Some Two-Dimensional Biharmonic Formulations

by

Hojjattollah Adibi, M.Sc.

Thesis submitted for the degree of

Doctor of Philosophy

Department of Mathematics

The City University, London

May 1989

Contents

		Page No.
Title		1
Acknowledgements		5
Abstract		6
List of symbols		7
Introduction		9
PART I: ALMANSI REPRESENTATION		
Chapter 1	Almansi Representation Theory	13
1.1	Introduction	13
1.2	Almansi representation: x, y, r^2-forms	13
1.3	Transformations into the x-or y-forms	15
1.4	Transformations into the r^2-form	17
Chapter 2	Biharmonic Boundary-Value Formulations	19
2.1	Introduction	19
2.2	Interior simply-connected domains	19
2.3	Infinite exterior domains	21
2.4	Ring-shaped domains	22
Chapter 3	Some Radially Symmetric Problems Via Almansi Representation	25
3.1	Introduction	25
3.2	Problem C-1	25
3.3	Problem C-2	27
3.4	Problem C-3	29
3.5	Limiting investigations	31

Page No.

Chapter 4	Some Non-Radially Symmetric Problems Via Almansi Representation	34
4.1	Introduction	34
4.2	Problem C-1	34
4.3	Problem C-2	38
4.4	Problem C-3	40

PART II: CHAKRABARTY REPRESENTATION

Chapter 5	Chakrabarty Representation Theory	45
5.1	Introduction	45
5.2	Biharmonic potentials	45
5.3	Continuity of Ω in $B+\partial B$	47
5.4	Investigation of the normal derivatives	51
Chapter 6	Expansions of the Biharmonic Potential Ω	54
6.1	Introduction	54
6.2	Expansions of Ω within infinite exterior and ring-shaped domains	54
6.3	Biharmonic Green Functions	60
6.4	Comparison between the Almansi and Chakrabarty Representations	62
6.5	Expansion of Ω in C-1	63
Chapter 7	Some Radially Symmetric Problems Via Chakrabarty Representation	65
7.1	Introduction	65
7.2	Problem C-1	66
7.3	Determination of the potentials Ω, ψ	67
7.4	Problem C-2	69

Page No.

7.5	Determination of the potentials Ω, ψ	71
7.6	Problem C-3	73
Chapter 8	Some Non-Radially Symmetric Problems Via Chakrabarty Representation	76
8.1	Introduction	76
8.2	Problem C-1	76
8.3	Problem C-2	79
8.4	Problem C-3	83

PART III: THE TWO-DIMENSIONAL PAPKOVICH-NEUBER FORMULA

Chapter 9	The two-dimensional Papkovich-Neuber formula	88
9.1	Introduction	88
9.2	Analysis of Papkovich-Neuber formula	88
9.3	Muskhelishvili's formula	90
9.4	Analysis of $\chi = x_1h_1 + x_2h_2 + f$	91
9.5	Introduction of stress function	94

APPENDICES

Appendix I	Almansi x, y-forms	98
Appendix II	Sternberg-Eubanks redundancy	100
Appendix III	Direct determination of the displacement components via Almansi	103

References and Bibliography 105

Acknowledgements

I should like to express my sincere thanks and appreciation to my dear supervisor Professor M.A. Jaswon for his assistance, guidance and encouragement in the preparation of this work. Also I take this opportunity to thank my wife Fatemeh for her continuing support, encouragement and patience. Thanks are also due to Vijaya Mauree, Secretary of Department of Mathematics, The City University, for her excellent typing.

Abstract

This thesis develops biharmonic analysis building upon the fundamental contributions which have been made by Almansi and Hadamard nearly one hundred years ago. The representations of Massonnet and Chakrabarty are discussed. It also makes a new analysis of the two-dimensional Papkovich-Neuber formula for elastostatic displacement fields, which is compared with Muskhelishvili's displacement formula for such fields.

List of Symbols

B	General domain
B_i	Interior domain
B_e	Infinite exterior domain
∂B	The boundary of domain B
$\partial B_1, \partial B_0$	The interior and exterior boundaries respectively of a ring-shaped domain B
χ	Biharmonic function/potential
$\underset{\sim}{p}$	Field point
$\underset{\sim}{q}$	Source point
R	Defined as $R = \lvert \underset{\sim}{p} - \underset{\sim}{q} \rvert$
r	Defined as $r = \lvert \underset{\sim}{p} \rvert$
δ	Dirac's delta function
$G(\underset{\sim}{p}, \underset{\sim}{q})$	Biharmonic Green function
θ, τ	The angles of source and field points with the positive x-axis respectively
ϕ, ψ	Harmonic functions/potentials
ϕ^*	Conjugate harmonic function to ϕ
σ, η, ζ	Source density distributions
$(\ldots)'_i = -\frac{d}{dn}$	Inward normal derivative
$(\ldots)'_e = \frac{d}{dn}$	Outward normal derivative
Ω	The Chakrabarty biharmonic potential
λ, μ	Lame's elastic constants
ν	Poisson's ratio
κ	Defined as $\kappa = \frac{1}{4(1-\nu)}$
$u_j;\ j = 1,2$	Two-dimensional displacement components
$\mathcal{H}, \mathcal{F}$	Analytic functions of z

$\overline{\mathcal{H}}, \overline{\mathcal{F}}$	Complex conjugate functions to $\mathcal{H}, \mathcal{F}$ respectively
Δ	Dilatation
ω	Rotation
$P_{\alpha\beta}$; $\alpha, \beta = 1,2$	Two-dimensional stress components
$e_{\alpha\beta}$; $\alpha, \beta = 1,2$	Two-dimensional strain components

Introduction

Biharmonic analysis was created at the end of the last century from two distinct sources: E. Almansi's representation of a biharmonic function in terms of two independent harmonic functions, and J. Hadamard's existence-uniqueness theorems for biharmonic boundary-value problems. These theorems proved indispensable in formulating two-dimensional elastostatics through a stress function and in the theory of transverse deflections of thin plates. However they do not, by themselves, enable appropriate solution of the biharmonic equation to be obtained except in the simplest cases. Almansi's representation marked a considerable step forward in the construction of solutions, since it effectively reduces the theory of biharmonic functions to that of harmonic functions. Some simple examples of boundary-value problems solved directly via Almansi's representation are given in Part I of this thesis. With the advent of fast digital computers in the early 1960's, it became the basis for a powerful method of numerical attack utilising boundary integral equations e.g. by Jaswon, M.A. and Symm, G.T. (1977).

An alternative representation to that of Almansi has been proposed by S.K. Chakrabarty in his Ph.D. thesis (1971). Here biharmonic potentials are generated from biharmonic sources on the boundary, so enabling biharmonic functions to be constructed which meet prescribed boundary conditions. This theory is outlined systematically in Part II of the thesis, which gives (apparently for the first time) the correct asymptotic expansion of a biharmonic potential. Dislocation contributions appear in the expansion, which makes Chakrabarty's representation particularly suited for ring-shaped domains. By contrast, Almansi's representation cannot accommodate dislocations as it stands without introducing multi-valued harmonic functions, so requiring a more complete representation involving only single-valued functions. All the problems solved in Part I are also solved in Part II, to compare the effectiveness of each representation for different types of domain.

Almansi's representation may be regarded as the real-variable analogue of Muskhelishvili's complex variable representation, as can be readily seen by separating Muskhelishvili's complex stress function into real and imaginary components. Alternatively we may separate Muskhelishvili's complex displacement formula into real and imaginary components and compare them with those obtained via Almansi. These formulae are discussed in Part III of the thesis and compared with the two-dimensional Papkovich-Neuber formula. Muskhelishvili's displacement formula (and therefore also Almansi's) proves to be a specialised variant of the Papkovich-Neuber formula, since the latter formula does not assume the existence of a stress function and reduces to the former when this constraint is introduced.

Parts I and II of the thesis have been accepted for publication (jointly with Professor M.A. Jaswon), in Proc. first European Boundary Element Meeting, Brussels, May 1988.

Also Part III of the thesis has been published as a paper (jointly with Professor M.A. Jaswon) in Proc. B.E.M. X, Vol.1, Southampton, September 1988. Copies of these papers are included at the end of Part II and of Part III.

PART I

ALMANSI REPRESENTATION

Chapter 1

Almansi Representation Theory

1.1 Introduction

This chapter will be devoted almost entirely to the introduction and identification of Almansi representations, which were introduced by Almansi as long ago as 1897. They play an important role in the boundary-value problems of two-dimensional elastostatics, because of their simplicity and also because they present biharmonic functions in terms of harmonic functions.

Almansi representations are in fact the analogue of Muskhelishvili's complex-variable formulation which will be explained later on.

1.2 Almansi representations: x, y, r^2-forms

Let χ be a harmonic function in some simply-connected domain B bounded by a contour ∂B, i.e. χ is continuous everywhere in $B+\partial B$, is differentiable to the fourth order in B, and satisfies the equation

$$\nabla^4\chi = \nabla^2(\nabla^2\chi) = 0 \qquad \text{in B.} \qquad (1.2.1)$$

It was shown by Almansi, E. (1897) and is also proved in Appendix I that we may always write

$$\chi = x\phi + \psi \text{ (or } y\phi + \psi) \quad \text{in } B+\partial B, \qquad (1.2.2)$$

where ϕ, ψ are harmonic functions in B. This representation effectively reduces the theory of biharmonic functions to that of harmonic functions. Note that ϕ, ψ are not unique for a given χ, since

$$x\phi + \psi = 0 \Rightarrow \phi = a + by,\ \psi = -ax - bxy, \qquad (1.2.3)$$

for arbitrary constants a, b, which is a serious limitation from the point of view of numerical attack.

An equivalent alternative to (1.2.2) is

$$\chi = r^2\phi + \psi; \quad r^2 = x^2 + y^2 \quad \text{in} \quad B+\partial B, \qquad (1.2.4)$$

since ϕ, ψ are now unique harmonic functions for a given χ. A formal proof of the uniqueness has been given by Jaswon, M.A. and Shidfar, A. (1980) who proved that the functional equation

$$r^2\phi + \psi = 0, \qquad (1.2.5)$$

has only the two independent non-trivial solutions

$$\left.\begin{array}{l} \phi = r^{-1}\cos\theta, \quad \psi = -r\cos\theta \\ \text{or} \\ \phi = r^{-1}\sin\theta, \quad \psi = -r\sin\theta \end{array}\right\}, \qquad (1.2.6)$$

which could not exist in B if the origin $(r = 0)$ lies in B.

It may be seen that (1.2.2) forms the real-variable analogue of Muskhelishvili's complex variable representation. This aspect will be studied in detail in Part II.

1.3 Transformations into the x-or y-forms

First it may be verified by direct analysis that each of the x-or y-forms can be transformed into the other. To see this let us start with the important identity

$$(x+iy)(\phi+i\phi^*) = (x\phi-y\phi^*) + i(x\phi^*+y\phi), \qquad (1.3.1)$$

in which ϕ, ϕ^* form a pair of conjugate harmonic functions. Now since $(x+iy)(\phi+i\phi^*)$ is an analytic function, therefore the expressions in brackets on the right-hand side are conjugate harmonic functions, i.e. writing

$$h = x\phi - y\phi^*; \quad \nabla^2 h = 0, \qquad (1.3.2)$$

we have

$$h^* = x\phi^* + y\phi; \quad \nabla^2 h^* = 0. \qquad (1.3.3)$$

It follows from (1.3.2), (1.3.3) that

$$\left.\begin{aligned} x\phi &= y\phi^* + h \\ y\phi &= -x\phi^* + h^* \end{aligned}\right\}, \qquad (1.3.4)$$

which are the desired transformations. For instance, if χ = xlogr, according to the transformation (1.3.4) we obtain

$$x\log r = y\theta + (x\log r - y\theta), \qquad (1.3.5)$$

in fact here we have

$$\phi = \log r, \ \phi^* = \theta, \ h = x\log r - y\theta; \ \nabla^2 h = 0. \qquad (1.3.6)$$

Similarly

$$y\log r = -x\theta + (y\log r + x\theta), \tag{1.3.7}$$

where in this case

$$\phi = \log r, \ \phi^* = \theta, \ h = y\log r + x\theta: \ \nabla^2 h^* = 0. \tag{1.3.8}$$

In order to transform the r^2-form into the x-or y-forms, note that

$$r^2\phi = x^2\phi + y^2\phi = x(x\phi - y\phi^*) + y(x\phi^* + y\phi)$$
$$= xh + yh^*, \tag{1.3.9}$$

where h, h* are defined in (1.3.2), (1.3.3) respectively. Now by virtue of (1.3.4) we have

$$yh^* = xh + (yh^* - xh), \tag{1.3.10}$$

so

$$r^2\phi = 2xh + (yh^* - xh)$$
$$= 2x(x\phi - y\phi^*) + \{(y^2 - x^2)\phi + 2xy\phi^*\}, \tag{1.3.11}$$

where

$$\nabla^2\{(y^2 - x^2)\phi + 2xy\phi^*\} = 0. \tag{1.3.12}$$

Similarly we obtain

$$r^2\phi = 2y(x\phi^* + y\phi) + \{(x^2 - y^2)\phi - 2xy\phi^*\}, \tag{1.3.13}$$

where also

$$\nabla^2\{x^2 - y^2)\phi - 2xy\phi^*\} = 0. \tag{1.3.14}$$

For instance*

$$r^2\log r = x\{2(x\log r - y\theta)\} + \{(y^2-x^2)\log r + 2xy\theta\}, \quad (1.3.15)$$

and also

$$r^2\log r = y\{2(y\log r + x\theta)\} + \{(x^2-y^2)\log r - 2xy\theta\}. \quad (1.3.16)$$

1.4 Transformation into the r^2-form

We may transform the x-and y-forms into the r^2-form. Thus for a given harmonic function ϕ, write

$$x\phi = r^2 h + f, \quad (1.4.1)$$

where h, f are harmonic functions to be determined. It may be verified that

$$\left.\begin{aligned} 2x\phi &= r^2\left(\frac{x\phi+y\phi^*}{r^2}\right) + (x\phi - y\phi^*) \\ 2y\phi &= r^2\left(\frac{y\phi-x\phi^*}{r^2}\right) + (y\phi + x\phi^*) \end{aligned}\right\}, \quad (1.4.2)$$

where all the expressions in brackets are harmonic functions, noting that

$$\frac{\phi+i\phi^*}{x+iy} = \left(\frac{x\phi+y\phi^*}{r^2}\right) - i\left(\frac{y\phi-x\phi^*}{r^2}\right). \quad (1.4.3)$$

For instance*

$$\left.\begin{aligned} 2x\log r &= r^2\left(\frac{x\log r+y\theta}{r^2}\right) + (x\log r - y\theta) \\ 2y\log r &= r^2\left(\frac{y\log r-x\theta}{r^2}\right) + (y\log r + x\theta) \end{aligned}\right\}, \quad (1.4.4)$$

---

* these are the real and imaginary components of the analytic functions $z^2\log z$, $\frac{\log z}{z}$ respectively

which shows that the relevant harmonic functions could be multi-valued.

To obtain (1.4.2) directly we operate upon both sides of (1.4.1) by ∇^2, which gives

$$\frac{\partial \phi}{\partial x} = 2\left(h + x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y}\right), \tag{1.4.5}$$

i.e.

$$h+r\frac{\partial h}{\partial r} = \frac{1}{2}\frac{\partial \phi}{\partial x}. \tag{1.4.6}$$

This is a form of the two-dimensional Bergman-Schiffer equation (Bergman, S. and Schiffer, M. 1953). A complete analysis and solution of equation (1.4.6) has been given by Jaswon, M.A. and Shidfar, A. (1980), which covers the formulas (1.4.2).

Chapter 2

Biharmonic Boundary-value Formulations

2.1 Introduction

In this chapter we develop more analysis for the Almansi representation. In particular we show how to exploit the Almansi representation in the solution of boundary-value problems, covering formulations for interior, infinite exterior and also ring-shaped domains. This development involves Hadamard's various existence-uniqueness theorems relating to biharmonic problems.

2.2 Interior simply-connected domains

The first systematic account of biharmonic boundary-value problems was given by Hadamard, J. (1908). In particular he showed that, if a set of values of a function χ and its normal derivative χ' are prescribed on ∂B, then χ can be uniquely determined within the interior domain B_i, such that χ is continuous and has continuous derivatives up to the fourth order within B_i and satisfies equation (1.2.1).

This is an existence-uniqueness theorem. To construct χ in B, we utilise the representation (1.2.4) and note that it holds on ∂B, since χ, ϕ, ψ remain continuous at ∂B, i.e.

$$\chi = r^2\phi + \psi \qquad \text{on} \quad \partial B. \tag{2.2.1}$$

Also an accompanying normal derivative holds on ∂B, i.e.

$$\chi' = (r^2\phi+\psi)' = r^2\phi' + 2rr'\phi + \psi' \quad \text{on} \quad \partial B. \tag{2.2.2}$$

These provide a pair of coupled functional relations for the four boundary functions ϕ, ψ, ϕ', ψ' in terms of χ, χ' on ∂B. However only two of these are independent since, in principle, ϕ' is known on ∂B if ϕ is known on ∂B and similarly for ψ, ψ'.

An effective way forward is to represent ϕ, ψ as simple-layer logarithmic potentials generated from sources on ∂B, i.e. we write

$$\left.\begin{aligned} \phi(\underset{\sim}{p}) &= \int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})\,dq \\ \psi(\underset{\sim}{p}) &= \int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})\eta(\underset{\sim}{q})\,dq \end{aligned}\right\} ;\ \underset{\sim}{p} \in B+\partial B,\ \underset{\sim}{q} \in \partial B \quad |\underset{\sim}{p}| \equiv r. \qquad (2.2.3)$$

Here $\underset{\sim}{p} = (p_1,p_2)$ is the field point, $\underset{\sim}{q} = (q_1,q_2)$ is the source point, dq is an elementary interval of ∂B at $\underset{\sim}{q}$; σ, η are source-density distributions to be determined, and

$$g(\underset{\sim}{p},\underset{\sim}{q}) = \log|\underset{\sim}{p}-\underset{\sim}{q}|. \qquad (2.2.4)$$

Also, Kellogg O.D. (1929)

$$\left.\begin{aligned} \phi'(\underset{\sim}{p}) &= \int_{\partial B} g'(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})\,dq + \pi\sigma(\underset{\sim}{p}) \\ \psi'(\underset{\sim}{p}) &= \int_{\partial B} g'(\underset{\sim}{p},\underset{\sim}{q})\eta(\underset{\sim}{q})\,dq + \pi\eta(\underset{\sim}{p}) \end{aligned}\right\} ;\ \underset{\sim}{p},\underset{\sim}{q} \in \partial B, \qquad (2.2.5)$$

where $g'(\underset{\sim}{p},\underset{\sim}{q})$ denotes the interior normal derivative of $g(\underset{\sim}{p},\underset{\sim}{q})$ at $\underset{\sim}{p}$ keeping $\underset{\sim}{q}$ fixed. Inserting (2.2.3), (2.2.5) into (2.2.1) and (2.2.2) yields a pair of coupled boundary integral equations for σ, η in terms of χ, χ'. With these known, we may generate ϕ, ψ and therefore χ throughout B. Exact or even approximate analytical solutions are generally out of the question. However numerical solutions to acceptable accuracy may be achieved by fast computers implementing well established discretisation procedures. This approach has been applied to deflection problems of thin plates, including a numerical refutation of Hadamard's celebrated conjecture, and in two-dimensional stress analysis, Brown, I.C. and Jaswon, E. (1971). More recently (Bhattacharyya, P.K. and Symm, G.T. 1979, 1984), it has been applied to two-dimensional displacement problems and to mixed boundary-value problems, though here the representation (1.2.2) is preferable to (1.2.4).

2.3 Infinite exterior domains

For infinite exterior domains, unlike interior domains, neither ϕ nor ψ as represented by (2.2.3) covers the possibility of a constant in χ. In fact it follows from (2.2.3) that

$$\phi(\underset{\sim}{p}) = \log|\underset{\sim}{p}|\int_{\partial B}\sigma(\underset{\sim}{q})dq-|\underset{\sim}{p}|^{-2}\int_{\partial B}(\underset{\sim}{p}.\underset{\sim}{q})\sigma(\underset{\sim}{q})dq+O(|\underset{\sim}{p}|^{-2}),$$

$$\text{as} \quad r \equiv |\underset{\sim}{p}| \to \infty, \qquad (2.3.1)$$

with the same formula for ψ, see Jaswon, M.A. and Symm, G.T. (1977). Thus the Almansi representation must be extended in this case by writing

$$\chi = r^2\phi + \psi + k; \qquad \underset{\sim}{p} \in B_e + \partial B, \qquad (2.3.2)$$

where ϕ, ψ are defined by (2.2.3) and k is an unknown constant balanced by the side condition

$$\int_{\partial B} \sigma(\underset{\sim}{q})dq = 0. \qquad (2.3.3)$$

This ensures the absence of $\log r$ in ϕ and therefore of $r^2\log r$ in χ, which may be justified by Hadamard's uniqueness requirement (Hadamard, J. 1908)

$$\chi = O(r) \qquad \text{as} \quad r \equiv |\underset{\sim}{p}| \to \infty. \qquad (2.3.4)$$

Now (2.3.2) holds on ∂B so that we may write

$$\chi = r^2\phi + \psi + k; \qquad \underset{\sim}{p} \in \partial B. \qquad (2.3.5)$$

Coupling (2.3.5) with

$$\chi_e'(\underset{\sim}{p}) = (r^2\phi + \psi + k)_e' ; \qquad \underset{\sim}{p} \in \partial B, \qquad (2.3.6)$$

and utilising (2.2.3), (2.2.5) together with the side condition (2.3.3), we have sufficient boundary equations for k, σ, η to be determined, so enabling ϕ and ψ, and hence χ to be determined in B_e.

2.4 Ring-shaped domains

For a ring-shaped domain B bounded externally by ∂B_0 and internally by ∂B_1 which encloses $r = 0$, Fig. 2.4.1, the Almansi representation must be extended by writing

$$\chi(\underset{\sim}{p}) = r^2\phi + \psi + ax\log r + by\log r;$$

$$\underset{\sim}{p} = (x,y) \in B + \partial B_0 + \partial B_1, \qquad (2.4.1)$$

where ϕ, ψ are defined by (2.2.3) and a, b are unknown coefficients balanced by two appropriate side conditions.

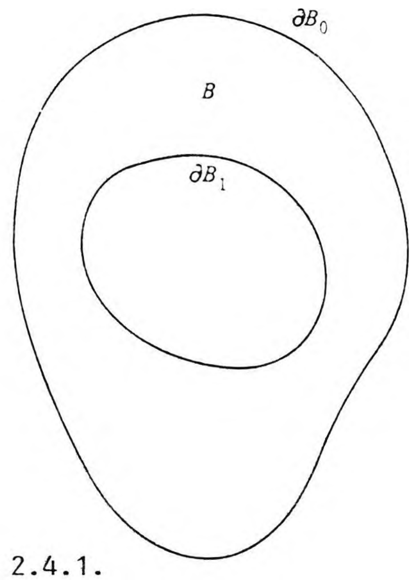


Fig. 2.4.1.

Note that

$$\phi(\underset{\sim}{p}) = \int_{\partial B_1} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq + \int_{\partial B_0} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq;$$

$$\underset{\sim}{p} \in B + \partial B_0 + \partial B_1, \quad g(\underset{\sim}{p},\underset{\sim}{q}) = \log|\underset{\sim}{p}-\underset{\sim}{q}|, \qquad (2.4.2)$$

where - see (2.3.1) -

$$\int_{\partial B_1} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq = \log|\underset{\sim}{p}|\int_{\partial B_1}\sigma(\underset{\sim}{q})dq$$

$$- |\underset{\sim}{p}|^{-2}\int_{\partial B_1}(\underset{\sim}{p}.\underset{\sim}{q})\sigma(\underset{\sim}{q})dq + O(|\underset{\sim}{p}|^{-2}) \qquad \text{as } r = |\underset{\sim}{p}| \longrightarrow \infty.$$

(2.4.3)

Now

$$\underset{\sim}{p}.\underset{\sim}{q} = p_1q_1 + p_2q_2; \quad \underset{\sim}{p} = (x,y) = (r\cos\theta,\ r\sin\theta), \qquad (2.4.4)$$

therefore

$$\frac{\underset{\sim}{p}\cdot\underset{\sim}{q}}{r^2} = \frac{\cos\theta}{r}q_1 + \frac{\sin\theta}{r}q_2 , \tag{2.4.5}$$

so that

$$\frac{1}{r^2}\int_{\partial B_1} (\underset{\sim}{p}\cdot\underset{\sim}{q})\sigma(\underset{\sim}{q})\,dq = \frac{\cos\theta}{r}\int_{\partial B_1} q_1\sigma(\underset{\sim}{q})\,dq$$

$$+ \frac{\sin\theta}{r}\int_{\partial B_1} q_2\sigma(\underset{\sim}{q})\,dq, \tag{2.4.6}$$

since $\underset{\sim}{p}$ is fixed in these integrations.

If so, the conditions

$$\int_{\partial B_1} q_1\sigma(\underset{\sim}{q})\,dq = 0, \qquad \int_{\partial B_1} q_2\sigma(\underset{\sim}{q})\,dq = 0, \tag{2.4.7}$$

ensure the absence of $r^{-1}\cos\theta$, $r^{-1}\sin\theta$ in ϕ, i.e. the absence of $r\cos\theta$, $r\sin\theta$ in $r^2\phi$, which are covered by ψ. Coupling (2.4.7) with the boundary conditions

$$\left.\begin{aligned} \chi &= r^2\phi + \psi + ax\log r + by\log r \\ \chi' &= (r^2\phi + \psi + ax\log r + by\log r)' \end{aligned}\right\} ; \quad (x,y) \in B+\partial B_0+\partial B_1 , \tag{2.4.8}$$

we have sufficient equations to determine a, b, σ, η.

