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Rapidly Convergent Series from Positive Term Series

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Abstract: In this paper we shall give description about the extraction of a rapidly decaying remainder from Euler series and telescoping series. Then we apply the procedure to generalised telescoping series. The new positive term series obtained with rapidly decaying remainder will converge faster than the original series. We shall apply the procedure to generalised telescoping series also. The introduction of such remainder will give a better approximation for the series.

Keywords: Remainder of a series, sequence of partial sums, Positive term series, rapidity of convergence, Euler series, Telescoping series.

I. INTRODUCTION

The theory of infinite series is an important branch of Mathematical Analysis.The historical development of infinite series can be divided into 3 periods. The period of Newton and Leibnitz is considered as the first period. The period of Euler is the second period and third period is the modern period, the period beginning with Gauss

Indian Mathematicians studied infinite series around 1350.In 1715, Brook Taylor provided a general method for constructing Taylor series for all functions. Later, the theory of hypergeometric series is developed by Leonhard Euler. The criteria of convergence and the questions of remainders and the rate of convergence were initially stated by Gauss.

The remainder of positive term series play a vital role in series approximation. By taking linear combination of the remainder with a telescoping series, a rapidly convergent remainder can be deduced. The new series will be faster convergent than the original series. That is the rate of convergence of the new series can be increased .

PRELIMINARIES

Definition 1;-

If $\sum_{n=1}^{\infty} c_n \sum_{n=1}^{\infty} c_n$ = s and $\sum_{n=1}^{\infty} c'_n \sum_{n=1}^{\infty} c'_n$ = $s' s'$ are two convergent series of positive terms whose partial sums are denoted by $s_n s_n$ and $s_n' s_n'$ a_n and if S - s_n S_n= r_n r_n and S - S_n' s_n' r_n' = r_n' denote the corresponding remainders, we say that the second series converges more or less rapidly than the first, according as $\lim_{n\to\infty} \frac{r_n'}{r_n} \lim_{n\to\infty} \frac{r_n'}{r_n}$ = 0 or $\lim_{n\to\infty} \frac{r_n'}{r_n} \lim_{n\to\infty} \frac{r_n'}{r_n}$ = $\infty\infty$

***Theorem** 2*

If $\frac{b_n}{a_n} \frac{b_n}{a_n} \to \to 0$ 0 or ∞ , then $\sum_{n=1}^{\infty} b_n \sum_{n=1}^{\infty} b_n$ converges more (less) rapidly than $\sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} a_n$

Proof:-

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Assume that $\frac{b_n}{a_n} \to 0$ as $n \to \infty$ and let $\epsilon > 0$ be given.

Then there exists a natural number K_1 such that $\frac{b_n}{a_n} < \epsilon$ for all $n \geq K_1$

That is $\frac{b_n}{a_n} < \epsilon$ for all $n \geq K_1$

Thus for all $n \geq K_1$, we have

$$\frac{r_n'}{r_n} < \epsilon \, \frac{a_{n+1}+a_{n+2}+a_{n+3}+\cdots\ldots\ldots}{a_{n+1}+a_{n+2}+a_{n+3}+\cdots\ldots\ldots}$$

$$= \epsilon$$

Thus $\frac{r_n'}{r_n} < \epsilon$ for all $n \geq K_1$

Thus $\frac{r_n'}{r_n} \to 0$ as $n \to \infty$ and so $\sum_{n=1}^{\infty} b_n$ converges more rapidly than $\sum_{n=1}^{\infty} a_n$

Now assume that $\frac{b_n}{a_n} \to \infty$ as $n \to \infty$ and let M >0 be arbitrary large number.

Then there exists a natural number K_2 such that $\frac{b_n}{a_n} > M$, for all $n \geq K_2$

Thus for all $n \geq K_2$, we have,

$$\frac{r_n'}{r_n} = \frac{b_{n+1}+b_{n+2}+b_{n+3}+\cdots\ldots\ldots}{a_{n+1}+a_{n+2}+a_{n+3}+\cdots\ldots\ldots}$$

$$> M \, \frac{a_{n+1}+a_{n+2}+a_{n+3}+\cdots\ldots\ldots}{a_{n+1}+a_{n+2}+a_{n+3}+\cdots\ldots\ldots}$$

$$= M,$$

$\frac{r_n'}{r_n} > M$ for all $n \geq K_2$

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Since M > 0, we have $\frac{r_n'}{r_n}$ $\frac{r_n{}'}{r_n}$ → → ∞ ∞ as → ∞ n → ∞ and so $\sum_{n=1}^{\infty} b_n \sum_{n=1}^{\infty} b_n$ converges less rapidly than $\sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} a_n$

Hence the proof.

