The continuous Redner–Ben-Avraham–Kahng coagulation system: well-posedness and asymptotic behaviour[1]

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ABSTRACT

This paper examines the existence of solutions to the continuous Redner-Ben-Avraham-Kahng coagulation system under specific growth conditions on unbounded coagulation kernels at infinity. Moreover, questions related to uniqueness and continuous dependence on the data are also addressed under additional restrictions. Finally, the large-time behaviour of solutions is also investigated.

KeyWords : Coagulation; Redner–Ben-Avraham–Kahng Coagulation System; Existence; Uniqueness; Large-time behaviour.

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1. INTRODUCTION

A particulate process is a kinetic process that involves the changing of the physical properties of particles. These processes are widely studied in a range of domains including engineering, chemistry, physics, astronomy, and in other disciplines. During a particulate process, particles may combine to form bigger particles or break down into smaller ones, with a change in size, shape, volume, etc. Coagulation (aggregation), fragmentation (breakage), nucleation, and growth are some of the different types of particulate processes.

Coagulation and fragmentation processes describe how clusters merge to form larger ones and break apart into smaller fragments, respectively, in the dynamics of cluster formation. In the most basic coagulation models, cluster particles are distinguished by their cluster size (or volume), which can be continuous [12], or discrete [14], depending on the physical circunstances. Many researchers have been interested in the mean field models of the coagulation-fragmentation type [15], of which Smoluchowski's coagulation system is a prototype. Smoluchowski coagulation equation governs a fundamental dynamical process which is illustrated by the binary interaction of a j-cluster (a cluster composed of j identical particles) and a k-cluster to produce a $(j+k)$-cluster where the average

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cluster size generally increases with time. In contrast to coagulation, the basic dynamic process of fragmentation involves the breakup of a given j-cluster into two or more smaller clusters in which the average cluster size decreases with time.

These mean-field models have received a great deal of attention among the various mathematical methods for modelling the dynamics of cluster formation.

We shall consider a coagulation system which models a different type of dynamics. This is nothing but a cluster eating model which governs a coagulation problem that, despite its primary mechanism being binary cluster reactions, exhibits cluster size development comparable to that of a fragmentation problem. Redner, Ben-Avraham and Kahng proposed a finite-dimensional model based on this mechanism in [13] (see also [7]), later referred to as the Redner–Ben–Avraham–Kahng (RBK) system. The process is essentially the following: when particles of size j collides with particles of size i, the result is the partial anhilation of the particles involved, with the formation of residue particles with size $|i-j|$ (or no particles left if $i=j$). Schematically, the process is

$$(\text{cluster with size } i) + (\text{cluster with size } j) \longrightarrow (\text{cluster with size } |i-j|). \tag{1.1}$$

The infinite-dimensional RBK model describing the time evolution of the concentrations of the various cluster sizes present in the system reads as (see [4])

$$\frac{d\wp_i}{dt} = \sum_{j=1}^{\infty} a_{i+j,j}\wp_{i+j}\wp_j - \sum_{j=1}^{\infty} a_{i,j}\wp_i\wp_j, \; i \in \mathbb{N}, \tag{1.2}$$

where $\wp_i(t)$ is the concentration of i-clusters at time $t \geq 0$ and $a_{i,j}$ is the non-negative and symmetric ($0 \leq a_{i,j} = a_{j,i} \; \forall \; i,j \in \mathbb{N}$) coagulation rate coefficient for the reaction (1.1). The first sum on the right-hand side of (1.2) corresponds to the formation of clusters of size i via the reaction $(i+j)+(j) \to (i)$, and the second sum to the destruction of i-clusters by their reaction with clusters of any size, as in (1.1). This process is similar to the coagulation-annihilation model with partial annihilation in which two or more species of clusters, say A and B, are present, and if a cluster A_i reacts with a cluster B_j the resulting cluster has size $|i-j|$ and is an A cluster if $i > j$, is a B cluster if $i < j$, and is an inert cluster, neither A or B, if $i = j$.

One of the major distinctions between the RBK and Smoluchowski coagulation models, developed in 1917 by Marian v. Smoluchowski [14] to describe the coagulation of colloids moving by Brownian motion, is the absence of the mass conservation property in the RBK model. Smoluchowski coagulation equation was transformed by Müller [12] into a continuous integral version in 1928. In a similar way, the continuous version of RBK coagulation system corresponding to the discrete case (1.2) can be given as:

$$\frac{\partial \wp}{\partial t} = \int_0^{\infty} a(\varsigma+\varrho,\varrho)\wp(\varsigma+\varrho,t)\wp(\varrho,t)d\varrho - \int_0^{\infty} a(\varsigma,\varrho)\wp(\varsigma,t)\wp(\varrho,t)d\varrho \tag{1.3}$$

with initial condition

$$\wp(\varsigma,0) = \wp^{\text{in}}(\varsigma) \geq 0, \tag{1.4}$$

where $\wp(\varsigma,t)$ denotes the concentration of particles of volume $\varsigma \in \mathbb{R}_+ := (0,\infty)$ at time $t \geq 0$. The non-negative quantity $a(\varsigma,\varrho)$ represents the coagulation rate at which particles of volume ς and particles of volume ϱ interact to produce particles of volume $|\varsigma - \varrho|$. The rate 'a' is also known as the coagulation kernel or coagulation coefficient, which, as in the discrete case, is assumed to be non-negative and symmetric i.e. $0 \leq a(\varsigma, \varrho) = a(\varrho, \varsigma), \ \forall \ (\varsigma, \varrho) \in \mathbb{R}_+^2$. The physical meaning of each of the integrals on the right hand side of (1.3) is also similar to what was described above for the discrete case (1.2): the first integral describes the creation of particles of size ς as a result of the coagulation of two particles with respective sizes ϱ and $\varsigma + \varrho$, while the second one represents the disappearance of particles of size ς due to coalescence with other particles.

Furthermore, we require some information on the time evolution of moments of solutions '$\wp$' to (1.3) throughout this article. For the concentration $\wp$ of particles, we define the r^{th} moment by:

$$M_r(\wp)(t) = M_r(t) := \int_0^\infty \varsigma^r \wp(\varsigma, t) d\varsigma, \text{ for } r \in \mathbb{R}. \tag{1.5}$$

For $r = 0$, the zeroth moment $M_0(t)$ represents the total number of particles while for $r = 1$, the first moment $M_1(t)$ denotes the total mass (volume) of particles at time t. Generally, both moments are assumed to be finite. For the RBK system, the zeroth moment $M_0(t)$ is a decreasing function because of annihilation of particles and the first moment $M_1(t)$ is not conserved but a decreasing function.

The primary goal of this research is to determine the existence and uniqueness of solutions to the continuous RBK model (1.3)-(1.4), which does not satisfy mass conservation property. Moreover, the continuous dependence on the input data and the large-time behaviour of weak solutions have also been investigated. On the topic of the existence and uniqueness of solutions to the Smoluchowski coagulation equation derived by using various techniques under varied growth conditions on the coagulation kernel, several articles have been published, see [1, 2, 3, 6, 9, 10, 11, 15, 16]. However, The RBK coagulation system has not been much explored. In [4], da Costa et al. established the existence and uniqueness of solutions to (1.2) under reasonably general conditions on the coagulation coefficients. Furthermore, they proved the differentiability of the solutions as well as their continuous dependence on the input data. In addition, certain remarkable invariance aspects were illustrated. Finally, a study of the long-term behaviour of solutions as well as a preliminary analysis of their scaling behaviour, were conducted. In [5] the authors have discussed the large-time behaviour of solutions $(\wp_i)$ to the RBK coagulation system (1.2) with nonnegative compactly supported initial data, which, due to the invariance properties alluded to above, turn system (1.2) into a finite-dimensional differential equation. However, to the best of our knowledge, the continuous version of the RBK model (1.3)-(1.4) has not yet been taken into account at all, and this is the first mathematical contribution for investigating the existence, uniqueness, continuous dependence and large-time behaviour of solutions to the continuous RBK model (1.3)-(1.4).

The article is organised in different sections. A few required definitions and assumptions, along with the main results, viz. existence, uniqueness and large-time behaviour of solutions to (1.3)-(1.4) are stated in Section 2. Section 3 is devoted to prove the existence of solutions to the continuous RBK system (1.3)-(1.4) for the initial data belongs to the space $X_{0,1}^+$ and to prove that the mass is not conserved and decreases with time. Next, the proof of Theorem 2.2 is described in Section 4 which shows the uniqueness of solutions to (1.3)-(1.4). Further, the continuous dependence of solution on the initial data is shown in the Section 5. At the end, the large-time behaviour of solutions to (1.3)-(1.4) is discussed in the last section.

2. PRELIMINARIES AND MAIN RESULTS

Let us define the following Banach space:

$$X_{0,1} := L^1(\mathbb{R}_+, (1+x)dx),$$

with norm

$$\|f\|_{X_{0,1}} := \int_0^\infty (1+x)|f(x)|dx.$$

Then we set

$$X_{0,1}^+ := \{y \in X_{0,1} : y \geq 0\},$$

which is the positive cone of the space $X_{0,1}$.

