  | - | $+$ | - | + | $\cdots$ |
|  | $\vdots$ | $\vdots$ | $\vdots$ | ! | : |

On the original $L_{11}$ matrix, we have:

$$
\cos \left(\left(i^{\prime}-j^{\prime}\right) \frac{2 \pi K}{N p S e g}\right)=\cos \left(x \frac{2 \pi K}{N p S e g}\right)
$$

where $x=2 m \times n p+i-2 n \times n p-j$.
For the permuted matrix, we will have the corresponding indexes given as:

$$
(2 m \times n p+n p-1-i, 2 n \times n p+n p-1-j)
$$

The cosine argument in the permuted matrix is:

$$
\begin{array}{r}
2 m \times n p+n p-1-i-(2 n \times n p+n p-1-j)= \\
2 m \times n p-i-2 n \times n p+j= \\
-(2 m \times n p+i-2 n \times n p-j)+4 m \times n p-4 n \times n p= \\
-x+4(m-n) \times n p
\end{array}
$$

Therefore:

$$
\begin{aligned}
\cos \left(\left(i^{\prime}-j^{\prime}\right) \frac{2 \pi K}{N p S e g}\right)=\cos \left((-x+4(m-n) \times n p) \frac{2 \pi K}{N p S e g}\right) & = \\
\cos \left(-x \frac{2 \pi K}{N p S e g}+4(m-n) \times n p \frac{2 \pi Q}{4 n p}\right) & = \\
\cos \left(-x \frac{2 \pi K}{N p S e g}+2 \pi(m-n) Q\right) & = \\
\cos \left(-x \frac{2 \pi K}{N p S e g}+2 k \pi\right) & = \\
\cos \left(x \frac{2 \pi K}{N p S e g}\right) & =
\end{aligned}
$$

So, the blocks of the original and permuted $L_{11}$ matrices are equal.
On the original $L_{12}$ matrix, we have:

$$
\cos \left(\left(i^{\prime}+j^{\prime}+1\right) \frac{2 \pi K}{N p S e g}\right)=\cos \left(x \frac{2 \pi K}{N p S e g}\right)
$$

where $x=2 m \times n p+i+2 n \times n p+j+1$.
For the permuted matrix, we will have the corresponding indexes given as:

$$
(2 m \times n p+n p-1-i, 2 n \times n p+n p-1-j)
$$

The cosine argument in the permuted matrix is:

$$
\begin{array}{r}
2 m \times n p+n p-1-i+(2 n \times n p+n p-1-j)+1= \\
2 m \times n p-i+2 n \times n p-j+2(n p-1)+1= \\
-(2 m \times n p+i+2 n \times n p+j+1)+4 m \times n p+4 n \times n p+2(n p-1)+2= \\
-x+4\left(m+n+\frac{1}{2}\right) \times n p
\end{array}
$$

Therefore:

$$
\begin{aligned}
\cos \left(\left(i^{\prime}+j^{\prime}+1\right) \frac{2 \pi K}{N p S e g}\right)=\cos \left(\left(-x+4\left(m+n+\frac{1}{2}\right) \times n p\right) \frac{2 \pi K}{N p S e g}\right) & = \\
\cos \left(-x \frac{2 \pi K}{N p S e g}+\left(4\left(m+n+\frac{1}{2}\right) \times n p \frac{2 \pi Q}{4 n p}\right)\right. & = \\
\cos \left(-x \frac{2 \pi K}{N p S e g}+2 \pi(m+n) Q+\pi Q\right) & =
\end{aligned}
$$

$$
\begin{aligned}
\cos \left(-x \frac{2 \pi K}{N p S e g}\right. & +2 k \pi+\pi Q) \stackrel{\text { as } \mathrm{Q} \text { is odd }}{=} \\
& -\cos \left(x \frac{2 \pi K}{N p S e g}\right)
\end{aligned}
$$

So, the blocks of the original and permuted $L_{12}$ matrices are symmetric.

- Cases $\left(\boldsymbol{i}^{\prime}, \boldsymbol{j}^{\prime}\right)=(\mathbf{2 m} \times \boldsymbol{n p}+\boldsymbol{i},(\mathbf{2 n}+\mathbf{1}) \times \boldsymbol{n p}+\boldsymbol{j}) \quad$ or $\quad\left(\boldsymbol{i}^{\prime}, \boldsymbol{j}^{\prime}\right)=((\mathbf{2 m}+\mathbf{1}) \times$ $\boldsymbol{n p}+\boldsymbol{i}, \mathbf{2 n} \times \boldsymbol{n} \boldsymbol{p}+\boldsymbol{j})$

These two cases are equal because $\cos \left((i-j) \frac{2 \pi K}{M}\right)=\cos \left((j-i) \frac{2 \pi K}{M}\right) \cos ((i+j+$ 1) $\left.\frac{2 \pi K}{M}\right)=\cos \left((j+i+1) \frac{2 \pi K}{M}\right)$, and so we will analyze the case $\left(i^{\prime}, j^{\prime}\right)=(2 m \times n p+$ $i,(2 n+1) \times n p+j)$.

From Figure C.1, these correspond to the case when the sign for the permutation subblock is negative, so $c_{i, j} \leftrightarrow-c_{n p-1-i, n p-1-j}$.


On the original $L_{11}$ matrix, we have:

$$
\cos \left(\left(i^{\prime}-j^{\prime}\right) \frac{2 \pi K}{N p S e g}\right)=\cos \left(x \frac{2 \pi K}{N p S e g}\right)
$$

where $x=2 m \times n p+i-(2 n+1) \times n p-j$.
For the permuted matrix, we will have the corresponding indexes given as:

$$
(2 m \times n p+n p-1-i,(2 n+1) \times n p+n p-1-j)
$$

The cosine argument in the permuted matrix is:

$$
\begin{array}{r}
2 m \times n p-i-(2 n+1) \times n p+j= \\
-(2 m \times n p+i-(2 n+1) \times n p-j)+4 m \times n p-2(2 n+1) \times n p= \\
-x+4\left(m-n-\frac{1}{2}\right) \times n p
\end{array}
$$

Therefore:

$$
\begin{aligned}
&-\cos \left(\left(i^{\prime}-j^{\prime}\right) \frac{2 \pi K}{N p S e g}\right)=-\cos \left(\left(-x+4\left(m-n-\frac{1}{2}\right) \times n p\right) \frac{2 \pi K}{N p S e g}\right)= \\
&-\cos \left(-x \frac{2 \pi K}{N p S e g}+4\left(m-n-\frac{1}{2}\right) \times n p \frac{2 \pi Q}{4 n p}\right)= \\
&-\cos \left(-x \frac{2 \pi K}{N p S e g}+2 \pi\left(m-n-\frac{1}{2}\right) Q\right)= \\
&-\cos \left(-x \frac{2 \pi K}{N p S e g}+2 k \pi-\pi Q\right) \stackrel{\text { as } Q \text { is odd }}{=} \\
& \cos \left(x \frac{2 \pi K}{N p S e g}\right)
\end{aligned}
$$

And so, the blocks of the original and permuted $L_{11}$ matrices are equal.
On the original $L_{12}$ matrix, we have:

$$
\cos \left(\left(i^{\prime}+j^{\prime}+1\right) \frac{2 \pi K}{N p S e g}\right)=\cos \left(x \frac{2 \pi K}{N p S e g}\right),
$$

where $x=2 m \times n p+i+(2 n+1) \times n p+j+1$.
For the permuted matrix, we will have the corresponding indexes given as:

$$
(2 m \times n p+n p-1-i,(2 n+1) \times n p+n p-1-j) .
$$

The cosine argument in the permuted matrix is:

$$
\begin{array}{r}
2 m \times n p+n p-1-i+(2 n+1) \times n p+n p-1-j+1= \\
-(2 m \times n p+i+(2 n+1) \times n p+j+1)+4 m \times n p+2(2 n+1) \times n p+2(n p-1)+2= \\
-x+4(m+n+1) \times n p
\end{array}
$$

Therefore:

$$
\begin{aligned}
-\cos \left(\left(i^{\prime}+j^{\prime}+1\right) \frac{2 \pi K}{N p S e g}\right)=-\cos \left((-x+4(m+n+1) \times n p) \frac{2 \pi K}{N p S e g}\right) & = \\
-\cos \left(-x \frac{2 \pi K}{N p S e g}+4(m+n+1) \times n p \frac{2 \pi Q}{4 n p}\right) & = \\
-\cos \left(-x \frac{2 \pi K}{N p S e g}+2 \pi(m+n+1) Q\right) & = \\
-\cos \left(-x \frac{2 \pi K}{N p S e g}+2 k \pi\right) & = \\
-\cos \left(x \frac{2 \pi K}{N p S e g}\right) &
\end{aligned}
$$

And so, the blocks of the original permuted $L_{12}$ matrices are symmetric.