Chapter 3

Some Radially Symmetric Problems via Almansi Representation

3.1 Introduction

Having familiarized ourselves with the two-dimensional Almansi representation and also the Hadamard existence - uniqueness theorem it is intended to solve some radially symmetric boundary-value problems by utilising this representation. We shall consider various representative types of domain, i.e. interior, infinite exterior and ring-shaped domains as follows:

C-1: Interior domain B_i of a circle of radius a with the boundary $\partial B(r=a)$,

C-2: Infinite exterior domain B_e of a circle of radius a with the boundary $\partial B(r=a)$,

C-3: Ring-shaped domain bounded by two concentric circles, i.e. internally by $\partial B_1(r=a)$ and externally by $\partial B_0(r=b)$.

3.2 Problem C-1

The problem is to identify a biharmonic function χ in B_i where χ, χ_i' are pre-assigned on ∂B as follows:

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= a^2 \qquad &(3.2.1a)\\ \chi_i'(\underset{\sim}{p}) &= -2a \qquad &(3.2.1b) \end{aligned}\right\} ; \quad \underset{\sim}{p} \in \partial B.$$

writing

$$\chi = r^2\phi + \psi; \qquad |\underset{\sim}{p}| = r \le a, \tag{3.2.2}$$

and noting that the problem is radially symmetric, it follows that

$$\phi = \alpha \text{ (a constant)}, \quad \psi = \beta \text{ (a constant)}, \tag{3.2.3}$$

since these are the only available radially symmetric harmonic functions inside $r \leq a$. If so

$$\chi = a^2\alpha + \beta; \qquad r \leq a. \tag{3.2.4}$$

By continuity representation (3.2.4) holds on the boundary, so giving from (3.2.1a) the boundary relation

$$\chi(a) = a^2\phi(a) + \psi(a) = a^2; \quad r = a, \tag{3.2.5}$$

i.e.

$$a^2\alpha + \beta = a^2; \qquad r = a. \tag{3.2.6}$$

Associated with (3.2.2) is the normal derivative relation

$$\chi_i'(a) = (r^2\phi+\psi)_i' \; ; \qquad r = a, \tag{3.2.7}$$

i.e.

$$-2a\alpha + a^2\phi_i' + \psi_i' = -2a, \tag{3.2.8}$$

on bearing in mind

$$\phi_i' = 0, \quad \psi_i' = 0, \quad r_i' = -\left(\frac{dr}{dn}\right)_{r=a} = -1. \tag{3.2.9}$$

Relations (3.2.6), (3.2.8) form a pair of linear equations for the constants α, β giving

$$\alpha = 1, \quad \beta = 0 \; ; \qquad r = a, \tag{3.2.10}$$

so yielding the representation

$$\chi = r^2 \quad ; \qquad r \le a, \tag{3.2.11}$$

as was anticipated.

3.3 Problem C-2

In this problem we propose to determine the biharmonic function χ in $B_e(r \ge a)$ subject to the boundary conditions

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= a^2 \\ \chi_e'(\underset{\sim}{p}) &= 2a \end{aligned}\right\} \quad ; \qquad \underset{\sim}{p} \in \partial B. \tag{3.3.1}$$

Pursuing the analysis of the preceding chapter, we adopt the representation

$$\chi(\underset{\sim}{p}) = r^2\phi + \psi \; ; \qquad |\underset{\sim}{p}| = r \ge a, \tag{3.3.2}$$

subject to Hadamard's uniqueness requirement

$$\chi = O(r) \qquad \text{as} \qquad r \longrightarrow \infty. \tag{3.3.3}$$

Since the problem is radially symmetric, therefore the only available harmonic functions ϕ, ψ in $r \ge a$ are

$$\phi = \alpha + \beta \log r, \qquad \psi = \gamma + \delta \log r, \tag{3.3.4}$$

where α, β, γ, δ are constants to be determined. However, because of (3.3.3) it follows that

$$\phi = 0 \qquad \text{in} \quad r \ge a. \tag{3.3.5}$$

If so

$$\chi(r) = \psi(r) = \gamma + \delta \log r; \quad r \geq a, \qquad (3.3.6)$$

i.e. χ is in fact a harmonic function in $r \geq a$.

By virtue of (3.3.1) we obtain

$$\chi(a) = (\gamma + \delta \log r)_{r=a} = \gamma + \delta \log a = a^2, \qquad (3.3.7)$$

$$\chi_e'(a) = \frac{d}{dr}(\gamma + \delta \log r)_{r=a} = \frac{\delta}{a} = 2a, \qquad (3.3.8)$$

giving

$$\gamma = a^2 - 2a^2 \log a, \qquad \delta = 2a^2. \qquad (3.3.9)$$

Consequently

$$\chi(r) = 2a^2 \log r - 2a^2 \log a + a^2; \quad r \geq a, \qquad (3.3.10)$$

which contrasts with the interior solution (3.2.11) obeying the same boundary conditions.

The exterior representation (3.3.2) differs slightly from the representation (2.3.2), i.e.

$$\chi(\underset{\sim}{p}) = r^2\phi + \psi + k; \qquad |\underset{\sim}{p}| = r \geq a. \qquad (3.3.11)$$

This is because in (2.3.2) the harmonic functions ϕ, ψ appear as the potentials

$$\left.\begin{aligned} \phi(\underset{\sim}{p}) &= \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\,\sigma(\underset{\sim}{q})\,dq \qquad (3.3.12a) \\ \psi(\underset{\sim}{p}) &= \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\,\eta(\underset{\sim}{q})\,dq \qquad (3.3.12b) \end{aligned}\right\} ; \quad |\underset{\sim}{p}| = r \geq a,$$

which do not cover a constant component in B_e. The constant k is balanced by the side condition

$$\int_{\partial B} \sigma(\underset{\sim}{q})dq = 0. \tag{3.3.13}$$

We may easily identify (3.3.6) with (3.3.11) by noting

$$\left.\begin{aligned} \alpha &= 0, \quad \beta = \int_{\partial B} \sigma(\underset{\sim}{q})dq = 0 \\ \gamma &= k, \quad \delta = \int_{\partial B} \eta(\underset{\sim}{q})dq \end{aligned}\right\} . \tag{3.3.14}$$

3.4 Problem C-3

In this section we deal with a ring-shaped domain B as described in C-3. Although in practice we will see that the Almansi representation for the ring-shaped domains is inferior to that of Chakrabarty which will be cited next, nevertheless it is of interest to investigate the application of the Almansi representation to ring-shaped domains.

Choosing the boundary conditions

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= a^2 \\ \chi_e'(\underset{\sim}{p}) &= 2a \end{aligned}\right\} ; \ \underset{\sim}{p} \in \partial B_1, \qquad \left.\begin{aligned} \chi(\underset{\sim}{p}) &= b^2 \\ \chi_i'(\underset{\sim}{p}) &= -2b \end{aligned}\right\} ; \ \underset{\sim}{p} \in \partial B_0, \tag{3.4.1}$$

the problem is radially symmetric and we may therefore write

$$\chi(p) = \alpha r^2 \log r + \beta r^2 + \gamma \log r + \delta \ ; \quad a \le r \le b, \tag{3.4.2}$$

where α, β, γ, δ are constants to be determined. These satisfy the two immediate boundary equations

$$\chi(a) = \alpha a^2 \log a + \beta a^2 + \gamma \log a + \delta = a^2, \tag{3.4.3}$$

$$\chi(b) = \alpha b^2 \log b + \beta b^2 + \gamma \log b + \delta = b^2. \tag{3.4.4}$$

They also satisfy the normal derivative equations

$$\chi_e'(a) = \frac{d}{dr}(\alpha r^2 \log r + \beta r^2 + \gamma \log r + \delta)_{r=a},$$

i.e.

$$(2a\log a + a)\alpha + 2a\beta + \frac{\gamma}{a} = 2a, \tag{3.4.5}$$

and also

$$\chi_i'(b) = -\frac{d}{dr}(\alpha r^2 \log r + \beta r^2 + \gamma \log r + \delta)_{r=a},$$

i.e.

$$(2b\log b + b)\alpha + 2b\beta + \frac{\gamma}{b} = 2b, \tag{3.4.6}$$

so providing four linear equations

$$\begin{bmatrix} a^2\log a & a^2 & \log a & 1 \\ b^2\log b & b^2 & \log b & 1 \\ 2a\log a + a & 2a & \frac{1}{a} & 0 \\ 2b\log b + b & 2b & \frac{1}{b} & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} a^2 \\ b^2 \\ 2a \\ 2b \end{bmatrix}, \tag{3.4.7}$$

which has the solutions

$$\beta = 1, \quad \alpha = \delta = \gamma = 0. \tag{3.4.8}$$

Accordingly we get

$$\chi(r) = r^2; \qquad a \le r \le b, \tag{3.4.9}$$

which satisfies the conditions of (3.4.1) as expected.

3.5 Limiting investigations

The results for C-1, C-2 can be achieved by considering the ring-shaped domain C-3 and letting either $a \to 0$, or $b \to \infty$, as appropriate. For instance we may obtain (3.3.10) by introducing

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= a^2 \\ \chi'_e(\underset{\sim}{p}) &= 2a \end{aligned}\right\} ; \ \underset{\sim}{p} \in \partial B_1, \qquad \left.\begin{aligned} \chi(\underset{\sim}{p}) &= 0 \\ \chi'_i(\underset{\sim}{p}) &= 0 \end{aligned}\right\} ; \ \underset{\sim}{p} \in \partial B_0, \tag{3.5.1}$$

i.e. the conditions on ∂B_1 remain unaltered. Now as before we have

$$\chi = \alpha r^2 \log r + \beta r^2 + \gamma \log r + \delta; \qquad a \le r \le b, \tag{3.5.2}$$

which yields the equations (3.4.3), (3.4.5) as they stand and equations (3.4.4), (3.4.6) with the right-hand sides replaced by 0,0 respectively. If so,

$$\alpha = \frac{4a^2b^2(\log a - \log b)}{\Delta},$$

$$\beta = \frac{-a^4 + a^2b^2 - 2a^2b^2\log a + 2a^2b^2\log b + 4a^2b^2\log^2 b - 4a^2b^2\log a\log b}{\Delta},$$

$$\gamma = \frac{2a^2b^2(a^2-b^2)}{\Delta},$$

$$\delta = a^2-\alpha a^2\log a-\beta a^2-\gamma\log a, \tag{3.5.3}$$

where

$$\Delta = -[(a^2-b^2)^2+4a^2b^2(\log a-\log b)^2] \neq 0; \quad a < b. \tag{3.5.4}$$

Now it follows that

$$\left.\begin{array}{l} \alpha,\ \beta \to 0, \\ \quad\gamma \to 2a^2, \\ \quad\delta \to a^2-2a^2\log a \end{array}\right\} \quad \text{as} \quad b \to \infty, \tag{3.5.5}$$

which yields

$$\chi \to 2a^2\log r + a^2 - 2a^2\log a \quad \text{as} \quad b \to \infty. \tag{3.5.6}$$

This conclusion is in agreement with (3.3.10).

Also (3.2.11) may be obtained by introducing

$$\left.\begin{array}{l} \chi(\underset{\sim}{p}) = 0 \\ \\ \chi_e'(\underset{\sim}{p}) = 0 \end{array}\right\} ; \quad \underset{\sim}{p} \in \partial B_1, \qquad \left.\begin{array}{l} \chi(\underset{\sim}{p}) = b^2 \\ \\ \chi_i'(\underset{\sim}{p}) = -2b \end{array}\right\} ; \quad \underset{\sim}{p} \in \partial B_0. \tag{3.5.7}$$

Now by utilising (3.5.2) we have the equations (3.4.4), (3.4.6) as they stand and equations (3.4.3), (3.4.5) with the right-hand sides replaced by 0, 0 respectively. If so, it follows that

$$\left.\begin{array}{r}\alpha,\ \gamma,\ \delta \longrightarrow 0 \\ \\ \beta \longrightarrow 1\end{array}\right\} \qquad \text{as} \quad a \longrightarrow 0, \tag{3.5.8}$$

which yields

$$\chi \longrightarrow r^2 \qquad \text{as} \quad a \longrightarrow 0, \tag{3.5.9}$$

in agreement with (3.2.11).

Chapter 4

Some Non-Radially Symmetric Problems Via Almansi Representation

4.1 Introduction

In this chapter we construct the analytical solutions for several types of non-radially symmetric problem, as an illustration of the importance of the Almansi representation in biharmonic boundary-value problems. As before we will be dealing with the three major categories of domain C-1, C-2, C-3. Clearly, in non-radially symmetric cases, if χ, χ' are pre-assigned on circular boundaries, then the source-density distributions σ, η will be functions of θ.

In this chapter, as in the preceding chapter we employ direct techniques to avoid solving the relevant boundary integral equations for the sources concerned.

4.2 Problem C-1

We propose to determine χ in B_i $(r \le a)$ subject to the boundary conditions:

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= \alpha x \\ \chi_i'(\underset{\sim}{p}) &= \beta x_i' \end{aligned}\right\} \; ; \qquad \underset{\sim}{p} \in \partial B, \tag{4.2.1}$$

where α, β are two given constants and $\underset{\sim}{p} = (x,y) = (r\cos\theta,\ r\sin\theta)_{r=a}$.

If so (4.2.1) can be written

$$\left.\begin{aligned} \chi(p) &= \alpha a\cos\theta \\ \chi_i'(p) &= -\beta\cos\theta \end{aligned}\right\} ; \qquad \underset{\sim}{p} \in \partial B. \tag{4.2.2}$$

Now introducing

$$\chi = r^2\phi + \psi ; \qquad r \le a, \tag{4.2.3}$$

the simplest possibilities for ϕ, ψ which could meet (4.2.2) are

$$\phi = Ar\cos\theta, \quad \psi = Br\cos\theta ; \quad r \le a, \tag{4.2.4}$$

where A, B are constants to be determined. Accordingly

$$\chi = r^2\phi + \psi = (Ar^3+Br)\cos\theta ; \quad r \le a, \tag{4.2.5}$$

so that (4.2.1) gives

$$\chi(a,\theta) = (Aa^3+Ba)\cos\theta = \alpha a\cos\theta, \tag{4.2.6a}$$

$$\begin{aligned} \chi_i'(a,\theta) &= -\frac{d}{dr}[(Ar^3+Br)\cos\theta]_{r=a} \\ &= -(3Aa^2+B)\cos\theta = -\beta\cos\theta, \end{aligned} \tag{4.2.6b}$$

yielding the solutions

$$A = \frac{\beta-\alpha}{2a^2}, \quad B = \frac{3\alpha-\beta}{2}. \tag{4.2.7}$$

Accordingly

$$\phi = \frac{\beta-\alpha}{2a^2} r\cos\theta, \quad \psi = \frac{3\alpha-\beta}{2} r\cos\theta ; \quad r \le a, \tag{4.2.8}$$

and hence

$$\chi = r^2\phi + \psi = [\frac{r^3(\beta-\alpha)}{2a^2} + \frac{r(3\alpha-\beta)}{2}]\cos\theta; \; r \le a, \qquad (4.2.9)$$

which has the preferable form

$$\chi = [\frac{\beta-\alpha}{2a^2} r^2 + \frac{3\alpha-\beta}{2}]x; \qquad r \le a. \qquad (4.2.10)$$

It can be easily checked this χ satisfies the boundary conditions (4.2.1).

Putting $\alpha = \beta$ in (4.2.10), we see that

$$\chi = \alpha x \; ; \qquad r \le a, \qquad (4.2.11)$$

which is the expected solution for a harmonic function having the compatible boundary data

$$\chi(\underset{\sim}{p}) = \alpha x, \quad \chi_i'(\underset{\sim}{p}) = \alpha x_i' \; ; \quad \underset{\sim}{p} \in \partial B. \qquad (4.2.12)$$

It is interesting to write

$$\left.\begin{aligned} \phi(\underset{\sim}{p}) &= \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\sigma(\underset{\sim}{q})dq \\ \psi(\underset{\sim}{p}) &= \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\eta(\underset{\sim}{q})dq \end{aligned}\right\} ; \quad \underset{\sim}{p} \in B_i+B_e+\partial B, \qquad (4.2.13)$$

since we can obtain σ, η immediately from the formulae

$$\left.\begin{aligned} \phi_i' + \phi_e' &= 2\pi\sigma \\ \psi_i' + \psi_e' &= 2\pi\eta \end{aligned}\right\} \quad \text{on} \quad \partial B, \qquad (4.2.14)$$

see Jaswon, M.A. and Symm, G.T. (1977). Note that corresponding with (4.2.8),

$$\phi = \frac{\beta-\alpha}{2r}\cos\theta, \quad \psi = \frac{(3\alpha-\beta)a^2}{2r}\cos\theta; \quad r \geq a, \tag{4.2.15}$$

so that

$$2\pi\sigma = \phi_i' + \phi_e' = -\frac{d}{dr}\left(\frac{\beta-\alpha}{2a^2} r\cos\theta\right)_{r=a} + \frac{d}{dr}\left(\frac{\beta-\alpha}{2r}\cos\theta\right)_{r=a}$$

$$= \frac{\alpha-\beta}{a^2}\cos\theta, \tag{4.2.16a}$$

$$2\pi\eta = \psi_i' + \psi_e' = -\frac{d}{dr}\left(\frac{3\alpha-\beta}{2} r\cos\theta\right)_{r=a} + \frac{d}{dr}\left(\frac{3\alpha-\beta}{2r}a^2\cos\theta\right)_{r=a}$$

$$= (\beta-3\alpha)\cos\theta. \tag{4.2.16b}$$

Accordingly

$$\sigma = \frac{\alpha-\beta}{2\pi a^2}\cos\theta, \quad \eta = \frac{\beta-3\alpha}{2\pi}\cos\theta \quad \text{on } \partial B. \tag{4.2.17}$$

To fix ideas, note that, by using the Hilbert integral formula - see Jaswon, M.A. and Symm, G.T. (1977) - we obtain

$$\left.\begin{aligned} \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\sigma(\underset{\sim}{q})dq &= \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\frac{(\alpha-\beta)}{2\pi a^2}\cos\theta.dq = \frac{\beta-\alpha}{2a}\cos\tau \\ \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\eta(\underset{\sim}{q})dq &= \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\frac{(\beta-3\alpha)}{2\pi}\cos\theta dq = \frac{3\alpha-\beta}{2}a\cos\tau \end{aligned}\right\} ;\ r=a, \tag{4.2.18}$$

where $\underset{\sim}{p} = (a\cos\tau,\ a\sin\tau)$; $\underset{\sim}{q} = (a\cos\theta,\ a\sin\theta)$, in agreement with (4.2.15), for $r=a$. From these results it may be deduced that the integrals define ϕ, ψ everywhere.

4.3 Problem C-2

We now determine χ in B_e subject to the boundary conditions

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= \alpha x \\ \chi_e'(\underset{\sim}{p}) &= \beta x_e' \end{aligned}\right\} \; ; \qquad \underset{\sim}{p} \in \partial B, \tag{4.3.1}$$

where α, β are given constants and $\underset{\sim}{p} = (x,y) = (a\cos\theta, a\sin\theta)$. If so the conditions (4.3.1) can be written

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= \alpha a\cos\theta \\ \chi_e'(\underset{\sim}{p}) &= \beta\cos\theta \end{aligned}\right\} \; ; \qquad \underset{\sim}{p} \in \partial B. \tag{4.3.2}$$

The representation

$$\chi = r^2\phi + \psi \; ; \qquad r \geq a \tag{4.3.3}$$

is now subject to Hadamard's uniqueness requirement

$$\chi = O(r) \qquad \text{as} \qquad r \longrightarrow \infty,$$

which suggests

$$\phi = \frac{A\cos\theta}{r} , \qquad \psi = \frac{B\cos\theta}{r} \; ; \qquad r \geq a, \tag{4.3.4}$$

where A, B are constants to be determined. Now proceeding as before we obtain

$$A = \frac{\alpha+\beta}{2} , \qquad B = \frac{\alpha-\beta}{2}\, a^2. \tag{4.3.5}$$

Accordingly

$$\phi = \frac{(\alpha+\beta)\cos\theta}{2r}, \qquad \psi = \frac{(\alpha-\beta)(a^2\cos\theta)}{2r}; \qquad r \geq a, \qquad (4.3.6)$$

and hence

$$\chi = r^2\phi + \psi = \left[\frac{\alpha+\beta}{2}r + \frac{a^2(\alpha-\beta)}{2r}\right]\cos\theta$$

$$= \left[\frac{\alpha+\beta}{2} + \frac{a^2(\alpha-\beta)}{2r^2}\right]x; \qquad r \geq a. \qquad (4.3.7)$$

It can be seen that:

1. χ is a biharmonic function satisfying the boundary conditions(4.3.2).

2. χ is a well-behaved biharmonic function, i.e.

$$\chi = O(r) \qquad \text{as} \qquad r \longrightarrow \infty.$$

3. Putting $\alpha = \beta$ in (4.3.7), we get

$$\chi = \alpha r\cos\theta = \alpha x; \qquad r \geq a,$$

which is an expected result as already noted in (4.2.11) for the corresponding interior problem.

Finally, writing

$$\left.\begin{aligned} \phi(\underset{\sim}{p}) &= \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\,\sigma(\underset{\sim}{q})\,dq \\ \psi(\underset{\sim}{p}) &= \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\,\eta(\underset{\sim}{q})\,dq \end{aligned}\right\}; \qquad \underset{\sim}{p} \in B_i+B_e+\partial B, \qquad (4.3.8)$$

where corresponding with (4.3.6)

$$\phi = \frac{(\alpha+\beta)r\cos\theta}{2a^2}, \qquad \psi = \frac{(\alpha-\beta)r\cos\theta}{2}; \quad r \le a, \qquad (4.3.9)$$

and utilising (4.2.14), we obtain

$$\sigma = \frac{\alpha+\beta}{-2\pi a^2}\cos\theta, \qquad \eta = \frac{\beta-\alpha}{2\pi}\cos\theta \quad \text{on} \quad \partial B. \qquad (4.3.10)$$

Consequently by using the Hilbert integral formula we have

$$\left.\begin{aligned} \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\sigma(\underset{\sim}{q})dq &= \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\frac{(\alpha+\beta)}{-2\pi a^2}\cos\theta.dq = \frac{\alpha+\beta}{2a}\cos\theta \\ \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\eta(\underset{\sim}{q})dq &= \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\frac{(\beta-\alpha)}{2\pi}\cos\theta dq = \frac{(\beta-\alpha)}{2}a\cos\theta \end{aligned}\right\}; \ r=a,$$

(4.3.11)

in agreement with (4.3.6) for $r=a$.

4.4 Problem C-3

Let χ, χ' be given on the boundary of the circular ring as follows:

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= x \\ \chi'_e(\underset{\sim}{p}) &= 0 \end{aligned}\right\}; \ \underset{\sim}{p} \in \partial B_1, \qquad \left.\begin{aligned} \chi(\underset{\sim}{p}) &= 0 \\ \chi'_i(\underset{\sim}{p}) &= 0 \end{aligned}\right\}; \ \underset{\sim}{p} \in \partial B_0. \qquad (4.4.1)$$

Utilising the Almansi representation (2.4.1) we write

$$\chi = r^2\phi + \psi + \alpha x\log r + \beta y\log r; \qquad a \le r \le b, \qquad (4.4.2)$$

where α, β are constants to be determined. It follows from the boundary conditions (4.4.1) that this takes the form

$$\chi = r^2(k+k_1 r\cos\theta+k_0\frac{\cos\theta}{r}+d\log r) + (K+K_1 r\cos\theta+K_0\frac{\cos\theta}{r}+D\log r) + \alpha r\cos\theta \log r, \qquad (4.4.3)$$

in fact the contribution $\beta y\log r$ is clearly not relevant here. We see that (4.4.3) involves 9 constants to be determined. However only 8 linear equations arise from meeting the boundary conditions (4.4.1), as will be developed below. The difficulty may be resolved by noting that the term

$$r^2(k_0\frac{\cos\theta}{r}), \text{ i.e. } k_0 r\cos\theta, \qquad (4.4.4)$$

is clearly covered by the term $K_1 r\cos\theta$. This means that we may omit the term $k_0\frac{\cos\theta}{r}$ without loss of generality, so leaving only 8 unknown coefficients to be determined from the 8 linear equations. These subdivide into two distinct sets of equations as follows:

$$\begin{bmatrix} a^2 & a^2\log a & \log a & 1 \\ b^2 & b^2\log b & \log b & 1 \\ 2a & a(1+2\log a) & \frac{1}{a} & 0 \\ 2b & b(1+2\log b) & \frac{1}{b} & 0 \end{bmatrix} \begin{bmatrix} k \\ d \\ D \\ K \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (4.4.5)$$

and

$$\begin{bmatrix} a^3 & a\log a & a & \frac{1}{a} \\ b^3 & b\log b & b & \frac{1}{b} \\ 3a^2 & 1+\log a & 1 & \frac{-1}{a^2} \\ 3b^2 & 1+\log b & 1 & \frac{-1}{b^2} \end{bmatrix} \begin{bmatrix} k_1 \\ \alpha \\ K_1 \\ K_0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (4.4.6)$$

The system of equations (4.4.5) is homogeneous; also noting that, since $0 < a < b$,

$$\det \begin{bmatrix} a^2 & a^2\log a & \log a & 1 \\ b^2 & b^2\log b & \log b & 1 \\ 2a & a(1+2\log a) & \frac{1}{a} & 0 \\ 2b & b(1+2\log b) & \frac{1}{b} & 0 \end{bmatrix}$$

$$= \frac{-1}{ab}[(a^2-b^2)^2 - 4a^2b^2(\log a-\log b)^2] \neq 0, \qquad (4.4.7)$$

it only has the trivial solution

$$k = d = D = K = 0. \qquad (4.4.8)$$

Also the system of (4.4.6) has always a unique solution since

$$\det \begin{bmatrix} a^3 & a\log a & a & \frac{1}{a} \\ b^3 & b\log b & b & \frac{1}{b} \\ 3a^2 & 1+\log a & 1 & \frac{-1}{a^2} \\ 3b^2 & 1+\log b & 1 & \frac{-1}{b^2} \end{bmatrix}$$

$$= 4a^2b^2(a^2-b^2)[(a^2-b^2) + (a^2+b^2)(\log b-\log a)] \neq 0, \qquad (4.4.9)$$

An interesting conclusion may be obtained from (4.4.6) by letting $b \to \infty$, keeping a fixed. We have

$$\left.\begin{aligned} k_1 &= O\left(\frac{1}{b^2\log b}\right) \to 0, & \alpha &= O\left(\frac{1}{\log b}\right) \to 0, \\ K_1 &\to \frac{1}{2}, & K_0 &\to \frac{a^2}{2} \end{aligned}\right\} \text{ as } b \to \infty. \qquad (4.4.10)$$

Accordingly from (4.4.3), (4.4.10),

$$\chi \longrightarrow \frac{1}{2}\left(1 + \frac{a^2}{r^2}\right)x \qquad \text{as} \quad b \longrightarrow \infty, \tag{4.4.11}$$

is agreement with (4.3.7), when $\alpha = 1$, $\beta = 0$.