Let us assume that the coagulation kernel 'a' is a non-negative, symmetric ($0 \leq a(\varsigma, \varrho) = a(\varrho, \varsigma),\ \forall\ (\varsigma, \varrho) \in \mathbb{R}_+^2$), and measurable function on $\mathbb{R}_+^2$ and that there are $\alpha \in [0,1)$ and a constant $k \geq 0$ such that

$$a(\varsigma, \varrho) \leq k(1+\varsigma)^\alpha (1+\varrho)^\alpha,\ \ (\varsigma, \varrho) \in \mathbb{R}_+^2. \tag{2.1}$$

We begin by defining the notion of the solution that will be used throughout the paper.

Definition 2.1. *Let $T \in (0, \infty]$, $\wp^{in} \in X_{0,1}^+$, and the coagulation kernel 'a' satisfy* (2.1). *A (mild) solution of* (1.3)-(1.4) *on* $[0, T)$ *is a solution of the corresponding integral equation, namely: it is a non-negative real valued function* $\wp : [0, T) \to X_{0,1}^+$ *such that for every* $t \in [0, T)$, *the following holds:*

$$\wp \in \mathcal{C}([0,T); L^1(0,\infty)) \cap L^\infty(0,T; X_{0,1}), \tag{2.2}$$

$$(\varrho, s) \mapsto a(\varsigma, \varrho)\wp(\varsigma, s)\wp(\varrho, s) \in L^1((0,\infty) \times (0,t)), \tag{2.3}$$

and for almost every $\varsigma \in \mathbb{R}_+$,

$$\begin{aligned} \wp(\varsigma, t) &= \wp^{in}(\varsigma) + \int_0^t \int_0^\infty a(\varsigma+\varrho, \varrho)\wp(\varsigma+\varrho, s)\wp(\varrho, s)d\varrho ds \\ &\quad - \int_0^t \int_0^\infty a(\varsigma, \varrho)\wp(\varsigma, s)\wp(\varrho, s)d\varrho ds. \end{aligned} \tag{2.4}$$

Our main contribution in this article is given by the following theorems:

Theorem 2.1. *Suppose the coagulation kernel 'a' satisfies* (2.1) *and consider any initial condition* $\wp^{in} \in X_{0,1}^+$. *Then, the initial value problem* (1.3)-(1.4) *has at least one solution '*$\wp$*' on* $[0, +\infty)$, *and it satisfies*

$$M_1(\wp)(t) \leq M_1(\wp^{in}),\ \ t \geq 0.$$

Theorem 2.2. *Assume the coagulation kernel 'a' satisfies* (2.1) *with* $\alpha \leq \frac{1}{2}$. *and take any non-negative initial condition* $\wp^{in} \in X_{0,1}^{+}$. *Then there exists a unique solution '*$\wp$*' of* (1.3)-(1.4) *on* $[0,\infty)$.

Theorem 2.3. *Let '*$\wp$*' be a solution to* (1.3)-(1.4) *with* $\wp^{in} \in X_{0,1}^{+}$. *Assume that for* $(\varsigma,\varrho) \in \mathbb{R}_+^2$, *there exists* $W > 0$ *such that the coagulation kernel 'a' satisfies* (2.1) *along with*

$$a(\varsigma,\varrho) \geq W. \tag{2.5}$$

Then

$$\lim_{t\to\infty} M_0(t) = 0. \tag{2.6}$$

Moreover, let us consider 'a' satisfying (2.1) *and*

$$a(\varsigma,\varrho) \geq C_0(\varsigma\varrho)^{\omega/2}, \ (\varsigma,\varrho) \in \mathbb{R}_+^2 \tag{2.7}$$

for some $\omega \in (1,2]$ *and* $C_0 > 0$. *Then*

$$\lim_{t\to\infty} M_1(t) = 0. \tag{2.8}$$

3. EXISTENCE OF SOLUTIONS

The aim of this section is to construct solutions for the continuous RBK system (1.3)-(1.4) having a initial condition $\wp^{\text{in}} \in X_{0,1}^{+}$. In order to attain this, we employ the weak L_1-compactness technique. This approach was first introduced in the classic work of Stewart [15]. For a very clear presentation of the method in the context of the continuous Smoluchowski's equation see [10].

3.1. The truncated problem.

The first step of this approach is the selection of an approximate system to the continuous RBK equations (1.3)-(1.4) with initial condition $\wp^{\text{in}} \in X_{0,1}^{+}$. For this, we consider the so called n-truncated system, i.e., the equation that results from considering that only clusters of sizes up to n are allowed to exist initially, and so, due to the RBK dynamics, clusters outside this size range will never be formed. The corresponding equation is

$$\frac{\partial \wp_n}{\partial t}(\varsigma,t) = \int_0^{n-\varsigma} a_n(\varsigma+\varrho,\varrho)\wp_n(\varsigma+\varrho,t)\wp_n(\varrho,t)d\varrho - \int_0^{n} a_n(\varsigma,\varrho)\wp_n(\varsigma,t)\wp_n(\varrho,t)d\varrho, \tag{3.1}$$

where $(\varsigma,t) \in (0,n)\times\mathbb{R}_+$, having initial condition

$$\wp_n(\varsigma,0) = \wp_n^{\text{in}}(\varsigma) = \wp^{\text{in}}(\varsigma)\mathbb{1}_{(0,n)}(\varsigma), \ \varsigma \in \mathbb{R}_+ \tag{3.2}$$

where $n \geq 1$ is a positive integer, and

$$a_n(\varsigma,\varrho) = a(\varsigma,\varrho)\mathbb{1}_{(0,n)}(\varsigma)\mathbb{1}_{(0,n)}(\varrho), \quad \text{for } (\varsigma,\varrho) \in \mathbb{R}_+^2, \tag{3.3}$$

and $\mathbb{1}_{(0,n)}$ is the characteristic function on the interval $(0,n)$.

Theorem 3.1. *Let us consider* (2.1) *holds and* $\wp_n^{in} \in X_{0,1}^+$. *Then there exists a unique non-negative solution* $\wp_n \in \mathcal{C}^1([0,\infty); L^1(0,n))$ *to* (3.1)-(3.2) *which satisfies, for all* $\phi \in L^\infty(0,n)$,

$$\frac{d}{dt}\int_0^n \phi(\varsigma)\wp_n(\varsigma,t)d\varsigma = \int_0^n\int_0^\varsigma \tilde{\phi}(\varsigma,\varrho)a_n(\varsigma,\varrho)\wp(\varsigma,t)\wp(\varrho,t)d\varsigma d\varrho, \tag{3.4}$$

where $\tilde{\phi}(\varsigma,\varrho) := \phi(\varsigma-\varrho) - \phi(\varsigma) - \phi(\varrho)$, *and*

$$\int_0^n \varsigma\wp_n(\varsigma,t)d\varsigma = \int_0^n \varsigma\wp_n^{in}d\varsigma - 2\int_0^t\int_0^n\int_0^\varsigma \varrho a_n(\varsigma,\varrho)\wp_n(\varsigma,s)\wp_n(\varrho,s)d\varrho d\varsigma ds, \tag{3.5}$$

for $t \geq 0$ *and* $n \geq 1$.

The proof of the above theorem easily follows from [3, Proposition 4.1].

3.2. **Weak compactness.** Firstly we computed the estimates for large and small sizes to rule out the escaping of matter as $\varsigma \to 0$ or $\varsigma \to \infty$. This is further complemented with the uniform integrability to prevents the concentration at a finite size. In view of Dunford-Pettis' theorem, these estimates ensure that the approximation sequence is weakly compact with respect to the volume variable.

Lemma 3.1. *Let* $T \in (0,\infty)$. *Let us assume that* (2.1) *holds true for coagulation kernel 'a', then the following are true*

(1) For each $t \in [0,T]$, *there exists* $L(T) > 0$ *such that*

$$\int_0^n (1+\varsigma)\wp_n(\varsigma,t)d\varsigma \leq L(T) \text{ for } n = 1,2,3,... \tag{3.6}$$

(2) Given $\epsilon > 0$, *there exists* $R > 0$ *such that for* $t \in [0,T]$

$$\sup_n \left\{ \int_R^\infty \wp_n(\varsigma,t)d\varsigma \right\} \leq \epsilon. \tag{3.7}$$

*(3) (**Uniform Integrability**) Choose* $\epsilon > 0$. *Then there exists* $\delta_\epsilon > 0$ *such that for every measurable set* $A \subset (0,\infty)$,

$$\int_A \wp_n(\varsigma,t)d\varsigma < \epsilon \text{ whenever } \mu(A) < \delta_\epsilon$$

for all $n = 1,2,3,...$ *and* $t \in [0,T]$.

Proof. (1) Let $n > 1$, $n \in \mathbb{N}$ and $t \in [0,T]$, where $T > 0$ is fixed. Now, by using (3.5), we obtain

$$\begin{aligned}\int_0^n (1+\varsigma)\wp_n(\varsigma,t)d\varsigma &= \int_0^n \wp_n(\varsigma,t)d\varsigma + \int_0^n \varsigma\wp_n(\varsigma,t)d\varsigma \\ &\leq \int_0^1 \wp_n(\varsigma,t)d\varsigma + 2\int_0^n \varsigma\wp_n(\varsigma,t)d\varsigma. \end{aligned} \tag{3.8}$$

Set $\phi(\varsigma) = \mathbb{1}_{(0,1)}(\varsigma)$ in (3.4) and note that

$$\tilde{\phi}(\varsigma,\varrho):=\begin{cases}1, & \text{if } (\varsigma,\varrho)\in R_7,\\ 0, & \text{if } (\varsigma,\varrho)\in R_4\cup R_6\cup R_8,\\ -1, & \text{if } (\varsigma,\varrho)\in R_1\cup R_3\cup R_5,\\ -2, & \text{if } (\varsigma,\varrho)\in R_2.\end{cases}$$

where the regions R_j are identified in Figure 1.