- Case $\left(i^{\prime}, \boldsymbol{j}^{\prime}\right)=((2 m+1) \times n p+i,(2 n+1) \times n p+\boldsymbol{j})$

From Figure C.1, these correspond to the case when the $\operatorname{sign}_{n p}$ for the permutation subblock is positive, so $c_{i, j} \leftrightarrow c_{n p-1-i, n p-1-j}$.
$n_{n p}\left\{\begin{array}{c|c|c|c|c|}\hline+ & - & + & - & \cdots \\ \hline- & + & - & + & \cdots \\ \hline+ & - & + & - & \cdots \\ \hline- & + & - & + & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline\end{array}\right.$

On the original $L_{11}$ matrix, we have:

$$
\cos \left(\left(i^{\prime}-j^{\prime}\right) \frac{2 \pi K}{N p S e g}\right)=\cos \left(x \frac{2 \pi K}{N p S e g}\right)
$$

where $x=(2 m+1) \times n p+i-(2 n+1) \times n p-j$.
For the permuted matrix, we will have the corresponding indexes given as:

$$
((2 m+1) \times n p+n p-1-i,(2 n+1) \times n p+n p-1-j)
$$

The cosine argument in the permuted matrix is:

$$
\begin{array}{r}
(2 m+1) \times n p+n p-1-i-((2 n+1) \times n p+n p-1-j)= \\
(2 m+1) \times n p-i-(2 n+1) \times n p+j= \\
-((2 m+1) \times n p+i-(2 n+1) \times n p-j)+2(2 m+1) \times n p-2(2 n+1) \times n p= \\
-x+4(m-n) \times n p
\end{array}
$$

Therefore:

$$
\begin{aligned}
\cos \left(\left(i^{\prime}-j^{\prime}\right) \frac{2 \pi K}{N p S e g}\right)=\cos \left((-x+4(m-n) \times n p) \frac{2 \pi K}{N p S e g}\right) & = \\
\cos \left(-x \frac{2 \pi K}{N p S e g}+4(m-n) \times n p \frac{2 \pi Q}{4 n p}\right) & = \\
\cos \left(-x \frac{2 \pi K}{N p S e g}+2 \pi(m-n) Q\right) & = \\
\cos \left(-x \frac{2 \pi K}{N p S e g}+2 k \pi\right) & = \\
\cos \left(x \frac{2 \pi K}{N p S e g}\right) & =
\end{aligned}
$$

So, the blocks of the original and permuted $L_{11}$ matrices are equal.
On the original $L_{12}$ matrix, we have:

$$
\cos \left(\left(i^{\prime}+j^{\prime}+1\right) \frac{2 \pi K}{N p S e g}\right)=\cos \left(x \frac{2 \pi K}{N p S e g}\right)
$$

where $x=(2 m+1) \times n p+i+(2 n+1) \times n p+j+1$.
For the permuted matrix, we will have the corresponding indexes given as:

$$
((2 m+1) \times n p+n p-1-i,(2 n+1) \times n p+n p-1-j)
$$

The cosine argument in the permuted matrix is:

$$
\begin{array}{r}
(2 m+1) \times n p+n p-1-i+((2 n+1) \times n p+n p-1-j)+1= \\
-((2 m+1) \times n p+i+(2 n+1) \times n p+j+1)+2(2 m+1) \times n p \\
+2(2 n+1) \times n p+2(n p-1)+2= \\
-x+4\left(m+n+\frac{3}{2}\right) \times n p
\end{array}
$$

Therefore:

$$
\begin{aligned}
& \cos \left(\left(i^{\prime}+j^{\prime}+1\right) \frac{2 \pi K}{N p S e g}\right)=\cos \left(\left(-x+4\left(m+n+\frac{3}{2}\right) \times n p\right) \frac{2 \pi K}{N p S e g}\right)= \\
& \cos \left(-x \frac{2 \pi K}{N p S e g}+4\left(m+n+\frac{3}{2}\right) \times n p \frac{2 \pi Q}{4 n p}\right)= \\
& \cos \left(-x \frac{2 \pi K}{N p S e g}+2 \pi(m+n) Q+3 \pi Q\right)= \\
& \cos \left(-x \frac{2 \pi K}{N p S e g}+2 k \pi+3 \pi Q\right) \stackrel{\text { as } Q \text { is odd }}{=} \\
&-\cos \left(x \frac{2 \pi K}{N p S e g}\right)
\end{aligned}
$$

So, the blocks of the original and permuted $L_{12}$ matrices are symmetric.

This proves that $P_{r} \times L_{1} \times P_{r}^{T}=P_{r} \times\left(L_{11}-L_{12}\right) \times P_{r}^{T}=P_{r} \times L_{11} \times P_{r}^{T}-P_{r} \times L_{12} \times P_{r}^{T}=$ $L_{11}+L_{12}=L_{2}$. Therefore, the eigenvalues of the quadratic form matrix, which are the eigenvalues of $L_{1}$ and $L_{2}$, come in pairs.

This result is valid for a rectangular data tapering window, an odd number of averaging segments $(P)$, and all values of $K$ when defining the base matrix $C_{K, 0}^{\prime}=\cos \left(\frac{2 \pi K}{N p S e g}(i-j)\right)$. If using $C_{K, 0}^{\prime}=\cos \left(\frac{\pi K}{N p S e g}(i-j)\right)$ the results are the same, but for even $K$ (or using half the number of sample points for each segment, NpSeg ).

When $N p S e g$ is high, although the eigenvalues are not exactly equal (in pairs) for $K$ odd, the resulting matrix will be very close to the one obtained with the previous (or next) value of $K$ which is even. Therefore one would expect the eigenvalues to be almost in pairs for $K$ odd, which was confirmed by simulation. Further, using other common data tapering windows will smooth the resulting spectral decomposition of the quadratic form matrix, but it should not be expected to spread the paired eigenvalues. This was also confirmed by simulation.

Further, and for other regions on the $(f ; \alpha)$ plane, when using a rectangular window, $K$ and $\Delta$ even, and an odd number of averaging segments, the final quadratic form matrix, $\boldsymbol{G}_{K, \Delta}$, will have the same eigenvalue structure as the base matrix $\boldsymbol{C}_{K, \Delta}^{\prime}$. For each additional segment, the resulting matrix will have a rank equal to the previous, added by the rank of the base matrix. As seen, when the base matrix has an eigenvalue structure $(+\lambda,+\lambda)$, the final matrix with $P$ segments (and $P$ must be odd) will have $2 P$ eigenvalues with structure $\left\{\left(+\lambda_{1},+\lambda_{1}\right),\left(+\lambda_{2},+\lambda_{2}\right) \ldots\left(+\lambda_{P},+\lambda_{P}\right)\right\}$.

Along the line $K=\Delta(K, \Delta \neq 0)$ and line $K=0(\Delta \neq 0)$, the base matrix has an eigenvalue structure $(+\lambda,-\lambda)$, so the final matrix with $P$ segments will have eigenvalues structure $\left\{\left(+\lambda_{1},-\lambda_{1}\right),\left(+\lambda_{2},-\lambda_{2}\right) \ldots\left(+\lambda_{P},-\lambda_{P}\right)\right\}$. In all other cases, the base matrix has an eigenvalue structure $(+\lambda,+\lambda,-\lambda,-\lambda)$, so the final matrix with $P$ segments will have eigenvalues structure $\left\{\left(+\lambda_{1},+\lambda_{1},-\lambda_{1},-\lambda_{1}\right),\left(+\lambda_{2},+\lambda_{2},-\lambda_{2},-\lambda_{2}\right) \ldots\left(+\lambda_{P},+\lambda_{P},-\lambda_{P},-\lambda_{P}\right)\right\}$.

## Appendix D - PDF from the Moment Generating Function

Using a partial fraction expansion (Oppenheim, Willsky and Nawab, 1996), the general MGF expressed by equation (4.40) can be written as $\left(h_{i}=\nu_{i} / 2 ; d_{i}=1 / 2 \lambda_{i} ; g_{1}=\prod_{i=1}^{D / 2}\left(d_{i}^{2}\right)^{h_{i}}=\right.$ $\left.\prod_{i=1}^{D / 2}\left(\left(2 \lambda_{i}\right)^{2}\right)^{-h_{i}}\right)$

$$
\begin{align*}
M(s) & =\prod_{i=1}^{D / 2}\left(\frac{d_{i}^{2}}{d_{i}^{2}-s^{2}}\right)^{h_{i}}=g_{1} \prod_{i=1}^{D / 2} \frac{1}{\left(d_{i}^{2}-s^{2}\right)^{h_{i}}} \\
& =g_{1} \sum_{i=1}^{D / 2} \sum_{j=1}^{h_{i}} \frac{B_{i j}}{\left(d_{i}^{2}-s^{2}\right)^{j}} \tag{D.1}
\end{align*}
$$

where $B_{i j}$ is given by

$$
\begin{equation*}
B_{i j}=\frac{1}{\left(h_{i}-j\right)!} \underbrace{\left.\frac{\partial^{h_{i}-j}}{\partial s^{h_{i}-j}}\left[\left(d_{i}^{2}-s^{2}\right)^{h_{i}} \frac{M(s)}{g_{1}}\right]\right|_{s=d_{i}}}_{D_{i}^{(m)}(s)} \tag{D.2}
\end{equation*}
$$

The derivatives in (D.2) can be expressed as:

$$
\begin{equation*}
D_{i}^{(m)}(s)=\frac{\partial^{m}}{\partial s^{m}} \prod_{k=1, k \neq i}^{D / 2} \frac{1}{\left(d_{k}^{2}-s^{2}\right)^{h_{k}}} \tag{D.3}
\end{equation*}
$$

Furthermore, recursively, these derivatives can be obtained as follows.

