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Free vibration analysis of FG plate with piezoelectric layers on elastic foundation using refined shear deformation theory

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Abstract
This study presents an analytical solution for free vibration analysis of functionally graded (FG) core integrated with piezoelectric layers and resting on elastic foundation. The four-variable refined plate theory is utilized which predicts parabolic variation of transverse shear stresses across the plate thickness, satisfies the zero traction on the plate surfaces and does not need the shear correction factor. Using both Hamilton's principle and Maxwell equation, the Equations of motions for simply supported rectangular plates resting on elastic foundation are obtained and the Navier method is adopted for solution of equations. Natural frequencies for different examples are obtained and they are compared with other common plate theories. It can be concluded that besides the simplicity of the presented formulation, this theory which does not need for shear correction factor, is very accurate in analysis of plates integrated with piezoelectric layers resting on elastic foundation and is comparable to other theories (the first order shear deformation theory (FSDT) and third order shear deformation theory). Also effects of power law index, thickness ratio and foundation parameter, on the natural frequency of plates have been investigated.

Keywords: piezoelectric layer; FG plate; four-variable theory; free vibration; elastic foundation.

Introduction

Functionally graded materials (FGMs) are a kind of composite, which their material properties change very smoothly and continuously from one surface to another. One of the most important FGMs is metal-ceramic combination which gains superior properties than each constituent. Functionally graded structures Due to their effective properties, are widely used in many industries such as light weight structures for aircrafts and space industries, high efficiency engine components, shipbuilding industries, medical instruments, biomechanics and automotive industries. Also piezoelectric materials due to their intrinsic coupled electromechanical properties are widely used in smart structures.

Plates rested on elastic foundations are very usual in structures. There exist a lot of various models of elastic foundations, and the simplest one is proposed by Winkler [1]. Many investigators have proposed various higher order shear deformation theories (HSDTs). A very recently developed HSDT is two-variable refined plate theory that contains only two unknown parameters, predict the parabolic transverse shear stresses across the thickness and satisfies zero traction conditions on free surfaces. Shimpi [2] developed this theory for isotropic plates and then extended to orthotropic plates by Shimpi and Patel [3] and Thai and Kim [4]. In two-variable refined plate theory the plate middle surface was assumed to be unstrained and therefore only the bending effects are considered. The four-variable refined plate theory was introduced by adding two other parameters regarding the in-plane displacements of plate middle surface. Benachour *et al* [5] presented analytical solution for free vibration of FG plates using this theory. There are various investigations on analyses of FGMs with embedded or surface bonded piezoelectric layers, acting as sensors and actuators. Askari Farsangi *et al* [6] and Askari Farsangi and Saidi [7] used Mindlin plate theory and derived an analytical solution for free vibration of hybrid piezoelectric laminated and FG plates. Mitchell and Reddy [8] presented a higher order shear deformation theory for composite laminates with piezoelectric layer. Hasani Baferani *et al* [9] proposed accurate solution for free vibration analysis of FG plates resting on elastic foundation. Thai and Choi [10] developed a refined shear deformation theory for free vibration of FG plates on elastic foundation, they investigated effects of boundary conditions and foundation parameters.

Theory and Formulations

Consider a simply supported rectangular plate of length a, width b and total thickness h_t with two bonded piezoelectric layer at top and bottom, resting on elastic foundation as shown in Figure 1. The thickness of elastic core is h and thickness of each piezoelectric layer is h_p. The right-handed Cartesian coordinate system is located at corner of the middle plane of plate. The four variable refined plate theories are employed for analysis of free vibration of the plate.

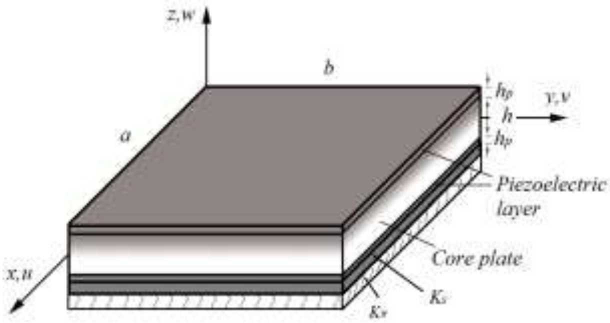

Figure 1. Geometry of FG plate resting on elastic foundation

According to assumptions of refined plate theory the displacement field (u in x-direction, v in y-direction and w in z-direction) is introduced as below [4]:

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w_b}{\partial x} - f\frac{\partial w_s}{\partial x}$$
$$v(x,y,z) = v_0(x,y) - z\frac{\partial w_b}{\partial y} - f\frac{\partial w_s}{\partial y} \quad (1)$$
$$w(x,y,z) = w_b(x,y) + w_s(x,y)$$

where u_0 and v_0 are the in-plane displacement of mid-plane in the x and y direction and w_b and w_s are bending and shear component of transverse displacement, respectively. The strain-displacement relationships are given by:

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_{xy}\end{Bmatrix} = \begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} + z\begin{Bmatrix}\chi_x^b\\ \chi_y^b\\ \chi_{xy}^b\end{Bmatrix} + f\begin{Bmatrix}\chi_x^s\\ \chi_y^s\\ \chi_{xy}^s\end{Bmatrix} \quad (2)$$
$$\begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} = g\begin{Bmatrix}\gamma_{yz}^s\\ \gamma_{xz}^s\end{Bmatrix}, \varepsilon_z = 0$$

where

$$\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} = \begin{Bmatrix}\frac{\partial u_0}{\partial x}\\ \frac{\partial v_0}{\partial y}\\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\end{Bmatrix}, \quad \begin{Bmatrix}\chi_x^b\\ \chi_y^b\\ \chi_{xy}^b\end{Bmatrix} = \begin{Bmatrix}-\frac{\partial^2 w_b}{\partial x^2}\\ -\frac{\partial^2 w_b}{\partial y^2}\\ -2\frac{\partial^2 w_b}{\partial x\,\partial y}\end{Bmatrix},$$
$$\begin{Bmatrix}\chi_x^s\\ \chi_y^s\\ \chi_{xy}^s\end{Bmatrix} = \begin{Bmatrix}-\frac{\partial^2 w_s}{\partial x^2}\\ -\frac{\partial^2 w_s}{\partial y^2}\\ -2\frac{\partial^2 w_s}{\partial x\,\partial y}\end{Bmatrix}, \quad \begin{Bmatrix}\gamma_{yz}^s\\ \gamma_{xz}^s\end{Bmatrix} = \begin{Bmatrix}\frac{\partial w_s}{\partial y}\\ \frac{\partial w_s}{\partial x}\end{Bmatrix}, \quad (3)$$
$$f = -\frac{1}{4}z + \frac{5}{3}z\left(\frac{z}{h_t}\right)^2, \quad g = \frac{5}{4} - 5\left(\frac{z}{h_t}\right)^2$$
$$h_t = h + 2h_p$$

The effective material properties of FG plates which change very smoothly and continuously from one surface to another can be expressed by following relation:

$$E(z) = E_m + (E_c - E_m)\left(\frac{1}{2} + \frac{z}{h}\right)^n$$
$$\rho(z) = \rho_m + (\rho_c - \rho_m)\left(\frac{1}{2} + \frac{z}{h}\right)^n \quad (4)$$

where n is power law index and subscripts m and c denote the property of metal and ceramic constituents, respectively. Linear constitutive equations for piezoelectric layer which couples the elastic and electric fields are given as below:

$$\{\sigma\} = [Q]\{\varepsilon\} - [e]\{E\}$$
$$\{D\} = [e]^t\{\varepsilon\} + [\Xi]\{E\} \quad (5)$$

where Q is the stress-reduced stiffness, e is the piezoelectric constants matrix, $,\Xi$ is the dielectric constant matrix, E is the electric field intensity vector and (σ, ε) are stress and strain tensors. The coefficients Q_{ij} for a FG plate can be written as:

$$Q_{11} = Q_{22} = \frac{E(z)}{1-\nu^2}, \quad Q_{12} = \frac{\nu E(z)}{1-\nu^2}$$
$$Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2(1+\nu)} \quad (5)$$