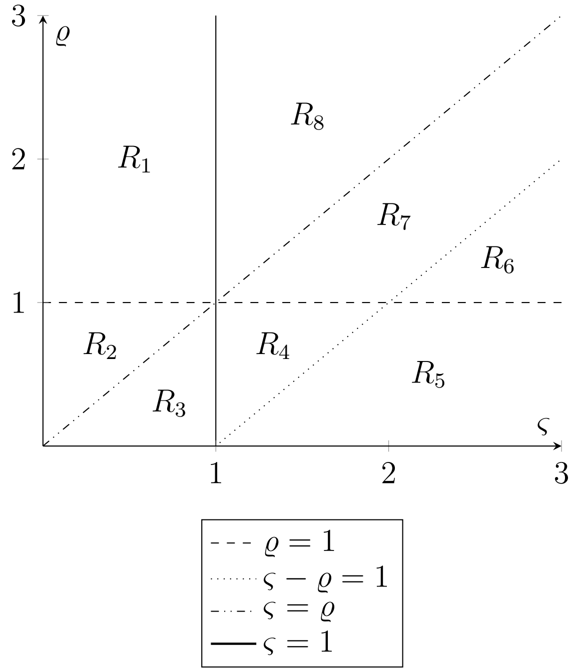

FIGURE 1 **Regions R_j where $\tilde{\phi}(\varsigma,\varrho)$ has constant values**

Since $\tilde{\phi}$ have positive values only in the region R_7 in which $\varsigma\in[1,\infty)$, $\varrho\in[1,\infty)$, $(\varsigma-\varrho)\in(0,1)$, we get

$$\begin{aligned}\frac{d}{dt}\int_0^1 \wp_n(\varsigma,t)d\varsigma &\le \int_1^n\int_{\varsigma-1}^{\varsigma} a_n(\varsigma,\varrho)\wp_n(\varsigma,t)\wp_n(\varrho,t)d\varrho d\varsigma\\ &\le 2\int_1^2\int_0^1 \varsigma\wp_n(\varsigma,t)\wp_n(\varrho,t)d\varrho d\varsigma+2\int_1^2\int_1^{\varsigma}\varsigma\varrho\wp_n(\varsigma,t)\wp_n(\varrho,t)d\varrho d\varsigma\\ &\quad+2\int_2^n\int_{\varsigma-1}^{\varsigma}\varsigma\varrho\wp_n(\varsigma,t)\wp_n(\varrho,t)d\varrho d\varsigma\\ &\le 2M_1(\wp^{\text{in}})\int_0^1\wp_n(\varsigma,t)d\varsigma+4(M_1(\wp^{\text{in}}))^2.\end{aligned}$$

Integrating both sides with respect to $t\in(0,T)$ and using Gronwall's lemma we obtain

$$\int_0^1 \wp_n(\varsigma,t)d\varsigma\le N(T),$$

where

$$N(T) = [\|\wp^{\rm in}\|_{L^1(\mathbb{R}_+),d\varsigma} + 4T(M_1(\wp^{\rm in}))^2]\exp(2TM_1(\wp^{\rm in})).$$

Next, from (3.5) we have

$$\int_0^n \varsigma\wp_n(\varsigma,t)d\varsigma \leq \int_0^n \varsigma\wp_n^{\rm in}d\varsigma \leq \int_0^\infty \varsigma\wp_n^{\rm in}d\varsigma = M_1(\wp^{\rm in}) \leq \|\wp^{\rm in}\|_{X_{0,1}},$$

using the above inequality in (3.8), and defining, with $L(T) := N(T) + 2M_1(\wp^{\rm in})$, we conclude the proof of part (1) of Lemma 3.1.

(2) Let $R_\epsilon > 0$ be such that $R_\epsilon > \dfrac{L(T)}{\epsilon}$ for some fixed $\epsilon > 0$. Then

$$\begin{aligned}\int_{R_\epsilon}^\infty \wp_n(\varsigma,t)d\varsigma &\leq \frac{1}{R_\epsilon}\int_{R_\epsilon}^\infty \varsigma\wp_n(\varsigma,t)d\varsigma\\ &\leq \frac{1}{R_\epsilon}L(T)\end{aligned}$$

for each $t \in [0,T]$ and positive integer $n \geq 1$. The proof of part (2) of Lemma 3.1 is completed by taking the supremum over all n in the above inequality.

(3) Choose $\epsilon > 0$. By (2.1), $0 \leq \alpha < 1$ and we can therefore choose $R > 1$ such that

$$\frac{L(T)}{R} + \frac{k(L(T))^2}{(1+R)^{1-\alpha}} \leq \epsilon. \tag{3.9}$$

Let $A \subset \mathbb{R}_+$. Using part (2) we have, for all $n \in \mathbb{N}$ sufficiently large and $t \in [0,T]$,

$$\int_R^\infty \mathbb{1}_A(\varsigma)\wp_n(\varsigma,t)d\varsigma \leq \int_R^\infty \wp_n(\varsigma,t)d\varsigma \leq \epsilon. \tag{3.10}$$

For $n \geq 1$, $\delta \in (0,1)$, and $t \in [0,T]$, let us introduce the following notations:

$$\mathcal{E}_{n,R}(t,\delta) = \sup\left\{\begin{array}{l}\int_0^R \mathbb{1}_A(\varsigma)\wp_n(\varsigma,t)d\varsigma : \\ A \text{ is a measurable subset of } \mathbb{R}_+ \text{ with } |A| \leq \delta\end{array}\right\},$$

and

$$\mathcal{E}^{\rm in}(t,\delta) = \sup\left\{\begin{array}{l}\int_0^R \mathbb{1}_A(\varsigma)\wp_0(\varsigma)d\varsigma : \\ A \text{ is a measurable subset of } \mathbb{R}_+ \text{ with } |A| \leq \delta\end{array}\right\},$$

where $\mathbb{1}_A$ is the indicator function on the set A.

For $T > 0$, $R > 1$ and $\delta \in (0,1)$, by Fubini's Theorem

$$\begin{aligned}&\frac{d}{dt}\int_0^R \mathbb{1}_A(\varsigma)\wp_n(\varsigma,t)d\varsigma\\ &\qquad\leq \int_0^R\int_0^{n-\varsigma} \mathbb{1}_A(\varsigma)a_n(\varsigma+\varrho,\varrho)\wp_n(\varsigma+\varrho,t)\wp_n(\varrho,t)d\varrho d\varsigma\\ &\qquad\leq \underbrace{\int_0^R\int_0^R \mathbb{1}_A(\varsigma)a_n(\varsigma+\varrho,\varrho)\wp_n(\varsigma+\varrho,t)\wp_n(\varrho,t)d\varsigma d\varrho}_{=:I_1^n}\end{aligned}$$

$$+\underbrace{\int_R^n \int_0^{n-\varrho} \mathbb{1}_A(\varsigma) a_n(\varsigma+\varrho,\varrho) \wp_n(\varsigma+\varrho,t) \wp_n(\varrho,t)\, d\varsigma d\varrho}_{=:I_2^n}. \tag{3.11}$$

Then, again by Fubini's theorem and applying the transformation $\varsigma + \varrho = \varsigma'$ and $\varrho = \varrho'$, the integral I_1^n can be estimated as follows (where we have dropped the $'$)

$$\begin{aligned} I_1^n &= \underbrace{\int_0^R \int_0^{\varsigma} \mathbb{1}_A(\varsigma-\varrho) a_n(\varsigma,\varrho) \wp_n(\varsigma,t) \wp_n(\varrho,t)\, d\varsigma d\varrho}_{=:I_{1,1}^n} \\ &\quad + \underbrace{\int_R^{2R} \int_{\varsigma-R}^{R} \mathbb{1}_A(\varsigma-\varrho) a_n(\varsigma,\varrho) \wp_n(\varsigma,t) \wp_n(\varrho,t)\, d\varsigma d\varrho}_{=:I_{1,2}^n}. \end{aligned}$$