$$
D_{i}^{(0)}(s)=\prod_{k=1, k \neq i}^{D / 2} \frac{1}{\left(d_{k}^{2}-s^{2}\right)^{h_{k}}}
$$

Using the log-derivative:

$$
\begin{aligned}
\frac{D_{i}^{(1)}(s)}{D_{i}^{(0)}(s)} & =\frac{\partial}{\partial s}\left[\log \left(D_{i}^{(0)}(s)\right)\right] \\
& =-\sum_{k=1, k \neq i}^{D / 2} h_{k} \frac{\partial}{\partial s}\left[\log \left(\left(d_{k}^{2}-s^{2}\right)\right)\right] \\
& =\sum_{k=1, k \neq i}^{D / 2} \frac{2 h_{k} s}{d_{k}^{2}-s^{2}}
\end{aligned}
$$

and following the same procedure for high-order derivatives, one arrives at:

$$
\begin{equation*}
D_{i}^{(m+1)}(s)=\sum_{n=0}^{m}\binom{m}{n} D_{i}^{(n)}(s) \sum_{k=1, k \neq i}^{D / 2} \frac{\partial^{m-n}}{\partial s^{m-n}}\left[\frac{2 h_{k} s}{d_{k}^{2}-s^{2}}\right] \tag{D.4}
\end{equation*}
$$

Expanding the inner term as:

$$
\frac{2 h_{k} s}{d_{k}^{2}-s^{2}}=h_{k}\left(\frac{1}{d_{k}-s}-\frac{1}{d_{k}+s}\right)
$$

and using the following derivatives

$$
\begin{aligned}
\frac{\partial^{n}}{\partial s^{n}}\left[\frac{1}{d_{k}-s}\right] & =\frac{n!}{\left(d_{k}-s\right)^{n+1}} \\
\frac{\partial^{n}}{\partial s^{n}}\left[\frac{1}{d_{k}+s}\right] & =\frac{(-1)^{n} n!}{\left(d_{k}+s\right)^{n+1}}
\end{aligned}
$$

equation (D.4) becomes:

$$
D_{i}^{(m+1)}(s)=\sum_{n=0}^{m}\binom{m}{n} D_{i}^{(n)}(s) \sum_{k=1, k \neq i}^{D / 2} h_{k}\left[\frac{(m-n)!}{\left(d_{k}-s\right)^{m-n+1}}-\frac{(-1)^{m-n}(m-n)!}{\left(d_{k}+s\right)^{m-n+1}}\right]
$$

and finally:

$$
\begin{equation*}
D_{i}^{(m+1)}(s)=\sum_{n=0}^{m} \frac{m!}{n!} D_{i}^{(n)}(s) \sum_{k=1, k \neq i}^{D / 2} h_{k}\left[\frac{1}{\left(d_{k}-s\right)^{m-n+1}}-\frac{(-1)^{m-n}}{\left(d_{k}+s\right)^{m-n+1}}\right] \tag{D.5}
\end{equation*}
$$

From (D.2), $B_{i j}$ is given as:

$$
\begin{aligned}
& B_{i j}=\frac{1}{\left(h_{i}-j\right)!}\left[D_{i}^{(m)}(s)\right]_{\substack{s=d_{i} \\
m=h_{i}-j}} \\
& =\frac{1}{\left(h_{i}-j\right)!} \sum_{n=0}^{h_{i}-j-1} \frac{\left(h_{i}-j-1\right)!}{n!} D_{i}^{(n)}\left(d_{i}\right) \sum_{k=1, k \neq i}^{D / 2} h_{k}\left[\frac{1}{\left(d_{k}-d_{i}\right)^{h_{i}-j-n}}-\frac{(-1)^{h_{i}-j-n-1}}{\left(d_{k}+d_{i}\right)^{h_{i}-j-n}}\right]
\end{aligned}
$$

and finally:

$$
B_{i j}=\frac{1}{\left(h_{i}-j\right)} \sum_{n=0}^{h_{i}-j-1} \frac{D_{i}^{(n)}\left(d_{i}\right)}{n!} \sum_{k=1, k \neq i}^{D / 2} h_{k}\left[\frac{1}{\left(d_{k}-d_{i}\right)^{h_{i}-j-n}}-\frac{(-1)^{h_{i}-j-n-1}}{\left(d_{k}+d_{i}\right)^{h_{i}-j-n}}\right]
$$

$$
\begin{equation*}
\text { for } j \neq h_{i} \tag{D.6}
\end{equation*}
$$

$$
B_{i, j}=\left.D_{i}^{(0)}(s)\right|_{s=d_{i}}=\prod_{k=1, k \neq i}^{D / 2} \frac{1}{d_{k}^{2}-d_{i}^{2}}, \quad \text { for } j=h_{i}
$$

The MGF and pdf pair of a r.v. $X$ with variance gamma distribution is (Gaunt, 2014):

$$
\begin{aligned}
M(s)_{V G} & =e^{\mu s}\left(\frac{\alpha^{2}-\beta^{2}}{\alpha^{2}-(\beta+s)^{2}}\right)^{\lambda} \\
f_{X}(x) & =\frac{\left(\alpha^{2}-\beta^{2}\right)^{\lambda}|x-\mu|^{\lambda-1 / 2}}{\sqrt{\pi} \Gamma(\lambda)(2 \alpha)^{\lambda-1 / 2}} K_{\lambda-1 / 2}(\alpha|x-\mu|) e^{\beta|x-\mu|}
\end{aligned}
$$

This MFG is equal to the inner part of the MGF expressed by (D.1), considering $\mu=0, \beta=0$, $\lambda=j$ and $\alpha=d_{i}$ (separating the $\alpha^{2 \lambda}$ term), so:

$$
\begin{equation*}
M(s)=g_{1} \sum_{i=1}^{D / 2} \sum_{j=1}^{h_{i}} \frac{B_{i j}}{d_{i}^{2 j}}\left(\frac{d_{i}^{2}}{d_{i}^{2}-s^{2}}\right)^{j} \tag{D.7}
\end{equation*}
$$

The pdf is therefore given as:

$$
\begin{equation*}
f_{X}(x)=g_{1} \sum_{i=1}^{D / 2} \sum_{j=1}^{h_{i}} B_{i j} \frac{|x|^{j-1 / 2}}{\sqrt{\pi} \Gamma(j)\left(2 d_{i}\right)^{j-1 / 2}} K_{j-1 / 2}\left(d_{i}|x|\right) \tag{D.8}
\end{equation*}
$$

## Appendix E-Simulations to obtain the pdf of the SCF and SCoF for a large number of averaging segments

This appendix presents extensive simulation results to assess the validity of the proposed statistical characterization of the SCoF and the SCF estimates when the input signal is just the noise component, modeled as a $\mathcal{N}\left(0, \sigma^{2}\right)$. We will first compare the histogram and empirical CDF for the estimates obtained along the ( $f ; \alpha$ ) plane for select points (as indicated in Figure E.1) and for varying $P$ and $N p S e g$, simulating $2^{20}$ runs for each point. Then, quantile-quantile plots (Q-Q plots) and probability plots (P-P plots) (Fox, 2002; Miles, 2011) are obtained for the SCoF and its Fisher z-transformation, alongside several Goodness-of-Fit tests, to assess the validity of approximating the estimates by a Gaussian distribution. This same procedure is then replicated for the SCF.


Figure E. 1 - Points on the $(f ; \alpha)$ plane to simulate the SCF and SCoF.

## Histograms and CDF plots for the SCoF

Figure E. 2 shows the histogram of $\operatorname{Re}\left\{C_{x}^{\alpha}(f)\right\}$ obtained for all $(f ; \alpha)$ points when $P=33$ and NpSeg $=64$, where the similaritude is evidenced, even when taking a closer look around $\operatorname{Re}\left\{C_{x}^{\alpha}(f)\right\}=0$. Figure E. 3 and Figure E. 4 present the empirical CDF obtained, in linear and logarithmic scales, respectively, showing a strong similarity for all cases again. The following figures plot the histogram and (logarithmic scale) CDF of $\operatorname{Re}\left\{C_{x}^{\alpha}(f)\right\}$ for various $P$ and NpSeg. Specifically, Figure E. 5 and Figure E. 6 shows the case $P=33, N p S e g=128$ and $N p S e g=256$. Figure E.7, Figure E. 8 and Figure E. 9 ilustrate the case $P=129, N p S e g=64,128,256$. We will not present images for the cases of $\operatorname{Im}\left\{C_{x}^{\alpha}(f)\right\}$, because they are similar to $\operatorname{Re}\left\{C_{x}^{\alpha}(f)\right\}$. As an
example, Figure E. 10 show the results obtained for $\operatorname{Im}\left\{C_{x}^{\alpha}(f)\right\}$ when $P=129$ and $N p S e g=256$, which are similar to the ones obtained for $\operatorname{Re}\left\{C_{x}^{\alpha}(f)\right\}$ (Figure E.9). Simulation results were also obtained for $P=257,513,1025$ and $N p S e g=64,128,256$. Figure E. 11 to Figure E.13, show these cases for $N p S e g=128$.

Figure E. 14 shows the histogram obtained for $\operatorname{Re}\left\{C_{x}^{\alpha}(f)\right\}$ for all $P=$ $5,9,17,33,65,129,257,513,1025$, and $N p S e g=128$, for all $(f ; \alpha)$ points simulated, and Figure E. 16 shows the same, but for $\operatorname{Im}\left\{C_{x}^{\alpha}(f)\right\}$. As can be seen, the results are almost identical.