Similarly by letting $a \longrightarrow 0$, keeping b fixed in (4.4.6), we get

$$k_1,\ \alpha,\ K_1,\ K_0 \longrightarrow 0 \qquad \text{as} \quad a \longrightarrow 0. \tag{4.4.12}$$

Accordingly from (4.4.3), (4.4.10),

$$\chi \longrightarrow 0 \qquad \text{as} \quad a \longrightarrow 0, \tag{4.4.13}$$

in agreement with (4.2.10), when $\alpha = \beta = 0$.

PART II

CHAKRABARTY REPRESENTATION

Chapter 5

Chakrabarty Representation Theory

5.1 Introduction

Despite its mathematical simplicity and ease of numerical implementation, the Almansi approach has not proved popular with engineers. Apart from competition with conventional BEM, it must be said that the sources concerned do not have clear physical significance. Indeed they only serve to generate the potentials ϕ, ψ which are themselves subsidiary to χ. We owe to Massonnet, C.H. (1948) the idea of sources on ∂B which generate directly the quantities of engineering interest in B. However the idea could hardly be carried much further at that time. A few years later (1956) there came a second paper in which he formulated the traction problem as a vector integral equation of the second kind for the Neumann problem of vector potential theory.

A two-dimensional approach utilising biharmonic potentials has been put forward by Chakrabarty, S.K. (1971). This seems closer than conventional BEM to the spirit of Massonnet's original paper.

5.2 Biharmonic potentials

Note that the function $r^2\log r$ is a singular biharmonic function. More precisely

$$\left.\begin{aligned} \nabla^2(r^2\log r) &= 4 + 4\log r \\ \nabla^4(r^2\log r) &= 4\nabla^2\log r = 8\pi\delta(r) \end{aligned}\right\} , \qquad (5.2.1)$$

where r denotes the radial distance from the origin and δ is Dirac's delta function defined by

$$\left.\begin{aligned} &\delta(\underset{\sim}{p}-\underset{\sim}{q}) = 0; && \underset{\sim}{p} \neq \underset{\sim}{q} \\ &\delta(\underset{\sim}{p}-\underset{\sim}{q}) = \infty; && \underset{\sim}{p} = \underset{\sim}{q} \quad ; \\ &\int_B \delta(\underset{\sim}{p}-\underset{\sim}{q})\,dp = 1 && \end{aligned}\right\} \underset{\sim}{p},\ \underset{\sim}{q} \in B, \tag{5.2.2}$$

where dp stands for the element of area at $\underset{\sim}{p}$. If so

$$\nabla^2(-r^2+r^2\log r) = 4\log r, \quad \nabla^4(-r^2+r^2\log r) = 8\pi\delta(r). \tag{5.2.3}$$

These properties suggest that a suitable biharmonic fundamental solution would be

$$G(\underset{\sim}{p},\underset{\sim}{q}) = -|\underset{\sim}{p}-\underset{\sim}{q}|^2 + |\underset{\sim}{p}-\underset{\sim}{q}|^2\log|\underset{\sim}{p}-\underset{\sim}{q}|, \tag{5.2.4}$$

which was first suggested by Chakrabarty, S.K. (1971). This allows us to construct the simple-layer biharmonic potential

$$\Omega(\underset{\sim}{p}) = \int_{\partial B} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})\,dq; \quad \underset{\sim}{p} \in B + \partial B,\ \underset{\sim}{q} \in \partial B, \tag{5.2.5}$$

where ζ is a source-density distribution to be determined. An interesting generalisation of (5.2.4) will be discussed in section 6.3.

An arbitrary χ in B may always be represented in the form

$$\chi = \Omega + \psi; \qquad \nabla^2\psi = 0 \qquad \text{in} \quad B. \tag{5.2.6}$$

To prove this we note that

$$\nabla^2\chi(\underset{\sim}{p}) = \nabla^2\Omega(\underset{\sim}{p}) = 4\int_{\partial B} \nabla^2 G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq$$

$$= 4\int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\zeta(\underset{\sim}{q})dq; \quad \left.\begin{matrix}\underset{\sim}{p} \in B + \partial B \\ \underset{\sim}{q} \in \partial B\end{matrix}\right\} . \tag{5.2.7}$$

Now $\nabla^2\chi$ is a known harmonic function in B, which may always be written in the form (2.2.3), i.e. ζ may be determined on ∂B in terms of $\nabla^2\chi$ on ∂B. If so

$$\nabla^2\{\chi(\underset{\sim}{p}) - 4\int_{\partial B} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq\} = 0; \quad \underset{\sim}{p} \in B + \partial B, \ \underset{\sim}{q} \in \partial B,$$

which implies the representation (5.2.6). This also holds on ∂B - see section 5.3 - so providing the boundary relation

$$\chi = \Omega + \psi \qquad \text{on} \quad \partial B. \tag{5.2.8}$$

5.3 Continuity of Ω in $B + \partial B$

The biharmonic potential

$$\Omega(\underset{\sim}{p}) = \int_{\partial B} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq;$$

$$G(\underset{\sim}{p},\underset{\sim}{q}) = -R^2 + R^2\log R; \qquad R = |\underset{\sim}{p}-\underset{\sim}{q}|,$$

is continuous at the boundary. Thus fixing a field point $\underset{\sim}{p}_0 \in \partial B$, we see that no problem arises from the contribution of R^2, i.e.

$$\int_{\partial B} |\underset{\sim}{p}-\underset{\sim}{q}|^2 \zeta(\underset{\sim}{q})\,dq \longrightarrow \int_{\partial B} |\underset{\sim}{p}_0-\underset{\sim}{q}|^2 \zeta(\underset{\sim}{q})\,dq \quad \text{as} \quad \underset{\sim}{p} \longrightarrow \underset{\sim}{p}_0 \in \partial B, \tag{5.3.1}$$

where $\underset{\sim}{p} \in \partial B$ or $\underset{\sim}{p} \in B$.

However it remains to prove the continuity of

$$U(\underset{\sim}{p}) = \int_{\partial B} \zeta(\underset{\sim}{q}) R^2 \log R\,dq \qquad \text{as} \quad \underset{\sim}{p} \longrightarrow \underset{\sim}{p}_0 \in \partial B. \tag{5.3.2}$$

To do this, first we show that $U(\underset{\sim}{p}_0)$ exists. Thus let us set up a local cartesian co-ordinate system with origin at $\underset{\sim}{p}_0$, where the x - and y - axes are the tangential and normal directions through $\underset{\sim}{p}_0$ respectively (Fig.5.3.1).

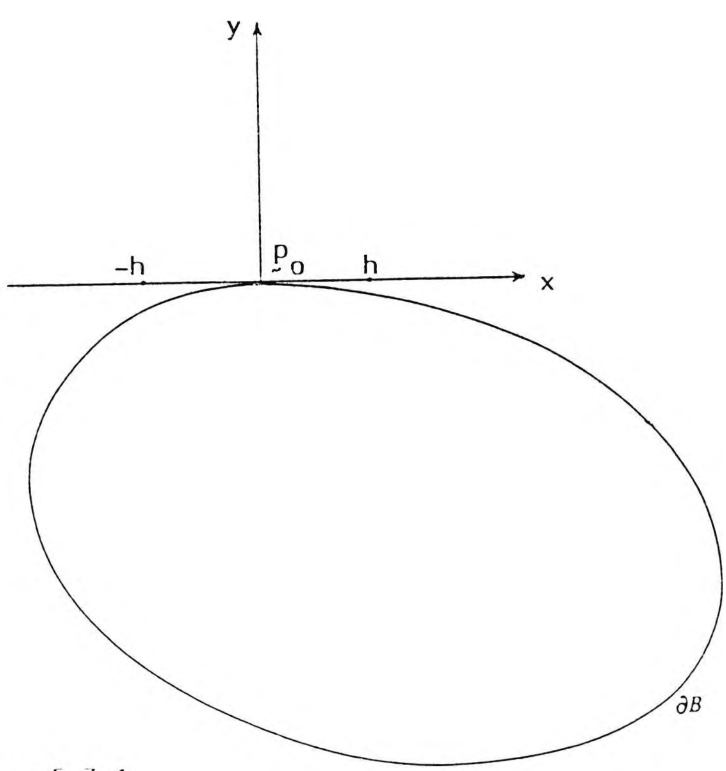


Fig. 5.3.1.

We only need to consider the contribution of the straight line interval $[-h,h]$, approximating a small arc of the curve on the local x-axis, which is

$$I(h) = \int_{-h}^{h} x^2 \log|x|dx = 2\int_0^h x^2 \log x dx = \frac{2h^3 \log h}{3} - \frac{2h^3}{9} . \tag{5.3.3}$$

Here we have assumed that $\zeta(\underset{\sim}{q}) \simeq \zeta(\underset{\sim}{q}_0)$ in the neighbourhood of $\underset{\sim}{p}_0$, so that ζ does not enter into the integral (5.3.3).

Now to prove the continuity of (5.3.2), it must be shown that, for any preassigned $\epsilon > 0$, the inequality

$$|U(\underset{\sim}{p}) - U(\underset{\sim}{p}_0)| < \epsilon; \qquad \underset{\sim}{p} \in B + \partial B, \tag{5.3.4}$$

holds for the distance $\underset{\sim}{p}\underset{\sim}{p}_0$ sufficiently small. Let

$$\partial B_0 = \{\underset{\sim}{q} \in \partial B; \quad |\underset{\sim}{q}-\underset{\sim}{p}_0| < \delta\}, \tag{5.3.5}$$

in which δ is chosen so that

$$N\delta^3 \log\delta^{-1} < \frac{\epsilon}{6} , \qquad \delta < e^{-\frac{1}{2}}, \tag{5.3.6}$$

where N satisfies the inqua(e)lity

$$|\zeta(\underset{\sim}{q})| \leq N; \qquad \underset{\sim}{q} \in \partial B. \tag{5.3.7}$$

Now writing

$$U = U_0 + U_1 \tag{5.3.8}$$

where

$$\left.\begin{aligned} U_0(\underset{\sim}{p}) &= \int_{\partial B_0} \zeta(\underset{\sim}{q}) R^2 \log R dq \\ \\ U_1(\underset{\sim}{p}) &= \int_{\partial B - \partial B_0} \zeta(\underset{\sim}{q}) R^2 \log R dq \end{aligned}\right\} \quad ; \quad \underset{\sim}{p} \in \partial B_0, \tag{5.3.9}$$

we have

$$|U_0(\underset{\sim}{p})| = \left| \int_{\partial B_0} \zeta(\underset{\sim}{q}) R^2 \log R dq \right| \le \int_{\partial B_0} |\zeta(q) R^2 \log R dq|$$

$$\le 2N\delta^3 \log \delta^{-1} < \frac{\epsilon}{3} , \tag{5.3.10}$$

by noting (5.3.7). Also

$$|U_0(\underset{\sim}{p}_0)| < \frac{\epsilon}{3} , \tag{5.3.11}$$

so that, no matter where $\underset{\sim}{p}$ is located,

$$|U_0(\underset{\sim}{p}) - U_0(\underset{\sim}{p}_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} . \tag{5.3.12}$$

But as $\underset{\sim}{p}_0 \notin \partial B - \partial B_0$, therefore the continuity of U_1 is obvious, hence

$$|U_1(\underset{\sim}{p}) - U_1(\underset{\sim}{p}_0)| < \frac{\epsilon}{3} , \tag{5.3.13}$$

for $\underset{\sim}{p}\underset{\sim}{p}_0$ sufficiently small. Now (5.3.4) follows from (5.3.12), (5.3.13).

5.4 Investigation of the normal derivative

By contrast with the simple-layer logarithmic potentials, the normal derivative of Ω has no jump at δB. In fact if $\underset{\sim}{p} \in \partial B$, then we have

$$\left.\begin{aligned} \Omega_i'(\underset{\sim}{p}) &= \int_{\partial B} G_i'(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq \\ \Omega_e'(\underset{\sim}{p}) &= \int_{\partial B} G_e'(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq \end{aligned}\right\} \; ; \quad \underset{\sim}{p},\ \underset{\sim}{q} \in \partial B, \qquad (5.4.1)$$

where G_i' signifies the inward normal derivative of G at $\underset{\sim}{p}$ keeping $\underset{\sim}{q}$ fixed, etc. Note that the property (5.4.1) emerges essentially from the presence of the term R^2 in $G(\underset{\sim}{p},\underset{\sim}{q})$.

Although (5.4.1) may be verified by a direct but straightforward method analogous to logarithmic potentials, we present a simple proof as follows. As before let us set up a local co-ordinate system with origin at some point 0 on ∂B, and the x - and y-axes through 0 are respectively tangential and normal directions, as illustrated in Fig.5.4.1.

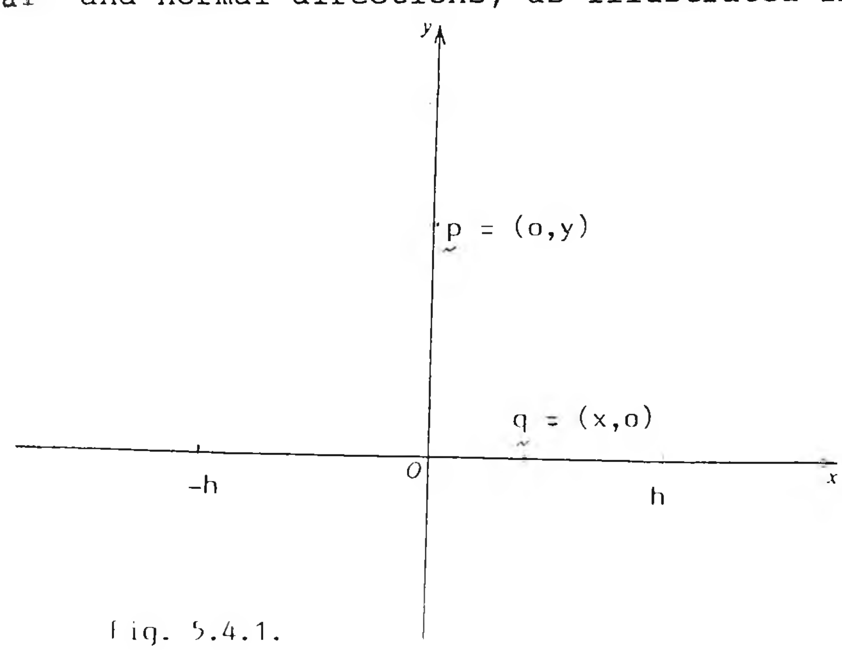


Fig. 5.4.1.

We also take $\underset{\sim}{p} = (o,y)$ and $\underset{\sim}{q} = (x,0)$ and $I = [-h,h]$ is a small interval on the x-axis, where $-h < x < h$. If so, without loss of generality we may take $\zeta(\underset{\sim}{q}) = 1$ in I, and thus the contribution of Ω to I may be written

$$\Omega_h = \int_{-h}^{h} f(x,y)dx; \qquad R = \sqrt{x^2+y^2} , \tag{5.4.2}$$

where $f(x,y) = -R^2+R^2\log R$. It follows that

$$\Omega_h = \frac{2}{3} h^3\log\sqrt{h^2+ y^2} + \frac{4}{3} y^3\tan^{-1}\frac{h}{y} - \frac{10}{3} y^2h$$

$$+ 2y^2h\log\sqrt{h^2+ y^2} - \frac{8}{9} h^3, \tag{5.4.3}$$

and that

$$\frac{\partial\Omega_h}{\partial y} = \frac{\partial}{\partial y}\int_{-h}^{h} f(x,y)dx = -6yh + 4yh\log\sqrt{h^2+y^2}$$

$$+ 4y^2\tan^{-1} \frac{h}{y} . \tag{5.4.4}$$

Also note that

$$\frac{\partial}{\partial y}f(x,y) = -y + 2y\log\sqrt{x^2 + y^2} , \tag{5.4.5}$$

and accordingly

$$\int_{-h}^{h} \frac{\partial}{\partial y}f(x,y)dx = -6yh + 4yh\log\sqrt{h^2 + y^2} + 4y^2\tan^{-1} \frac{h}{y} . \tag{5.4.6}$$

We now examine (5.4.4), (5.4.6) as $y \to 0$. It is easily seen from (5.4.4) that

$$\frac{\partial}{\partial y} \int_{-h}^{h} f(x,y)\,dx \to 0 \qquad \text{as} \quad y \to 0. \tag{5.4.7}$$

Also it is seen from (5.4.5) that

$$\frac{\partial f(x,y)}{\partial y} \to 0 \qquad \text{as} \quad y \to 0, \tag{5.4.8}$$

so that

$$\int_{-h}^{h} \frac{\partial f(x,y)}{\partial y}\,dx \to 0 \qquad \text{as} \quad y \to 0. \tag{5.4.9}$$

Consequently from (5.4.4), (5.4.6), (5.4.7) and (5.4.9)

$$\Omega_e'(\underset{\sim}{p}) = \int_{\partial B} G_e'(\underset{\sim}{p},\underset{\sim}{q})\,\zeta(\underset{\sim}{q})\,dq.$$

Similarly we conclude that

$$\Omega_i'(\underset{\sim}{p}) = \int_{\partial B} G_i'(\underset{\sim}{p},\underset{\sim}{q})\,\zeta(\underset{\sim}{q})\,dq.$$

Chapter 6

Expansions Of The Biharmonic Potential Ω

6.1 Introduction

So far we have stated some fundamental properties of the biharmonic potential Ω. In this chapter we carry out more analysis for Ω by obtaining its exterior expansions. Because of Hadamard's existence - uniqueness theorem, we have to impose some side conditions for infinite exterior domains. Also we consider ring-shaped domains. We shall then be able to compare the Almansi representation with that of Chakrabarty, and explain the advantages and disadvantages of each. This will allow us to choose the more suitable representation for any specific problem.

6.2 Expansions of Ω within infinite exterior and ring-shaped domains

In order to determine the expansion of Ω within B_e exterior to ∂B, we write

$$\Omega(\underset{\sim}{p}) = \int_{\partial B} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq; \quad \underset{\sim}{p} \in B_e + \partial B, \ \underset{\sim}{q} \in \partial B, \qquad (6.2.1)$$

where $\underset{\sim}{p} = (x,y)$ in B_e, $\underset{\sim}{q} = (q_1,q_2)$ on ∂B, and let τ, θ be the angles that $\underset{\sim}{p}$, $\underset{\sim}{q}$ make with the positive x-axis respectively as illustrated in Fig. 6.2.1. Now by noting the expansion - see Durell, C.V. and Robson, A. (1936) -

$$\tfrac{1}{2}\log(1-2\rho\cos\theta+\rho^2) = \sum_{n=1}^{\infty} \frac{\rho^n \cos n\theta}{-n} ; \quad 0 \le \rho < 1, \qquad (6.2.2)$$

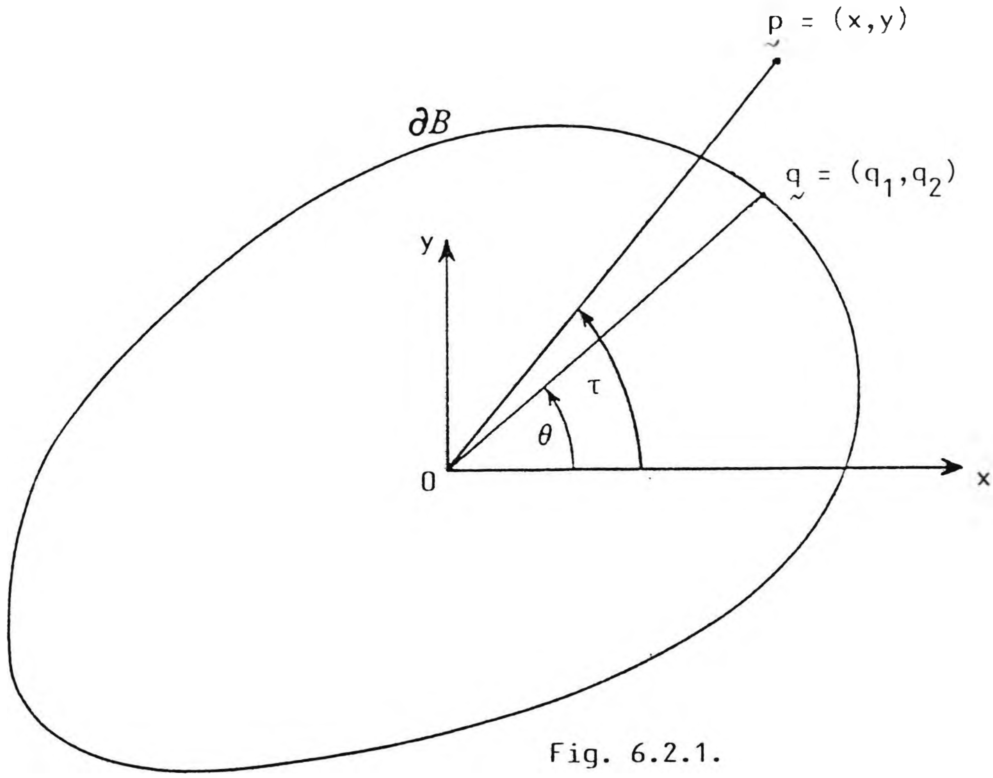


Fig. 6.2.1.

we have

$$\nabla^2\Omega(\underset{\sim}{p}) = 4\int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\zeta(\underset{\sim}{q})dq; \qquad \underset{\sim}{p} \in B_e,\ \underset{\sim}{q} \in \partial B,$$

$$= 4\int_{\partial B} \tfrac{1}{2}\log(|\underset{\sim}{p}|^2+|\underset{\sim}{q}|^2-2\underset{\sim}{p}.\underset{\sim}{q})\zeta(\underset{\sim}{q})dq$$

$$= 4\int_{\partial B} \tfrac{1}{2}\log[|\underset{\sim}{p}|^2(1+|\tfrac{\underset{\sim}{q}}{\underset{\sim}{p}}|^2-2|\tfrac{\underset{\sim}{q}}{\underset{\sim}{p}}|\cos(\theta-\tau))]\zeta(\underset{\sim}{q})dq$$

$$= 4\int_{\partial B} [\log|\underset{\sim}{p}| + \sum_{n=1}^{\infty} \frac{|q|^n|p|^{-n}\cos n(\theta-\tau)}{-n}]\zeta(\underset{\sim}{q})dq$$

$$\text{as} \quad |\underset{\sim}{p}| \longrightarrow \infty. \qquad (6.2.3)$$

Accordingly, using the property

$$\int_{\partial B} \sum_{n=1}^{\infty} \cdots\cdots = \sum_{n=1}^{\infty} \int_{\partial B} \cdots\cdots , \qquad (6.2.4)$$

which can be justified by the theory of uniform convergence (Wylie, C.R. and Barrett, L.C. 1983), we obtain

$$\nabla^2\Omega(\underset{\sim}{p}) = 4\log r\int_{\partial B} \zeta(\underset{\sim}{q})dq - 4r^{-1}\cos\tau\int_{\partial B} q_1\zeta(\underset{\sim}{q})dq$$

$$- 4r^{-1}\sin\tau\int_{\partial B} q_2\zeta(\underset{\sim}{q})dq + 4\sum_{n=-2}^{-\infty} \frac{r^n(b_{nc}\cos n\tau + b_{ns}\sin n\tau)}{n}$$

$$\text{as } |\underset{\sim}{p}| = r \longrightarrow \infty, \qquad (6.2.5)$$

in which $q_1 = |\underset{\sim}{q}|\cos\theta$, $q_2 = |\underset{\sim}{q}|\sin\theta$.

Now using

$$\nabla^2(-r^2+r^2\log r) = 4\log r,\ \nabla^2(x\log r) = 2r^{-1}\cos\tau,$$

$$\nabla^2(y\log r) = 2r^{-1}\sin\tau, \qquad (6.2.6)$$

we immediately see that (6.2.5) implies

$$\Omega(\underset{\sim}{p}) = (-r^2+r^2\log r)\int_{\partial B} \zeta(\underset{\sim}{q})dq - 2x\log r\int_{\partial B} q_1\zeta(\underset{\sim}{q})dq$$

$$- 2y\log r\int_{\partial B} q_2\zeta(\underset{\sim}{q})dq + r^2\sum_{n=-2}^{-\infty} \frac{r^n}{n(n+1)}(b_{nc}\cos n\tau + b_{ns}\sin n\tau)$$

$$+h \qquad \text{as } |\underset{\sim}{p}| = r \longrightarrow \infty, \qquad (6.2.7)$$

in which h is a harmonic function whose form will be determined next.