II. RESULTS AND DISCUSSIONS

In this section, we are discussing about the methods applied to deduce a rapidly convergent remainder from Euler series and Telescoping series.

1. *RAPIDLY CONVERGENT REMAINDER FROM EULER SERIES*

Euler series is a positive term convergent series which converges to $\frac{\pi^2}{6}$

That is , $\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \frac{\pi^2}{6}$

Theorem 3

The remainder after k-1 terms for the Euler series is

$R_{k-1} = \sum \frac{(k-1)!}{n^2(n+1)(n+2)\ldots\ldots(n+k-1)} \sum \frac{(k-1)!}{n^2(n+1)(n+2)\ldots\ldots(n+k-1)}$.

Proof:-

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \left\{\frac{1}{n^2} \sum_{n=1}^{\infty} \left\{\frac{1}{n^2} - \frac{1}{n(n+1)}\right\} - \frac{1}{n(n+1)}\right\}$$

$$= \sum_{n=1}^{\infty} \left\{\frac{1}{n} \sum_{n=1}^{\infty} \left\{\frac{1}{n} - \frac{1}{n+1}\right\} - \frac{1}{n+1}\right\} + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

The remainder term $R_1 = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$ is of order ($\frac{1}{n^2} \frac{1}{n^2}$) and so it converges more rapidly.

Now we iterate this process.

We shall represent the new series $\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$ as a telescoping series plus another series which is smaller order of magnitude.

Thus $1 + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} + \sum_{n=1}^{\infty} \left\{\frac{1}{n^2(n+1)} \sum_{n=1}^{\infty} \left\{\frac{1}{n^2(n+1)} - \frac{1}{n(n+1)(n+2)}\right\} \frac{1}{n(n+1)(n+2)}\right\}$

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$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} + \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)} \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)}$$

$$= 1 + \sum_{n=1}^{\infty} \{\frac{1}{2n(n+1)} - \frac{1}{2(n+1)(n+2)}\} \sum_{n=1}^{\infty} \{\frac{1}{2n(n+1)} - \frac{1}{2(n+1)(n+2)}\} + \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)} \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)}$$

$$= 1 + \frac{11}{44} + \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)} \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)}$$

The remainder $R_2 = \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)} \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)}$ and it converges more rapidly.
Now we iterate this process. At each stage we will get a telescoping series which sums to exactly the corresponding element of the original series , plus another series which would be the remainder term.
Now

$$1 + \frac{11}{44} + \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)} \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)}$$

$$= 1 + \frac{11}{44} + \sum_{n=1}^{\infty} \frac{2}{n(n+1)(n+2)(n+3)} \sum_{n=1}^{\infty} \frac{2}{n(n+1)(n+2)(n+3)} + \sum_{n=1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)} \sum_{n=1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)}$$

$$= 1 + \frac{1}{4} + \sum_{n=1}^{\infty} \{\frac{2}{3n(n+1)(n+2)} - \frac{2}{3(n+1)(n+2)(n+3)}\} + \sum_{n=1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)} \frac{1}{4} + \sum_{n=1}^{\infty} \{\frac{2}{3n(n+1)(n+2)} - \frac{2}{3(n+1)(n+2)(n+3)}\} + \sum_{n=1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)}$$

$$= 1 + \frac{1}{4} + \frac{1}{9} + \sum_{n=1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)} \frac{1}{4} + \frac{1}{9} + \sum_{n=1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)}$$

Here first three terms are the terms of the original series. The remainder
$R_3 = \sum_{n=1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)} \sum_{n=1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)}$ and converges more rapidly.

Continuing this process, by mathematical induction, we have

$$\frac{\pi^2 \pi^2}{6 \; 6} = \sum_{n=1}^{k-1} \frac{1}{n^2} \sum_{n=1}^{k-1} \frac{1}{n^2} + \sum \frac{(k-1)!}{n^2(n+1)(n+2)\ldots\ldots(n+k-1)} \sum \frac{(k-1)!}{n^2(n+1)(n+2)\ldots\ldots(n+k-1)}$$

where

$R_{k-1} = \sum \frac{(k-1)!}{n^2(n+1)(n+2)\ldots\ldots(n+k-1)} \sum \frac{(k-1)!}{n^2(n+1)(n+2)\ldots\ldots(n+k-1)}$ which is rapidly convergent.

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$\frac{\pi^2\pi^2}{6\ \ 6}$ = $S_{k-1}+R_{k-1}$.

Here R_{k-1}. = $\sum \frac{(k-1)!}{n^2(n+1)(n+2)\ldots\ldots(n+k-1)}\sum \frac{(k-1)!}{n^2(n+1)(n+2)\ldots\ldots(n+k-1)}$ and it converges more rapidly.
Also by definition , the new series will converge more rapidly than the original series.
Hence the proof. □

Now we shall apply the same procedure to telescoping series to obtain a rapidly convergent remainder.