Figure E. 16 shows the empirical CDF obtained for these same parameters, alongside the expected CDF if the estimates were modeled as Gaussian with zero mean and variance equal to the one estimated from the simulated samples.

Finally, Figure E. 17 and Figure E. 18 show the averaged histogram obtained for all ( $f ; \alpha$ ) points, for $P=129$ and $P=257$, respectively, and $N p S e g=64,128,256$.

From the simulation results presented, one can conclude that all cases show similar behavior for the ( $f ; \alpha$ ) point considered, especially for high $P$. Also, for fixed $P$, the histograms are similar for varying $N p S e g$, as seen in Figure E. 17 and Figure E.18. Further, the hypothesis of modeling the estimates as Gaussian seems quite accurate for large $P$ (from Figure E.16, this would be the case for $P \geq 129$ or, in a more conservative estimate, for $P \geq 257$ ).

A note should be given for the case $\left(f_{i} ; \alpha_{i}\right)=\left(f_{s} / 4 ; f_{s} / 2\right)$. In this case, and as shown in Section 4.2.3.1, the imaginary component of the SCF and the SCoF is zero, so they have only the real component. Therefore, these cases are not included in the plots; they would result in similar curves but would have double variance compared to the other cases.


Figure E. 2 - Histogram of the real part of the SCoF, $P=33, N p S e g=64$.


Figure E. 3 - Empirical CDF of the real part of the $\mathrm{SCoF}, P=33, N p S e g=64$.


Figure E. 4 - Empirical CDF of the real part of the SCoF (logarithmic scale), $P=33, N p S e g=64$.


Figure E. 5 - Histogram and Empirical CDF of the real part of the SCoF (logarithmic scale), $P=33$, $N p S e g=128$.


Figure E. 6 - Histogram and Empirical CDF of the real part of the SCoF (logarithmic scale), $P=33$,
$N p S e g=256$.


Figure E. 7 - Histogram and Empirical CDF of the real part of the SCoF (logarithmic scale), $P=129$, $N p S e g=64$.


Figure E. 8 - Histogram and Empirical CDF of the real part of the SCoF (logarithmic scale), $P=129$, $N p S e g=128$.


Figure E. 9 - Histogram and Empirical CDF of the real part of the SCoF (logarithmic scale), $P=129$, $N p S e g=256$.


Figure E. 10 - Histogram and Empirical CDF of the imaginary part of the SCoF (logarithmic scale), $P=$ $129, N p S e g=256$.


Figure E. 11 - Histogram and Empirical CDF of the real part of the SCoF (logarithmic scale), $P=257$, $N p S e g=128$.


Figure E. 12 - Histogram and Empirical CDF of the real part of the SCoF (logarithmic scale), $P=513$,
$N p S e g=128$.


Figure E. 13 - Histogram and Empirical CDF of the real part of the SCoF (logarithmic scale), $P=1025$, $N p S e g=128$.


Figure E. 14 - Histogram of the real part of the SCoF $P=5,9,17,33,65,129,257,513,1025$ and $N p S e g=128$.


Figure E. 15 - Histogram of the imaginary part of the $\mathrm{SCoF} P=5,9,17,33,65,129,257,513,1025$ and $N p S e g=128$.


Figure E. 16 -Empirical and expected CDF of the real part of the SCoF (logarithmic scale), $P=$ $5,9,17,33,65,129,257,513,1025, N p S e g=128$.


Figure E. 17 - Mean histogram of the real part of the $\operatorname{SCoF}$ for all $(f ; \alpha)$ points, $P=129, N p S e g=$ $64,128,256$.


Figure E. 18 - Mean histogram of the real part of the SCoF for all $(f ; \alpha)$ points, $P=257, N p S e g=$ 64, 128, 256.

## Q-Q and P-P Plots for the SCoF and its Fisher z-transformation

We will now obtain Quantile-Quantile plots (Q-Q plots) and Probability-Probability plots (P-P plots) along with Goodness-of-Fit tests to justify the Gaussian assumption of the Fisher-z transformation of the SCoF (see Section 4.3), and the SCoF itself when $P$ is large.

Quantile-quantile (Q-Q) plots are a powerful tool for visually assessing some distribution assumption of the sample data. Given $F(x)$ the theoretical CDF, and $F_{n}(x)$ the empirical CDF, the Q-Q plot is obtained for a sample $x_{1} \ldots x_{n}$, by plotting the theoretical quantiles of the assumed distribution, $F^{-1}\left(F_{n}\left(x_{i}\right)\right)$, against the sample quantiles. If the empirical CDF is consistent with the theoretical CDF, the points obtained in the Q-Q plot will fall on a straight line. Plotting the empirical quantiles of a normally distributed sample $x \sim \mathcal{N}\left(\mu, \sigma^{2}\right)$ against the standard normal $(\mathcal{N}(0,1))$ quantiles will result in a straight line, where the slope is an estimate of the standard deviation, $\sigma$, and the y-axis interception estimates the mean, $\mu$ (Loy, Follett and Hofmann, 2016).

Probability-Probability (P-P) plots are obtained by plotting the theoretical CDF against the empirical CDF (a variant plot of the empirical and expected CDF of the real part of the SCoF as illustrated in Figure E.16). If the P-P plot is approximately linear, it indicates that the assumed true distribution gives a reasonably good fit to the given data distribution. Q-Q plots have a better resolution at the tails but less in the center, whereas P-P plots have a better resolution in the center of distribution but less at the tails (Wilk and Gnanadesikan, 1968).

In the following, the histogram, Q-Q plot, and P-P plot of the SCoF (real part) and its Fisher ztransformation will be plotted for all $(f ; \alpha)$ points, for varying (increasing) $P$ and for $N p S e g=$ 128.

Figure E. 19 till Figure E. 32 show the sample histogram alongside the theoretical normal distribution (with the mean and variance estimated from the input signal), the Q-Q plot, and the P-P plot of $\operatorname{Re}\left\{C_{x}^{\alpha}(f)\right\}$ and its Fisher z-transformation, for $P=5,17,65,129,257,512,1024$.

Clearly, for low $P$, the sample distribution does not fit the theoretical normal distribution, and the same conclusion can be reached for the input signal Fisher z-transformation. For example, for $P=$ 65 - and as shown in Figure E. 23 and Figure E. 24 - even though the histograms might induce a possible fit to the Gaussian assumption, the Q-Q Plots and the P-P plots clearly show a deviation from that distribution, with points deviating from the straight line. Nevertheless, for the Fisher ztransformation, the results show a closer agreement.

As before, for large $P$, typically, for $P \geq 129$ (or again, in a more conservative estimate, for $P \geq$ 257), all three forms of visual analysis corroborate the Gaussian assumption, with a close agreement of the sample histogram with the normal distribution, and points on the Q-Q plots and P-P plots distributed on the straight line.

However, for the input signal Fisher z-transformation, the Gaussian assumption seems plausible even for $P \geq 65$.


Figure E. 19 - Histogram, Q-Q plot, and P-P plot of the real part of the $\mathrm{SCoF}, P=5, N p S e g=128$.




Figure E. 20 - Histogram, Q-Q plot, and P-P plot of the real part of the SCoF Fisher z-transformation, $P=5, N p S e g=128$.




Figure E. 21 - Histogram, Q-Q plot, and P-P plot of the real part of the $\mathrm{SCoF}, P=17, N p S e g=128$.




Figure E. 22 - Histogram, Q-Q plot, and P-P plot of the real part of the SCoF Fisher z-transformation,

$$
P=17, \quad N p S e g=128
$$




Figure E. 23 - Histogram, Q-Q plot, and P-P plot of the real part of the $\mathrm{SCoF}, P=65, N p S e g=128$.




Figure E. 24 - Histogram, Q-Q plot, and P-P plot of the real part of the SCoF Fisher z-transformation, $P=65, N p S e g=128$.




Figure E. 25 - Histogram, Q-Q plot, and P-P plot of the real part of the SCoF, $P=129, N p S e g=128$.




Figure E. 26 - Histogram, Q-Q plot, and P-P plot of the real part of the SCoF Fisher z-transformation, $P=129, N p S e g=128$.




Figure E. 27 - Histogram, Q-Q plot, and P-P plot of the real part of the SCoF, $P=257, N p S e g=128$.




Figure E. 28 - Histogram, Q-Q plot, and P-P plot of the real part of the SCoF Fisher z-transformation,

$$
P=257, \quad N p S e g=128
$$




Figure E. 29 - Histogram, Q-Q plot, and P-P plot of the real part of the SCoF, $P=513, N p S e g=128$.




Figure E. 30 - Histogram, Q-Q plot, and P-P plot of the real part of the SCoF Fisher z-transformation,

$$
P=513, \quad N p S e g=128
$$





Figure E. 31 - Histogram, Q-Q plot, and P-P plot of the real part of the $\mathrm{SCoF}, P=1025, N p S e g=128$.




Figure E. 32 - Histogram, Q-Q plot, and P-P plot of the real part of the SCoF Fisher z-transformation,

$$
P=1025, \quad N p S e g=128
$$

## Goodness-of-Fit tests the SCoF and its Fisher z-transformation

Finally, we will use standard Goodness-of-Fit tests to assess the validity of approximating the SCoF and its Fisher z-transformation by a Gaussian distribution. Specifically, we will use the Chisquare, Anderson-Darling, Kolmogorov-Smirnov, Shapiro-Wilks and Shapiro-Francia tests, which are, probably, the most widely used in practice (Arnastauskaitė, Ruzgas and Bražėnas, 2021).