Also (6.2.7) may be obtained directly by utilising

$$G(\underset{\sim}{p},\underset{\sim}{q}) = -|\underset{\sim}{p}-\underset{\sim}{q}|^2 + |\underset{\sim}{p}-\underset{\sim}{q}|^2\log|\underset{\sim}{p}-\underset{\sim}{q}| = -[|\underset{\sim}{p}|^2 - 2\underset{\sim}{p}.\underset{\sim}{q} + |\underset{\sim}{q}|^2]$$

$$+ [|\underset{\sim}{p}|^2 - 2\underset{\sim}{p}.\underset{\sim}{q} + |\underset{\sim}{q}|^2][\log|\underset{\sim}{p}| + \sum_{n=1}^{\infty} \frac{|q|^n|p|^{-n}\cos n(\theta-\tau)}{-n}]$$

$$\text{as} \quad |\underset{\sim}{p}| = r \longrightarrow \infty, \qquad (6.2.8)$$

this follows from (6.2.2), which yields

$$G(\underset{\sim}{p},\underset{\sim}{q}) = -|\underset{\sim}{p}|^2 + \underset{\sim}{p}.\underset{\sim}{q} + |\underset{\sim}{p}|^2\log|\underset{\sim}{p}| + |\underset{\sim}{q}|^2\log|\underset{\sim}{p}|$$

$$- 2\underset{\sim}{p}.\underset{\sim}{q}\log|\underset{\sim}{p}| + |\underset{\sim}{p}|^2 \sum_{n=-2}^{-\infty} \frac{|p|^n|q|^{-n}\cos n(\theta-\tau)}{n(n+1)}$$

$$+ |\underset{\sim}{q}|^2 \sum_{n=1}^{\infty} \frac{|p|^{-n}|q|^n\cos n(\theta-\tau)}{-n(n+1)},$$

$$\text{as} \quad |\underset{\sim}{p}| = r \longrightarrow \infty. \qquad (6.2.9)$$

Accordingly

$$\Omega(\underset{\sim}{p}) = \int_{\partial B} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq = (-r^2+r^2\log r)\int_{\partial B}\zeta(\underset{\sim}{q})dq$$

$$- 2x\log r\int_{\partial B} q_1\zeta(\underset{\sim}{q})dq - 2y\log r\int_{\partial B} q_2\zeta(\underset{\sim}{q})dq$$

$$+ r^2\int_{\partial B}\left[\sum_{n=-2}^{-\infty}\frac{|p|^n|q|^{-n}\cos n(\theta-\tau)}{n(n+1)}\right]\zeta(\underset{\sim}{q})dq + h$$

$$\text{as} \quad r \longrightarrow \infty, \tag{6.2.10}$$

i.e.

$$\Omega(r) = (-r^2+r^2\log r)\int_{\partial B}\zeta(\underset{\sim}{q})dq - 2x\log r\int_{\partial B} q_1\zeta(\underset{\sim}{q})dq$$

$$- 2y\log r\int_{\partial B} q_2\zeta(\underset{\sim}{q})dq$$

$$+ r^2\sum_{n=-2}^{-\infty}\frac{r^n}{n(n+1)}(b_{nc}\cos n\tau + b_{ns}\sin n\tau) + h$$

$$\text{as} \quad r \longrightarrow \infty, \tag{6.2.11}$$

in which h is the harmonic function

$$h = \int_{\partial B}(\underset{\sim}{p}.\underset{\sim}{q} + |\underset{\sim}{q}|^2\log|\underset{\sim}{p}|)\zeta(\underset{\sim}{q})dq$$

$$+ \int_{\partial B}\left[|\underset{\sim}{q}|^2\sum_{n=1}^{\infty}\frac{|q|^n|p|^{-n}\cos n(\theta-\tau)}{-n(n+1)}\right]\zeta(\underset{\sim}{q})dq, \tag{6.2.12}$$

i.e.

$$h = \alpha x + \beta y + \delta\log r + O(r^{-1}) \quad \text{as } r \longrightarrow \infty. \tag{6.2.13}$$

It may be shown that the constants α, β, δ are as follows:

$$\alpha = \int_{\partial B} q_1 \zeta(\underset{\sim}{q})\, dq, \quad \beta = \int_{\partial B} q_2 \zeta(\underset{\sim}{q})\, dq, \quad \delta = \int_{\partial B} |\underset{\sim}{q}|^2 \zeta(\underset{\sim}{q})\, dq, \tag{6.2.14}$$

i.e. α, β are essentially the coefficients of $-2x\log r$, $-2y\log r$ in (6.2.10). Note that the results (6.2.14) sharpen the expression (6.2.13) given in Jaswon, M.A. and Symm, G.T (1977).

Within a ring-shaped domain B bounded externally by ∂B_0 and internally by ∂B_1, which encloses $r = 0$, - see Fig. 6.2.2 -

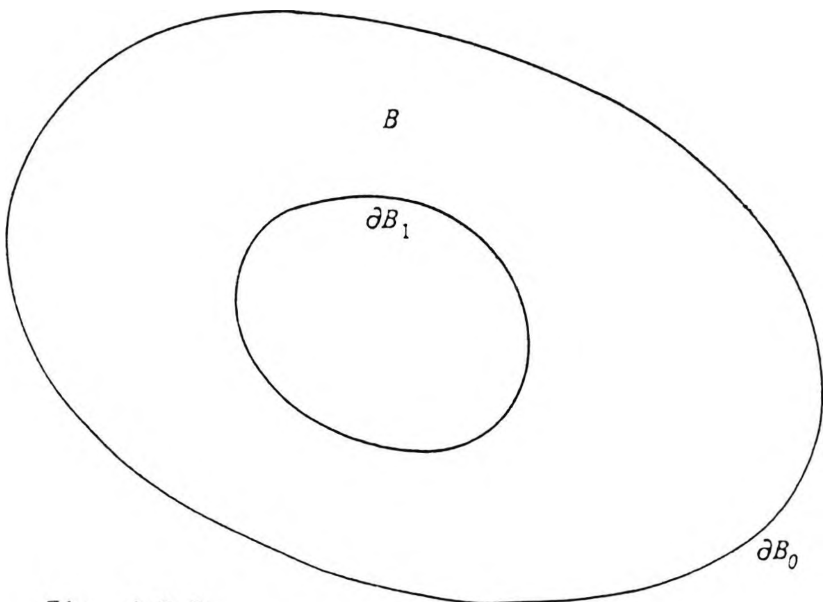


Fig. 6.2.2.

we write

$$\Omega(\underset{\sim}{p}) = \int_{\partial B_0} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq + \int_{\partial B_1} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq;$$

$$\underset{\sim}{p} \in B + \partial B_0 + \partial B_1, \qquad (6.2.15)$$

where the contribution of $\int_{\partial B_1} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq$ has the form (6.2.10), and the contribution of $\int_{\partial B_0} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq$ for a circle will be discussed in section 6.5.

6.3 Biharmonic Green functions

Let

$$G(\underset{\sim}{p},\underset{\sim}{q}) = AR^2 + BR^2\log R; \quad \underset{\sim}{p} \in B_e + \partial B, \ \underset{\sim}{q} \in \partial B, \qquad (6.3.1)$$

where $R = |\underset{\sim}{p}-\underset{\sim}{q}|$; $\underset{\sim}{q} = (|\underset{\sim}{q}| \cos\theta, |\underset{\sim}{q}| \sin\theta)$,
$\underset{\sim}{p} = (|\underset{\sim}{p}| \cos\tau, |\underset{\sim}{p}| \sin\tau)_{r\geq a}$, see Fig. 6.2.1.

For any choice of A and B, G is a biharmonic Green's function, i.e. it is a fundamental solution of the biharmonic equation

$$\nabla^4\chi = 0. \qquad (6.3.2)$$

In (5.2.4) we have chosen A = -1, B = 1 following Chakrabarty's suggestion. An alternative choice is A = -1, B = 2 which gives

$$G(\underset{\sim}{p},\underset{\sim}{q}) = -R^2 + 2R^2\log R, \qquad (6.3.3)$$

and satisfies the relations

$$\left.\begin{aligned} \nabla^2 G(\underset{\sim}{p},\underset{\sim}{q}) &= 4 + 8\log R \\ \nabla^4 G(\underset{\sim}{p},\underset{\sim}{q}) &= 16\pi\delta(R) \end{aligned}\right\} . \tag{6.3.4}$$

Now introducing the biharmonic potential

$$\Omega(\underset{\sim}{p}) = \int_{\partial B} [-R^2 + 2R^2\log R]\zeta(\underset{\sim}{q})dq, \tag{6.3.5}$$

we obtain the following exterior expansion for Ω:

$$\begin{aligned} \Omega(\underset{\sim}{p}) = {} & (-r^2+2r^2\log r)\int_{\partial B} \zeta(\underset{\sim}{q})dq - 4x\log r\int_{\partial B} q_1\zeta(\underset{\sim}{q})dq \\ & - 4y\log r\int_{\partial B} q_2\zeta(\underset{\sim}{q})dq \\ & + 2r^2 \sum_{n=-2}^{-\infty} \frac{r^n}{n(n+1)}(b_{nc}\cos n\tau + b_{ns}\sin n\tau) + h, \\ & \text{as} \quad |\underset{\sim}{p}| = r \longrightarrow \infty, \end{aligned} \tag{6.3.6}$$

in which h is a harmonic function of the form

$$h = \delta\log r + o(r^{-1}) + \gamma. \tag{6.3.7}$$

where δ is given by (6.2.14) and γ is a constant.

Clearly (6.3.6), (6.3.7) differ slightly from (6.2.10), (6.2.13) respectively. In particular (6.3.7) does not cover the term $\alpha x + \beta y$, however it includes a constant which may be convenient in some specialised problems.

6.4 Comparison between the Almansi and Chakrabarty representations

We have already introduced some useful expansions of the biharmonic potential Ω; however, regarding similar expansions of the Almansi representation, it will be of importance to have a comparison between these two kinds of representation and utilise either of them as appropriate. For an infinite exterior domain the biharmonic functions $x\log r$, $y\log r$, $r^2\log r$ must be excluded from Ω by imposing suitable side conditions. In this case the Chakrabarty representation must be extended to

$$\chi = \Omega + \psi + ax + by + c; \quad (x,y) \in B_e + \partial B, \qquad (6.4.1)$$

where a, b, c are unknown constants balanced by the three equations

$$\int_{\partial B} q_1\zeta(\underset{\sim}{q})dq = 0, \quad \int_{\partial B} q_2\zeta(\underset{\sim}{q})dq = 0, \quad \int_{\partial B} \zeta(\underset{\sim}{q})dq = 0, \qquad (6.4.2)$$

bearing in mind that these conditions ensure the absence respectively of $x\log r$, $y\log r$ and $r^2\log r$ in Ω, as required by Hadamard's uniqueness - existence theorem. However, it follows from (6.2.13), (6.2.14) that the linear terms $\alpha x+\beta y$ in h are also eliminated by conditions (6.4.2), so requiring the explicit introduction of $ax+by$ in (6.4.1). Clearly the constant c compensates for the absence of $r^2\log r$ in Ω.

As we see from the above analysis and by comparison with the Almansi representation (2.3.2), for infinite exterior domains the Chakrabarty representation is inferior to that of Almansi by reference to Hadamard's requirement.

However for ring-shaped domains the Chakrabarty approach is superior to that of Almansi, because the singular biharmonic functions xlogr, ylogr, r^2logr could exist within such domains but are not covered by the Almansi representation, unless ϕ, ψ become multi-valued harmonic functions.

6.5 Expansion of Ω in C-1

So far we have determined the exterior expansion of Ω, i.e. the asymptotic expansion (6.2.11) when ∂B happens to be a circle of radius a. It is also interesting to determine its interior expansion in C-1, which is needed in Chapter 8. We write

$$\Omega(\underset{\sim}{p}) = \int_{\partial B} [-R^2+R^2\log R]\zeta(\underset{\sim}{q})dq$$

$$R = |\underset{\sim}{p}-\underset{\sim}{q}|;\ \underset{\sim}{p} = (x,y) = (r\cos\tau,\ r\sin\tau),$$

$$\underset{\sim}{q} = (q_1,q_2) = (a\cos\theta,\ a\sin\theta);$$

$$r \le a, \tag{6.5.1}$$

where $\zeta(\underset{\sim}{q}) = \zeta(\theta)$. Now it can be shown - see (6.2.9) - that

$$G(\underset{\sim}{p},\underset{\sim}{q}) = -R^2+R^2\log R = -|\underset{\sim}{q}|^2 + \underset{\sim}{p}.\underset{\sim}{q} + |\underset{\sim}{p}|^2\log|\underset{\sim}{q}|$$

$$+ |\underset{\sim}{q}|^2\log|\underset{\sim}{q}| - 2\underset{\sim}{p}.\underset{\sim}{q}\log|\underset{\sim}{q}|$$

$$+ |\underset{\sim}{q}|^2 \sum_{n=2}^{\infty} \frac{|\underset{\sim}{p}|^n|\underset{\sim}{q}|^{-n}\cos n(\theta-\tau)}{n(n-1)}$$

$$+ |\underset{\sim}{p}|^2 \sum_{n=1}^{\infty} \frac{|\underset{\sim}{p}|^n|\underset{\sim}{q}|^{-n}\cos n(\theta-\tau)}{-n(n+1)}\ ;\ |\underset{\sim}{p}| < |\underset{\sim}{q}|. \tag{6.5.2}$$

Accordingly

$$\Omega(\underset{\sim}{p}) = (-a^2+r^2\log a)\int_{\partial B} \zeta(\underset{\sim}{q})dq - 2x\log a\int_{\partial B} q_1\zeta(\underset{\sim}{q})dq$$

$$- 2y\log a\int_{\partial B} q_2\zeta(\underset{\sim}{q})dq$$

$$+ a^2\int_{\partial B} [\sum_{n=2}^{\infty} \frac{r^n a^{-n}\cos n(\theta-\tau)}{n(n-1)}]\zeta(\underset{\sim}{q})dq + h; \quad r < a,$$

$$(6.5.3)$$

in which h is the harmonic function

$$h = a^2\log a\int_{\partial B} \zeta(\underset{\sim}{q})dq + \int_{\partial B} (\underset{\sim}{p}\cdot\underset{\sim}{q})\zeta(\underset{\sim}{q})dq$$

$$+ r^2\int_{\partial B} [\sum_{n=1}^{\infty} \frac{r^n a^{-n}\cos n(\theta-\tau)}{-n(n+1)}]\zeta(\underset{\sim}{q})dq; \quad r < a, \qquad (6.5.4)$$

Note that expansion (6.5.3) is also true for $r = a$, in fact (6.2.10), (6.5.3) provide the exterior and interior expansions respectively of the biharmonic potential Ω given on $r = a$.

It is interesting to introduce

$$\zeta(\underset{\sim}{q}) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n\cos n\theta + B_n\sin n\theta); \quad \underset{\sim}{q} \in \partial B, \qquad (6.5.5)$$

since inserting this into (6.2.10), (6.5.3), we can obtain the exterior and interior expansions in terms of $\sin n\tau$, $\cos n\tau$; $n = 0, 1, 2, \ldots .$

Chapter 7

Some Radially Symmetric Problems Via Chakrabarty Representation

7.1 Introduction

In the last two chapters we have developed the analysis which enables us to solve boundary value problems by utilising the Chakrabarty representation. However since this representation involves a combination of biharmonic and harmonic potentials, our first step is to find the source densities concerned. In this chapter we shall embark on the determination of the Chakrabarty potentials relevant to the biharmonic boundary-value problems C-1, C-2, C-3.

As the Chakrabarty representation is quite new and little known, its approach has not yet been applied to any non-trivial problem. However we will consider how the Chakrabarty representation can be exploited to provide a formulation of biharmonic boundary-value problems theoretically competitive with that of Almansi. The Chakrabarty representation provides the boundary relations

$$\left.\begin{aligned} \chi &= \Omega + \psi \\ \chi' &= \Omega' + \psi' \end{aligned}\right\} \quad ; \qquad \text{on} \quad \partial B \qquad (7.1.1)$$

where Ω, ψ are defined in (5.2.5), (2.2.3) and Ω', ψ' are defined in (5.4.1), (2.2.5) respectively. Accordingly given χ, χ' on ∂B, we have a pair of coupled boundary integral equations for ζ, η, which may be solved numerically to provide Ω, ψ, and therefore also χ, in the domain concerned.

This aspect will be studied analytically through this chapter and the next.

7.2 Problem C-1

In this problem we propose to find a biharmonic function χ in C-1, where χ, χ_i' are given on ∂B as follows:

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= a^2 \qquad (7.2.1a) \\ \chi_i'(\underset{\sim}{p}) &= -2a \qquad (7.2.1b) \end{aligned}\right\} ; \quad \underset{\sim}{p} \in \partial B.$$

Writing

$$\chi = \Omega + \psi; \qquad r \leq a, \tag{7.2.2}$$

and noting the radial symmetry we have*

$$\Omega = Ar^2 + B, \quad \psi = C, \quad r \leq a, \tag{7.2.3}$$

where A, B, C are constants. If so, from (7.2.2), (7.2.3)

$$\chi = Ar^2 + D, \quad D = B + C; \quad r \leq a, \tag{7.2.4}$$

and accordingly from (7.2.1a)

$$\chi(a) = Aa^2 + D = a^2; \qquad r = a. \tag{7.2.5}$$

Also from (7.2.1b),

$$\chi_i'(a) = -\frac{d}{dr}(Ar^2+D)_{r=a} = -2aA = -2a. \tag{7.2.6}$$

---

* Note the absence of $r^2 \log r$ since this becomes singular at $r = 0$.

Hence from (7.2.5), (7.2.6),

$$A = 1, \qquad D = 0, \tag{7.2.7}$$

which yields

$$\chi = r^2 ; \qquad\qquad r \le a. \tag{7.2.8}$$

7.3 Determination of the potentials Ω, ψ

It is of considerable interest to obtain the contributions of Ω, ψ to χ in (7.2.2). Thus we write

$$\left.\begin{aligned} \Omega(\underset{\sim}{p}) &= \int_{\partial B} [-R^2+R^2\log R]\zeta(\underset{\sim}{q})\,dq \qquad (7.3.1a) \\ \psi(\underset{\sim}{p}) &= \int_{\partial B} \eta(\underset{\sim}{q})\log R\,dq \qquad (7.3.1b) \end{aligned}\right\} ; \quad \underset{\sim}{p} \in B_i + \partial B,\ \underset{\sim}{q} \in \partial B,$$

here

$$R = |\underset{\sim}{p}-\underset{\sim}{q}|;\ \underset{\sim}{q} = (a\cos\theta,\ a\sin\theta),\ \underset{\sim}{p} = (r\cos\tau,\ r\sin\tau)_{r\le a}, \tag{7.3.1c}$$

and ζ, η are sources to be determined. Because of radial symmetry, these are constants, say ζ_0, η_0 respectively. If so

$$\begin{aligned} \psi(r) &= \psi_0(\text{a constant}) \\ &= \eta_0\int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\,dq = 2\pi a\eta_0\log a; \qquad r \le a, \end{aligned} \tag{7.3.2}$$

where η_0 remains to be evaluated. Consequently

$$\chi_i'(\underset{\sim}{p}) = \Omega_i'(\underset{\sim}{p}); \qquad \underset{\sim}{p} \in \partial B. \qquad (7.3.3)$$

From (7.3.1a) we have - see (6.5.3), (6.5.4) for $\zeta = \zeta_0$,

$$\Omega(r) = 2a\pi\zeta_0[(a^2+r^2)\log a - a^2]; \qquad r \leq a, \qquad (7.3.4)$$

therefore from (7.2.1b)

$$\Omega_i'(a) = -\frac{d}{dr}[\Omega(r)]_{r=a} = -4\pi a^2\zeta_0 \log a = -2a, \qquad (7.3.5)$$

so that

$$\zeta_0 = \frac{1}{2\pi a \log a}; \qquad r = a. \qquad (7.3.6)$$

Now from (7.3.4), (7.3.6) it follows that

$$\Omega(r) = r^2 + \frac{a^2\log a - a^2}{\log a}; \qquad r \leq a. \qquad (7.3.7)$$

Accordingly from (7.2.2), (7.3.2),

$$\chi(r) = r^2 + \frac{a^2\log a - a^2}{\log a} + \psi_0$$

$$= r^2 + \frac{a^2\log a - a^2}{\log a} + 2\pi a\eta_0 \log a; \qquad r \leq a. \qquad (7.3.8)$$

By virtue of (7.2.1a) we obtain

$$\chi(a) = a^2 + \frac{a^2\log a - a^2}{\log a} + 2\pi a\eta_0 \log a = a^2, \qquad (7.3.9)$$

which gives

$$\eta_0 = \frac{a(1-\log a)}{2\pi\log^2 a}\ ; \qquad r = a, \qquad (7.3.10)$$

therefore

$$\psi_0(\underset{\sim}{p}) = \eta_0\int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|dq = \frac{a^2-a^2\log a}{\log a}\ ; \qquad r \le a. \qquad (7.3.11)$$

This contribution cancels out the constant term in (7.3.8), to yield

$$\chi(r) = r^2\ ; \qquad r \le a. \qquad (7.3.12)$$

7.4 Problem C-2

We now determine χ in C-2 by utilising the Chakrabarty representation, where the boundary conditions are as follows:

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= a^2 \qquad (7.4.1a) \\ \chi_e'(\underset{\sim}{p}) &= 2a \qquad (7.4.1b) \end{aligned}\right|\ ; \qquad \underset{\sim}{p} \in \partial B.$$

We write

$$\chi = \Omega + \psi + k; \qquad r \ge a, \qquad (7.4.2)$$

where k is a constant to be determined subject to Hadamard's uniqueness requirement,

$$\chi = O(r) \qquad \text{as} \quad r \longrightarrow \infty. \qquad (7.4.3)$$

Because of radial symmetry if follows from (6.2.10) and (6.2.12) that

$$\Omega = A(-r^2+r^2\log r) + B\log r; \qquad r \ge a. \qquad (7.4.4)$$

Also from (2.3.1), the harmonic function ψ will be

$$\psi = C\log r; \qquad r \geq a, \tag{7.4.5}$$

where A, B, C are constants to be determined. It follows from (7.4.2), (7.4.4), (7.4.5) that

$$\chi = A(-r^2+r^2\log r) + D\log r + k; \quad D = B + C, \quad r \geq a. \tag{7.4.6}$$

Now because of the requirement (7.4.3), we have

$$A = 0, \tag{7.4.7}$$

and therefore

$$\chi = D\log r + k; \qquad r \geq a. \tag{7.4.8}$$

If so, from (7.4.1a),

$$\chi(a) = D\log a + k = a^2, \tag{7.4.9}$$

and from (7.4.1b)

$$\chi_e'(a) = \frac{d}{dr}(D\log r+k)_{r=a} = \frac{D}{a} = 2a. \tag{7.4.10}$$

Hence

$$D = 2a^2, \quad k = a^2 - 2a^2\log a, \tag{7.4.11}$$

so that

$$\chi(r) = 2a^2\log r + a^2 - 2a^2\log a; \qquad r \geq a. \tag{7.4.12}$$

Clearly it is not possible to determine B, C separately using symmetry arguments.

7.5 Determination of the potentials Ω, ψ

Writing

$$\left.\begin{aligned}\Omega(\underset{\sim}{p}) &= \int_{\partial B} [-R^2+R^2\log R]\zeta(\underset{\sim}{q})dq \\ \psi(\underset{\sim}{p}) &= \int_{\partial B} \eta(\underset{\sim}{q})\log R dq\end{aligned}\right\} ; \ \underset{\sim}{p} \in B_e + \partial B, \ \underset{\sim}{q} \in \partial B, \qquad (7.5.1b)$$

here

$$R = |\underset{\sim}{p}-\underset{\sim}{q}|; \ \underset{\sim}{q} = (a\cos\theta, \ a\sin\theta), \ \underset{\sim}{p} = (r\cos\tau, \ r\sin\tau)_{r \geq a}, \qquad (7.5.1c)$$

and ζ, η are constant sources, say ζ_0, η_0 to be determined. It follows from (7.4.3), that

$$\int_{\partial B} \zeta_0 dq = 0,$$

i.e.

$$\zeta_0 = 0, \qquad (7.5.2)$$

so that

$$\Omega = 0; \qquad r \geq a. \qquad (7.5.3)$$

Accordingly from (7.4.2),

$$\chi = \psi + k; \qquad r \geq a. \qquad (7.5.4)$$

By virtue of (7.4.1a), (7.4.1b) we get

$$\chi(a) = \psi(a) + k = a^2, \qquad (7.5.5)$$

and also

$$\chi_e'(a) = \psi_e'(a) = 2a. \tag{7.5.6}$$

But

$$\psi(a) = \eta_0 \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\,dq = 2\pi\eta_0 \log a, \tag{7.5.7}$$

and also

$$\psi_e'(a) = \eta_0 \int_{\partial B} \log_e'|\underset{\sim}{p}-\underset{\sim}{q}|\,dq + \pi\eta_0$$

$$= \eta_0 \int_0^{2\pi} \frac{1}{2a}(a\,d\theta) + \pi\eta_0 = 2\pi\eta_0. \tag{7.5.8}$$

Consequently from (7.5.5)-(7.5.8) we have

$$\left.\begin{aligned} 2\pi a\eta_0 \log a + k &= a^2 \\ 2\pi\eta_0 &= 2a \end{aligned}\right\}, \tag{7.5.9}$$

i.e.

$$\eta_0 = \frac{a}{\pi}, \qquad k = a^2 - 2a^2 \log a. \tag{7.5.10}$$

Therefore

$$\psi(r) = \eta_0 \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\,dq = 2\pi a\eta_0 \log r; \qquad r \geq a, \tag{7.5.11}$$

and now from (7.5.4), (7.5.10) we deduce

$$\chi(r) = 2a^2 \log r + a^2 - 2a^2 \log a; \qquad r \geq a, \tag{7.5.12}$$

in agreement with (7.4.12).