Since $(A+\varrho) \cap (0,\varsigma) \subset (0,R)$ and $|(A+\varrho) \cap (0,\varsigma)| \leq |A+\varrho| = |A| \leq \delta$ then it follows by (2.1) and (3.6) that

$$I_{1,1}^n \leq kL(T)(1+R)^{\alpha} \mathcal{E}_{n,R}(t,\delta).$$

Next, for $I_{1,2}^n$, since $(\varsigma - A) \cap (\varsigma - R, R) \subset (0,R)$ and $|(\varsigma - A) \cap (\varsigma - R, R)| \leq |\varsigma - A| = |A| \leq \delta$ it follows by a similar way to above that

$$I_{1,2}^n \leq kL(T)(1+R)^{\alpha} \mathcal{E}_{n,R}(t,\delta).$$

Therefore, we have

$$I_1^n \leq 2kL(T)(1+R)^{\alpha} \mathcal{E}_{n,R}(t,\delta).$$

Now, for $\epsilon > 0$, from part (2), we have

$$\begin{aligned} \int_R^n \wp_n(\varsigma,t) d\varsigma &= \int_0^n \int_0^{\varsigma} [\mathbb{1}_{(R,n)}(\varsigma-\varrho) - \mathbb{1}_{(R,n)}(\varsigma) - \mathbb{1}_{(R,n)}(\varrho)] \times \\ &\qquad \times a_n(\varsigma,\varrho) \wp_n(\varsigma,t) \wp_n(\varrho,t) d\varrho d\varsigma \\ &\leq \frac{L(T)}{R}, \end{aligned}$$

and so by (2.1), (3.9), Fubini's Theorem and part (1) of the present lemma, we get

$$\begin{aligned} &\int_0^n \int_0^{\varsigma} \mathbb{1}_{(R,n)}(\varsigma-\varrho) a_n(\varsigma,\varrho) \wp_n(\varsigma,t) \wp_n(\varrho,t) d\varrho d\varsigma \\ &\qquad \leq \frac{L(T)}{R} + \int_0^n \int_0^{\varsigma} \mathbb{1}_{(R,n)}(\varsigma) a_n(\varsigma,\varrho) \wp_n(\varsigma,t) \wp_n(\varrho,t) d\varrho d\varsigma \\ &\qquad\quad + \int_0^n \int_0^{\varsigma} \mathbb{1}_{(R,n)}(\varrho) a_n(\varsigma,\varrho) \wp_n(\varsigma,t) \wp_n(\varrho,t) d\varrho d\varsigma \\ &\qquad \leq \frac{L(T)}{R} + \int_0^n \int_0^n \mathbb{1}_{(R,n)}(\varsigma) a_n(\varsigma,\varrho) \wp_n(\varsigma,t) \wp_n(\varrho,t) d\varrho d\varsigma \end{aligned}$$

$$\begin{aligned} &\leq \frac{L(T)}{R} + k\int_R^n (1+\varsigma)^\alpha \wp_n(\varsigma,t)d\varsigma \int_0^n (1+\varrho)^\alpha \wp_n(\varrho,t)d\varrho \\ &\leq \frac{L(T)}{R} + \frac{k(L(T))^2}{(1+R)^{1-\alpha}} \leq \epsilon. \end{aligned}$$

Therefore, from above inequality and Fubini's theorem, we obtain

$$\begin{aligned} I_2^n &= \int_R^n \int_0^{n-\varrho} \mathbb{1}_A(\varsigma) a_n(\varsigma+\varrho,\varrho)\wp_n(\varsigma+\varrho,t)\wp_n(\varrho,t)d\varsigma d\varrho \\ &= \int_R^n \int_R^{\varsigma} \mathbb{1}_A(\varsigma-\varrho) a_n(\varsigma,\varrho)\wp_n(\varsigma,t)\wp_n(\varrho,t)d\varrho d\varsigma \leq \epsilon. \end{aligned}$$

Hence, it follows from (3.11) and the bounds on I_1^n and I_2^n, that

$$\frac{d}{dt}\mathcal{E}_{n,R}(t,\delta) \leq 2kL(T)(1+R)^\alpha \mathcal{E}_{n,R}(t,\delta) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that

$$\frac{d}{dt}\mathcal{E}_{n,R}(t,\delta) \leq 2kL(T)(1+R)^\alpha \mathcal{E}_{n,R}(t,\delta).$$

Finally, by Gronwall's lemma we get

$$\mathcal{E}_{n,R}(t,\delta) \leq \mathcal{E}^{\text{in}}(\delta)\exp(2kTL(T)(1+R)^\alpha),$$

and the proof of part (3) is completed by taking the limit as $\delta \to 0$ and using the uniform integrability of $\wp^{\text{in}}$.

□

Hence by the Dunford-Pettis theorem, for each $t \in [0,T]$, we have a sequence $\{\wp_n\}_{n>1}$ which lies in a weakly compact subset of $L^1(\mathbb{R}_+, dz)$ with respect to the volume variable.

3.3. **Time equicontinuity.** The weak compactness issue has been resolved by the results of the previous section. Now, we will look for the time variable. For this, we need the following lemma.

Lemma 3.2. *Suppose that $t_1, t_2 \in [0,T]$, $T < \infty$, and the assumptions of Lemma 3.1 still hold. Then, for $\lambda \in (1,+\infty)$ we have*

$$\int_0^\lambda |\wp_n(\varsigma,t_2) - \wp_n(\varsigma,t_1)|d\varsigma \leq \mathcal{L}(\lambda,T)|t_2-t_1|, \tag{3.12}$$

where $\mathcal{L}(\cdot) > 0$ is a function dependent on the parameters k and $\wp^{in}$.

***Proof*.** Let $t > 0$. By (3.1) we have

$$\begin{aligned} &\int_0^\lambda |\wp_n(\varsigma,t_2) - \wp_n(\varsigma,t_1)|d\varsigma \\ &\qquad \leq \int_{t_1}^{t_2} \Big[\underbrace{\int_0^n \int_0^{n-\varsigma} \mathbb{1}_{(0,\lambda)}(\varsigma) a_n(\varsigma+\varrho,\varrho)\wp_n(\varsigma+\varrho,t)\wp_n(\varrho,t)d\varrho d\varsigma}_{=:L_{1,n}} \end{aligned}$$

$$+\underbrace{\int_0^n \int_0^n \mathbb{1}_{(0,\lambda)}(\varsigma) a_n(\varsigma,\varrho)\wp_n(\varsigma,t)\wp_n(\varrho,t)d\varrho d\varsigma}_{=:L_{2,n}}\Bigg] dt \tag{3.13}$$

Using Fubini's theorem,

$$\begin{aligned}
L_{1,n}(\wp_n)(\lambda,t) &= \int_0^n \int_0^{n-\varsigma} \mathbb{1}_{(0,\lambda)}(\varsigma) a_n(\varsigma+\varrho,\varrho)\wp_n(\varsigma+\varrho,t)\wp_n(\varrho,t)d\varrho d\varsigma \\
&= \int_0^n \int_\varrho^n \mathbb{1}_{(0,\lambda)}(\varsigma-\varrho) a_n(\varsigma,\varrho)\wp_n(\varsigma,t)\wp_n(\varrho,t)d\varsigma d\varrho \\
&\leq \int_0^n \int_0^\varsigma a_n(\varsigma,\varrho)\wp_n(\varsigma,t)\wp_n(\varrho,t)d\varrho d\varsigma \leq 2kL^2(T).
\end{aligned} \tag{3.14}$$

Next, for $\varsigma \in (0,\lambda)$ and $\lambda \in (1,n)$,

$$a_n(\varsigma,\varrho) \leq k(2+\lambda)(1+\varrho)^\alpha,$$

and using the above inequality we get

$$\begin{aligned}
L_{2,n}(\wp_n)(\lambda,t) &= \int_0^\lambda \int_0^n a_n(\varsigma,\varrho)\wp_n(\varsigma,t)\wp_n(\varrho,t)d\varrho d\varsigma \\
&\leq k(2+\lambda)\int_0^\lambda \int_0^n (1+\varrho)^\alpha \wp_n(\varsigma,t)\wp_n(\varrho,t)d\varrho d\varsigma \\
&\leq 2k(2+\lambda)L^2(T).
\end{aligned} \tag{3.15}$$

Now, using (3.14) and (3.15) in (3.13) and defining

$$\mathcal{L}(\lambda,T) := 2k(3+\lambda)L^2(T),$$

the proof of Lemma 3.2 is completed. □

For any given $\epsilon > 0$ and $\phi \in L^\infty(\mathbb{R}_+)$ there exists $\lambda_\epsilon > 1$ such that

$$\frac{L(T)}{\lambda_\epsilon} < \frac{\epsilon}{4\|\phi\|_{L^\infty}}.$$

Next, for $n \geq 1$ and $t \in [0,T]$, using (3.7), we have

$$\int_{\lambda_\epsilon}^\infty \wp_n(\varsigma,t)d\varsigma < \frac{\epsilon}{4\|\phi\|_{L^\infty}}. \tag{3.16}$$

Therefore, for $t_1, t_2 \in [0,T]$, by using (3.12) and (3.16), we get

$$\begin{aligned}
\left|\int_0^\infty \phi(\varsigma)[\wp_n(\varsigma,t_2)-\wp_n(\varsigma,t_1)]d\varsigma\right| &\leq \|\phi\|_{L^\infty}\int_0^{\lambda_\epsilon} |\wp_n(\varsigma,t_2)-\wp_n(\varsigma,t_1)|d\varsigma + \frac{\epsilon}{2} \\
&\leq \|\phi\|_{L^\infty}\mathcal{L}(\lambda_\epsilon,T)|t_2-t_1| + \frac{\epsilon}{2} \leq \epsilon,
\end{aligned} \tag{3.17}$$

provided

$$|t_2-t_1| \leq \delta(\epsilon,T) := \frac{\epsilon}{2\|\phi\|_{L^\infty}\mathcal{L}(\lambda_\epsilon,T)}.$$