2. RAPIDLY CONVERGENT REMAINDER FOR TELESCOPING SERIES

We know that the Telescoping series is convergent positive term series. By taking linear combination of the remainder of telescoping series with another telescoping series, a rapidly decaying remainder can be obtained. This method is discussed in the following theorem.

Theorem 4 :-

The rapidly convergent remainder for telescoping series

$\sum_{n=1}^{\infty} \frac{1}{n(n+p)}$ where $p \epsilon Z^+$

$= \frac{1}{p+1} + \frac{1}{2(p+2)} + \frac{1}{3(p+3)} \ldots . + \frac{1}{k(p+k)} + \sum_{k=1}^{\infty} \frac{(p+1)(p+2)(p+3)\ldots\ldots(p+k)}{(n)(n+p)(n+p+1)(n+p+2)\ldots.(n+p+k)}$

Proof:-

We have $\sum_{n=1}^{\infty} \frac{1}{n(n+p)} = \sum_{n=1}^{\infty} \frac{1}{(n+p)(n+p+1)} + \sum_{n=1}^{\infty}[\frac{1}{n(n+p)} - \frac{1}{(n+p)(n+p+1)}]$

$= \frac{1}{p+1} + \sum_{n=1}^{\infty}[\frac{(p+1)}{n(n+p)(n+p+1)}]$

Now$\sum_{n=1}^{\infty}$ = $\sum \frac{(p+1)}{(n+p)(n+p+1)(n+p+2)}$

$+\sum_{n=1}^{\infty}[\frac{(p+1)}{n(n+p)(n+p+1)} - \frac{(p+1)}{(n+p)(n+p+1)(n+p+2)}]$

$= \frac{1}{2(p+2)} + \sum_{n=1}^{\infty}[\frac{(p+1)(p+2)}{n(n+p)(n+p+1)(n+p+2)}]$

Thus $\sum_{n=1}^{\infty} \frac{1}{n(n+p)} = \frac{1}{p+1} + \frac{1}{2(p+2)} + \sum_{n=1}^{\infty}[\frac{(p+1)(p+2)}{n(n+p)(n+p+1)(n+p+2)}]$

Similarly we can prove that

$\sum_{n=1}^{\infty}$ =

$\frac{1}{3(p+3)} + \sum_{n=1}^{\infty}[\frac{(p+1)(p+2)(p+3)}{n(n+p)(n+p+1)(n+p+2)(n+p+3)}]$

Thus $\sum_{n=1}^{\infty} \frac{1}{n(n+p)} = \frac{1}{p+1} + \frac{1}{2(p+2)} + \frac{1}{3(p+3)} +$

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$$\sum_{n=1}^{\infty}\left[\frac{(p+1)(p+2)(p+3)}{n(n+p)(n+p+1)(n+p+2)(n+p+3)}\right]$$

Continuing like this , after k terms we get ,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+p)} = \frac{1}{p+1} + \frac{1}{2(p+2)} + \frac{1}{3(p+3)} \dots + \frac{1}{k(p+k)} + \sum_{k=1}^{\infty} \frac{(p+1)(p+2)(p+3)\dots(p+k)}{(n)(n+p)(n+p+1)(n+p+2)\dots(n+p+k)}$$

Thus the remainder after k terms ,

$$R_k = \sum_{k=1}^{\infty} \frac{(p+1)(p+2)(p+3)\dots(p+k)}{(n)(n+p)(n+p+1)(n+p+2)\dots(n+p+k)} \quad \square$$

Note

For the telescoping series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

the remainder term $R_k = \sum_{k=1}^{\infty} \frac{(k+1)!}{(n)(n+1)(n+2)(n+3)\dots(n+k+1)}$ $\square$.

Now we shall generalise the result obtained above.

Theorem 5

The rapidly convergent remainder term for the series $\sum_{n=1}^{\infty} \frac{1}{p(p+d)}$ where p= an+b ,

a,b, p$\in Z^+$ with $0 \neq a \neq d$ is $R_k = \sum_{k=1}^{\infty} \frac{(a+d)(2a+d)(3a+d)\dots(ka+d)}{(p)(p+d)(p+d+a)(p+d+2a)\dots(p+d+ka)}$

Proof;-

We have $\sum_{n=1}^{\infty} \frac{1}{p(p+d)} = \sum_{n=1}^{\infty} \frac{1}{(p+d)(p+d+a)} + \sum_{n=1}^{\infty}\left[\frac{1}{p(p+d)} - \frac{1}{(p+d)(p+d+a)}\right]$

$$= \frac{1}{a}\left\{\frac{1}{a+b+d}\right\} + \sum_{n=1}^{\infty}\left[\frac{(a+d)}{p(p+d)(p+d+a)}\right]$$