7.6 Problem C-3

We now turn our attention to the ring-shaped domain C-3, and we will see that the fundamental idea is closely related to (3.4.2). Thus considering the boundary conditions

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= a^2 \\ \chi_e'(\underset{\sim}{p}) &= 2a \end{aligned}\right\}; \ \underset{\sim}{p} \in \partial B_1, \qquad \left.\begin{aligned} \chi(\underset{\sim}{p}) &= b^2 \\ \chi_i'(\underset{\sim}{p}) &= -2b \end{aligned}\right\}; \ \underset{\sim}{p} \in \partial B_0, \tag{7.6.1}$$

we utilise the Chakrabarty representation

$$\chi = \Omega + \psi; \qquad a \le r \le b, \tag{7.6.2}$$

where

$$\Omega(\underset{\sim}{p}) = \int_{\partial B} [-R^2+R^2\log R]\zeta(\underset{\sim}{q})dq \tag{7.6.3a}$$

$$; \ \underset{\sim}{p} \in B + \partial B, \ \underset{\sim}{q} \in \partial B;$$

$$\psi(\underset{\sim}{p}) \int_{\partial B} \eta(\underset{\sim}{q})\log R dq \tag{7.6.3b}$$

$$\partial B = \partial B_1 + \partial B_0; \quad R = |\underset{\sim}{p}-\underset{\sim}{q}|; \quad \underset{\sim}{q} = (r\cos\theta, \ r\sin\theta)_{r=a,b};$$

$$\underset{\sim}{p} = (r\cos\tau, \ r\sin\tau)_{a\le r\le b} .$$

Now let us split Ω, ψ as follows:

$$\left.\begin{aligned} \Omega &= \Omega_0 + \Omega_1 \\ \psi &= \psi_0 + \psi_2 \end{aligned}\right\}; \qquad a \le r \le b, \tag{7.6.4}$$

here

$$\left.\begin{aligned}\Omega_j(\underset{\sim}{p}) &= \int_{\partial B_j} [-R^2+R^2\log R]\zeta_j(\underset{\sim}{q})dq \qquad (7.6.5a)\\ \psi_j(\underset{\sim}{p}) &= \int_{\partial B_j} \eta_j(\underset{\sim}{q})\log R dq \qquad (7.6.5b)\end{aligned}\right\} ;\ j = 0,1;\ p \in B + \partial B,\ q \in \partial B.$$

As the problem is radially symmetric, therefore ζ_j, η_j; $j = 1, 2$, are constant sources to be determined. If so, from (7.6.5a), (7.6.5b) we get

$$\left.\begin{aligned}\Omega_0(r) &= 2\pi b\zeta_0[(b^2+r^2)\log b-b^2]\\ \Omega_1(r) &= 2\pi a\zeta_1[-r^2+r^2\log r+a^2\log r]\\ \psi_0(r) &= 2\pi b\eta_0\log b\\ \psi_1(r) &= 2\pi a\eta_1\log r\end{aligned}\right\} ;\ a \le r \le b. \qquad (7.6.6)$$

Since the theory is linear and radially symmetric, we can superpose independent solutions determined from C-1, C-2. This gives

$$\chi(r) = 2\pi b\zeta_0[(b^2+r^2)\log b-b^2] + 2\pi a\zeta_1[-r^2+r^2\log r+a^2\log r]$$
$$+ 2\pi b\eta_0\log b + 2\pi a\eta_1\log r; \quad a \le r \le b. \qquad (7.6.7)$$

So that

$$\chi_e'(a) = \frac{d}{dr}[\chi(r)]_{r=a}$$
$$= 4\pi ab\zeta_0\log b + 4\pi a^2\zeta_1\log a + 2\pi\eta_1, \qquad (7.6.8)$$

$$\chi_i'(b) = -\frac{d}{dr}[\chi(r)]_{r=b}$$

$$= -4\pi b\zeta_0 \log b + 2\pi a\zeta_1[b-2b\log b - \frac{a^2}{b}]$$

$$- \frac{2\pi a}{b}\eta_1 \ . \tag{7.6.9}$$

Now by virtue of (7.6.1), we get four linear equations for ζ_0, ζ_1, η_0, η_1, with the solutions

$$\zeta_1 = \eta_1 = 0, \quad \zeta_0 = \frac{1}{2\pi b\log b}, \quad \eta_0 = \frac{b(1-\log b)}{2\pi\log^2 b} \ . \tag{7.6.10}$$

Accordingly it follows that

$$\left.\begin{aligned} &\Omega_1(r) = 0, \quad \psi_1(r) = 0, \\ &\Omega_0(r) = r^2 + b^2 - \frac{b^2}{\log b}, \quad \psi_0(r) = \frac{b^2}{\log b} - b^2 \end{aligned}\right\} ; \ a \le r \le b. \tag{7.6.11}$$

Finally from (7.6.2) we obtain

$$\chi(r) = r^2; \qquad a \le r \le b. \tag{7.6.12}$$

Note that Ω_0, ψ_0 introduced in (7.6.11), are as Ω, ψ in (7.3.7), (7.3.11) respectively.

Chapter 8

Some Non-Radially Symmetric Problems Via Chakrabarty Representation

8.1 Introduction

This chapter provides an interesting application of Chakrabarty's representation to some familiar non-radially symmetric problems. Also of significance is the exploitation of the source-densities in solving these problems.

In conclusion this chapter reproduces results already obtained by utilising the Almansi representation.

8.2 Problem C-1

Let us determine the biharmonic function χ in C-1 where the boundary conditions are as follows:

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= \alpha x \\ \chi_i'(\underset{\sim}{p}) &= \beta x_i' \end{aligned}\right\} ; \quad \underset{\sim}{p} = (x,y) \in \partial B, \tag{8.2.1}$$

here $\underset{\sim}{p} = (a\cos\tau,\ a\sin\tau)$ and α, β are two given constants. If so (8.2.1) can be written as

$$\chi(\underset{\sim}{p}) = \alpha a\cos\tau \tag{8.2.2a}$$

$$; \quad \underset{\sim}{p} \in \partial B.$$

$$\chi_i'(\underset{\sim}{p}) = -\beta\cos\tau \tag{8.2.2b}$$

We utilise the Chakrabarty representation

$$\chi = \Omega + \psi, \tag{8.2.3}$$

where

$$\left.\begin{aligned}\Omega(\underset{\sim}{p}) &= \int_{\partial B} [-R^2+R^2\log R]\zeta(\underset{\sim}{q})dq \\ \psi(\underset{\sim}{p}) &= \int_{\partial B} \eta(\underset{\sim}{q})\log R dq\end{aligned}\right\} ; \underset{\sim}{p} \in B_i + \partial B,\ \underset{\sim}{q} \in \partial B, \tag{8.2.4}$$

here $R= |\underset{\sim}{p}-\underset{\sim}{q}|$; $\underset{\sim}{q} = (a\cos\theta,\ a\sin\theta)$, $\underset{\sim}{p} = (r\cos\tau,\ r\sin\tau)_{,r\le a}$, and ζ, η are source-densities to be determined. Now since only the coefficients of $\cos\tau$ in the expansion (6.5.3) survive, because of the boundary conditions (8.2.1), therefore the source-densities ζ, η must be as follows:

$$\zeta(\underset{\sim}{q}) = A\cos\theta, \quad \eta(\underset{\sim}{q}) = B\cos\theta; \quad \underset{\sim}{q} \in \partial B, \tag{8.2.5}$$

where A, B are constants to be determined. If so, it appears from the interior expansion (6.5.3), and from

$$\psi(\underset{\sim}{p}) = \log a\int_{\partial B} \eta(\underset{\sim}{q})dq - a^{-2}\int_{\partial B} (\underset{\sim}{p}.\underset{\sim}{q})\eta(\underset{\sim}{q})dq$$

$$+ \int_{\partial B} [\sum_{n=2}^{\infty} \frac{r^n a^{-n}\cos n(\theta-\tau)}{-n}]\eta(q)dq ; \underset{\sim}{p} \in B_i + \partial B,\ \underset{\sim}{q} \in \partial B, \tag{8.2.6}$$

that the corresponding potentials Ω, ψ reduce to

$$\Omega(r) = (-\frac{r^3}{2} + ra^2 - 2a^2 r\log a)\pi A\cos\tau ; \quad r \le a, \tag{8.2.7}$$

$$\psi(r) = -Br\pi\cos\tau ; \quad r \le a. \tag{8.2.8}$$

Superposing (8.2.7), (8.2.8) gives

$$\chi(r) = [(-\frac{r^3}{2} + ra^2 - 2a^2 r\log a)A\pi - Br\pi]\cos\tau \;; \qquad r \le a. \tag{8.2.9}$$

By virtue of (8.2.2a) we have

$$\chi(a) = [(\frac{a^3}{2} - 2a^3\log a)A - Ba]\pi\cos\tau = \alpha a\cos\tau \;; \tag{8.2.10}$$

also from (8.2.2b)

$$\begin{aligned}\chi_i'(a) = -\frac{d}{dr}[\chi(r)]_{r=a} &= [(\frac{a^2}{2} + 2a^2\log a)A + B]\pi\cos\tau \\ &= -\beta\cos\tau.\end{aligned} \tag{8.2.11}$$

Solving for A, B:

$$A = \frac{\alpha-\beta}{\pi a^2}\,, \quad B = \frac{-(\alpha+\beta+4\alpha\log a-4\beta\log a)}{2\pi}\,. \tag{8.2.12}$$

Accordingly

$$\left.\begin{aligned}\zeta(\underset{\sim}{q}) &= \frac{\alpha-\beta}{\pi a^2}\cos\theta \\ \eta(\underset{\sim}{q}) &= -\frac{1}{2\pi}(\alpha+\beta+4\alpha\log a-4\beta\log a)\cos\theta\end{aligned}\right\} ;\ \underset{\sim}{q} \in \partial B. \tag{8.2.13}$$

So we finally obtain

$$\chi(r) = [\frac{r^3(\beta-\alpha)+ra^2(3\alpha-\beta)}{2a^2}]\cos\tau, \qquad r \le a, \tag{8.2.14}$$

i.e.

$$\chi(r) = [\frac{r^2(\beta-\alpha)}{2a^2} + \frac{3\alpha-\beta}{2}]x \; ; \qquad r \le a, \tag{8.2.15}$$

in agreement with (4.2.10). Note that $A = 0$ when $\alpha = \beta$, i.e. $\Omega = 0$ from (8.2.7), therefore $\chi = \psi$.

8.3 Problem C-2

We now determine a biharmonic function χ in C-2 subject to the boundary conditions

$$\left.\begin{aligned} \chi(\underset{\sim}{p}) &= \alpha x \\ \chi'_e(\underset{\sim}{p}) &= \beta x'_e \end{aligned}\right\} ; \qquad \underset{\sim}{p} = (x,y) \in \partial B, \tag{8.3.1}$$

where $\underset{\sim}{p} = (a\cos\tau,\ a\sin\tau)$ and α, β are two given constants. These conditions can be written as

$$\chi(\underset{\sim}{p}) = \alpha a\cos\tau \tag{8.3.2a}$$

$$\chi'_e(\underset{\sim}{p}) = \beta\cos\tau \tag{8.3.2b}$$

$$; \qquad \underset{\sim}{p} \in \partial B.$$

Now we adopt the extended Chakrabarty representation

$$\chi = \Omega + \psi + Ax + By + C; \quad \underset{\sim}{p} = (x,y) \in B_e + \partial B, \tag{8.3.3}$$

where

$$\Omega(\underset{\sim}{p}) = \int_{\partial B} [-R^2+R^2\log R]\zeta(\underset{\sim}{q})dq \tag{8.3.4a}$$

$$\psi(\underset{\sim}{p}) = \int_{\partial B} \eta(\underset{\sim}{q})\log R dq \tag{8.3.4b}$$

$$; \qquad \underset{\sim}{p} \in B_e + \partial B, \quad \underset{\sim}{q} \in \partial B,$$

here $R = |\underset{\sim}{p}-\underset{\sim}{q}|$; $\underset{\sim}{q} = (q_1, q_2) = (a\cos\theta, a\sin\theta)$, $\underset{\sim}{p} = (r\cos\tau, r\sin\tau)_{r \geq a}$, and A, B, C are constants to be determined subject to Hadamard's requirements

$$\chi = O(r) \qquad \text{as} \quad |\underset{\sim}{p}| = r \longrightarrow \infty. \tag{8.3.5}$$

If so, by noting the expansion - see (6.2.11) -

$$\begin{aligned}\Omega(\underset{\sim}{p}) = {} & (-r^2+r^2\log r)\int_{\partial B} \zeta(\underset{\sim}{q})dq - 2x\log r\int_{\partial B} q_1\zeta(\underset{\sim}{q})dq \\ & - 2y\log r\int_{\partial B} q_2\zeta(\underset{\sim}{q})dq \\ & + r^2 \sum_{n=-2}^{-\infty} \frac{r^n}{n(n+1)}(b_{nc}\cos n\tau + b_{ns}\sin n\tau) \\ & + x\int_{\partial B} q_1\zeta(\underset{\sim}{q})dq + y\int_{\partial B} q_2\zeta(\underset{\sim}{q})dq + \delta\log r + O(r^{-1}) \\ & \qquad \text{as} \quad |\underset{\sim}{p}| = r \longrightarrow \infty,\end{aligned} \tag{8.3.6}$$

we must have

$$\int_{\partial B} \zeta(\underset{\sim}{q})dq = \int_{\partial B} q_1\zeta(\underset{\sim}{q})dq = \int_{\partial B} q_2\zeta(\underset{\sim}{q})dq = 0. \tag{8.3.7}$$

These side conditions show that the biharmonic potential Ω cannot cover the term $O(r)$ in C-2 - see (6.2.13), (6.2.14), therefore Ω no longer makes any contribution to (8.3.3), i.e.

$$\Omega(\underset{\sim}{p}) = 0, \qquad \underset{\sim}{p} \in B_e + \partial B. \tag{8.3.8}$$

Consequently (8.3.3) simplifies to

$$\chi = \psi + Ax + By + C \; ; \qquad \underset{\sim}{p} = (x,y) \in B_e + \partial B. \qquad (8.3.9)$$

On the other hand introducing

$$\eta(\underset{\sim}{q}) = D\cos\theta \; ; \qquad \underset{\sim}{q} \in \partial B, \qquad (8.3.10)$$

where D is a constant to be determined, it follows from (8.3.4b) that

$$\psi(r) = \int_{\partial B} \eta(\underset{\sim}{q}) \log R dq = \int_{\partial B} D\cos\theta \log[|\underset{\sim}{p}|^2 + |\underset{\sim}{q}|^2 - 2\underset{\sim}{p}.\underset{\sim}{q}]dq$$

$$= \int_{\partial B} D\cos\theta[\log|\underset{\sim}{p}| + \sum_{n=1}^{\infty} \frac{|q|^n |p|^{-n} \cos n(\theta-\tau)}{-n}]ad\theta$$

$$= -\frac{Da^2\pi}{r}\cos\tau \; ; \qquad r \geq a, \qquad (8.3.11)$$

see (6.2.3) for $r > a$, and also Hilbert's integral formula, Jaswon, M.A. and Symm, G.T. (1977), for $r = a$. Accordingly from (8.3.9) we have

$$\chi(r) = -\frac{Da^2\pi}{r}\cos\tau + Ar\cos\tau + Br\sin\tau + C \; ; \qquad r \geq a. \qquad (8.3.12)$$

Furthermore by noting the boundary conditions (8.3.1) it follows that only the coefficients A, D survive, i.e. we have

$$B = C = 0, \qquad (8.3.13)$$

which yields

$$\chi(r) = (-\frac{Da^2\pi}{r} + Ar)\cos\tau \; ; \qquad r \geq a. \qquad (8.3.14)$$

Now by virtue of (8.3.2a) we have

$$\chi(a) = (-D a\pi + Aa)\cos\tau = \alpha a\cos\tau. \tag{8.3.15}$$

Also, from (8.3.2b),

$$\chi_e'(a) = \frac{d}{dr}[\chi(r)]_{r=a} = (D\pi + A)\cos\tau = \beta\cos\tau. \tag{8.3.16}$$

Consequently

$$A = \frac{\alpha+\beta}{2}, \qquad D = \frac{\beta-\alpha}{2\pi}. \tag{8.3.17}$$

It follows that

$$\eta(\underset{\sim}{q}) = \frac{\beta-\alpha}{2\pi}\cos\theta\ ; \qquad \underset{\sim}{q} \in \partial B, \tag{8.3.18}$$

and therefore

$$\psi(r) = \frac{a^2(\alpha-\beta)}{2\pi}\cos\tau\ ; \qquad r \geq a. \tag{8.3.19}$$

Accordingly

$$\chi(r) = \left[\frac{a^2(\alpha-\beta)}{2r} + \frac{r(\alpha+\beta)}{2}\right]\cos\tau\ ; \qquad r \geq a, \tag{8.3.20}$$

or

$$\chi(r) = \left[\frac{a^2(\alpha-\beta)}{2r^2} + \frac{\alpha+\beta}{2}\right]x\ ; \qquad r \geq a, \tag{8.3.21}$$

in agreement with (4.3.7).

8.4 Problem C-3

In order to determine the biharmonic function χ such that

$$\left.\begin{aligned}\chi(\underset{\sim}{p}) &= x\\ \chi'_e(\underset{\sim}{p}) &= 0\end{aligned}\right\}; \ \underset{\sim}{p} \in \partial B_1, \qquad \left.\begin{aligned}\chi(\underset{\sim}{p}) &= 0\\ \chi'_i(\underset{\sim}{p}) &= 0\end{aligned}\right\}; \ \underset{\sim}{p} \in \partial B_0, \tag{8.4.1}$$

we utilise the Chakrabarty representation

$$\chi(\underset{\sim}{p}) = \Omega(\underset{\sim}{p}) + \psi(\underset{\sim}{p}); \qquad \underset{\sim}{p} \in B + \partial B, \tag{8.4.2}$$

where

$$\left.\begin{aligned}\Omega(\underset{\sim}{p}) &= \int_{\partial B} [-R^2+R^2\log R]\zeta(\underset{\sim}{q})dq\\ \psi(\underset{\sim}{p}) &= \int_{\partial B} \eta(\underset{\sim}{q})\log R dq\end{aligned}\right\}; \quad \underset{\sim}{p} \in B + \partial B, \ \underset{\sim}{q} \in \partial B, \tag{8.4.3}$$

$\partial B = \partial B_1 + \partial B_0$; $R = |\underset{\sim}{p}-\underset{\sim}{q}|$, $\underset{\sim}{q} = (r\cos\theta,\ r\sin\theta)_{r=a,b}$

$\underset{\sim}{p} = (x,y) = (r\cos\tau,\ r\sin\tau)_{a\le r\le b}$.

Now consider the four potentials

$$\left.\begin{aligned}\Omega_j(\underset{\sim}{p}) &= \int_{\partial B_j} [-R^2+R^2\log R]\zeta_j(\underset{\sim}{q})dq\\ \psi_j(\underset{\sim}{p}) &= \int_{\partial B_j} \eta_j(\underset{\sim}{q})\log R dq\end{aligned}\right\}; \ j = 0,1; \ \underset{\sim}{p} \in B + \partial B, \ \underset{\sim}{q} \in \partial B, \tag{8.4.4}$$

where ζ_j, η_j; $j = 0, 1$ are source densities to be determined subject to the boundary conditions (8.4.1). If so, since only the constants and also only the coefficients of $\cos\tau$ survive, therefore ζ_j, η_j; $j = 0, 1$ are as follows:

$$\zeta_j = A_0^{(j)} + A_1^{(j)}\cos\theta, \quad \eta_j = B_0^{(j)} + B_1^{(j)}\cos\theta; \; j = 0,1, \tag{8.4.5}$$

where $A_0^{(j)}$, $A_1^{(j)}$, $B_0^{(j)}$, $B_1^{(j)}$; $j = 0,1$, are 8 constants to be determined from the expansions (6.5.3), (6.2.10), (2.3.1), (8.2.6) respectively we get

$$\left.\begin{aligned}
\Omega_0(r) &= 2\pi b A_0^{(0)}(b^2\log b + r^2\log b - b^2) \\
&\quad + \pi A_1^{(0)}\left(-\frac{r^3}{2} + rb^2 - 2rb^2\log b\right)\cos\tau \\
\Omega_1(r) &= 2\pi a A_0^{(1)}(a^2\log r - r^2 + r^2\log r) \\
&\quad + \pi a^2 A_1^{(1)}\left(-\frac{a^2}{2r} + r - 2r\log r\right)\cos\tau \\
\psi_0(r) &= 2\pi b B_0^{(0)}\log b - r\pi B_1^{(0)}\cos\tau \\
\psi_1(r) &= 2\pi a B^{(1)}\log r - \frac{a^2}{r}\pi B_1^{(1)}\cos\tau
\end{aligned}\right\} ; \; a \le r \le b. \tag{8.4.6}$$

Superposing these independent potentials gives

$$\chi(r) = \Omega_0(r) + \Omega_1(r) + \psi_0(r) + \psi_1(r); \quad a \le r \le b, \tag{8.4.7}$$

accordingly we can now determine

$$\chi_e'(a) = -\frac{d}{dr}[\chi(r)]_{r=a}, \quad \chi_i'(b) = \frac{d}{dr}[\chi(r)]_{r=b}. \tag{8.4.8}$$

Applying the boundary conditions (8.4.1), we obtain 8 linear equations, which subdivide into two distinct sets of equations as follows:

$$\begin{bmatrix} \pi a^2(b-2b\log b-\frac{a^2}{2b}) & \pi b^3(\frac{1}{2}-2\log b) & -b\pi & -\frac{a^2}{b}\pi \\ \pi a^3(\frac{1}{2}-2\log a) & \pi(b^2-2b^2\log b-\frac{a^2}{2}) & -b\pi & -a\pi \\ \pi a^2(-\frac{1}{2}-2\log a) & \pi(b^2-2b^2\log b-\frac{3a^2}{2}) & -\pi & \pi \\ \pi a^2(\frac{a^2}{2b^2}-1-2\log b) & \pi b^2(-\frac{1}{2}-2\log b) & -\pi & \frac{a^2}{b^2}\pi \end{bmatrix} \begin{bmatrix} A_1^{(0)} \\ A_1^{(1)} \\ B_1^{(0)} \\ B_1^{(1)} \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{8.4.9}$$

and

$$\begin{bmatrix} 2\pi b(a^2\log b+b^2\log b-b^2) & 2\pi a^3(2\log a-1) & 2\pi b\log b & 2\pi a\log a \\ 2\pi b^3(2\log b-1) & 2\pi a(a^2\log b+b^2\log b-b^2) & 2\pi b\log b & 2\pi a\log b \\ 4\pi ab\log b & 4\pi a^2\log a & 0 & 2\pi \\ 4\pi b^2\log b & 2\pi a(2b\log b-b+\frac{a^2}{b}) & 0 & \frac{2\pi a}{b} \end{bmatrix} \begin{bmatrix} A_0^{(0)} \\ A_0^{(1)} \\ B_0^{(0)} \\ B_0^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{8.4.10}$$

It can be shown that the system of homogeneous equations (8.4.10) has no non-trivial solutions, because its discriminating determinant has the value

$$-16\pi^4 ab\ \log b[(a^2-b^2)^2 - 4a^2b^2(\log a-\log b)^2] \neq 0; \quad 0 < a < b. \tag{8.4.11}$$

Also the system of equations (8.4.9) has a unique solution, since its discriminating determinant has the value

$$\frac{4\pi^4 a^2(a^2-b^2)}{b}[(a^2-b^2) + (a^2+b^2)(\log b-\log a)] \neq 0; \quad 0 < a < b. \tag{8.4.12}$$

It should be noted that (8.4.11), (8.4.12) match (4.4.7), (4.4.9) respectively.

Proceedings of the first

European Boundary Element Meeting

Brussels, Hyatt Regency Hotel, 8 - 10th may 1988

editor:

Jean-Louis Migeot
Structural Dynamics Research Corporation, Paris
Continuum Mechanics Department, Université Libre de Bruxelles

Complimentary copies of these proceedings can be obtained at the price of 1,200 BEF. Please send your cheque or credit card number to:

D.B.M. sprl
203 avenue Louise
1050 Bruxelles - Belgium
telephone: +32.2.640.89.09.

Foreword

You will find in the following pages the proceedings of the European Boundary Element Meeting. This includes several very interesting papers on Stress and Displacement Analysis, Potential Problems and Flow simulations presented at the sunday afternoon's 'Problem Oriented Sessions'.

As far as the chairman's presentation at the four 'Technique Oriented sessions' are concerned, only one summary has been received until now. It is included at the end of this volume. The other summaries will be released later as soon as we get them.

I take the opportunity of this foreword to thank one more time all attendants. I appreciated the great interest of most of the presentations and their high scientific level and I was pleased to see that both young and experienced researchers were keen on exchanging their experience. In this matter, I think EBEM has fully played the role of exchange forum I had in mind when I decided to organize that meeting.

Jean-Louis Migeot
conference director

THE CONFERENCE STAFF ...

Jean-Louis Migeot, conference director

Structural Dynamics Research Corporation
31 Boulevard des Bouvets - 92000 Nanterre
FRANCE - +33.1.47.78.14.45.

and

Université Libre de Bruxelles
Continuum Mechanics Department - CP 194/5
50 avenue F.D. Roosevelt - 1050 Bruxelles
BELGIUM - +32.2.642.27.56.

Véronique Lemmens, Pascal Escoyez,
Waldo Vandenhende, conference secretariat

Data Processing for Business and Management (DBM)
203 avenue Louise - 1050 Bruxelles
BELGIUM - +32.2.640.89.09.