The time equicontinuity of the family $\{\wp_n(t), t \in [0,T]\}$ in $L^1(\mathbb{R}_+)$ is implied from the estimate (3.17). Thus, a refined version of the *Arzelà−Ascoli Theorem* (see [15, Theorem 2.1]) implies the existence of a subsequence $(\wp_n)$ (not relabelled) and a function $\wp \in L^\infty((0,T); L^1(\mathbb{R}_+))$ such that

$$\wp_n \to \wp \text{ in } C([0,T]; L^1(\mathbb{R}_+)_w), \tag{3.18}$$

which means

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\{ \left| \int_0^\infty \{\wp_n(\varsigma,t) - \wp(\varsigma,t)\}\phi(\varsigma)d\varsigma \right| \right\} = 0, \tag{3.19}$$

for all $T > 0$ and $\phi \in L^\infty(\mathbb{R}_+)$. Non-negativity of $\wp_n(.,t), \forall\, n \in \mathbb{N}$, implies that, for every $t \in [0,T]$,

$$\wp(.,t) \geq 0 \text{ a.e. in } \mathbb{R}_+.$$

Finally, applying the weak convergence of $\wp_n(t_2) - \wp_n(t_1)$ to $\wp(t_2) - \wp(t_1)$ from (3.18), Lemma 3.1, and taking $\phi(\varsigma) = \text{sign}(\wp_n(\varsigma,t_2) - \wp_n(\varsigma,t_1))$ in (3.17), we conclude that

$$\|\wp(t_1) - \wp(t_2)\|_{L^1(\mathbb{R}_+)} \leq \varepsilon.$$

Hence, we have

$$\wp \in \mathcal{C}([0,T]; L^1(\mathbb{R}_+)), \tag{3.20}$$

where $\mathcal{C}([0,T]; L^1(\mathbb{R}_+))$ is the space of all continuous functions from $[0,T]$ to $L^1(\mathbb{R}_+)$.

3.4. **Convergence of the approximations of the integrals.** We are now in a position to complete the proof of the existence of at least one solution to the continuous RBK model. For this, we show that the function $\wp$ is indeed a solution to (1.3)-(1.4) in the sense of Definition 2.1. In order to prove our claim, we demonstrate that the truncated integrals on the right-hand side to (3.1) converge weakly to the integrals on the right-hand to (1.3).

Let $T \in (0,+\infty)$ and $W > 0$. For $W \leq n$, by using (3.6) we obtain

$$\begin{aligned} \int_0^T \int_0^W \int_0^W a(\varsigma,\varrho)\wp_n(\varsigma,s)\wp_n(\varrho,s)d\varrho d\varsigma ds &\leq k \int_0^T \left(\int_0^W (1+\varsigma)^\alpha \wp_n(\varsigma,s)d\varsigma \right)^2 ds \\ &\leq k \int_0^T \left(\int_0^n (1+\varsigma)\wp_n(\varsigma,s)d\varsigma \right)^2 ds \\ &\leq kL(T)^2 T =: C_1(T). \end{aligned}$$

It thus follows from (3.18) and the Lebesgue dominated convergence theorem that

$$\int_0^T \int_0^W \int_0^W a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,s)d\varrho d\varsigma ds \leq C_1(T),$$

and, as $W > 0$ is arbitrary, we have

$$\int_0^T \int_0^\infty \int_0^\infty a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,s)d\varrho d\varsigma ds \leq C_1(T). \tag{3.21}$$

From (3.6) we conclude that

$$\sup_{t\in[0,T]} \|\wp(t)\|_{X_{0,1}} \le C_2(T). \tag{3.22}$$

Therefore, (2.3) holds by using (2.1), (3.21), (3.22) and Fubini theorem.

Let us consider a function $\phi \in L^\infty(\mathbb{R}_+)$ and $t \in [0, +\infty)$. Using (3.2) and (3.18) we obtain

$$\lim_{n\to\infty} \int_0^\infty \phi(\varsigma)(\wp_n(\varsigma, t) - \wp_n^{\text{in}}(\varsigma))d\varsigma = \int_0^\infty \phi(\varsigma)(\wp(\varsigma, t) - \wp^{\text{in}}(\varsigma))d\varsigma. \tag{3.23}$$

Next, choose $b > 0$. For $n \ge 1$ and $s \in (0, t)$, we set

$$\begin{aligned} \mathcal{Z}_{1,n}(b, s) &:= \int_0^b \int_0^\varsigma \tilde{\phi}(\varsigma, \varrho) a_n(\varsigma, \varrho)\wp_n(\varsigma, s)\wp_n(\varrho, s)d\varrho d\varsigma \\ \mathcal{Z}_{2,n}(b, s) &:= \int_b^\infty \int_0^\varsigma \tilde{\phi}(\varsigma, \varrho) a_n(\varsigma, \varrho)\wp_n(\varsigma, s)\wp_n(\varrho, s)d\varrho d\varsigma \end{aligned}$$

and

$$\begin{aligned} \mathcal{Z}_1(b, s) &:= \int_0^b \int_0^\varsigma \tilde{\phi}(\varsigma, \varrho) a(\varsigma, \varrho)\wp(\varsigma, s)\wp(\varrho, s)d\varrho d\varsigma \\ \mathcal{Z}_2(b, s) &:= \int_b^\infty \int_0^\varsigma \tilde{\phi}(\varsigma, \varrho) a(\varsigma, \varrho)\wp(\varsigma, s)\wp(\varrho, s)d\varrho d\varsigma \end{aligned}$$

where $\tilde{\phi}(\varsigma, \varrho) = \phi(\varsigma - \varrho) - \phi(\varsigma) - \phi(\varrho)$.

Now, for $n \ge b$ and $(\varsigma, \varrho) \in (0, b)^2$, we have $a_n(\varsigma, \varrho) = a(\varsigma, \varrho)$. Using this along with (3.18) and [9, Lemma 2.9] it follows that

$$\lim_{n\to+\infty} \mathcal{Z}_{1,n}(b, s) = \mathcal{Z}_1(b, s).$$

From (3.7) and the Lebesgue dominated convergence theorem together with the above identity it follows that

$$\lim_{n\to+\infty} \int_0^t \mathcal{Z}_{1,n}(b, s)ds = \int_0^t \mathcal{Z}_1(b, s)ds. \tag{3.24}$$

Next, by using (2.1), (3.4) and (3.6), we obtain

$$\begin{aligned} &\int_0^t \mathcal{Z}_{2,n}(b, s)ds \\ &\le \frac{3k}{(1+b)^{1-\alpha}} \|\phi\|_{L^\infty(0,+\infty)} \int_0^t \left(\int_0^\infty (1+\varsigma)\wp_n(\varsigma, s)d\varsigma\right)\left(\int_0^\infty (1+\varrho)^\alpha \wp_n(\varrho, s)d\varrho\right)ds \\ &\le \frac{6kTL(T)^2}{(1+b)^{1-\alpha}} \|\phi\|_{L^\infty(0,+\infty)} \le \frac{C_3(T)}{(1+b)^{1-\alpha}} \|\phi\|_{L^\infty(0,+\infty)}, \end{aligned} \tag{3.25}$$

where $C_3(T) := 6kL^2(T)$. Similarly, let $C_4(T)$ be a constant depending on T, k and $\wp^{\text{in}}$. Then

$$\int_0^t \mathcal{Z}_2(b,s)ds \leq \frac{C_4(T)}{(1+b)^{1-\alpha}}\|\phi\|_{L^\infty(0,+\infty)}. \tag{3.26}$$

Hence, from (3.24)-(3.26), we get

$$\begin{aligned}&\limsup_{n\to+\infty}\left|\int_0^t \Big[(\mathcal{Z}_{1,n}(b,s)+\mathcal{Z}_{2,n}(b,s))-(\mathcal{Z}_1(b,s)+\mathcal{Z}_2(b,s))\Big]ds\right.\\ \leq\ & \frac{C_5(T)}{(1+b)^{1-\alpha}}\|\phi\|_{L^\infty(0,+\infty)}.\end{aligned}$$

The right hand side of the previous inequality is independent of $b>0$ and so it is valid for each $b>0$. Hence, we get

$$\begin{aligned}\lim_{n\to\infty}&\int_0^t\int_0^\infty\int_0^\varsigma \tilde{\phi}(\varsigma,\varrho)a_n(\varsigma,\varrho)\wp_n(\varsigma,s)\wp_n(\varrho,t)d\varrho d\varsigma ds\\ &=\int_0^t\int_0^\infty\int_0^\varsigma \tilde{\phi}(\varsigma,\varrho)a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,t)d\varrho d\varsigma ds.\end{aligned} \tag{3.27}$$

Next, by using (3.23), (3.27) and take $n\to\infty$ in (3.4), we obtain that $\wp$ satisfies

$$\int_0^\infty \phi(\varsigma)(\wp(\varsigma,t)-\wp^{\text{in}}(\varsigma))d\varsigma=\int_0^t\int_0^\infty\int_0^\varsigma \tilde{\phi}(\varsigma,\varrho)a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,t)d\varrho d\varsigma ds. \tag{3.28}$$

Here, $\tilde{\phi}$ is defined as in (3.4). However, due to (2.3), the Fubini theorem allows us to rephrase the right-hand side term of (3.28) as

$$\begin{aligned}\int_0^\infty \phi(\varsigma)\int_0^t\Big(&\int_0^\infty a(\varsigma+\varrho,\varrho)\wp(\varsigma+\varrho,s)\wp(\varrho,s)d\varrho\\ &-\int_0^\infty a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,s)d\varrho\Big)dsd\varsigma.\end{aligned}$$

Hence, (3.28) can also be written as

$$\begin{aligned}\int_0^\infty &\phi(\varsigma)(\wp(\varsigma,t)-\wp^{\text{in}}(\varsigma))d\varsigma\\ &=\int_0^\infty \phi(\varsigma)\int_0^t\Big(\int_0^\infty a(\varsigma+\varrho,\varrho)\wp(\varsigma+\varrho,s)\wp(\varrho,s)d\varrho\\ &\qquad-\int_0^\infty a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,s)d\varrho\Big)dsd\varsigma.\end{aligned}$$

As this equality holds for every $\phi\in L^\infty(\mathbb{R}_+)$ and by recalling (3.21) and (3.22), we have shown that $\wp$ being a weak solution to (1.3)-(1.4) on $[0,\infty)$ in the sense of Definition 2.1.