Now $\sum_{n=1}^{\infty}\left[\frac{(a+d)}{p(p+d)(p+d+a)}\right] = \sum \frac{(a+d)}{(p+d)(p+d+a)(p+d+2a)}$

$$+\sum_{n=1}^{\infty}\left[\frac{(a+d)}{p(p+d)(p+d+a)} - \frac{(a+d)}{(p+d)(p+d+a)(p+d+2a)}\right]$$

$$= \frac{a+d}{2a}\left\{\frac{1}{(a+b+d)(2a+b+d)}\right\} + \sum_{n=1}^{\infty}\left[\frac{(a+d)(2a+d)}{p(p+d)(p+d+a)(p+d+2a)}\right]$$

Thu $\sum_{n=1}^{\infty} \frac{1}{p(p+d)} = \frac{1}{a}\left\{\frac{1}{a+b+d}\right\} + \frac{a+d}{2a}\left\{\frac{1}{(a+b+d)(2a+b+d)}\right\} + \sum_{n=1}^{\infty}\left[\frac{(a+d)(2a+d)}{p(p+d)(p+d+a)(p+d+2a)}\right]$

Similarly we can prove that

$\sum_{n=1}^{\infty}\left[\frac{(a+d)(2a+d)}{p(p+d)(p+d+a)(p+d+2a)}\right] =$

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$\frac{(a+d)(2a+d)}{3a}\{\frac{1}{(a+b+d)(2a+b+d)(3a+b+d)}\}+\sum_{n=1}^{\infty}[\frac{(a+d)(2a+d)(3a+d)}{p(p+d)(p+d+a)(p+d+2a)(p+d+3a)}]$

Thus

$\sum_{n=1}^{\infty} \quad \frac{1}{p(p+d)}=\frac{1}{a}\{\frac{1}{a+b+d}\}+\frac{a+d}{2a}\{\frac{1}{(a+b+d)(2a+b+d)}\}+$

$\frac{(a+d)(2a+d)}{3a}\{\frac{1}{(a+b+d)(2a+b+d)(3a+b+d)}\}+$
$\sum_{n=1}^{\infty}[\frac{(a+d)(2a+d)(3a+d)}{p(p+d)(p+d+a)(p+d+2a)(p+d+3a)}]$

Continuing like this , after k terms we get ,

$\sum_{n=1}^{\infty} \quad \frac{1}{p(p+d)}= \frac{1}{a}\{\frac{1}{a+b+d}\}+\frac{a+d}{2a}\{\frac{1}{(a+b+d)(2a+b+d)}\}+$

$\frac{(a+d)(2a+d)}{3a}\{\frac{1}{(a+b+d)(2a+b+d)(3a+b+d)}\}+\frac{(a+d)(2a+d)(3a+d)}{4a}\{\frac{1}{(a+b+d)(2a+b+d)(3a+b+d)(4a+b+d)}\}+$

…………………………………………………………………………………………………..
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$+\{\ (a+d)(2a+d)(3a+d)\ \dots\dots\dots\{\frac{1}{(a+b+d)(2a+b+d)(3a+b+d)\dots\dots(.ka+d)}\ \}$

$+\sum_{k=1}^{\infty}[\frac{(a+d)(2a+d)(3a+d)\dots\dots(ka+d)}{p(p+d)(p+d+a)(p+d+2a)\dots\dots.(p+d+ka)}]$

Thus the remainder after k terms of the series is

$R_k = \sum_{k=1}^{\infty}[\frac{(a+d)(2a+d)(3a+d)\dots\dots(ka+d)}{p(p+d)(p+d+a)(p+d+2a)\dots\dots.(p+d+ka)}]$

Since the ratio between remainder term of deduced series and remainder term of original series is dacaying, the deduced series will converge more faster than the original series.

Hence the proof.

Note

When a =d , we have $R_k = \sum_{k=1}^{\infty}[\frac{(k+1)!a^k}{p(p+d)(p+2d)(p+3d)\dots\dots.[p+(k+1)d]}]$ □

Remark :-

Since the ratio of the remainder term between deduced series to the original series by the above procedure is decaying to zero, the deduced series will be rapidly convergent than the original series, by Definition 1.

III. CONCLUSION

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In this paper we have discussed about the deduction of a rapidly convergent seriesfor Alternating Harmonic series and Gregory series. Also we have generalised the result. This method can be applied to other alternating series also. Since the ratio of remainder of deduced series and original series is decaying to zero, the deduced series will converge more faster than the original series.

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