TABLE DES MATIERES

I. STRESS AND DISPLACEMENT ANALYSIS PROBLEMS

Three Dimensional Stress Analysis Using an Indirect Discrete Boundary Method (J.L. Wearing and M.A. Sheikh, Department of Mechanical Engineering, University of Sheffield, Britain) -1.1-

BEM versus F.E. in Turbomachinery Structural Application (R.N. Tadros and M. Hongoh, Pratt & Wihtney of Canada, Canada) -1.13-

A Green's Function Method for Nonlinear Analysis of Plates (J.T. Katsikadelis and M.S. Nerantzaki, National Technical University, Athen, Greece) -1.23-

A Boundary Integral Equation Approach To Friction Contact Problems of Classical Elastostatics (Bruno Deshoullières and D. Fortuné, Université de Poitiers, Laboratoire de Mécanique Théorique, France) -1.43-

On Boundary Variational Principles and B.E.M. for Frictionless Elastic Contact Problems (C. Alessandri, S. Alliney and A. Tralli, University of Florence and University of Udine, Italy) -1.53-

Development of High Order Elements For Circular Discretization In The Displacement Discontinuity Method (Y. Belkacemi, A. Bouhaddane, J.P. Henry, Laboratoire de Mécanique de Lille, Département Mécanique des Matériaux Fragiles, Ecole Universitaire d'Ingénieurs de Lille, France) -1.75-

Development of High Order Element To The Displacement Discontinuity Method Applied To An Elastic Anisotropic Body (E. Morel, J.P. Henry, Laboratoire de Mécanique de Lille, Département Mécanique des Matériaux Fragiles, Ecole Universitaire d'Ingénieurs de Lille, France) -1.85-

Two and Three-dimensional Coupling of BEM in Elasticity (E. Schnack, R. Carmine, I. Becker, N. Karaosmanoglu, Institut of Solid Mechanics, Karlsruhe University, Germany) -1.93-

On the Computation of Three-Dimensional Singularities of Elastic Fields near Vertices (Klaus Volk, Hermann Schmitz, Wolfgang Wendland, Universität Stuttgart, Mathematisches Institut A, Germany) -1.105-

II. FLOW PROBLEMS

Integral Equation Solution of the Full Potential Equation for Three-Dimensional, Steady, Transonic Wing Flows
(Li-Chuang Chu, PRC Kentron Corp. Hampton, Virginia, U.S.A., Osama A. Kandil, Old Dominion University, Norfolk, Virginia, U.S.A) -2.1-

A Boundary Element Approach to the Inverse Design of Airfoils
(J.L. Monteiro Fernandes, Dept. Engenharia Mecânica, Instituto Superior Técnico, Lisboa, Portugal) -2.27-

BEM Solution of Oseen Flow Past a Circular Cylinder
(Mark Kelmanson, School of Mathematics, University of Bradford, England) -2.39-

Poiseuille Flow With Surface Tension in an Open Channel
(Erik B. Hansen, Laboratory of Applied Mathematical Physics, The Technical University of Denmark, Lyngby, Denmark) -2.51-

An Effective Solution of The Singularities at Multi-Domain Points For Laplace Problems
(E. Bruch, Laboratories of Hydrodynamics, Applied Hydraulics and Hydraulics Constructions of The University of Liège, Belgium) -2.65-

III. POTENTIAL PROBLEMS

Some Two-Dimensional Biharmonic Formulations
(M.A. Jaswon and H. Adibi, Dept. of Mathematics, City University, London, England) -3.1-

Boundary Element Method for Contact Heat Transfer
(L.J. Gray, A.L. Askew and G.E. Giles, Bergen Scientific Center, Norway, Oak Ridge National Laboratory, USA and Martin Marietta Energy Systems, Inc, USA. -3.21-

IV. INDUSTRIAL SOFTWARES

CA.ST.OR 3D - An industrial B.I.E. software
(A. Chaudouet-Miranda and M. Cristescu, CETIM, Senlis, France) -4.1-

V. TECHNIQUE ORIENTED SESSIONS: Chairman's summary

Some aspects of numerical integration techniques in the boundary element method
(H.L.G. Pina, Instituto Superior Tecnico, Lisbon, Portugal) -5.1-

Some Two-Dimensional Biharmonic Formulations

by

M.A. JASWON and H.ADIBI

Department of Mathematics

City University

London

January 1988

Abstract

Useful contributions has been made by Almansi, Massonnet, and Chakrabarty to the development of the biharmonic theory. Their formulations are reviewed and compared and interesting directions of future research work are identified.

1. Introduction

Two-dimensional elastostatics offers scope for specialised boundary formulations which are not applicable to three-dimensional problems. Everybody is familiar with Muskhelishvili's complex variable approach (1953b). However this only works well for domains which can be mapped analytically onto the unit circle. There exists a real-variable analogue of Muskhelishvili's formulations which was introduced by Almansi as long ago as 1897. With the advent of digital computers, it became the basis for a practicable method of attacking biharmonic boundary - value problems through the numerical solution of integral equations. A brief account of the theory appears in Section 2.

A significant conceptual advance was made in 1948 by Massonnet, who introduced 'fictitive' vector sources on the curved boundary of a plate which generated appropriate stress components in the interior. This may be regarded as a primitive ancestor of the three-dimensional vector integral equations later on constructed by Massonnet himself (1956), and more systematically by Kupradze (1965), which worked with hypothetical vector sources on the boundary. It would certainly be possible to discretise these equations over a curved surface and so create a BEM approach theoretically competitive with the established BEM approach. These issues are discussed more fully in Section 3.

A new and little known biharmonic representation has been introduced by Chakrabarty (1971), and further considered by Jaswon and Symm (1977). This works with biharmonic potentials generated directly from scalar sources on the boundary, and it can be exploited to provide a formulation of biharmonic boundary - value problems theoretically competitive with that of Almansi. An account of the theory appears in Section 4.

To summarize, biharmonic theory continues to flourish, with the aim of providing new and more adaptable boundary - value formulations concerned with two-dimensional elastostatics.

2. Almansi Representation

Let χ be a biharmonic function in some simply-connected domain B bounded by a contour ∂B, i.e. χ is continuous everywhere in $B+\partial B$, is differentiable to the fourth order in B, and satisfies the equation

$$\nabla^4\chi = \nabla^2(\nabla^2\chi) = 0 \quad \text{in } B. \tag{1}$$

It was shown by Almansi (1897) that we may always write

$$\chi = x\phi + \psi \quad (\text{or } y\phi + \psi) \quad \text{in } B+\partial B, \tag{2}$$

where ϕ, ψ are harmonic functions in B. This representation effectively reduces the theory of biharmonic functions to that of harmonic functions.

Note that ϕ, ψ are not unique for a given χ, since

$$x\phi + \psi = 0 \quad - \quad \phi = a + by, \quad \psi = -ax - bxy, \tag{3}$$

which is a serious limitation from the point of view of a numerical attack. An equivalent alternative to (2) is

$$\chi = r^2\phi + \psi \; ; \quad r^2 = x^2 + y^2 \quad \text{in} \quad B+\partial B, \tag{4}$$

but ϕ, ψ are now unique for a given χ. Since

$$r^2\phi + \psi = 0 \quad - \quad \begin{cases} \phi = r^{-1}\cos\theta, & \psi = -r\cos\theta \\ \text{or} & \\ \phi = r^{-1}\sin\theta, & \psi = -r\sin\theta, \end{cases} \tag{5}$$

which could not exist in B if $r = 0$ lies in B.

The first systematic account of biharmonic boundary-value problems was given by Hadamard (1908). In particular he showed that:

χ and $\frac{d\chi}{dn}$ ($\equiv\chi'$) given on ∂B – χ is uniquely determined in $B+\partial B$ (6)

this is just an existence theorem. To construct χ in B, we utilise the representation (4) and note that it holds on ∂B since χ, ϕ, ψ remain continuous at ∂B; i.e.

$$\chi = r^2\phi + \psi \quad \text{on} \quad \partial B. \tag{7}$$

Also an accompanying normal derivative relation holds on ∂B, i.e.

$$\chi' = (r^2\phi + \psi)' = r^2\phi' + 2rr'\phi + \psi' \quad \text{on} \quad \partial B. \tag{8}$$

These provide a pair of coupled functional relations for the four boundary quantities ϕ, ψ, ϕ', ψ' in terms of χ, χ' on ∂B. However only two of these are independent since, in principle, ϕ' is known on ∂B if ϕ is known on ∂B and similarly for ψ.

An effective way forward is to represent ϕ, ψ as simple – layer logarithmic potentials generated for sources on ∂B, i.e., we write

$$\begin{aligned} \phi(\underset{\sim}{p}) &= \int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq \quad ; \quad \underset{\sim}{p} \in B+\partial B \\ \Psi(\underset{\sim}{p}) &= \int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})\eta(\underset{\sim}{q})dq \quad ; \quad \underset{\sim}{q} \in \partial B. \end{aligned} \tag{9}$$

Here $\underset{\sim}{p}$ is the field point, $\underset{\sim}{q}$ is the source point, dq is an elementary interval of ∂B at $\underset{\sim}{q}$, σ, η are source – density distributions to be determined, and

$$g(\underset{\sim}{p},\underset{\sim}{q}) = \log|\underset{\sim}{p}-\underset{\sim}{q}|. \tag{10}$$

Also, Kellogg (1929),

$$\left.\begin{aligned} \phi'(\underset{\sim}{p}) &= \int_{\partial B} g'(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq + \pi\sigma(\underset{\sim}{p}) \\ \psi'(\underset{\sim}{p}) &= \int_{\partial B} g'(\underset{\sim}{p},\underset{\sim}{q})\eta(\underset{\sim}{q})dq + \pi\eta(\underset{\sim}{p}) \end{aligned}\right\} \quad \underset{\sim}{p},\underset{\sim}{q} \in \partial B, \tag{11}$$

where $g'(\underset{\sim}{p},\underset{\sim}{q})$ denotes the interior normal derivative of $g(\underset{\sim}{p},\underset{\sim}{q})$ at $\underset{\sim}{p}$ keeping $\underset{\sim}{q}$ fixed. Inserting (9), (11) into (7) and (8) yields a pair of coupled boundary integral equations for σ, η in terms of χ, χ'. With these known, we may generate ϕ, ψ and therefore χ throughout B. Exact or even approximate analytical solutions are out of the questions. However numerical solutions to acceptable accuracy may be achieved by fast computers implementing well established discretisation procedures. This approach has been applied to deflection problems of thin plates, including a numerical refutation of Hadamard's celebrated conjecture, and in two-dimensional stress analysis (see App I). More recently (Bhattacharyya & Symm 1979, 1984), it has been applied to two-dimensional displacement problems and to mixed boundary value problems, though here the representation (2) is preferable to (4).

3. Massonnet Approach

Despite its mathematical simplicity and ease of numerical implementation, the Almansi approach has not proved popular with engineers. Apart from competition with conventional BEM methods, it must be said that the sources concerned do not have any clear physical significance. Indeed they only serve to generate the potentials ϕ, ψ which are themselves subsidiary to χ. We owe to Massonnet (1948) the idea of sources on ∂B which generate directly the quantities of engineering interest in B. However the idea could hardly be carried much further at that time. A few years later (1956) there came a second paper in which he formulated the traction problem as a vector integral equation of the second kind, analogous to Fredholm's classical integral equation of the second kind for the Neumann problem of scalar potential theory. At about the same time, and independently, Kupradze (1965) produced similar equations building upon his systematic formulation of elasticity in terms of vector simple-layer and double-layer potentials. However these formulations were not taken up, partly because they were not too well understood and partly because few people had any experience with the numerical solution of such formidable equations. Note that they are vector integral equations over curved boundaries involving highly singular kernels. By the early 1960's, however, it became possible to attack successfully the corresponding scalar integral equations of classical potential theory, so providing numerical solutions of hitherto intractable problems concerned with torsion, capacitance, conformal mapping, and potential fluid flow.

These successes encouraged people to attack the more difficult boundary-value problems of elastostatics, both two - and three - dimensional, so eventually creating the field of BEM as we know it today. However modern BEM is based essentially upon the Betti-Somigliana formula, which works directly with boundary displacements and tractions, in preference to the Kupradze - Massonnet formulation which employs hypothetical source densities having no immediate physical significance. It is well understood that the two formulations are mathematically equivalent, but their comparative numerical properties have not been tested. The results of such tests would be of great interest.

4. Chakrabarty Representation

A two-dimensional approach utilising biharmonic potentials has been put forward by Chakrabarty (1971). This seems closer than conventional BEM to the spirit of Massonnet's original paper. First note that $r^2 \log r$ is a singular biharmonic function. More precisely

$$\left.\begin{aligned} \nabla^2(r^2 \log r) &= 4 + 4 \log r \\ \nabla^4(r^2 \log r) &= 4\nabla^2 \log r = 8\pi\delta(r) \end{aligned}\right\} . \tag{12}$$

If so

$$\left.\begin{aligned} \nabla^2(-r^2 + r^2 \log r) &= 4 \log r \\ \nabla^4(-r^2 + r^2 \log r) &= 8\pi\delta(r) \end{aligned}\right\} . \tag{13}$$

These properties suggest that a suitable biharmonic fundamental solution would be

$$G(\underset{\sim}{p},\underset{\sim}{q}) = -|\underset{\sim}{p}-\underset{\sim}{q}|^2 + |\underset{\sim}{p}-\underset{\sim}{q}|^2 \log |\underset{\sim}{p}-\underset{\sim}{q}| , \tag{14}$$

so allowing us to construct the biharmonic potential

$$\Omega(\underset{\sim}{p}) = \int_{\partial B} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq ; \qquad \underset{\sim}{p} \in B+\partial B, \quad \underset{\sim}{q} \in \partial B, \tag{15}$$

where ζ is scalar source - density distribution to be determined. It may be shown (Adibi, 1989) that:

(1) Ω exists and is continuous in B and is differentiable to the fourth order in B;

(2) Ω exists on ∂B and remains continuous as we approach ∂B from B;

(3) $\nabla^4\Omega = 0$ in B;

$$(4)\quad \frac{d\Omega(\underset{\sim}{p})}{dn} = \int_{\partial B} G'(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq, \qquad \underset{\sim}{p} \in \partial B.$$

An arbitrary χ in B may always be represented in the form

$$\chi = \Omega + \psi ; \qquad \nabla^2\psi = 0 \quad \text{in} \quad B. \tag{16}$$

To prove this, note that

$$\begin{aligned} \nabla^2\chi(\underset{\sim}{p}) = \nabla^2\Omega(\underset{\sim}{p}) &= 4\int_{\partial B} \nabla^2 G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq \\ &= 4\int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\zeta(\underset{\sim}{q})dq ; \qquad \left.\begin{matrix} \underset{\sim}{p} \in B+\partial B \\ \underset{\sim}{q} \in \partial B \end{matrix}\right\} . \end{aligned} \tag{17}$$

Now $\nabla^2\chi$ is a harmonic function in B, which may always be written in the form (9), i.e. may be determined on ∂B in terms of $\nabla^2\chi$ on ∂B. If so

$$\nabla^2 \{\chi(\underset{\sim}{p}) - 4\int_{\partial B} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq\} = 0,$$

which implies the representation (16). This holds on ∂B, so providing the boundary relation

$$\chi = \Omega + \psi \quad \text{on} \quad \partial B. \tag{18}$$

Also,

$$\chi' = \Omega' + \psi' \quad \text{on} \quad \partial B. \tag{19}$$

Accordingly given χ, χ' on ∂B and writing ψ in the form (9), we have a pair of coupled boundary integral equations for ζ, η, which may be solved numerically to provide Ω, ψ and therefore also χ in B.

The Chakrabarty approach has not yet been applied to any non-trivial problem. However it appears more suited than the Almansi approach for dealing with ring-shaped domains. This is because the singular biharmonic functions

$$x \log r, \quad y \log r, \tag{20}$$

could exist within such domains but are not covered by (7) unless ϕ, ψ become multi-valued, i.e.

$$\left.\begin{aligned} 2x \log r &= r^2\left(\frac{x \log r + y\theta}{r^2}\right) + (x \log r - y\theta) \\ 2y \log r &= r^2\left(\frac{y \log r - x\theta}{r^2}\right) + (y \log r + x\theta) \end{aligned}\right\} . \tag{21}$$

Accordingly we must extend the Almansi representation in this case by writing

$$\chi = r^2\phi + \psi + a\,x \log r + b\,y \log r, \tag{22}$$

where a, b are unknown coefficients balanced by two appropriate side conditions. Note that (see fig.1)

$$\phi(\underset{\sim}{p}) = \int_{\partial B_1} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq + \int_{\partial B_0} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq; \quad \underset{\sim}{p} \in B+\partial B_1+\partial B_0 , \tag{23}$$

so that

$$\int_{\partial B_1} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq = \log r \int_{\partial B_1} \sigma(\underset{\sim}{q})dq - r^{-2} \int_{\partial B_1} (\underset{\sim}{p}\cdot\underset{\sim}{q})\sigma(\underset{\sim}{q})dq + O(r^{-2})$$

$$\text{as} \quad r = |\underset{\sim}{p}| \to \infty . \tag{24}$$

If so the conditions

$$\int_{\partial B_1} q_1 \sigma(\underset{\sim}{q}) dq = 0, \quad \int_{\partial B_1} q_2 \sigma(\underset{\sim}{q}) dq = 0 \; ; \quad \underset{\sim}{q} = (q_1 \, , \, q_2), \tag{25}$$

ensure the absence of $\frac{\cos\theta}{r}$, $\frac{\sin\theta}{r}$ in ϕ, i.e. the absence of $r\cos\theta(\equiv x)$, $r\sin\theta(\equiv y)$ in $r^2\phi$, which are covered by ψ. By contrast, if we write

$$\Omega(\underset{\sim}{p}) = \int_{\partial B_1} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq + \int_{\partial B_0} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq; \quad \underset{\sim}{p} \in B+\partial B_1+\partial B_0 \, , \tag{26}$$

then

$$\int_{\partial B_1} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})dq = (-r^2 + r^2 \log r) \int_{\partial B_1} \zeta(\underset{\sim}{q})dq - 2x\log r \int_{\partial B_1} q_1 \zeta(\underset{\sim}{q})dq$$

$$- 2y\log r \int_{\partial B_1} q_2 \zeta(q)dq + \frac{r^2}{4} \sum_{n=-2}^{-\infty} \frac{r^n}{n+1}(b_{nc}\cos n\theta + b_{ns}\sin n\theta)$$

$$+ \psi \quad \text{as} \quad r = |\underset{\sim}{p}| \to \infty \, , \tag{27}$$

where ψ is a harmonic function of the form $\alpha x + \beta y + \delta\log r + O(r^{-1})$. Therefore $x\log r$, $y\log r$ are automatically accounted for by Ω as defined.

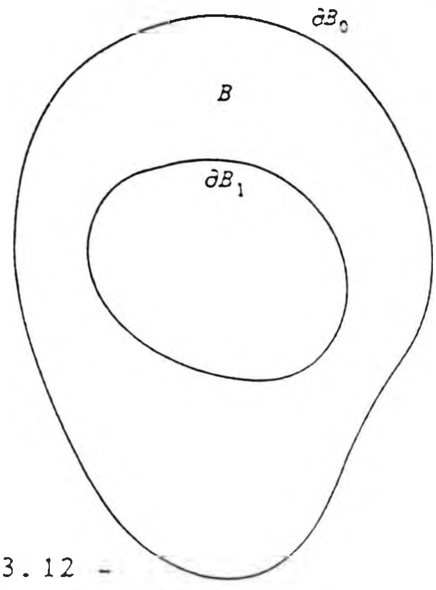


Fig.1

For infinite exterior domains the position is reversed since $x \log r$, $y \log r$ (and also $r^2 \log r$) must now be excluded from Ω by appropriate side conditions. Thus we now write

$$\chi = \Omega + \psi + \alpha x + \beta y + \gamma \tag{28}$$

where α, β, γ are unknown constants balanced by the three conditions

$$\int_{\partial B} q_1 \zeta(\underset{\sim}{q})dq = 0, \qquad \int_{\partial B} q_2 \zeta(\underset{\sim}{q})dq = 0, \qquad \int \zeta(\underset{\sim}{q})dq = 0. \tag{29}$$

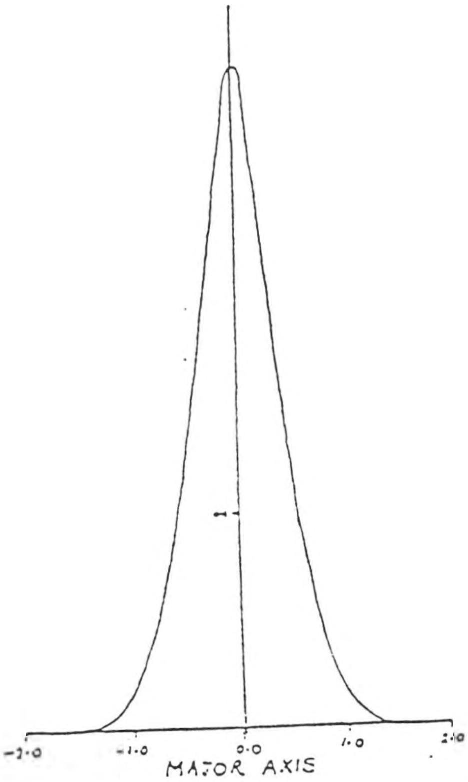
These may be readily understood by reference to the expansion (27), which shows that they ensure the absence respectively of $x \log r$, $y \log r$, and $r^2 \log r$ in Ω, as required by Hadamard's uniqueness–existence theorem for exterior domains. In place of these we have the linear terms $\alpha x + \beta y + \gamma$ which are not covered by either Ω or ψ. Note that the Almansi representation (7) must be trivially extended in this case by writing

$$\chi = r^2 \phi + \psi + k, \tag{30}$$

where k is an unknown constant balanced by the side condition $\int_{\partial B} \sigma(\underset{\sim}{q})dq = 0$, which ensures the absence of $\log r$ in ϕ and therefore of $r^2 \log r$ in χ.

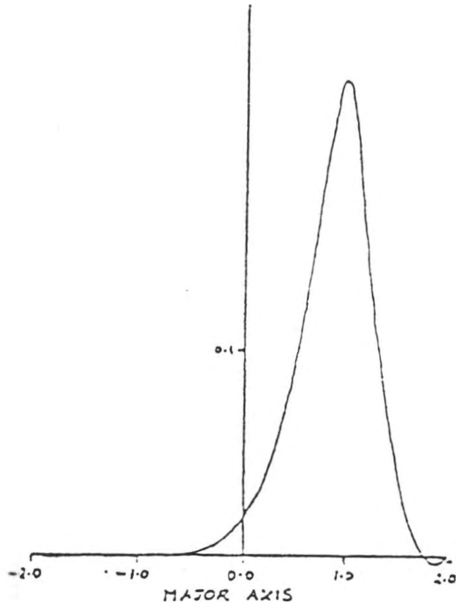
Appendix I. The clamped elliptic plate under a concentrated transverse load

Hadamard conjectured that a thin clamped plate under concentrated transverse loading suffers a transverse deflection, which everywhere has the same direction as the load. This conjecture is known to be true for a circle, but it may be demonstrated that the conjecture fails for a clamped elliptic plate of axial ratio greater than about $\sqrt{2}:1$ (Brown and E.Jaswon, 1971). For details see figures 1–4, noting that the vertical axis measures the deflection, where the dimensionless quantity $\frac{ka}{8\pi D}$ (a=2, k is a constant and D being the flexural rigidity), is taken as unit. The source point is indicated by an arrow on the major axis.



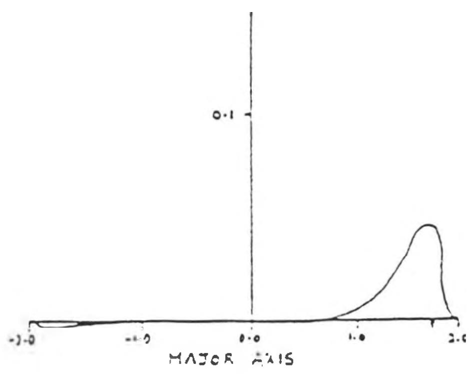
2:1/3 ELLIPSE. LOAD AT 0.00

FIG.1.



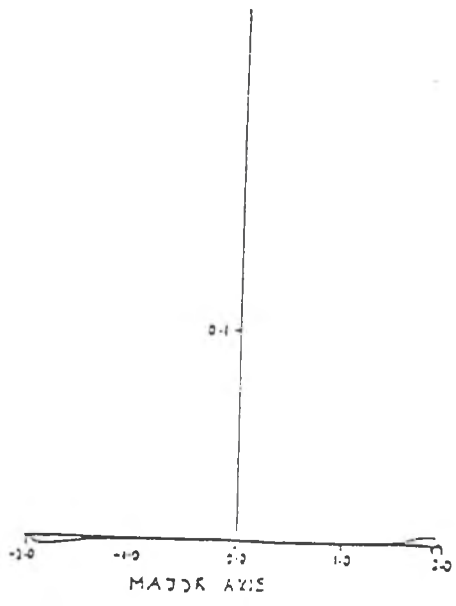
2:1/3 ELLIPSE. LOAD AT 1.00

FIG.2



2:1/3 ELLIPSE. LOAD AT 1.75

FIG.3



2:1/3 ELLIPSE. LOAD AT 1.90

FIG.4

Appendix II. Biharmonic Green's Function

Let

$$G = Ar^2 + Br^2 \log r, \qquad (1)$$

r being the radial distance from the origin. For any choice of A and B, G is a Biharmonic Green's Function, i.e. it is a fundamental solution of the biharmonic equation

$$\nabla^4 \chi = 0. \qquad (2)$$

In (13) we have chosen $A = -1$ and $B = 1$, i.e.

$$G = -r^2 + r^2 \log r, \qquad (3)$$

which has the convenient properties

$$\left.\begin{aligned} \nabla^2 G &= 4 \log r \\ \nabla^4 G &= 8\pi\delta(r) \end{aligned}\right\}, \qquad (4)$$

see (15).

It should be noted that if we choose $A = -1$ and $B = 2$, i.e.

$$G = -r^2 + 2r^2 \log r \qquad (5)$$

or more generally

$$G(\underset{\sim}{p},\underset{\sim}{q}) = -|\underset{\sim}{p}-\underset{\sim}{q}|^2 + 2|\underset{\sim}{p}-\underset{\sim}{q}|\log|\underset{\sim}{p}-\underset{\sim}{q}|,$$

then the exterior expansion of the biharmonic potential

$$\Omega(\underset{\sim}{p}) = \int_{\partial B} G(\underset{\sim}{p},\underset{\sim}{q})\zeta(\underset{\sim}{q})d\underset{\sim}{q} \quad , \qquad (6)$$

does not cover the term $\alpha x + \beta y$. However it covers a constant, which may be utilise in some specialised problems, for further details see (Adibi, 1983).

<u>References</u>

1. Adibi, H. (1989). "Some two-dimensional biharmonic formulations". Ph.D. Thesis, City University, London.

2. Almansi, E. (1897). Sull' integrazione dell' equazione differentiale $\Delta^{2n} = 0$. Annali di mathematica pura et applicate, Series III, 2, 1-51.

3. Bhattacharyya, P.K. and Symm, G.T. (1979). A novel formulation and solution of the plane elastostatic displacement problem. Comp & Maths, with Appls. vol.6, pp. 443-448.

4. Bhattacharyya, P.K. and Symm, G.T. (1984). New formulation and solution of the plane elastostatic traction problem. Appl. Math. Modelling, 1984, vol.B, August.