Next, we prove that any weak solution '$\wp$' to (1.3)-(1.4) on $[0,T)$, $T\in\mathbb{R}_+$, in the sense of Definition 2.1 is not mass-conserving and it satisfies

$$M_1(\wp)(t)\leq M_1(\wp^{\text{in}}),\ t\geq 0. \tag{3.29}$$

To prove this let $\wp_n$ be the solution to the (3.1)-(3.2) for $\varsigma \in \mathbb{R}_+$. Set $\phi(\varsigma) = 1$ and $\phi(\varsigma) = \varsigma$ in (3.4), respectively, then we conclude that the total number of particles and total mass of particles decreases with time ($t \geq 0$). Thus

$$\begin{aligned}\int_0^n \varsigma\wp_n(\varsigma, t)d\varsigma &\leq \int_0^n \varsigma\wp_n(\varsigma, 0)d\varsigma \\ &= \int_0^n \varsigma\wp(\varsigma, 0)d\varsigma \\ &\leq \int_0^\infty \varsigma\wp_n(\varsigma, 0)d\varsigma = M_1(\wp^{\text{in}}).\end{aligned}$$

Taking $n \to \infty$ in the above inequality we obtain (3.29) for each $t \in (0, T]$.
This complete the proof of Theorem 2.1.

4. UNIQUENESS OF SOLUTIONS

The construction of solutions to the continuous RBK system using weak compactness arguments has drawback that uniqueness has to be evaluated independently. Thus, the goal of the present section is to prove Theorem 2.2. For this, we need following lemma:

Lemma 4.1. *Let $\wp$ be the solution of (1.3). Consider g be a measurable function on $\mathbb{R}_+$. For $t_2 \in (0, \infty)$ and $t_1 \in [0, t_2)$, there holds*

$$\begin{aligned}&\int_0^n g(\varsigma)[\wp(\varsigma, t_2) - \wp(\varsigma, t_1)d\varsigma \\ &\quad = \int_{t_1}^{t_2} \Bigg[\int_0^n \int_0^\varsigma [g(\varsigma - \varrho) - g(\varsigma) - g(\varrho)]a(\varsigma, \varrho)\wp(\varsigma, s)\wp(\varrho, s)d\varrho d\varsigma \\ &\qquad + \int_n^\infty \int_{\varsigma-n}^\varsigma g(\varsigma - \varrho)a(\varsigma, \varrho)\wp(\varsigma, s)\wp(\varrho, s)d\varrho d\varsigma \\ &\qquad - \int_0^n \int_n^\infty g(\varsigma)a(\varsigma, \varrho)\wp(\varsigma, s)\wp(\varrho, s)d\varrho d\varsigma\Bigg] ds. \qquad (4.1)\end{aligned}$$

Proof. By Fubini's theorem (2.4), we get

$$\begin{aligned}&\int_0^n g(\varsigma)[\wp(\varsigma, t_2) - \wp(\varsigma, t_1)d\varsigma \\ &\quad = \int_{t_1}^{t_2} \Bigg[\int_0^\infty \int_0^n g(\varsigma)a(\varsigma, \varrho)\wp(\varsigma, s)\wp(\varrho, s)d\varrho d\varsigma \\ &\qquad - \int_0^n \int_0^\infty g(\varsigma)a(\varsigma, \varrho)\wp(\varsigma, s)\wp(\varrho, s)d\varrho d\varsigma\Bigg] ds.\end{aligned}$$

Next, set $\varsigma + \varrho = \varsigma'$ and $\varrho = \varrho'$ and use Fubini's theorem again to we conclude (4.1). □

Proof. (of Theorem 2.2) Let d_1 and d_2 in $X_{0,1}^+$ be two solutions to (1.3)-(1.4) on $[0, T]$, where $T > 0$, with $d_1(\varsigma, 0) = d_2(\varsigma, 0)$. Let $G = d_1 - d_2$,

$$\begin{aligned}\mathfrak{X}(\varsigma, \varrho, t) &= a(\varsigma, \varrho)[d_1(\varsigma, t)d_1(\varrho, t) - d_2(\varsigma, t)d_2(\varrho, t)] \\ &= a(\varsigma, \varrho)[d_1(\varsigma, t)G(\varrho, t) + G(\varsigma, t)d_2(\varrho, t)],\end{aligned}$$

and define $Y(t)$ as

$$Y(t) = \int_0^n (1+\varsigma)^\beta |G(\varsigma,t)| d\varsigma,$$

with $\alpha \leq \beta \leq 1-\alpha$. By using $|G(\varsigma,t)| = \text{sign}(G(\varsigma,t))G(\varsigma,t)$ in above identity, we get

$$Y(t) = \int_0^n (1+\varsigma)^\beta \text{sign}(G(\varsigma,t))(d_1(\varsigma,t) - d_2(\varsigma,t))d\varsigma. \tag{4.2}$$

Now, from (4.1) with $g(\varsigma) = (1+\varsigma)^\beta \text{sign}(Y(\varsigma,t))$ and let

$B(\varsigma,\varrho,s) = (1+\varsigma-\varrho)^\beta \text{sign}(G(\varsigma-\varrho,s)) - (1+\varsigma)^\beta \text{sign}(G(\varsigma,s)) - (1+\varrho)^\beta \text{sign}(G(\varrho,s))$,

we obtain

$$\begin{aligned} Y(t) &= \int_0^\infty (1+\varsigma)^\beta \text{sign}(G(\varsigma,t))d_1(\varsigma,t)d\varsigma - \int_0^\infty (1+\varsigma)^\beta \text{sign}(G(\varsigma,t))d_2(\varsigma,t)d\varsigma \\ &= \underbrace{\int_0^t \int_0^\infty \int_0^\varsigma B(\varsigma,\varrho,s)\mathfrak{X}(\varsigma,\varrho,s)d\varrho d\varsigma ds}_{=:A_1(t)} \\ &\quad + \underbrace{\int_0^t \int_n^\infty \int_{\varsigma-n}^\varsigma (1-\varsigma-\varrho)^\beta \text{sign}(Y(\varsigma-\varrho,s))\mathfrak{X}(\varsigma,\varrho,s)d\varrho d\varsigma ds}_{=:A_2(t)} \\ &\quad - \underbrace{\int_0^t \int_0^n \int_n^\infty (1-\varsigma)^\beta \text{sign}(Y(\varsigma,s))\mathfrak{X}(\varsigma,\varrho,s)d\varrho d\varsigma ds}_{=:A_3(t)}. \end{aligned} \tag{4.3}$$

We can easily see that in A_1, A_2 and A_3

$$\varrho \leq \varsigma \implies \varrho \leq 1+\varsigma \implies 1+\varsigma-\varrho \geq 0,$$

$$\alpha \leq \beta \implies (1+\varsigma)^\alpha \leq (1+\varsigma)^\beta,$$

and

$$1+\varsigma-\varrho \leq 1+\varsigma \implies (1+\varsigma-\varrho)^\beta \leq (1+\varsigma)^\beta.$$

Using the properties of signum function, we estimate B as

$$\begin{aligned} & B(\varsigma,\varrho,s)G(\varrho,s) \\ \leq\ & [(1+\varsigma-\varrho)^\beta + (1+\varsigma)^\beta - (1+\varrho)^\beta]|G(\varrho,s)| \\ \leq\ & [(1+\varsigma-\varrho)^\beta + (1+\varsigma)^\beta]|G(\varrho,s)| \leq 2(1+\varsigma)^\beta |G(\varrho,s)|, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} & B(\varsigma,\varrho,s)G(\varsigma,s) \\ \leq\ & [(1+\varsigma-\varrho)^\beta - (1+\varsigma)^\beta + (1+\varrho)^\beta]|G(\varsigma,s)| \\ \leq\ & (1+\varrho)^\beta |G(\varsigma,s)|. \end{aligned} \tag{4.5}$$

Next, from (2.1), (4.4) and (4.5), we evaluate A_1 as

$$\begin{aligned} A_1(t) &\leq 2k\int_0^t \left[\int_0^n (1+\varsigma)d_1(\varsigma,s)\int_0^{\varsigma}(1+\varrho)^{\alpha}|G(\varrho,s)|d\varrho d\varsigma\right]ds \\ &\quad +k\int_0^t \left[\int_0^n (1+\varsigma)^{\alpha}|G(\varsigma,s)|\int_0^{\varsigma}(1+\varrho)d_2(\varrho,s)d\varrho d\varsigma\right]ds \\ &= 3kL(T)\int_0^t Y(s)ds. \end{aligned} \tag{4.6}$$