The two hypotheses of a goodness-of-fit are the null and alternative hypotheses defined as, respectively,

$$
H_{0} \text { : The distribution is normal; }
$$

$$
H_{A}: \text { The distribution is not normal. }
$$

Although none of the tests assures that the sample distribution follows a specific test distribution, it will decide if the null hypothesis can or can not be rejected. Therefore, if the calculated $p$-value of the test is bigger than the significance level (typically, 0.05 , which is the value used here), the $H_{0}$ can not be rejected (so we assume that the sample data are not significantly different from a normal population) ${ }^{48}$.

[^34]It is well known that the $p$-value estimation in a large samples size, $n$, will lead to $\lim _{n \rightarrow \infty} p=0$ (Lin, Lucas and Shmueli, 2013). Therefore, for the complete samples simulated $\left(2^{20}\right)$, and, as suggested in (Lin, Lucas and Shmueli, 2013; Arnastauskaitè, Ruzgas and Bražėnas, 2021), we will randomly draw a sample of reasonable size for fitting the model and estimate the $p$-value for the tests considered. Here, we used 2500 randomly drawn samples and repeated the process $1,000,000$ times (Arnastauskaitė, Ruzgas and Bražènas, 2021), obtaining the mean, minimum and maximum estimated $p$-value, along with a consistency estimate, measuring the percentage of the draws that did not reject the null hypothesis ( $p \geq 0.05$ ).

When the samples come from a Gaussian distribution (specifically from a standard normal distribution), the simulation results obtained from the described methodology, as shown in Table E.1, give the basis for some analysis of the results obtained for the SCoF and its Fisher ztransformation. As can be seen, the consistency is typically around 0.95 for all tests, the mean $p$ value around 0.5 , and the minimum and maximum can reach almost 0 and 1 , respectively.

| Test | Consistency | $\operatorname{mean}(\boldsymbol{p}$-value $)$ | $\min (p$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.94959 | 0.49245 | $2.6629 \mathrm{e}-07$ | 0.99999 |
| Anderson-Darling | 0.95028 | 0.50016 | $5.0000 \mathrm{e}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.95005 | 0.50022 | $9.8046 \mathrm{e}-07$ | 1 |
| Shapiro-Wilks | 0.95261 | 0.49824 | $6.4022 \mathrm{e}-06$ | 0.99997 |
| Shapiro-Francia | 0.94964 | 0.50221 | $5.6409 \mathrm{e}-06$ | 0.99999 |

Table E. 1 - p-values obtained from a standard normal sample distribution.
A similar procedure was done using the simulation results for the SCoF and its Fisher ztransformation, for varying $P(P=5,9,17,33,65,129,257,513,1025)$ and for the chosen case of NpSeg $=128$. Table E. 2 till Table E. 10 show the results obtained for the SCoF, and Table E. 11 till Table E. 19 show the results for its Fisher z-transformation.

| Test | Consistency | mean $(\boldsymbol{p}$-value $)$ | $\min (p$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0 | $3.1484 \mathrm{E}-07$ | $1.3199 \mathrm{E}-33$ | 0.01017 |
| Anderson-Darling | 0 | $5.0000 \mathrm{E}-04$ | $5.0000 \mathrm{E}-04$ | $5.0000 \mathrm{E}-04$ |
| Kolmogorov-Smirnov | 0.06981 | 0.01418 | $2.3157 \mathrm{E}-13$ | 0.73031 |
| Shapiro-Wilks | 0 | $1.4663 \mathrm{E}-12$ | 0 | $4.9654 \mathrm{E}-09$ |
| Shapiro-Francia | 0 | $1.4029 \mathrm{E}-11$ | $3.3307 \mathrm{E}-16$ | $3.4109 \mathrm{E}-08$ |

Table E. 2 - p-values obtained for the SCoF, for $P=5$.

| Test | Consistency | $\operatorname{mean}(p$-value $)$ | $\min (p$-value $)$ | $\max (p$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.00562 | 0.00149 | $2.7113 \mathrm{E}-19$ | 0.82990 |
| Anderson-Darling | 0.00029 | 0.00074 | $5.0000 \mathrm{E}-04$ | 0.25710 |
| Kolmogorov-Smirnov | 0.55612 | 0.11769 | $5.5171 \mathrm{E}-09$ | 0.99914 |
| Shapiro-Wilks | 0 | $3.1773 \mathrm{E}-06$ | $1.1429 \mathrm{E}-12$ | 0.01340 |
| Shapiro-Francia | 0 | $1.1470 \mathrm{E}-05$ | $1.2035 \mathrm{E}-11$ | 0.03416 |

Table E. 3 - p-values obtained for the SCoF, for $P=9$.

| Test | Consistency | mean $(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.45323 | 0.11044 | $6.5067 \mathrm{E}-11$ | 0.99966 |
| Anderson-Darling | 0.33584 | 0.07074 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.85243 | 0.31164 | $1.0714 \mathrm{E}-07$ | 1 |
| Shapiro-Wilks | 0.10991 | 0.02162 | $1.4099 \mathrm{E}-08$ | 0.99815 |
| Shapiro-Francia | 0.18925 | 0.03755 | $7.2798 \mathrm{E}-08$ | 0.99983 |

Table E. 4 - p-values obtained for the SCoF, for $P=17$.

| Test | Consistency | mean( $\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.86560 | 0.35547 | $2.9564 \mathrm{E}-08$ | 0.99999 |
| Anderson-Darling | 0.82394 | 0.31558 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.92690 | 0.43947 | $9.7840 \mathrm{E}-07$ | 0.99999 |
| Shapiro-Wilks | 0.75076 | 0.24970 | $6.0596 \mathrm{E}-07$ | 0.99983 |
| Shapiro-Francia | 0.82814 | 0.31562 | $2.1754 \mathrm{E}-06$ | 0.99995 |

Table E. 5 - p-values obtained for the SCoF, for $P=33$.

| Test | Consistency | mean $(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.93225 | 0.45174 | $4.2157 \mathrm{E}-07$ | 0.99998 |
| Anderson-Darling | 0.92127 | 0.44134 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.94326 | 0.48112 | $1.2502 \mathrm{E}-06$ | 1 |
| Shapiro-Wilks | 0.91349 | 0.41801 | $5.0968 \mathrm{E}-06$ | 1 |
| Shapiro-Francia | 0.94043 | 0.46867 | $5.4162 \mathrm{E}-06$ | 0.99999 |

Table E. 6 - p-values obtained for the SCoF, for $P=65$.

| Test | Consistency | $\operatorname{mean}(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.94532 | 0.48043 | $1.2891 \mathrm{E}-06$ | 0.99999 |
| Anderson-Darling | 0.94189 | 0.48175 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.94767 | 0.49409 | $2.5396 \mathrm{E}-08$ | 1 |
| Shapiro-Wilks | 0.94475 | 0.47541 | $3.0419 \mathrm{E}-06$ | 0.99998 |
| Shapiro-Francia | 0.95742 | 0.50744 | $9.3599 \mathrm{E}-06$ | 0.99999 |

Table E. 7 - p-values obtained for the SCoF, for $P=129$.

| Test | Consistency | $\operatorname{mean}(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.95035 | 0.49118 | $1.4669 \mathrm{E}-06$ | 1 |
| Anderson-Darling | 0.94913 | 0.49803 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.94935 | 0.49884 | $2.0313 \mathrm{E}-06$ | 1 |
| Shapiro-Wilks | 0.95551 | 0.50096 | $2.4911 \mathrm{E}-06$ | 0.99998 |
| Shapiro-Francia | 0.95974 | 0.51667 | $4.3861 \mathrm{E}-06$ | 0.99999 |

Table E. 8 - p-values obtained for the SCoF , for $P=257$.

| Test | Consistency | $\operatorname{mean}(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.95038 | 0.49328 | $3.9984 \mathrm{E}-07$ | 0.99999 |
| Anderson-Darling | 0.95017 | 0.50007 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.94999 | 0.50007 | $5.2989 \mathrm{E}-08$ | 1 |
| Shapiro-Wilks | 0.95062 | 0.49739 | $9.0632 \mathrm{E}-07$ | 0.99999 |
| Shapiro-Francia | 0.94927 | 0.50430 | $1.7496 \mathrm{E}-06$ | 0.99999 |

Table E. 9 - p-values obtained for the SCoF, for $P=513$.

| Test | Consistency | $\operatorname{mean}(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.95039 | 0.49345 | $5.2988 \mathrm{E}-07$ | 1 |
| Anderson-Darling | 0.95063 | 0.50150 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.95042 | 0.50098 | $2.5434 \mathrm{E}-06$ | 1 |
| Shapiro-Wilks | 0.95377 | 0.50124 | $5.0218 \mathrm{E}-06$ | 1 |
| Shapiro-Francia | 0.95290 | 0.50640 | $7.2928 \mathrm{E}-06$ | 0.99998 |

Table E. 10 - p-values obtained for the SCoF, for $P=1025$.