5. Brown, I.C. and Jaswon, E. (1971). "The Clamped Elliptic plate under A concentrated Transverse LOAD". Research Memorandum, The City University, London.

6. Chakrabarty, S.K. (1971), "Numerical Solutions of some integral equations related to biharmonic problems." Ph.D. Thesis, Indian Institute of Technology, Kharagpur.

7. Hadamard, J. (1908). Memoire sur le probleme d'analyse relatial equilibre des plaques elastiques encastrees. Mem. pres. diu. sav., Acad. Sci. Inst. France, 33(4), 1-126.

8. Jaswon, M.A. and Symm, G.T. (1977). Integral equation methods in potential theory and elastostatics.

9. Kellogg, O.D. (1929). "Foundations of Potential Theory". Springer, Berlin.

10. Kupradze, V.D. (1965). "Potential Methods in the Theory of Elasticity". Israel program for Scientific Translations, Jerusalem.

11. Massonnet, C.E. (1948). Resolution graphomecanique des problems géneraux de l'elasticite plane. Bull. Centre Et. Rech. Essais Sc. Genie civil. vol.4, 169-180.

12. Massonnet, C.E. (1956). Solution generale du probleme aux tensions de l'elasticite tridimensionnell le Congres Int. Mec. Appl., Bruxelles, vol.8, 168-180.

13. Muskhelishvili, N.I. (1953b). "Some Basic Problems of The Mathematical Theory of Elasticity." Noordhoff, Groningen.

PART III

THE TWO-DIMENSIONAL PAPKOVICH-NEUBER FORMULA

Chapter 9

The Two-Dimensional Papkovich-Neuber Formula

9.1 Introduction

Throughout this chapter it will be assumed that the bodies under the action of external forces are elastic and isotropic. The three-dimensional Papkovich-Neuber formula provides a general solution of the homogeneous Cauchy-Navier equation in terms of four independent harmonic functions. The corresponding two-dimensional formula provides a general two-dimensional solution in terms of three independent harmonic functions. On the other hand, Muskhelishvili's complex variable formalism provides a general two-dimensional solution which involves two independent analytic functions, i.e. equivalent to only two independent harmonic functions. A reconciliation between these two formulations may be effected by considering the role played by the Airy stress function, Airy, G.B. (1862).

9.2 Analysis of Papkovich-Neuber formula

According to Papkovich, P.F. (1932) and Neuber, H. (1934), an arbitrary linear elastic displacement field u_j; $j = 1,2,3$ has the representation

$$2\mu u_j = h_j-\kappa \frac{\partial}{\partial x_j}(x_1h_1+x_2h_2+x_3h_3+f); \; j = 1,2,3 \qquad (9.2.1)$$

where x_1, x_2, x_3 are rectangular cartesian co-ordinates and h_1, h_2, h_3, f are harmonic functions; μ is the shear modulus and $\kappa^{-1} = 4(1-\nu)$ where ν is Poisson's ratio. The generality of this representation has been proved by Stokes-Helmholtz (Sommerfeld, 1964). Note that the scalar harmonic function f may not always be necessary. This possibility has been discussed by Eubanks, R.A. and Sternberg, E. (1956), Jaswon, M.A. and Symm, G.T. (1977) and Millar, R.F. (1984). The two-dimensional case of redundancy has been discussed by Jaswon, M.A. and Shidfar, A. (1980), see Appendix II.

Choosing

$$h_1 = \frac{F_1}{r}, \quad h_2 = h_3 = f = 0; \qquad r = |\underset{\sim}{r}|, \tag{9.2.2}$$

where F_1 is a constant having the dimension of force and $\underset{\sim}{r} = (x_1, x_2, x_3)$ is the position vector of a field point, we obtain

$$u_j = \frac{F_1}{2\mu r}\,\delta_{1j} - \frac{\kappa}{2\mu}\frac{\partial}{\partial x_j}\left(\frac{F_1 x_1}{r}\right); \quad j = 1,2,3 \tag{9.2.3}$$

which is Kelvin's solution for the displacement field generated by a concentrated force of magnitude $2\pi F_1$ acting in the x_1-direction at $x_1 = x_2 = x_3 = 0$.

Putting $x_3 = 0$, $h_3 = 0$ in (9.2.1) we obtain the two-dimensional formula

$$2\mu u_j = h_j - \kappa\frac{\partial}{\partial x_j}(x_1 h_1 + x_2 h_2 + f); \quad j = 1,2 \tag{9.2.4}$$

where h_1, h_2, f are two-dimensional harmonic functions.

This immediately invites comparison with the real-variable analogue of Muskhelishvili's complex-variable formula. However the latter only involves two independent harmonic functions, whilst (9.2.4) involves three independent harmonic functions. Our essential aim is to investigate this apparent inconsistency and explain how it may be resolved.

9.3 Muskhelishvili's formula

Muskhelishvili, N.I. (1953) has introduced the complex representation

$$
\begin{aligned}
2\mu(u_1+iu_2) &= (3-4\nu)\mathcal{H} - z\overline{\mathcal{H}'} - \overline{\mathcal{F}'} \\
&= 4(1-\nu)\mathcal{H} - [\overline{\mathcal{H}} + z\overline{\mathcal{H}'}] - \overline{\mathcal{F}'} ,
\end{aligned}
\tag{9.3.1}
$$

where $z = x_1 + ix_2$, $\mathcal{H}$ and $\mathcal{F}$ are analytic functions of z, $\overline{\mathcal{H}}$ and $\overline{\mathcal{F}}$ their complex conjugate functions, and primes denote differentiation with respect to z. To break this down we write

$$
\left.
\begin{aligned}
\mathcal{H} &= H_1 + iH_2, & \overline{\mathcal{H}} &= H_1 - iH_2 \\
\mathcal{F} &= F_1 + iF_2, & \overline{\mathcal{F}} &= F_1 - iF_2
\end{aligned}
\right\} ,
\tag{9.3.2}
$$

and note that

$$
\overline{\mathcal{H}'} = \frac{\partial H_1}{\partial x_1} - i\,\frac{\partial H_2}{\partial x_1} , \qquad \overline{\mathcal{F}'} = \frac{\partial F_1}{\partial x_1} - i\,\frac{\partial F_2}{\partial x_1} . \tag{9.3.3}
$$

If so

$$z\overline{\mathcal{H}}' = (x_1+ix_2)\left(\frac{\partial H_1}{\partial x_1} - i\,\frac{\partial H_2}{\partial x_1}\right)$$

$$= \left(x_1\,\frac{\partial H_1}{\partial x_1} + x_2\,\frac{\partial H_2}{\partial x_1}\right) + i\left(x_2\,\frac{\partial H_1}{\partial x_1} - x_1\,\frac{\partial H_2}{\partial x_1}\right)$$

$$= \left(x_1\,\frac{\partial H_1}{\partial x_1} - x_2\,\frac{\partial H_2}{\partial x_1}\right) + i\left(x_2\,\frac{\partial H_2}{\partial x_2} + x_1\,\frac{\partial H_1}{\partial x_2}\right),$$

which gives from (9.3.1)

$$\left.\begin{aligned} 2\mu u_1 &= 4(1-\nu)H_1 - \frac{\partial}{\partial x_1}(x_1H_1+x_2H_2+F_1) \\ 2\mu u_2 &= 4(1-\nu)H_2 - \frac{\partial}{\partial x_2}(x_1H_1+x_2H_2+F_1) \end{aligned}\right\} . \qquad (9.3.4)$$

This formula may also be obtained by direct real analysis, see Appendix III. On incorporating the factor $4(1-\nu)$ into H_1, etc., formula (9.3.4) appears identical with (9.2.4) except that the conjugate harmonic functions H_1, H_2 replace the uncoupled harmonic functions h_1, h_2. Since (9.2.4), (9.3.4) have the same degree of generality, there immediately arises the question as to why (9.2.4) involves three independent harmonic functions whilst (9.3.4) involves only two. This will be considered in the next section.

9.4 Analysis of $\chi = x_1h_1+x_2h_2+f$

It is convenient to write h_1^*, h_2^* as the conjugate harmonic functions to h_1, h_2 respectively, and we note that

$(h_1{}^*) = -h_1$, etc. Clearly $(h_1 - h_2{}^*)$, $(h_1{}^* + h_2)$ form a pair of conjugate harmonic functions, so allowing us to write

$$\chi = x_1 h_1 + x_2 h_2 + f = x_1\left(\frac{h_1 - h_2{}^*}{2}\right) + x_2\left(\frac{h_1{}^* + h_2}{2}\right)$$

$$+ x_1\left(\frac{h_1 + h_2{}^*}{2}\right) - x_2\left(\frac{h_1{}^* - h_2}{2}\right) + f, \qquad (9.4.1)$$

with the immediate identifications

$$H_1 = \frac{h_1 - h_2{}^*}{2}, \qquad H_2 = \frac{h_1{}^* + h_2}{2}. \qquad (9.4.2)$$

Also the remaining terms in (9.4.1) - see (9.4.4) below - form a harmonic function on bearing in mind

$$\nabla^2(x_1 h_1) = 2\frac{\partial h_1}{\partial x_1}, \quad \nabla^2(x_2 h_1{}^*) = 2\frac{\partial h_1{}^*}{\partial x_2} = 2\frac{\partial h_1}{\partial x_1}, \quad \text{etc.,} \qquad (9.4.3)$$

or alternatively noting that - see (1.3.1) -

$$(x_1 + ix_2)(H_1 + iH_2) = (x_1 H_1 - x_2 H_2) + i(x_1 H_2 + x_2 H_1), \qquad (9.4.4)$$

which shows that the expressions in brackets are harmonic functions, so providing the identification

$$F_1 = x_1\left(\frac{h_1 + h_2{}^*}{2}\right) - x_2\left(\frac{h_1{}^* - h_2}{2}\right) + f. \qquad (9.4.5)$$

Accordingly χ really involves only two independent harmonic functions by virtue of the transformation (9.4.1), giving the formula

$$2\mu u_j = 4(1-\nu)H_j - \frac{\partial \chi}{\partial x_j}\ ; \quad j = 1,2 \qquad (9.4.6)$$

in place of (9.2.4). This is in agreement with (9.3.4) and has the advantage of connecting H_1, H_2, F_1 with h_1, h_2, f.

The formula (9.2.4) always remains valid but suffers from an excessive generality. For example, choosing

$$h_1 = \phi, \quad h_2 = -\phi^*, \quad f = -(x_1\phi - x_2\phi^*), \tag{9.4.7}$$

we obtain

$$\chi = x_1\phi - x_2\phi^* - (x_1\phi - x_2\phi^*) \equiv 0, \tag{9.4.8}$$

which provides a pure shear field

$$2\mu u_1 = \phi, \quad 2\mu u_2 = -\phi^*, \tag{9.4.9}$$

i.e. a non-trivial field characterised by null dilatation and rotation components:

$$\left.\begin{aligned}
2\mu\Delta &= 2\mu\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = \left(\frac{\partial \phi}{\partial x_1} - \frac{\partial \phi^*}{\partial x_2}\right) = 0 \\
2\mu\omega &= -\mu\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = -\frac{1}{2}\left(\frac{\partial \phi}{\partial x_2} + \frac{\partial \phi^*}{\partial x_1}\right) = 0 \\
2\mu e_{12} &= \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial \phi}{\partial x_2} - \frac{\partial \phi^*}{\partial x_1}\right) = \frac{\partial \phi}{\partial x_2}
\end{aligned}\right\} . \tag{9.4.10}$$

By contrast formula (9.4.6) gives a null displacement field since $H_1 = 0$, $H_2 = 0$, $F_1 = 0$ as follows from putting $h_1 = \phi$, $h_2 = -\phi^*$ in (9.4.2) and noting that $\chi = 0$.

There is no difficulty in representing the displacement field (9.4.9) by formula (9.4.6), since we may always write a pair of conjugate harmonic functions in the form

$$\phi = \frac{\partial h}{\partial x_1}, \quad \phi^* = -\frac{\partial h}{\partial x_2}\ ; \quad \nabla^2 h = 0, \tag{9.4.11}$$

so allowing us to choose $H_1 = 0$, $H_2 = 0$, $F_1 = -h$ (i.e. $\chi = -h$). It may be verified that ∇, ω are as given in (9.4.10) and that $e_{12} \neq 0$ for this field.

9.5 Introduction of stress function

The function $\chi = x_1h_1 + x_2h_2 + f$ is biharmonic as follows from the property

$$\nabla^2\chi = 2\left(\frac{\partial h_1}{\partial x_1} + \frac{\partial h_2}{\partial x_2}\right), \tag{9.5.1}$$

and it could therefore qualify as a stress function. To examine this possibility we compute the stress components associated with formula (9.2.4):

$$\left.\begin{aligned} p_{11} &= \frac{1}{2}\left(\frac{\partial h_1}{\partial x_1} - \frac{\partial h_2}{\partial x_2}\right) + \kappa\,\frac{\partial^2\chi}{\partial x_2^2} \\ p_{22} &= \frac{1}{2}\left(\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_1}\right) + \kappa\,\frac{\partial^2\chi}{\partial x_1^2} \\ p_{12} &= \frac{1}{2}\left(\frac{\partial h_1}{\partial x_2} + \frac{\partial h_2}{\partial x_1}\right) - \kappa\,\frac{\partial^2\chi}{\partial x_1\partial x_2} \end{aligned}\right\} . \tag{9.5.2}$$

Clearly χ would be the Airy stress function provided

$$\frac{\partial h_1}{\partial x_1} = \frac{\partial h_2}{\partial x_2}, \qquad \frac{\partial h_1}{\partial x_2} = -\frac{\partial h_2}{\partial x_1}, \tag{9.5.3}$$

i.e. h_1, h_2 must be conjugate harmonic functions, in which case

$$H_1 = \frac{h_1 - h_2^*}{2} = h_1, \qquad H_2 = \frac{h_1^* + h_2}{2} = h_2, \tag{9.5.4}$$

so may also be seen directly. Therefore the redundant generality of (9.2.4) has been traced to the absence of a stress function, by contrast with the formula (9.4.6).

It may be remarked that Muskhelishvili's formulation is based upon a complex stress function

$$\bar{z}\mathcal{H} + \mathcal{F}, \tag{9.5.5}$$

where all the symbols have already been defined. Breaking (9.5.5) down into its components yields

$$\chi = \mathrm{Re}\{\bar{z}\mathcal{H} + \mathcal{F}\} = x_1 H_1 + x_2 H_2 + F_1, \tag{9.5.6}$$

as expected.

Finally we remark that

$$x_1 H_1 + x_2 H_2 = 2x_1 H_1 + f, \tag{9.5.7}$$

on bearing in mind that

$$x_1 H_1 - x_2 H_2 = -f\ ;\ \nabla^2 f = 0. \tag{9.5.8}$$

This immediately provides the Almansi representation for χ exploited by Jaswon, M.A. and Symm, G.T. (1977), i.e.

$$\chi = 2x_1 H_1 + g \ ; \quad \nabla^2 g = 0, \quad g = F_1 + f, \tag{9.5.9}$$

with the associated displacement formula

$$\left.\begin{aligned} 2\mu u_1 &= 4(1-\nu)H_1 - \frac{\partial\chi}{\partial x_1} \\ 2\mu u_2 &= 4(1-\nu)H_2 - \frac{\partial\chi}{\partial x_2} \end{aligned}\right\} . \tag{9.5.10}$$

Boundary Elements X

Vol.1: Mathematical and Computational Aspects

Editor:
C.A. Brebbia

Computational Mechanics Publications
Southampton Boston

Springer-Verlag Berlin Heidelberg New York
London Paris Tokyo

FOREWORD

The success of the Boundary Element Method in the last ten years has been mainly due to the development of the technique as a practical engineering tool. The year 1978 was a milestone in this regard, as it was then that classical boundary integral equations were interpreted in a different way. The emphasis then on the use of quasi-variational concepts and transformations of the type already known in finite elements gave origin to a new methodology which was well adapted to the direct boundary integral formulation. In spite of the rapid advances in boundary element research in subsequent years, these definitions are still accepted as the basis of the method.

One of the most versatile interpretations of the technique is provided by the use of Lagrangian multipliers, which permits a simple deduction of the boundary integrals starting with the differential equations governing the problem. Unfortunately this concept, which is not only mathematically correct but very elegant, has been misunderstood, perhaps because classical boundary integral scientists were not familiar with variational calculus. This branch of mathematics is one of the most difficult to comprehend, possibly because it entails understanding a series of philosophical concepts rather than purely algebraic expressions.

Another area of confusion has been the relationship between boundary elements and other numerical techniques. Erroneous concepts about symmetry and the correct interpretation of the integral equations in terms of energy in the nineteen seventies were followed by a better understanding of the basic principles in the nineteen eighties. Many of the original problems seem to have been simply due to attempts to force the boundary element method to conform to finite element concepts.

The most important development in the last 10 years has been the awareness of the engineering and scientific community that boundary elements is a new and more powerful technique than finite elements, and that the latter can be seen as a particular case of BEM rather than the other way round.

From the history of science point of view it is interesting to point out that while finite elements was a method predominantly based on approximations, boundary elements combine them with powerful analytical solutions. This combination, which in a way was a revaluation of past work is the more powerful aspect of boundary elements and the one that gives the method great accuracy of results and versatility.

From the engineering point of view, boundary elements can be seen in many cases as a computational technique which is better conditioned than finite elements for analysis and design. While FEM analysis demands an unne essary discretization of the domain, boundary elements is a function of tl surface configuration only. Further advances are still required to elimina the need to discretize surfaces by treating each of them as one large eleme instead. The concept of elements is in itself an FEM idea and the next sta should be to develop boundary surfaces or patches.

The work carried out in non-linear and time dependent problems has up now also suffered from the application of old FE concepts. Undue emphas is put on the subdivision of the continuum into a fixed grid consisting the so-called cells. With the exception of some papers such as those dealir with moving internal boundaries and others related to the dual reciproci method. most applications of BEM in non-linear and time dependent prol lems are too closely related to similar work previously done using fini elements. In this regard it is necessary for BEM scientists to learn to b more audacious and innovative in their thinking in order to realize the fu potentialities of the new method.

These ten years of BEM research have been particularly rewarding for th editor who has seen the technique developing from its humble beginning into a powerful engineering method. What in 1978 was an eccentric ob session which was deemed to be unnecessary, has now become not only a established research technique but a powerful tool for engineering analysis

Carlos Brebbia

CONTENTS

SECTION 1 - BASIC FORMULATIONS

INVITED PAPER
The Generalized Boundary Element Method for Nonlinear Problems 3
N. Tosaka and K. Kakuda

INVITED PAPER
Variational Principles for Boundary Element Formulations in Structural Mechanics 19
C. Polizzotto

A Hybrid Displacement Variational Formulation of BEM 33
T.G.B. DeFigueiredo and C.A. Brebbia

The Hybrid Boundary Element Method in Elastostatics: Overview of the Theory and Examples 43
N.A. Dumont

INVITED PAPER
Modern Coupling of FEM and BEM with Mixed Variational Formulations 59
E. Schnack, I. Becker and N. Karaosmanoglu

Some Computational Aspects of the BEM Variational Formulation 75
P. D'Aguanno, T. Panzeca and M. Zito

SECTION 2 - MATHEMATICAL ASPECTS

INVITED PAPER
The Boundary Element Method for Linear Elliptic Equations with Variable Coefficients 91
D.L. Clements

New Expressions of the Fundamental Solution of Axisymmetric Helmholtz-Type Equations 97
M. Tsuchimoto and T. Honma

Boundary Element Analysis System Based on a Formulation with Relative Quantity 111
H. Kisu and T. Kawahara

INVITED PAPER
Further Applications of P-Adaptive Boundary Elements 123
M. Cerrolaza and E. Alarcon

Lagrangian Node Generation in BEM using Half-Plane Formulation 137
Dj. Modjtahedi and M. Rahbari

Estimates of Errors Arising in the Computation of Forces Acting on Rigid Bodies in Stokes and Potential Flows due to Variations in Geometry 147
E. Gavze

Equations with Discontinuous Coefficients 161
F. Hartmann and P. Schoepp

Spline Integral Equation Method for Rotating Disc of Variable Thickness 167
X-y. Liu and J-j. Zheng

INVITED PAPER
The Two-Dimensional Papkovich-Neuber Formula 177
M.A. Jaswon and H. Adibi

INVITED PAPER
An Accuracy Postprocessor for Boundary Element Analysis 187
J.J. Rencis and K-Y. Jong

INVITED PAPER
The H-P Version of the Galerkin Boundary Element Method for Integral Equations on Polygons and Open Arcs 205
E.P. Stephan

SECTION 3 - NUMERICAL ASPECTS

INVITED PAPER
Integration Methods for Singular Boundary Element Integrands 219
W.S. Hall

INVITED PAPER
Quadrature Methods for Singular and Nearly Singular Integrals in 3-D Boundary Element Method 237
K. Hayami and C.A. Brebbia

A Flexible Quadrature Method over Cauchy Singularities in Boundary Integral Equations 265
H. Niessner

Accurate Numerical Integration of Singular Kernels in the Two-Dimensional Boundary Element Method 279
M. Sato, S. Yoshioka, K. Tsukui and R. Yuuki

A Numerical Integration Scheme for the Galerkin Approach in Boundary Elements 297
P. Parreira

The Boundary Representation Angle in 2D BEM Potential Problems 313
C.F. Flood and W.S. Hall

A Simple Procedure to Solve the Mild-Slope Equation using BEM and Perturbation Technique - Part I 331
R. Rangogni

SECTION 4 - COMPUTATIONAL ASPECTS

An Efficient Technique for Reducing Domain Integrals to the Boundary 347
J.P.S. Azevedo and C. A. Brebbia

Numerical Integration for Supercomputers in Three-Dimensional Potential Problems 363
H. Tsuboi and Y. Ishii

SECTION 5 - EDGE FUNCTION METHOD

INVITED PAPER
The Edge Function Method (EFM) for 2-D Regions with Arbitrary Boundaries, in Fracture Elastodynamics, Thin Plates and Laplace/Poisson Problems 381
P.M. Quinlan

INVITED PAPER
The Edge-Function Method for Eigenvalues and Eigenfunctions for Two-Dimensional Elastodynamics in Polygonal Regions 397
J.J. Grannell and M.J.A. O'Callaghan

INVITED PAPER
The Lap-Edge Method for 2-D Problems in Heat Conduction for Loaded Polygonal Regions with Circular Cavities 417
B.G.Mc Enery and M.H. Quinlan

SECTION 6 - COUPLING

INVITED PAPER
The Coupling of BEM and FEM - A Brief Review 431
G.C. Hsiao

INVITED PAPER
Mixed Methods and the Marriage between Finite Elements and Boundary Elements 447
A. Bossavit

INVITED PAPER
BEM-FEM Coupling for Screen and Infinite Domain Potential Problems 459
J.J. Grannell

A Coupled Formulation of Finite and Boundary Element Methods in Two-Dimensional Elasticity 477
C.K. Choi and J.S. Ahn

INVITED PAPER
A Combined Finite Element Boundary Element Technique for Stress Analysis 493
J.L. Wearing, M.A. Sheikh and A.J. Hickson

Boundary Element and Finite Element Methods for the Coupled Fluid-Structure Interaction Problem 509
S. Amini and P.J. Harris

Fluid-Structure Coupling Boundary Element Method for Analyzing Free Vibration of Axisymmetric Thick-Walled Tanks 521
Y-Y. Huang, S-K. Wang and W-M. Cheng

A Combined Boundary Integral FE Method for Large Domain Problems 535
D. Givoli

INVITED PAPER
The Modelling of Highly Nonlinear Water Waves: A step toward a Numerical Wave Tank 549
S.T. Grilli, J. Skourup and I.A. Svendsen

SECTION 7 - INVERSE PROBLEMS

INVITED PAPER
Some Recent Advances in Boundary Element Research for Inverse Problems 567
M. Tanaka

Shape Identification of a Free Surface with Tractions of Constant Magnitude 583
R.A. Meric

Source Inversion of Acoustic Emission based on the Boundary Element Formulation 593
M. Ohtsu

INVITED PAPER
Shape Optimization of Electrode and Insulator 603
H. Tsuboi

SECTION 8 - RESEARCH IN PROGRESS

INVITED PAPER
One Decade of Engineering Research on BIE-BEM in China 621
Q. Du and Z. Yao

INVITED PAPER
Boundary Element Research in Brazil 631
J.A. Costa Jr.

INVITED PAPER

The Two-Dimensional Papkovich-Neuber Formula

M.A. Jaswon and H. Adibi
Department of Mathematics, The City University, London. EC1V 0HB, UK

The Papkovich-Neuber formula provides a general solution of the homogeneous Navier-Cauchy equation in terms of four independent harmonic functions. The corresponding two-dimensional formula provides a general two-dimensional solution in terms of three independent harmonic functions. On the other hand, Muskhelishvili's complex variable formalism provides a general two-dimensional solution which involves two independent analytic functions, i.e. equivalent to only two independent harmonic functions. A reconciliation between these two formulations may be effected by considering the role played by the Airy stress function.

1. Introduction

According to Papkovich (1932) and Neuber (1934), an arbitrary linear elastic displacement field u_j; $j = 1, 2, 3$ has the representation

$$2\mu u_j = h_j - \kappa \frac{\partial}{\partial x_j}(x_1 h_1 + x_2 h_2 + x_3 h_3 + f); \; j = 1, 2, 3 \qquad (1)$$

where x_1, x_2, x_3 are rectangular cartesian co-ordinates and h_1, h_2, h_3, f are harmonic functions; μ is the shear modulus and $\kappa^{-1} = 4(1-\nu)$ where ν is Poisson's ratio. For instance choosing

$$h_1 = \frac{F_1}{r}, \quad h_2 = h_3 = f = 0, \qquad (2)$$

where F_1 is a constant having the dimension of force, we obtain

$$u_j = \frac{F_1}{2\mu r} \delta_{1j} - \frac{\kappa}{2\mu} \frac{\partial}{\partial x_j} \left(\frac{F_1 x_1}{r}\right); \qquad j = 1, 2, 3 \qquad (3)$$

which is Kelvin's solution for the displacement field generated by a concentrated force of magnitude $2\pi F_1$ acting in the x_1-direction at $x_1 = x_2 = x_3 = 0$.