For A_2, again by using (2.1) and $|Y(\varsigma,s)| \leq d_1(\varsigma,s)+d_2(\varsigma,s)$, we obtain

$$\begin{aligned} A_2(t) &\leq k(1+n)^{\beta}\int_0^t \left[\int_n^{\infty}(1+\varsigma)^{\alpha}d_1(\varsigma,s)\int_{\varsigma-n}^{\varsigma}(1+\varrho)^{\alpha}|G(\varrho,s)|d\varrho d\varsigma\right]ds \\ &\quad +k(1+n)^{\beta}\int_0^t \left[\int_n^{\infty}(1+\varsigma)^{\alpha}|G(\varsigma,s)|\int_{\varsigma-n}^{\varsigma}(1+\varrho)^{\alpha}d_2(\varrho,s)d\varrho d\varsigma\right]ds \\ &\leq kL(T)\int_0^t \left[\int_0^{\infty}(1+\varsigma)^{\beta}|G(\varsigma,s)|d\varsigma\right]ds \\ &\quad +kL(T)\int_0^t \left[\int_n^{\infty}(1+\varsigma)d_1(\varsigma,s)d\varsigma+\int_n^{\infty}(1+\varsigma)d_2(\varsigma,s)d\varsigma\right]ds. \end{aligned} \tag{4.7}$$

And finally, for A_3, we see that

$$-(1+\varsigma)^{\beta}\mathrm{sign}(G(\varsigma,s))G(\varsigma,s) \leq -(1+\varsigma)^{\beta}|G(\varsigma,s)| \leq 0,$$

and

$$-(1+\varsigma)^{\beta}\mathrm{sign}(G(\varsigma,s))G(\varrho,s) \leq (1+\varsigma)^{\beta}|G(\varsigma,s)| \geq 0.$$

Then

$$\begin{aligned} -A_3(t) &\leq \int_0^t \left[\int_0^n (1-\varsigma)d_1(\varsigma,s)\int_n^{\infty}(1-\varrho)^{\beta}|G(\varrho,s)|d\varrho d\varsigma\right]ds \\ &\leq kL(T)\int_0^t \left[\int_0^{\infty}(1+\varsigma)^{\beta}|G(\varsigma,s)|d\varsigma\right]ds. \end{aligned} \tag{4.8}$$

Inserting the estimates on $A_1(t)$, $A_2(t)$ and $A_3(t)$ from (4.6)–(4.7) into (4.3), we obtain

$$Y(t) \leq 3kL(T)\int_0^t Y(s)ds + 2kL(T)\int_0^t\int_0^{\infty}(1+\varsigma)^{\beta}|G(\varsigma,s)|d\varsigma ds \tag{4.9}$$

$$+kL(T)\int_0^t \left[\int_n^{\infty}(1+\varsigma)d_1(\varsigma,s)d\varsigma+\int_n^{\infty}(1+\varsigma)d_2(\varsigma,s)d\varsigma\right]ds. \tag{4.10}$$

By the definition of solution, we have

$$\int_n^{\infty}(1+\varsigma)d_1(\varsigma,s)d\varsigma \to 0, \text{ as } n\to\infty \text{ pointwise},$$

and

$$\int_n^{\infty}(1+\varsigma)d_1(\varsigma,s)d\varsigma \leq \int_n^{\infty}1d_1(\varsigma,s)d\varsigma + \int_n^{\infty}\varsigma d_1(\varsigma,s)d\varsigma \leq M_0(0)+M_1(0).$$

Thus, by the dominated convergence theorem, we get

$$\int_0^t \int_n^\infty (1+\varsigma) d_1(\varsigma, s) d\varsigma ds \to 0, \text{ as } n \to \infty.$$

Therefore, letting $n \to \infty$ in (4.9), we conclude that

$$\int_0^\infty (1+\varsigma)^\beta |G(\varsigma, s)| d\varsigma \leq 5kL(T) \int_0^t \int_0^\infty (1+\varsigma)^\beta |G(\varsigma, s)| d\varsigma ds.$$

Since $G(.,0) = 0$, Gronwall's lemma implies

$$\int_0^\infty (1+\varsigma)^\beta |G(\varsigma, s)| d\varsigma = 0,$$

and so $d_1(\varsigma, t) = d_2(\varsigma, t)$ a.e. for each $\varsigma \in (0, \infty)$ and $t \in [0, T)$, which confirms the uniqueness of solutions to (1.3)-(1.4). This completes the proof of Theorem 2.2. □

5. CONTINUOUS DEPENDENCE

In order to show the continuous dependence relatively to the initial condition we prove the following partial result:

Theorem 5.1. *Consider '$\wp_1$' and '$\wp_2$' are the solutions of* (1.3) *with initial conditions $\wp_1(0) = \wp_1^{in}$ and $\wp_2(0) = \wp_2^{in}$. Also, assume that the coagulation kernel 'a' satisfies* (2.1) *with $0 \leq \alpha \leq 1/2$. Then, for each $t \geq 0$, there is a positive constant $V(t, \|\wp_1^{in}\|_{X_{0,1}}, \|\wp_2^{in}\|_{X_{0,1}})$ such that*

$$\|\wp_1(t) - \wp_2(t)\|_{(L^1(\mathbb{R}_+);(1+\varsigma)^\beta d\varsigma)} \leq V(t, \|\wp_1^{in}\|_{X_{0,1}}, \|\wp_2^{in}\|_{X_{0,1}}) \|\wp_1^{in} - \wp_2^{in}\|_{(L^1(\mathbb{R}_+);(1+\varsigma)^\beta d\varsigma)}, \quad (5.1)$$

where $\alpha \leq \beta \leq 1 - \alpha$.

Proof. According to the Definition 2.1, $\wp_1$ and $\wp_2$ are given as

$$\begin{aligned} \wp_1(\varsigma, t) &= \wp_1^{\text{in}}(\varsigma) + \int_0^t \int_0^\infty a(\varsigma+\varrho, \varrho) \wp_1(\varsigma+\varrho, s) \wp_1(\varrho, s) d\varrho ds \\ &\quad - \int_0^t \int_0^\infty a(\varsigma, \varrho) \wp_1(\varsigma, s) \wp_1(\varrho, s) d\varrho ds, \end{aligned}$$

and

$$\begin{aligned} \wp_2(\varsigma, t) &= \wp_2^{\text{in}}(\varsigma) + \int_0^t \int_0^\infty a(\varsigma+\varrho, \varrho) \wp_2(\varsigma+\varrho, s) \wp_2(\varrho, s) d\varrho ds \\ &\quad - \int_0^t \int_0^\infty a(\varsigma, \varrho) \wp_2(\varsigma, s) \wp_2(\varrho, s) d\varrho ds. \end{aligned}$$

Now, for all $\varsigma \in (0, \infty)$ and $t \geq 0$, let us define $f = \wp_1 - \wp_2$ and

$$g(t) = \int_0^\infty (1+\varsigma)^\beta |f(\varsigma, t)| d\varsigma, \quad (5.2)$$

and by the same calculations as in the proof of Theorem 2.2, we obtain

$$g(t) \leq g(0) + k(2\|\wp_1^{\text{in}}\|_{X_{0,1}} + 3\|\wp_2^{\text{in}}\|_{X_{0,1}}) \int_0^t g(s) ds.$$

Applying Gronwall's lemma, we obtain

$$g(t) \leq V(t, \|\wp_1^{\text{in}}\|_{X_{0,1}}, \|\wp_2^{\text{in}}\|_{X_{0,1}})g(0),$$

where,

$$V(t, \|\wp_1^{\text{in}}\|_{X_{0,1}}, \|\wp_2^{\text{in}}\|_{X_{0,1}}) := \exp[kt(2\|\wp_1^{\text{in}}\|_{X_{0,1}} + \|\wp_2^{\text{in}}\|_{X_{0,1}})].$$

This completes the proof of Theorem 5.1. □

6. LARGE-TIME BEHAVIOUR

In this section we discuss aspects of the large-time behaviour of solutions to the continuous RBK equation (1.3)-(1.4) which is motivated from [8]. Because of the cluster eating mechanism the total number of particles, as well as the total mass of the system are expected to go to zero when time tends to infinity. This is stated in Theorem 2.3. Our aim of this section is to prove this theorem, for which we need the following lemmas. Throughout this section we consider $\wp^{\text{in}} \in X_{0,1}^+$, $\wp^{\text{in}} \neq 0$ and $\wp$ denotes a solution to (1.3)-(1.4) on $(0, +\infty)$.

Lemma 6.1. *For $s_2 \geq s_1 \geq 0$, we have*

$$\int_0^\infty \varsigma^j \wp(\varsigma, s_2) d\varsigma \leq \int_0^\infty \varsigma^j \wp(\varsigma, s_1) d\varsigma \tag{6.1}$$

where $j \in [0,1]$.