As can be concluded from these results, the hypothesis of approximating the SCoF by a Gaussian distribution is not verified for low $P$. Nevertheless, for $P \geq 129$ or $P \geq 257$, the results obtained are similar to those obtained for the standard normal distribution sample.

As referenced, the following tables show the results obtained for the Fisher z-transformation of the SCoF.

| Test | Consistency | mean $(\boldsymbol{p}$-value $)$ | $\boldsymbol{\operatorname { m i n } ( \boldsymbol { p } \text { -value } )}$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.69534 | 0.24083 | $1.2320 \mathrm{E}-12$ | 0.99973 |
| Anderson-Darling | 0.58880 | 0.17932 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.89043 | 0.36604 | $2.5123 \mathrm{E}-06$ | 0.99998 |
| Shapiro-Wilks | 0.51982 | 0.16162 | $2.3054 \mathrm{E}-09$ | 0.99884 |
| Shapiro-Francia | 0.46177 | 0.13678 | $3.7402 \mathrm{E}-09$ | 0.99893 |

Table E. 11 - p-values obtained for the SCoF z-transformation, for $P=5$.

| Test | Consistency | mean( $\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.87609 | 0.39120 | $4.6347 \mathrm{E}-08$ | 0.99995 |
| Anderson-Darling | 0.85262 | 0.36245 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.93615 | 0.45579 | $7.4979 \mathrm{E}-06$ | 0.99999 |
| Shapiro-Wilks | 0.81073 | 0.34019 | $1.2705 \mathrm{E}-07$ | 0.99997 |
| Shapiro-Francia | 0.76765 | 0.30588 | $8.3297 \mathrm{E}-08$ | 0.99989 |

Table E. 12 - p-values obtained for the SCoF z-transformation, for $P=9$.

| Test | Consistency | mean $(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.92110 | 0.44729 | $4.4839 \mathrm{E}-07$ | 0.99992 |
| Anderson-Darling | 0.91876 | 0.44252 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.94569 | 0.48631 | $7.3061 \mathrm{E}-06$ | 1 |
| Shapiro-Wilks | 0.87894 | 0.41366 | $3.6697 \mathrm{E}-07$ | 0.99991 |
| Shapiro-Francia | 0.84475 | 0.38203 | $2.2150 \mathrm{E}-07$ | 0.99968 |

Table E. 13 - p-values obtained for the SCoF z-transformation, for $P=17$.

| Test | Consistency | mean $(\boldsymbol{p}$-value $)$ | $\boldsymbol{m i n}(\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.93797 | 0.47464 | $4.2181 \mathrm{E}-08$ | 0.99994 |
| Anderson-Darling | 0.93884 | 0.47773 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.95016 | 0.49516 | $2.4746 \mathrm{E}-06$ | 0.99999 |
| Shapiro-Wilks | 0.92514 | 0.46649 | $8.2928 \mathrm{E}-06$ | 0.99985 |
| Shapiro-Francia | 0.90708 | 0.45057 | $6.4738 \mathrm{E}-06$ | 0.99973 |

Table E. 14 - p-values obtained for the SCoF z-transformation, for $P=33$.

| Test | Consistency | mean $(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | max $(\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.94581 | 0.48851 | $4.5415 \mathrm{E}-06$ | 0.99998 |
| Anderson-Darling | 0.94841 | 0.49689 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.95060 | 0.50076 | $1.5456 \mathrm{E}-05$ | 0.99997 |
| Shapiro-Wilks | 0.94687 | 0.49317 | $3.1814 \mathrm{E}-05$ | 0.99993 |
| Shapiro-Francia | 0.93848 | 0.48668 | $6.7862 \mathrm{E}-05$ | 0.99995 |

Table E. 15 - p-values obtained for the SCoF z-transformation, for $P=65$.

| Test | Consistency | $\operatorname{mean}(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.94944 | 0.48925 | $1.6519 \mathrm{E}-05$ | 0.99994 |
| Anderson-Darling | 0.94995 | 0.49872 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.95120 | 0.49989 | $5.3761 \mathrm{E}-05$ | 0.99997 |
| Shapiro-Wilks | 0.95195 | 0.49760 | $3.8363 \mathrm{E}-05$ | 0.99997 |
| Shapiro-Francia | 0.94869 | 0.49873 | $4.7810 \mathrm{E}-05$ | 0.99990 |

Table E. 16 - p-values obtained for the SCoF z-transformation, for $P=129$.

| Test | Consistency | mean $(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.94904 | 0.49159 | $1.4680 \mathrm{E}-06$ | 0.99996 |
| Anderson-Darling | 0.95016 | 0.50025 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.94981 | 0.49993 | $2.4367 \mathrm{E}-05$ | 0.99999 |
| Shapiro-Wilks | 0.95315 | 0.50265 | $1.1578 \mathrm{E}-05$ | 0.99997 |
| Shapiro-Francia | 0.94907 | 0.50160 | $1.2280 \mathrm{E}-05$ | 0.99985 |

Table E. 17 - p-values obtained for the SCoF z-transformation, for $P=257$.

| Test | Consistency | $\operatorname{mean}(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.94963 | 0.49225 | $1.2423 \mathrm{E}-05$ | 0.99996 |
| Anderson-Darling | 0.95023 | 0.50017 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.95075 | 0.50003 | $4.2352 \mathrm{E}-06$ | 0.99998 |
| Shapiro-Wilks | 0.94904 | 0.49555 | $1.2632 \mathrm{E}-05$ | 0.99998 |
| Shapiro-Francia | 0.94237 | 0.49500 | $8.6855 \mathrm{E}-06$ | 0.99999 |

Table E. 18 - p-values obtained for the SCoF z-transformation, for $P=513$.

| Test | Consistency | mean $(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.94917 | 0.49349 | $8.2459 \mathrm{E}-07$ | 0.99999 |
| Anderson-Darling | 0.94973 | 0.50088 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.94903 | 0.49922 | $1.1771 \mathrm{E}-05$ | 0.99999 |
| Shapiro-Wilks | 0.95054 | 0.49917 | $1.6347 \mathrm{E}-06$ | 0.99991 |
| Shapiro-Francia | 0.94600 | 0.50169 | $2.8429 \mathrm{E}-06$ | 0.99982 |

Table E. 19 - p-values obtained for the SCoF z-transformation, for $P=1025$.

From the results obtained for the SCoF Fisher z-transformation (Table E. 11 till Table E.19), and even for low $P$, the Gaussian approach seems feasible. Specifically, for $P \geq 33$ or $P \geq 65$, the results obtained are in line with the ones obtained for the standard normal distribution sample. This is also in line with the conclusions obtained from the analysis of the histograms, Q-Q plots, and P-P plots.

As the Fisher z-transformation is given as $z=\operatorname{atanh}\left(C_{x}^{\alpha}\right)$, then, when $P$ is high, its pdf will be concentrated around $z=0$. In this case, atanh $\left(C_{x}^{\alpha}\right) \approx C_{x}^{\alpha}$, so that the pdf of both the SCoF and its Fisher z-transformation will be approximated by a Gaussian distribution.

## Q-Q and P-P Plots for the SCF

For the SCF, and with no overlapping segments, the Central Limit Theorem could be invoked to approximate the sample distribution as Gaussian, as it would be obtained by the mean of independent segments. Without the exhaustiveness made for the SCoF , we will present a short
sample of simulation results obtained. We will obtain the histograms, Q-Q and P-P plots, and Goodness-of-fit test for varying $P$, again for $N p S e g=128$.

Figure E. 33 till Figure E. 38 show the histograms, Q-Q and P-P plots for $P=$ 5, 33,128,257,513, 1024 .

As obtained for the SCoF, for low $P$, the sample distribution of the SCF does not fit the theoretical normal distribution. However, for large $P$, typically, for $P \geq 129$ (or, in a more conservative estimate, for $P \geq 257$ or even $P \geq 513$ ), all three forms of visual analysis corroborate the Gaussian assumption, with a close agreement of the sample histogram with the normal distribution, and points on the Q-Q plots and P-P plots distributed on the straight line.




Figure E. 33 - Histogram, Q-Q plot, and P-P plot of the real part of the SCF, $P=5, N p S e g=128$.




Figure E. 34 - Histogram, Q-Q plot, and P-P plot of the real part of the $\mathrm{SCF}, P=33, N p S e g=128$.




Figure E. 35 - Histogram, Q-Q plot, and P-P plot of the real part of the SCF, $P=129, N p S e g=128$.




Figure E. 36 - Histogram, Q-Q plot, and P-P plot of the real part of the SCF, $P=257, N p S e g=128$.



Figure E. 37 - Histogram, Q-Q plot, and P-P plot of the real part of the SCF, $P=513, N p S e g=128$.


Figure E. 38 - Histogram, Q-Q plot, and P-P plot of the real part of the $\mathrm{SCF}, P=1025, N p S e g=128$.