Putting $x_3 = 0$, $h_3 = 0$ in (1) we obtain the two-dimensional formula

$$2\mu u_j = h_j - \kappa \frac{\partial}{\partial x_j}(x_1 h_1 + x_2 h_2 + f); \quad j = 1, 2 \qquad (4)$$

where h_1, h_2, f are two-dimensional harmonic functions.

This immediately invites comparison with the real-variable analogue of Muskhelishvili's complex variable formula. However the latter only involves two independent harmonic functions, whilst (4) involves three independent harmonic functions. Our essential aim is to investigate this apparent inconsistency and explain how it may be resolved.

2. Muskhelishvili's Formula

Muskhelishvili (1953) has introduced the complex representation

$$2\mu(u_1 + iu_2) = (3 - 4\nu)\mathcal{H} - z\overline{\mathcal{H}'} - \overline{\mathcal{F}'} \tag{5}$$

$$= 4(1 - \nu)\mathcal{H} - [\mathcal{H} + z\overline{\mathcal{H}'}] - \overline{\mathcal{F}'}, \tag{6}$$

where $z = x_1 + ix_2$, $\mathcal{H}$ and $\mathcal{F}$ are analytic functions of z, $\overline{\mathcal{H}}$ and $\overline{\mathcal{F}}$ their complex conjugate functions, and primes denote differentiation with respect to z. To break this down we write

$$\left.\begin{aligned} \mathcal{H} &= H_1 + iH_2, & \overline{\mathcal{H}} &= H_1 - iH_2 \\ \mathcal{F} &= F_1 + iF_2, & \overline{\mathcal{F}} &= F_1 - iF_2 \end{aligned}\right\}, \tag{7}$$

and note that

$$\left.\begin{aligned} \overline{\mathcal{H}'} &= \frac{\partial H_1}{\partial x_1} - i\frac{\partial H_2}{\partial x_1} = \frac{\partial H_2}{\partial x_2} + i\frac{\partial H_1}{\partial x_2} \\ \overline{\mathcal{F}'} &= \frac{\partial F_1}{\partial x_1} - i\frac{\partial F_2}{\partial x_1} = \frac{\partial F_1}{\partial x_1} + i\frac{\partial F_1}{\partial x_2} \end{aligned}\right\}. \tag{8}$$

If so

$$z\overline{\mathcal{H}'} = (x_1 + ix_2)(\frac{\partial H_1}{\partial x_1} - i\frac{\partial H_2}{\partial x_1})$$

$$= (x_1\frac{\partial H_1}{\partial x_1} + x_2\frac{\partial H_2}{\partial x_1}) + i(x_2\frac{\partial H_1}{\partial x_1} - x_1\frac{\partial H_2}{\partial x_1})$$

$$= (x_1\frac{\partial H_1}{\partial x_1} + x_2\frac{\partial H_2}{\partial x_1}) + i(x_2\frac{\partial H_2}{\partial x_2} + x_1\frac{\partial H_1}{\partial x_2}),$$

which gives from (6):

$$\left.\begin{aligned} 2\mu u_1 &= 4(1-\nu)H_1 - \frac{\partial}{\partial x_1}(x_1H_1 + x_2H_2 + F_1) \\ 2\mu u_2 &= 4(1-\nu)H_2 - \frac{\partial}{\partial x_2}(x_1H_1 + x_2H_2 + F_1) \end{aligned}\right\} . \qquad (9)$$

On incorporating the factor $4(1-\nu)$ into H_1, etc, formula (9) appears identical with (4) except that the conjugate harmonic functions H_1, H_2 replace the uncoupled harmonic functions h_1, h_2. Since (4), (9) have the same degree of generality, there immediately arises the question as to why (4) involves three independent harmonic functions whilst (9) involves only two. This will be considered in the next section.

3. Analysis of $\chi = x_1h_1 + x_2h_2 + f$

It is convenient to write $h_1{}^*$, $h_2{}^*$ as the conjugate harmonic functions to h_1, h_2 respectively, and we note that $(h_1{}^*)^* = -h_1$, etc. Clearly $(h_1-h_2{}^*)$, $(h_1{}^*+h_2)$ form a pair of conjugate harmonic functions, so allowing us to write

$$\chi = x_1h_1 + x_2h_2 + f = x_1\left(\frac{h_1 - h_2{}^*}{2}\right) + x_2\left(\frac{h_1{}^* + h_2}{2}\right) \qquad (10)$$

$$+ x_1\left(\frac{h_1 + h_2{}^*}{2}\right) - x_2\left(\frac{h_1{}^* - h_2}{2}\right) + f,$$

with the immediate identifications

$$H_1 = \frac{h_1 - h_2{}^*}{2}, \quad H_2 = \frac{h_1{}^* + h_2}{2}. \qquad (11)$$

Also the remaining terms in (10) form a harmonic function on bearing in mind (or see footnote p.8)

$$\nabla^2(x_1h_1) = 2\frac{\partial h_1}{\partial x_1}, \quad \nabla^2(x_2h_1{}^*) = 2\frac{\partial h_1{}^*}{\partial x_2} = 2\frac{\partial h_1}{\partial x_1}, \text{ etc,} \qquad (12)$$

so providing the identification

$$F_1 = x_1\left(\frac{h_1 + h_2{}^*}{2}\right) - x_2\left(\frac{h_1{}^* - h_2}{2}\right) + f. \qquad (13)$$

Accordingly χ involves only two independent harmonic functions by virtue of the transformation (10), giving the formula

$$2\mu u_j = 4(1-\nu)H_j - \frac{\partial\chi}{\partial x_j}; \quad j = 1, 2 \qquad (14)$$

in place of (4). This is consistent with (9) but has the advantage of connecting H_1, H_2, F_1 with h_1, h_2, f.

The formula (4) always remains valid but suffers from an excessive generality. For example, choosing

$$h_1 = \phi, \qquad h_2 = -\phi^*, \qquad f = -(x_1\phi - x_2\phi^*), \tag{15}$$

we obtain

$$\chi = x_1\phi - x_2\phi^* - (x_1\phi - x_2\phi^*) \equiv 0, \tag{16}$$

which provides a pure shear field

$$2\mu u_1 = \phi, \quad 2\mu u_2 = -\phi^*, \tag{17}$$

i.e. a non-trivial field characterised by null dilatation and rotation components:

$$\left.\begin{aligned}
2\mu\Delta &= 2\mu\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = \left(\frac{\partial\phi}{\partial x_1} - \frac{\partial\phi^*}{\partial x_2}\right) = 0,\\
2\mu\omega &= -\mu\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = -\tfrac{1}{2}\left(\frac{\partial\phi}{\partial x_2} + \frac{\partial\phi^*}{\partial x_1}\right) = 0,\\
2\mu e_{12} &= \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \tfrac{1}{2}\left(\frac{\partial\phi}{\partial x_2} - \frac{\partial\phi^*}{\partial x_1}\right) = \frac{\partial\phi}{\partial x_2}.
\end{aligned}\right\} \tag{18}$$

By contrast formula (14) gives a null displacement field since $H_1 = 0$, $H_2 = 0$ as follows from putting $h_1 = \phi$, $h_2 = -\phi^*$ in (11) and noting that $\chi = 0$ under the transformation (10).

There is no difficulty in representing the displacement field (17) by formula (14), since we may always write a pair of conjugate harmonic functions in the form

$$\phi = \frac{\partial h}{\partial x_1}, \quad \phi^* = -\frac{\partial h}{\partial x_2}; \quad \nabla^2 h = 0, \tag{19}$$

so allowing us to choose $H_1 = 0$, $H_2 = 0$, $F_1 = -h$ (i.e. $\chi = -h$). It may be verified that Δ, ω are as given in (18) and that $e_{12} \neq 0$ for this field.

4. Introduction of stress function

The function $\chi = x_1 h_1 + x_2 h_2 + f$ is biharmonic as follows from the property*

$$\nabla^2 \chi = 2\left(\frac{\partial h_1}{\partial x_1} + \frac{\partial h_2}{\partial x_2}\right), \tag{20}$$

and it could therefore qualify as a stress function. To examine the possibility we compute the stress components associated with formula (4):

$$\left.\begin{aligned} p_{11} &= \tfrac{1}{2}\left(\frac{\partial h_1}{\partial x_1} - \frac{\partial h_2}{\partial x_2}\right) + \kappa \frac{\partial^2 \chi}{\partial x_2^2} \\ p_{22} &= \tfrac{1}{2}\left(\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_1}\right) + \kappa \frac{\partial^2 \chi}{\partial x_1^2} \\ p_{12} &= \tfrac{1}{2}\left(\frac{\partial h_1}{\partial x_2} + \frac{\partial h_2}{\partial x_1}\right) - \kappa \frac{\partial^2 \chi}{\partial x_1 \partial x_2} \end{aligned}\right\} . \tag{21}$$

Clearly χ would be the Airy stress function provided

$$\frac{\partial h_1}{\partial x_1} = \frac{\partial h_2}{\partial x_2}, \quad \frac{\partial h_1}{\partial x_2} = -\frac{\partial h_2}{\partial x_1}, \tag{22}$$

---

* Note that $(x_1 + ix_2)(H_1 + iH_2) = (x_1H_1 - x_2H_2) + i(x_1H_2 - x_2H_1)$ showing that the expressions in brackets are harmonic functions

i.e. h_1, h_2 must be conjugate harmonic functions, in which case

$$H_1 = \frac{h_1 - h_2^*}{2} = h_1, \qquad H_2 = \frac{h_1^* + h_2}{2} = h_2, \tag{23}$$

as may also be seen directly. Therefore the redundant generality of (4) has been traced to the absence of a stress function, by contrast with the formula (14).

It may be remarked that Muskhelishvili's formulation is based upon a complex stress function

$$\bar{z}\Phi + \Psi, \tag{24}$$

where all the symbols have already been defined. Breaking (24) down into its components yields

$$\chi = \operatorname{Re}\{\bar{z}\Phi + \Psi\} = x_1H_1 + x_2H_2 + F_1, \tag{25}$$

as expected.

Finally we remark that

$$x_1H_1 + x_2H_2 = 2x_1H_1 + f, \tag{26}$$

on bearing in mind that *

$$x_1H_1 - x_2H_2 = -f; \quad \nabla^2 f = 0.$$

This immediately provides the Almansi representation for χ exploited by Jaswon and Symm (1977), i.e.

$$\chi = 2x_1H_1 + g; \quad \nabla^2 H_1 = 0, \quad g = F_1 + f,$$

with the associated displacement formula

$$\left.\begin{aligned} 2\mu u_1 &= 4(1-\nu)H_1 - \kappa \frac{\partial \chi}{\partial x_1} \\ 2\mu u_2 &= 4(1-\nu)H_2 - \kappa \frac{\partial \chi}{\partial x_2} \end{aligned}\right\} \quad . \tag{27}$$

Acknowledgements

Thanks are due to Prof. D.A. Spence, Imperial College, London, for having pointed out the inconsistency between the Papkovich-Neuber and Muskhelishvili formulations.

References

1. Jaswon, M.A. and Symm, G.T. (1977). Integral equation methods in potential theory and elastostatics.

2. Muskhelishvili, N.I. (1953b). "Some Basic problems of The Mathematical Theory of Elasticity". Noordhoff, Groningen.

3. Neuber, H. (1934). Ein neuer Ansatz zür losung räumlicher probleme der Elastizitätstheorie.j. Appl. Math. Mech. (ZAMM), 14, 203-312.

4. Papkovich. P.F. (1932). Solution général des équations différentielles fondamentales d'élasticité exprimée par trois fonctions harmoniques. C.R. Acad. Sci. Paris, 195, 513-515.

APPENDICES

Appendix I

Almansi x, y-forms

Given a harmonic function h, there exists another harmonic function ϕ such that

$$h = \frac{\partial \phi}{\partial x} \; ; \quad \nabla^2 \phi = 0, \tag{AI.1}$$

(Bhattacharyya, 1975). Now if χ be a biharmonic function, $\nabla^2\chi$ is harmonic and accordingly it may be written as

$$\nabla^2 \chi = 2 \frac{\partial \phi}{\partial x} \; ; \quad \nabla^2 \phi = 0. \tag{AI.2}$$

If so, clearly $x\phi$ is a particular biharmonic solution, therefore this equation has the general solution

$$\chi = x\phi + \psi \; ; \quad \nabla^2\phi = \nabla^2\psi = 0. \tag{AI.3}$$

Similarly we may obtain

$$\chi = y\phi + \psi \; ; \quad \nabla^2\phi = \nabla^2\psi = 0. \tag{AI.4}$$

It may be shown that neither the x - nor y-forms are unique (see Chapter 1).

For instance let $\chi = r^2 \log r$; $r^2 = x^2 + y^2$, if so

$$\nabla^2\chi = 4 + 4\log r = 2 \frac{\partial \phi}{\partial x} \; ; \quad \nabla^2\phi = 0, \tag{AI.5}$$

in which ϕ is a harmonic function to be determined up to an arbitrary harmonic function of y. In this case

$$\phi = 2\int(\log r + 1)dx + \eta(y) \; ; \quad \nabla^2\eta(y) = 0, \tag{AI.6}$$

which gives

$$\phi = 2(x\log r - y\theta) + \eta(y), \tag{AI.7}$$

as may be verified by differentiation. The simplest possibility for $\eta(y)$ is $\eta = 0$.

If so, from (AI.3)

$$\chi = x[2(x\log r - y\theta)] + \psi \ ; \quad \nabla^2\psi = 0, \tag{AI.8}$$

in which case

$$\begin{aligned}\psi &= \chi - x\phi \\ &= r^2\log r - 2x^2\log r + 2xy\theta \\ &= (y^2-x^2)\log r + 2xy\theta,\end{aligned} \tag{AI.9}$$

which is a harmonic function, since

$$\psi = -R_e(z^2\log z); \quad z = x + iy. \tag{AI.10}$$

Appendix II

Sternberg-Eubanks redundancy

It was first pointed out by Eubanks, R.A. and Sternberg, E., (1956) that the harmonic function f, in the three-dimensional Papkovich-Neuber formula

$$2\mu \underset{\sim}{U} = \underset{\sim}{h} - \kappa\nabla(\underset{\sim}{r}.\underset{\sim}{h}+f); \quad \underset{\sim}{U} = (u_1, u_2, u_3),$$
$$\underset{\sim}{h} = (h_1, h_2, h_3), \underset{\sim}{r} = (x_1, x_2, x_3), \kappa^{-1} = 4(1-\nu) \qquad \text{(AII.1)}$$

is essentially redundant. A simplified argument by Jaswon, M.A. and Symm, G.T. (1977) runs as follows. To say that f is redundant means that we can find a harmonic vector $\underset{\sim}{V}$ which satisfies the equation

$$-\kappa\nabla f = \underset{\sim}{V} - \kappa\nabla(\underset{\sim}{r}.\underset{\sim}{V}), \qquad \text{(AII.2)}$$

for an arbitrary choice of harmonic function f. If $\underset{\sim}{V}$ exists then it may clearly be written $\underset{\sim}{V} = \nabla S$ where S is a scalar harmonic function, so providing the vector equation

$$-\kappa\nabla f = \nabla S - \kappa\nabla(\underset{\sim}{r}.\nabla S), \qquad \text{(AII.3)}$$

for S, which yields the scalar equation

$$-\kappa f = S - \kappa\underset{\sim}{r}.\nabla S, \qquad \text{(AII.4)}$$

for S in terms of f. Equation (AII.4) may be transformed into the spherical polar form

$$kS + r\frac{dS}{dr} = f; \quad x_1, x_2, x_3 \longrightarrow r, \theta, \psi$$
$$k = -\kappa^{-1} = 4(\nu-1) \qquad \text{(AII.5)}$$

which is a variant of the Bergman-Schiffer equation (1953) with the particular solution

$$S_0 = r^{-k} \int^r \rho^{k-1} f(\rho,\theta,\psi)\,d\rho. \tag{AII.6}$$

Now the integral in the right-hand side of (AII.6) is generally a harmonic function, e.g. if

$$f = r^n P_n{}^m(\cos\theta)\exp(im\psi);\ |m| \le n;\ n = 0,1,2,\ldots, \tag{AII.7}$$

then

$$S_0 = \frac{r^n}{k+n} P_n{}^m(\cos\theta)\exp(im\psi);\ n \ne -k, \tag{AII.8}$$

which is a harmonic function in the same domain as f. Similarly if

$$f = r^{-n-1} P_n{}^m(\cos\theta)\exp im\psi;\ |m| \le n;\ n = 0,1,2,\ldots, \tag{AII.9}$$

then

$$S_0 = \frac{r^{-n-1}}{k-n-1} P_n{}^m(\cos\theta)\exp(im\psi);\quad n \ne k-1. \tag{AII.10}$$

Evidently a breakdown occurs in (AII.8) if k is an integer and $n = -k \ge 0$, and similarly for (AII.10) if $n = k-1 \ge 0$. On physical grounds ν lies within the range $0 < \nu \le \frac{1}{2}$, so that k lies within the range $-4 < k \le -2$, i.e. $k = -2$ or -3 for the breakdown possibilities. Apart from these possibilities, we conclude that f may always be eliminated without loss of generality.

This argument can be readily adapted to the two-dimensional Papkovich-Neuber formula (9.2.4). Thus (AII.5) gets replaced

$$kS + r\frac{dS}{dr} = f; \quad x_1, x_2 \longrightarrow r, \theta, \tag{AII.11}$$

yielding the particular solution

$$S_0 = r^{-k}\int_0^r \rho^{k-1} f(\rho,\theta)\,d\rho. \tag{AII.12}$$

For instance if

$$f = r^n \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta\ ; \quad n = 0, \pm1, \pm2, \ldots, \tag{AII.13}$$

then

$$S_0 = \frac{r^n}{n+k}\begin{matrix}\cos\\ \sin\end{matrix}\, n\theta; \quad n = 0, \pm1, \pm2, \ldots, \tag{AII.14}$$

yielding the breakdown possibilities $n = -k = 2, 3$ as before.

The above analysis in principle holds for Muskhelishvili's displacement formula - see (9.3.4) -

$$\left.\begin{aligned} 2\mu u_1 &= 4(1-\nu)H_1 - \frac{\partial}{\partial x_1}(x_1H_1 + x_2H_2 + F) \\ 2\mu u_2 &= 4(1-\nu)H_2 - \frac{\partial}{\partial x_2}(x_1H_1 + x_2H_2 + F) \end{aligned}\right\} . \tag{AII.15}$$

However it would not be relevant here, since H_1, H_2 form a pair of conjugate harmonic functions which become uncoupled if we eliminate f by virtue of (AII.2). Clearly the advantage of losing f is always outweighed by the disadvantage of losing the stress function associated with (AII.15).

Appendix III

Direct determination of the displacement components via Almansi

Consider the strain-displacement relations

$$e_{11} = \frac{\partial u_1}{\partial x_1}, \quad e_{22} = \frac{\partial u_2}{\partial x_2}, \quad e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \qquad \text{(AIII.1)}$$

and stress-strain relation

$$p_{\alpha\beta} = 2\mu e_{\alpha\beta} + \lambda\Delta\delta_{\alpha\beta}; \; \alpha,\beta = 1,2; \Delta = e_{11} + e_{22}, \qquad \text{(AIII.2)}$$

(Love, 1927); if so, utilising

$$p_{11} = \frac{\partial^2 U}{\partial x_2^{\,2}}, \quad p_{22} = \frac{\partial^2 U}{\partial x_1^{\,2}}, \quad p_{12} = -\frac{\partial^2 U}{\partial x_1 \partial x_2}, \qquad \text{(AIII.3)}$$

in which U is Airy's stress function, we get

$$p_{11} = 2\mu e_{11} + \lambda\Delta = 2\mu e_{11} + \lambda(e_{11}+e_{22})$$

$$= 2\mu e_{11} + \frac{\lambda}{2(\lambda+\mu)}(p_{11}+p_{22})$$

$$= 2\mu \frac{\partial u_1}{\partial x_1} + \frac{\lambda}{2(\lambda+\mu)} \nabla^2 U. \qquad \text{(AIII.4)}$$

Accordingly

$$2\mu \frac{\partial u_1}{\partial x_1} = p_{11} - \frac{\lambda}{2(\lambda+\mu)}\nabla^2 U = \nabla^2 U - \frac{\partial^2 U}{\partial x_1^{\,2}} - \frac{\lambda}{2(\lambda+\mu)}\nabla^2 U,$$

i.e.

$$2\mu \frac{\partial u_1}{\partial x_1} = -\frac{\partial^2 U}{\partial x_1^{\,2}} + \frac{\lambda + 2\mu}{2(\lambda+\mu)} \nabla^2 U. \qquad \text{(AIII.5)}$$

Integrating, we obtain

$$2\mu u_1 = -\frac{\partial U}{\partial x_1} + \frac{\lambda + 2\mu}{2(\lambda+\mu)} \int \nabla^2 U dx_1, \qquad \text{(AIII.5a)}$$

and similarly

$$2\mu u_2 = -\frac{\partial U}{\partial x_2} + \frac{\lambda + 2\mu}{2(\lambda+\mu)} \int \nabla^2 U dx_2. \qquad \text{(AIII.5b)}$$

On the other hand noting Appendix 1, there exists a harmonic function H such that

$$\nabla^2 U = 4 \frac{\partial H}{\partial x_1} = 4 \frac{\partial H^*}{\partial x_2} , \qquad \text{(AIII.6)}$$

where H^* is the conjugate harmonic function to H, accordingly from (AIII.5a), (AIII.5b) we obtain

$$\left.\begin{aligned} 2\mu u_1 &= -\frac{\partial U}{\partial x_1} + \frac{2(\lambda+2\mu)}{(\lambda+\mu)} H \\ 2\mu u_2 &= -\frac{\partial U}{\partial x_2} + \frac{2(\lambda+2\mu)}{(\lambda+\mu)} H^* \end{aligned}\right\} , \qquad \text{(AIII.7)}$$

i.e.

$$\left.\begin{aligned} 2\mu u_1 &= -\frac{\partial U}{\partial x_1} + 4(1-\nu)H \\ 2\mu u_2 &= -\frac{\partial U}{\partial x_2} + 4(1-\nu)H^* \end{aligned}\right\} , \qquad \text{(AIII.8)}$$

in agreement with (9.3.4).

References and Bibliography

Airy, G.B. (1862). "On the strains in the interior of beams". Phil. Trans. Roy. Soc., 153, 49.

Almansi, E. (1897). "Sull' integrazione dell' equazione differentiale $\Delta^{2n} = 0$". Annali di matematica pura et applicate, series III, 2, 1-51.

Bergman, S. and Schiffer, M. (1953). Kernel Functions and Elliptic Differential Equations in Mathematical Physics". Academic Press, New York and London.

Bhattacharyya, P.K. (1975). "Biharmonic Potential Theory and its Applications". Ph.D. Thesis, The City University, London.

Bhattacharyya, P.K. and Symm, G.T. (1979). "A novel formulation and solution of the plane elastostatic displacement problem". Comp & Maths, with Appls. Vol. 6, pp. 443-448.

Bhattacharyya, P.K. and Symm, G.T. (1984). "New formulation and solution of the plane elastostatic traction problem". Appl. Math. Modelling, 1984, Vol. B, August.

Brown, I.C. and Jaswon, E. (1971). "The clamped elliptic plate under a concentrated transverse load". Research Memorandum, The City University, London.

Chakrabarty, S.K. (1971). "Numerical solutions of some integral equations related to biharmonic problems". Ph.D. Thesis, Indian Institute of Technology, Kharagpur.

Durell, C.V. and Robson, A. (1936). "Advanced Trigonometry". G. Bell and Sons, Ltd. London.

Eubanks, R.A. and Sternberg, E. (1956). "On the completeness of the Boussinesq-Papkovich stress functions". J. Rational Mech. Anal., Vol. 5, No.5.

Hadamard, J. (1908). "Memoire sur le probleme d'analyse relatif a l'equilibre des plaques elastiques encastrees". Mem. Pres. Div., Sav., Acad. Sci. Inst. France, 33(4), 1-126.

Jaswon, M.A. and Shidfar, A. (1980). "On a differential equation of Bergman and Schiffer". J. Maths. & Phys. Madras., Vol. 14, No.3. M.A. Biot Commemorative Volume.

Jaswon, M.A. and Symm, G.T. (1977). "Integral Equation Methods in Potential Theory and Elastostatics". Academic Press, London, New York, San Francisco.

Kellogg. O.D. (1929). "Foundations of potential theory". Springer, Berlin.

Love, A.E.H. (1927). "A Treatise on the Mathematical Theory of Elasticity". Fourth Edition. Cambridge University Press.

MacMillan, W.D. (1958). "The Theory of the Potential". Dover.

Massonnet, C.H. (1948). "Résolution graphomécanique des problèms generaux de l'élasticité plane". Bull. Centre Et. Rech. Essais Sc. Génie Civil. Vol. 4, 169-180.

Massonnet, C.H. (1956). "Solution générale du probleme aux tensions de l'élasticite tridimensionnelle". ge Congrès Int. Méc. Appl., Bruxelles, Vol. 8, 168-180.

Millar, R.F (1984). "On the completeness of the Papkovich potentials". Quarterly of Applied Mathematics, Vol. XLI, 385-393.

Muskhelishvili, N.I. (1953). "Some Basic Problems in the Mathematical Theory of Elasticity". Noordhoff, Groningen.

Neuber, H. (1934). "Ein neuer Ansatz zür Losung räumlicher probleme der Elastizitätstheorie". Z.A.M.M., 14, 203-212.

Papkovich, P.F. (1932). "Solution générale des équations, differentielles fondamentales d'élasticité exprimée par trois fonctions harmoniques". C.R. Acad. Sci. Paris, 195, 513-515.

Sommerfeld, A. (1964). "Mechanics of Deformable Bodies". Academic Press.

Wylie, C.R. and Barrett, L.C. (1983). "Advanced Engineering Mathematics", International Student Edition. McGraw-Hill Book Company Japan, Ltd.