Proof. Let $R \geq 1$ and $t = s_1$ and $t = s_2$. Then from (2.4) and Fubini's theorem, we get

$$\begin{aligned}
&\int_0^\infty \wp(\varsigma, s_2) \min\{\varsigma^j, R^j\} d\varsigma - \int_0^\infty \wp(\varsigma, s_1) \min\{\varsigma^j, R^j\} d\varsigma \\
= &\int_{s_1}^{s_2} \Big[\int_0^\infty \int_0^\infty \min\{\varsigma^j, R^j\} a(\varsigma + \varrho, \varrho) \wp(\varsigma + \varrho, s) \wp(\varrho, s) d\varrho d\varsigma \\
&- \int_0^\infty \int_0^\infty \min\{\varsigma^j, R^j\} a(\varsigma, \varrho) \wp(\varsigma, s) \wp(\varrho, s) d\varrho d\varsigma \Big] ds \\
= &\int_{s_1}^{s_2} \int_0^\infty \int_0^\varsigma [\min\{(\varsigma - \varrho)^j, R^j\} - \min\{\varsigma^j, R^j\} - \min\{\varrho^j, R^j\}] a(\varsigma, \varrho) \wp(\varsigma, s) \wp(\varrho, s) d\varrho d\varsigma ds.
\end{aligned}$$

Since $j \in [0,1]$, we have $\min((\varsigma - \varrho)^j, R^j) - \min(\varsigma^j, R^j) - \min(\varrho^j, R^j) \leq 0$ for $0 \leq \varrho \leq \varsigma$. Using this we obtain, for $R \geq 1$,

$$\int_0^\infty \wp(\varsigma, s_2) \min\{\varsigma^j, R^j\} d\varsigma \leq \int_0^\infty \wp(\varsigma, s_2) \min\{\varsigma^j, R^j\} d\varsigma.$$

Owing to Definition 2.1, we may let $R \to \infty$ in above inequality and get (6.1). This completes proof of Lemma 6.1. □

Lemma 6.2. *For $s_2 \geq s_1 \geq 0$ and $Q > 0$, we have*

$$\int_{s_1}^{s_2} \int_0^\infty \int_0^\infty a(\varsigma, \varrho) \wp(\varsigma, s) \wp(\varrho, s) d\varrho d\varsigma ds \leq \int_0^\infty \wp(\varsigma, s_1) d\varsigma \tag{6.2}$$

and

$$\int_{s_1}^{s_2}\int_{Q}^{\infty}\int_{Q}^{\infty} a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,s)d\varrho d\varsigma ds \leq \frac{2}{Q}\int_{0}^{\infty}\varsigma\wp(\varsigma,s_1)d\varsigma \tag{6.3}$$

Proof. From (2.4) and Fubini's theorem, we obtain, as above,

$$\int_{s_1}^{s_2}\int_{0}^{\infty}\int_{0}^{\varsigma} a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,s)d\varrho d\varsigma ds = \int_{0}^{\infty}\wp(\varsigma,s_1)d\varsigma - \int_{0}^{\infty}\wp(\varsigma,s_2)d\varsigma.$$

Since the above inequality is true for each $\varsigma \in (0,\infty)$, so we get (6.2).

Next, for proving (6.3), multiplying both side of (2.4) by $\min\{\varsigma, Q\}$ then integrating with respect to ς and using Fubini's theorem, we notice that

$$\begin{aligned}
&\int_{0}^{\infty}\wp(\varsigma,s_2)\min\{\varsigma,Q\}d\varsigma - \int_{0}^{\infty}\wp(\varsigma,s_1)\min\{\varsigma,Q\}d\varsigma \\
= &\int_{s_1}^{s_2}\Big[\int_{0}^{\infty}\int_{0}^{\infty}\min\{\varsigma,Q\}a(\varsigma+\varrho,\varrho)\wp(\varsigma+\varrho,s)\wp(\varrho,s)d\varrho d\varsigma \\
&-\int_{0}^{\infty}\int_{0}^{\infty}\min\{\varsigma,Q\}a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,s)d\varrho d\varsigma\Big]ds \\
= &\int_{s_1}^{s_2}\int_{0}^{\infty}\int_{0}^{\varsigma}[\min\{(\varsigma-\varrho\},Q\}-\min\{\varsigma,Q\}-\min\{\varrho,Q\}]a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,s)d\varrho d\varsigma ds,
\end{aligned}$$

where

$$\min\{\varsigma-\varrho,Q\}-\min\{\varsigma,Q\}-\min\{\varrho,Q\} \leq \begin{cases} -2\varrho, & \text{if } \varsigma \leq Q \\ -\varrho, & \text{if } \varsigma > Q,\ \varrho < Q \\ -Q, & \text{if } \varsigma > Q,\ \varrho > Q, \end{cases}$$

and using these values in the integral above we conclude that

$$\int_{s_1}^{s_2}\int_{Q}^{\infty}\int_{Q}^{\infty} Qa(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,s)d\varrho d\varsigma ds \leq \int_{0}^{\infty}\varsigma\wp(\varsigma,s_1)d\varsigma - \int_{0}^{\infty}\varsigma\wp(\varsigma,s_2)d\varsigma.$$

Using (6.1) with $j = 1$ we obtain (6.3). □

We are now ready to prove Theorem 2.3.

To prove the first assertion of the theorem, consider

$$Ł(\nu,t) := \int_{\nu}^{\infty}\wp(\varsigma,t)d\varsigma, \text{ where } \nu > 0, \text{ and } t \geq 0.$$

Then, for $s_2 \geq s_1 \geq 0$

$$\begin{aligned}
&Ł(\nu,s_2) - Ł(\nu,s_1) \\
\leq\ & -\int_{s_1}^{s_2}\int_{\nu}^{2\nu}\int_{\varsigma-\nu}^{\nu} a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,s)d\varrho d\varsigma ds \\
&-\int_{s_1}^{s_2}\int_{\nu}^{\infty}\int_{\nu}^{\varsigma} a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,s)d\varrho d\varsigma ds \leq 0
\end{aligned} \tag{6.4}$$

From (6.4), we can easily see that $Ł(\nu,.)$ is a non-increasing and non-negative function of time, which implies there exists $Ł(\nu) \geq 0$ such that

$$\lim_{t\to+\infty} Ł(\nu,t) = Ł(\nu). \tag{6.5}$$

Again from (6.2), we deduce that (2.5) gives, for $U \in (\nu,\infty)$, that

$$\begin{aligned}\int_0^t \left(\int_\nu^U \wp(\varsigma,s)d\varsigma\right)^2 ds &\leq \frac{1}{W}\int_0^t\int_0^U\int_0^U a(\varsigma,\varrho)\wp(\varsigma,s)\wp(\varrho,s)d\varrho d\varsigma ds\\ &\leq \frac{1}{W}M_0(0),\end{aligned}$$

and so

$$t \longmapsto Ł(\nu,t) - Ł(U,t) \in L^2(0,+\infty). \tag{6.6}$$

Next, from (6.5), (6.6) and time monotonicity of $Ł(U,.)$ we have

$$0 \leq Ł(\nu) = Ł(U) \leq Ł(U,0),\ U \in (\nu,\infty).$$

Since $\wp^{\text{in}} \in L^1(\mathbb{R}_+)$, we pass to the limit as $U \to \infty$ in the above identity to conclude that $Ł(\nu) = 0$ for each $\nu > 0$. We next observe that

$$\|\wp(t)\|_{L^1(0,+\infty)} = \int_0^\nu \wp(\varsigma,t)d\varsigma + Ł(\nu,t).$$

Now, letting $t \to +\infty$ and then $\nu \to 0$, we obtain

$$\lim_{t\to+\infty}\int_0^\infty \wp(\varsigma,t)d\varsigma = 0.$$

This implies (2.6).

Now, to prove the second assertion of the theorem, we argue as in [8]. For this, let us take $\vartheta(x) = (x^{1-\omega/2}-1)_+,\ x \in \mathbb{R}_+$, where $\omega \in (1,2)$. Since $\omega+1 > 2$ and ϑ vanishes in a neighbourhood of $x = 0$, we notice that

$$\Im = \int_0^\infty \vartheta'(x)x^{-1/2}dx < +\infty,$$

Next, for $s_2 \geq s_1 \geq 0$, Hölder inequality, (2.7), and (6.3) imply that

$$\begin{aligned}&\int_{s_1}^{s_2}\left(\int_0^\infty \vartheta'(Q)\int_Q^\infty \varsigma^{\omega/2}\wp(\varsigma,s)d\varsigma dQ\right)^2 ds\\ \leq\ & \frac{2\Im^2}{C_0}M_1(s_1).\end{aligned}$$

Noticing that

$$\int_0^\infty \vartheta'(Q)\int_Q^\infty \varsigma^{\omega/2}\wp(\varsigma,s)dudQ \geq K(\omega)\int_2^\infty \varsigma\wp(\varsigma,s)d\varsigma,$$

and combining the previous inequalities we get

$$\int_{s_1}^{s_2}\left(\int_2^\infty \varsigma\wp(\varsigma,s)d\varsigma\right)^2 ds \leq KM_1(s_1) \tag{6.7}$$

where K is a positive constant which depends only on ω and C_0. Now, from (2.7) and (6.2), we notice that

$$\int_{s_1}^{s_2}\left(\int_0^2 \varsigma\wp(\varsigma,s)d\varsigma\right)^2 ds \leq KM_0(s_1). \tag{6.8}$$

Next, using (6.1), (6.7), (6.8) and Young's inequality, we conclude that

$$\begin{aligned}\int_0^\infty M_1(s)^2 ds &\leq K\int_0^\infty (1+\varsigma)\wp(\varsigma,s_1)d\varsigma \\ &\leq K(M_0(0)+M_1(0)).\end{aligned} \tag{6.9}$$

which implies $M_1(t)\in L^2(0,\infty)$. Also, (6.1) with $j=1$ implies that the total mass is a decreasing and non-negative function. Therefore,

$$\lim_{t\to\infty} M_1(t) = 0.$$

Finally, for $\omega = 2$ also, from (2.7) and (6.2), the identity (6.9) holds true. This completes proof of Theorem 2.3.

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