## Goodness-of-Fit tests the SCF

As for the SCoF case, we will use standard Goodness-of-Fit tests to assess the validity of approximating the SCF by a Gaussian distribution. Again, we will use the Chi-square, AndersonDarling, Kolmogorov-Smirnov, Shapiro-Wilks, and Shapiro-Francia tests. Table E. 20 till Table E. 25 show the results obtained for $P=5,33,129,257,513,1025$.

| Test | Consistency | mean( $\boldsymbol{p}$-value $)$ | $\min ($ p-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | $4.5000 \mathrm{E}-04$ | $6.8495 \mathrm{E}-05$ | $8.8679 \mathrm{E}-89$ | 0.69547 |
| Anderson-Darling | 0 | $5.0000 \mathrm{E}-04$ | $5.0000 \mathrm{E}-04$ | $5.0000 \mathrm{E}-04$ |
| Kolmogorov-Smirnov | 0 | $4.4776 \mathrm{E}-08$ | $1.2750 \mathrm{E}-21$ | $1.7965 \mathrm{E}-04$ |
| Shapiro-Wilks | 0 | $1.3323 \mathrm{E}-20$ | 0 | $9.9920 \mathrm{E}-16$ |
| Shapiro-Francia | 0 | $1.2990 \mathrm{E}-19$ | 0 | $6.2172 \mathrm{E}-15$ |

Table E. 20 - p-values obtained for the SCF for $P=5$.

| Test | Consistency | mean( $\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.23207 | 0.05378 | $9.0142 \mathrm{E}-17$ | 0.99294 |
| Anderson-Darling | 0.11245 | 0.02356 | $5.0000 \mathrm{E}-04$ | 0.96870 |
| Kolmogorov-Smirnov | 0.75827 | 0.21626 | $2.6788 \mathrm{E}-07$ | 0.99987 |
| Shapiro-Wilks | 0.05465 | 0.01200 | $2.3759 \mathrm{E}-14$ | 0.99621 |
| Shapiro-Francia | 0.04091 | 0.00887 | $7.3941 \mathrm{E}-14$ | 0.98878 |

Table E. 21 - p-values obtained for the SCF for $P=33$.

| Test | Consistency | $\operatorname{mean}(\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.89357 | 0.41039 | $1.3432 \mathrm{E}-10$ | 0.99998 |
| Anderson-Darling | 0.88270 | 0.39380 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.94147 | 0.46920 | $2.6057 \mathrm{E}-06$ | 0.99998 |
| Shapiro-Wilks | 0.81824 | 0.35365 | $3.1054 \mathrm{E}-07$ | 0.99994 |
| Shapiro-Francia | 0.77160 | 0.31903 | $2.3889 \mathrm{E}-07$ | 0.99994 |

Table E. 22 - p-values obtained for the SCF for $P=129$.

| Test | Consistency | mean( $\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.93314 | 0.46679 | $5.9300 \mathrm{E}-07$ | 0.99994 |
| Anderson-Darling | 0.93398 | 0.47054 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.94909 | 0.49352 | $3.1143 \mathrm{E}-05$ | 1 |
| Shapiro-Wilks | 0.91542 | 0.45387 | $1.1015 \mathrm{E}-06$ | 0.99995 |
| Shapiro-Francia | 0.89295 | 0.43190 | $1.6073 \mathrm{E}-06$ | 0.99982 |

Table E. 23 - p-values obtained for the SCF for $P=257$.

| Test | Consistency | mean( $\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.94425 | 0.48321 | $6.5508 \mathrm{E}-06$ | 0.99999 |
| Anderson-Darling | 0.94558 | 0.49000 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.94930 | 0.49808 | $2.1056 \mathrm{E}-05$ | 0.99998 |
| Shapiro-Wilks | 0.93489 | 0.47743 | $6.0595 \mathrm{E}-06$ | 0.99995 |
| Shapiro-Francia | 0.91822 | 0.46138 | $4.3883 \mathrm{E}-06$ | 0.99991 |

Table E. 24 - p-values obtained for the SCF for $P=513$.

| Test | Consistency | mean( $\boldsymbol{p}$-value $)$ | $\min (\boldsymbol{p}$-value $)$ | $\max (\boldsymbol{p}$-value $)$ |
| :--- | ---: | ---: | ---: | ---: |
| Chi-square | 0.94842 | 0.48981 | $1.8132 \mathrm{E}-05$ | 0.99997 |
| Anderson-Darling | 0.94790 | 0.49651 | $5.0000 \mathrm{E}-04$ | 0.99000 |
| Kolmogorov-Smirnov | 0.95152 | 0.50066 | $5.6498 \mathrm{E}-06$ | 0.99997 |
| Shapiro-Wilks | 0.94712 | 0.49362 | $5.2327 \mathrm{E}-06$ | 0.99987 |
| Shapiro-Francia | 0.94121 | 0.49269 | $2.1374 \mathrm{E}-06$ | 0.99989 |

Table E. 25 - p-values obtained for the SCF for $P=1025$.

From these results, the hypothesis of approximating the SCF by a Gaussian distribution is not verified for low $P$. Nevertheless, for $P \geq 257$ or $P \geq 513$, the results obtained are similar to those obtained for the standard normal distribution sample.

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WIDEBAND SPECTRUM SENSING
FOR DYNAMIC SPECTRUM SHARING


[^0]:    ${ }^{1}$ 2013/752/EU, Amending Decision 2006/771/EC on harmonization of the radio spectrum for use by short-range devices and repealing Decision 2005/928/EC (https://eur-lex.europa.eu/legal-content/EN/TXT/?uri=CELEX\%3A32013D0752)

[^1]:    ${ }^{2}$ http://dynamicspectrumalliance.org The Dynamic Spectrum Alliance is a global, cross-industry, not-for-profit organization advocating for laws, regulations, and economic best practices that will lead to more efficient utilization of spectrum and affordable broadband for all.

[^2]:    ${ }^{3}$ In January 2020, the FCC certified CommScope, Federated Wireless, Inc. (Federated), Google and Sony, Inc. (Sony) as CBRS Spectrum Access System (SAS) Administrators (FCC, 2020).
    ${ }^{4}$ There are 15 covering areas in the west coast and 26 in the East and Gulf Coast.

[^3]:    ${ }^{5}$ https://www.multefire.org/ The MulteFire Alliance is an independent, diverse, and international member-driven consortium defining and promoting MulteFire - a cellular-based technology for operating in unlicensed and shared spectrum.

[^4]:    ${ }^{6}$ In the USA the two allowed bands are $5,150-5,350 \mathrm{GHz}$ and $5,470-5,850 \mathrm{GHz}$, with a total of 580 MHz bandwidth. In Europe the allowed bands are $5,150-5,350 \mathrm{GHz}$ and $5,470-5,725 \mathrm{GHz}$, with a total of 455 MHz bandwidth.

[^5]:    ${ }^{7}$ With work starting in 2012, the standard was expected to be completed in 2020, having parts finalized and published earlier. Despite the COVID-19 pandemic, there has been only a slightly time shift, and the "Detailed specifications of the radio interfaces of IMT-2020" standard was published in February 2021 (https://www.itu.int/en/ITU-R/study-groups/rsg5/rwp5d/imt-2020).
    ${ }^{8}$ ECC Decision (14)02-Harmonized technical and regulatory conditions for the use of the band $2300-2400 \mathrm{MHz}$ for Mobile/Fixed Communications Networks (MFCN)

[^6]:    ${ }^{9}$ Available at https://www.etsi.org/standards.
    ${ }^{10}$ All information regarding the Portugal LSA test pilot can be found at the following URL:
    https://www.anacom.pt/render.jsp?contentId=1631361.
    ${ }^{11}$ In Portugal, this entity is the "Autoridade Nacional de Comunicações - ANACOM", and the national table of radio spectrum allocations can be found at the following URL: https://www.anacom.pt/render.jsp?categoryId=150422.

[^7]:    ${ }^{12}$ https://www.etsi.org/membership (the two continents with no members are Antarctica and South America)
    ${ }^{13}$ Decision No 243/2012/EU of the European Parliament and of the Council of 14 March 2012 establishing a multiannual radio spectrum policy programme (https://eur-lex.europa.eu/legal-content/EN/ALL/?uri=CELEX:32012D0243).

[^8]:    ${ }^{14}$ The Digital Single Market Strategy adopted by the EC in May, 2015 is built on three pillars: (1) better access for consumers and businesses to digital goods and services across Europe; (2) creating the right conditions and a level playing field for digital networks and innovative services to flourish; (3) maximizing the growth potential of the digital economy (http://eur-lex.europa.eu/legalcontent/EN/TXT/?qid=1447773803386\&uri=CELEX:52015DC0192).
    ${ }^{15}$ The known as "RSPP bands" are: $694-862 \mathrm{MHz} ; 880-915 \mathrm{MHz} ; 925-960 \mathrm{MHz} ; 1,427-1,517 \mathrm{MHz} ; 1,710-1,785 \mathrm{MHz} ; 1,805-$ 1,880 MHz; 1,920-1,980 MHz; 1,920-1,880 MHz; 2,110 - 2,170 MHz; 2,300-2400 MHz; 2,500-2,690 MHz; 3,400-3,800 MHz
    ${ }^{16}$ Radio Spectrum Decisions - Decisions adopted by the Commission pursuant to the Radio Spectrum Decision in view of the harmonization of the technical conditions for the availability and efficient use of spectrum (https://ec.europa.eu/digital-single-market/en/news/radio-spectrumdecisions).

[^9]:    ${ }^{17}$ RSPG Opinion on the ITU-R World Radiocommunication Conference 2019 (http://rspg-spectrum.eu/wp-content/uploads/2018/10/RSPG18-038final-RSPG_opinion_on_WRC19.pdf).

[^10]:    ${ }^{18}$ In Portugal, the first successful LSA implementation and trial was finalized by the Portuguese Regulator (ANACOM) in March 2019 (see https://www.cept.org/ecc/topics/lsa-implementation).

[^11]:    ${ }^{19}$ https://www.wirelessinnovation.org/

[^12]:    ${ }^{20}$ The term Software Defined Radio was used for a radio communication system where components that had been typically implemented in hardware (like filters) where instead implemented by means of software in a digital system. It is said that "Mitola objected to Blust's term, but finally accepted it as a pragmatic pathway towards the ideal software radio."

[^13]:    ${ }^{21}$ A Spectrum Hole, also known as White Space is a band of frequencies assigned to a PU, but, at a specific time or geographic location, is not beeing used by that PU ( PU is in an idle state).
    ${ }^{22}$ http://grouper.ieee.org/groups/dyspan/

[^14]:    ${ }^{23}$ https://www.govinfo.gov/content/pkg/FR-2007-05-30/pdf/E7-10337.pdf

[^15]:    ${ }^{24}$ In this algorithm, modelling the input signal as Gaussian (as done in the case of the MED) would not make sense, because the probability of detection would always be equal to the probability of false alarm, as $\operatorname{Pr}\left\{\lambda_{\max }\left(\boldsymbol{R}_{r}\left(N_{s}\right)>\gamma \lambda_{\min }\left(\boldsymbol{R}_{r}\left(N_{s}\right)\right\}\right.\right.$ would be equal to $\operatorname{Pr}\left\{\lambda_{\max }\left(\boldsymbol{R}_{n}\left(N_{s}\right)>\gamma \lambda_{\text {min }}\left(\boldsymbol{R}_{n}\left(N_{s}\right)\right\}\right.\right.$.

[^16]:    ${ }^{25}$ If the asymmetric CAF is defined by $\widetilde{R}_{x}^{\alpha}(\tau)=\lim _{T \rightarrow \infty} \frac{1}{T} \int_{-T / 2}^{+T / 2} x(t) x^{*}(t-\tau) e^{-j 2 \pi \alpha t} d t$, the relation to the symmetric CAF is obtained similarly as $\widetilde{R}_{x}^{\alpha}(\tau)=e^{-j \pi \alpha \tau} R_{x}^{\alpha}(\tau)$.

[^17]:    ${ }^{26}$ Given $F(f)=\operatorname{FT}\{f(t)\}$ and $G(f)=\operatorname{FT}\{g(t)\}$, then $\int_{-\infty}^{+\infty} f(t) g^{*}(t) d t=\int_{-\infty}^{+\infty} F(f) G^{*}(f) d f$
    ${ }^{27} \lim _{\Delta t \rightarrow \infty, \Delta f \rightarrow 0}\{\cdot\}$ means $\lim _{\Delta f \rightarrow 0}\left\{\lim _{\Delta t \rightarrow \infty}\{\cdot\}\right\}$

[^18]:    ${ }^{28}$ Assuming these variables have zero mean, which holds if the signal has no additive sin wave component with frequency $f \pm \alpha / 2$.

[^19]:    ${ }^{29}$ Generally, the smoothed cyclic cross periodogram of two signals $x(t)$ and $y(t)$ is given by
    $S_{x y_{T}}^{\alpha}(n, f)_{\Delta t \Delta f}=\left\langle X_{T}(n, f+\alpha / 2) Y_{T}^{*}(n, f-\alpha / 2)\right\rangle_{\Delta t}$.

[^20]:    ${ }^{30}$ Using $\sum_{r} X_{T}\left(r, f_{k}\right) X_{T}^{*}\left(r, f_{l}\right) g(n-r)=\sum_{r} X_{T}\left(n-r, f_{k}\right) X_{T}^{*}\left(n-r, f_{l}\right) g(r)$ and noting that $X_{T}(n-r, f)=\sum_{k} a(k-r) x(n-$ $k) e^{-j 2 \pi(n-k) f}$.

[^21]:    ${ }^{31}$ In this context, the Strip Spectral Correlation Analyzer (SSCA) (Brown and Loomis, 1993) is probably the algorithm most referenced along the FAM algorithm.
    ${ }^{32} N_{t} \approx P \times L$ and $\Delta t=\frac{1}{\Delta \alpha}=\frac{N_{t}}{f_{s}}$, so $\Delta \alpha \approx \frac{f_{s}}{P \times L}$.

[^22]:    ${ }^{33}$ In this section, $\Delta f$ references the zoomed frequency region, and $\Delta \alpha$ references the zoomed cycle frequency region of interest. Therefore, here, $\Delta \alpha$ should not be misinterpreted as the CSA cell or FAM cycle frequency resolution, as used in section 3.3 or 3.4.

[^23]:    ${ }^{35}$ We will use this $\left(f_{i} ; \alpha_{i}\right)$ definition, providing $N p S e g$ points for $\left[0 ; \frac{f_{s}}{2}\right]$ frequency plane and $\left[0 ; f_{s}\right]$ cyclic frequency plane. The all plane FFT definition would use $f_{i}=K \frac{f_{s}}{N p S e g}$ and $\alpha_{i}=\Delta \frac{2 f_{s}}{N p S e g}$.

[^24]:    ${ }^{36}$ Here, authors use the standard FFT $\left(N p S e g\right.$ points for $\left.\left[0, f_{s}\right]\right)$ so $\frac{\left(C_{K, \Delta}\right)_{m, n}+\left(\boldsymbol{C}_{K, \Delta}\right)^{T}}{2}=\cos \left(\frac{2 \pi K}{N p S e g}(n-m)\right)$.

[^25]:    ${ }^{37}$ Commonly, the definition of $M(s=j w)=\varphi(w)$ is also used.

[^26]:    ${ }^{38}$ When considering $M(s=-j w)=\varphi(w)$ the r.v. moments are obtained from the CF as $E\left[Y^{k}\right]=\left.\frac{1}{(-j)^{k}} \frac{\partial^{k} \varphi(w)}{\partial w^{k}}\right|_{w=0}$

[^27]:    ${ }^{39}$ For a gamma r.v., $x$, with shape parameter $\alpha$ and rate parameter $\beta, E[x]=\alpha / \beta$.

[^28]:    ${ }^{40} \alpha_{x}=\beta$ and $\lambda_{x}=\alpha$, where $\alpha$ and $\beta$ are given by equation (4.31).

[^29]:    ${ }^{41}$ The given ML estimator is biased, and $E\left[\hat{\sigma}_{N}^{2}\right]=\frac{N-1}{N} \sigma^{2}$. The unbiased estimator is therefore given by $\hat{\sigma}_{N}=\sqrt{\frac{1}{N-1} \sum_{i=0}^{N-1}\left(x_{i}-\hat{\mu}_{N}\right)^{2}}$.

[^30]:    ${ }^{43}$ http://www.cortexlab.fr/

[^31]:    ${ }^{45}{ }_{1} F_{1}(a, b ; x)=\sum_{k=0}^{\infty} \frac{(a)_{k} x^{k}}{(b)_{k} k!}$, so $\sum_{m=0}^{\infty} \frac{(\beta)_{m}}{m!(\gamma)_{m}} x^{m}{ }_{1} F_{1}\left(\beta^{\prime}, \gamma+k ; y\right)=\sum_{m=0}^{\infty} \frac{(\beta)_{m}}{m!(\gamma)_{m}} x^{m} \sum_{m=0}^{\infty} \frac{\left(\beta^{\prime}\right)_{n}}{n!(\gamma+k)_{n}} y^{n}=\sum_{m, n=0}^{\infty} \frac{(\beta)_{m}\left(\beta^{\prime}\right)_{n}}{m!n!(\gamma)_{m}(\gamma+k)_{n}} x^{m} y^{n}=$ $\sum_{m, n=0}^{\infty} \frac{(\beta)_{m}\left(\beta^{\prime}\right)_{n}}{m!n!(\gamma)_{m+n}} x^{m} y^{n}=\Phi_{2}\left(\beta, \beta^{\prime}, \gamma, x, y\right)$

[^32]:    ${ }^{46} \frac{1}{2}\left[\begin{array}{cc}J & -I \\ J & I\end{array}\right]\left[\begin{array}{cc}J D J & B^{T} \\ B & D\end{array}\right]\left[\begin{array}{cc}J & J \\ -I & I\end{array}\right]=\frac{1}{2}\left[\begin{array}{cc}D-B J-J B^{T}+D & D-B J+J B^{T}-D \\ D+B J-J B^{T}-D & D+B J+J B^{T}+D\end{array}\right]=\frac{1}{2}\left[\begin{array}{cc}2 D-2 B J & 0 \\ 0 & 2 D+2 B J\end{array}\right]=\left[\begin{array}{cc}D-B J & 0 \\ 0 & D+B J\end{array}\right]$

[^33]:    ${ }^{47}$ When considering $C_{K, 0}^{\prime}==\cos \left(\frac{\pi K}{N p S e g}(i-j)\right)$, the code line "np $=2 \wedge(\mathrm{n}) ;$ " must be replaced by "np $=2^{\wedge}(\mathrm{n}+1)$;"

[^34]:    ${ }^{48}$ With a significance level $\alpha=0.05$, if the $p$-value obtained is less than this value, the null hypothesis is rejected, meaning that the probability that the null hypothesis is true is less than 0.05 ; if the $p$-value is bigger than $\alpha$, then the null hypothesis can not be rejected, so we can state that no significant departure from a normal distribution was found.