The electric field E is derivable from an electrostatic potential ϕ as following equation:

$$E_i = -\phi_{,i} \quad i = 1,2,3 \quad (6)$$

where electrostatic potential through the thickness of the piezoelectric layer is defined as [11]:

$$\phi(x,y,z,t) = \begin{cases}\varphi(x,y,t)\left[1 - \left(\frac{z-h-h_p/2}{h_p/2}\right)^2\right], & h \le z \le h + h_p\\ \varphi(x,y,t)\left[1 - \left(\frac{-z-h-h_p/2}{h_p/2}\right)^2\right], & -h-h_p \le z \le -h\end{cases} \quad (7)$$

The governing equations will be obtained using the principle of minimum potential energy:

$$\delta(U + V - T) = 0 \quad (8)$$

The equations of motion can be obtained by minimizing the total potential energy with respect to u_0, v_0, w_b and w_s:

$$\delta u_o: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0\ddot{u} - I_1\frac{\partial \ddot{w}_b}{\partial x} - J_1\frac{\partial \ddot{w}_s}{\partial x}$$
$$\delta v_o: \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = I_0\ddot{v} - I_1\frac{\partial \ddot{w}_b}{\partial y} - J_1\frac{\partial \ddot{w}_s}{\partial y}$$
$$\delta w_b: \frac{\partial^2 M_x^b}{\partial x^2} + \frac{\partial^2 M_y^b}{\partial y^2} + 2\frac{\partial^2 M_{xy}^b}{\partial x\,\partial y} - K_w(w_b + w_s) + K_s\nabla^2(w_b + w_s) = I_0(\ddot{w}_b + \ddot{w}_s) + I_1\left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}\right) - I_2\nabla^2\ddot{w}_b - J_2\nabla^2\ddot{w}_s \quad (9)$$
$$\delta w_s: \frac{\partial^2 M_x^s}{\partial x^2} + \frac{\partial^2 M_y^s}{\partial y^2} + 2\frac{\partial^2 M_{xy}^s}{\partial x\,\partial y} + \frac{\partial Q_{yz}}{\partial y} + \frac{\partial Q_{xz}}{\partial x} - K_w(w_b + w_s) + K_s\nabla^2(w_b + w_s) = I_0(\ddot{w}_b + \ddot{w}_s) + J_1\left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}\right) - J_2\nabla^2\ddot{w}_b - K_2\nabla^2\ddot{w}_s$$

The parameters K_w and K_s are the Winkler and Pasternak parameters for elastic foundation. The stress resultants N, M and Q are as bellow:

$$(N_x, N_y, N_{xy}) = \int_{-h-h_p}^{h+h_p}(\sigma_x, \sigma_y, \sigma_{xy})dz,$$
$$(M_x^b, M_y^b, M_{xy}^b) = \int_{-h-h_p}^{h+h_p}(\sigma_x, \sigma_y, \sigma_{xy})z\,dz,$$
$$(M_x^s, M_y^s, M_{xy}^s) = \int_{-h-h_p}^{h+h_p}(\sigma_x, \sigma_y, \sigma_{xy})f\,dz, \quad (10)$$
$$(Q_{yz}, Q_{xz}) = \int_{-h-h_p}^{h+h_p}(\sigma_{yz}, \sigma_{xz})dz,$$

the mass moments of inertia are defined as**:**

$$I_i = \int_{-h-h_p}^{h+h_p} \rho z^i \, dz\,, \quad J_i = -\frac{1}{4} I_i + \frac{5}{3{h_t}^2} I_{i+2}$$
$$K_2 = \frac{1}{16} I_2 - \frac{5}{6{h_t}^2} I_4 + \frac{25}{9{h_t}^4} I_6, \tag{11}$$

The electric potential in piezoelectric layer satisfies Maxwell's equation in the following integral form:

$$\int_{h}^{h+h_p} \vec{\nabla}\cdot\vec{D}\, dz + \int_{-h-h_p}^{-h} \vec{\nabla}\cdot\vec{D}\, dz =$$
$$\int_{h}^{h+h_p} \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) dz + \tag{12}$$
$$\int_{-h-h_p}^{-h} \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) dz = 0$$

Substituting Eq. (5b) in Eq. (16) yields:

$$\lambda_1 \nabla^2 w_s - \lambda_2 \nabla^2 \varphi - \lambda_3 \nabla^2 w_b + \lambda_4 \varphi = 0 \tag{13}$$

where

$$\lambda_1 =$$
$$-\frac{1}{6} \frac{h_p e_{31} \left(57h^4 + 174h^3 h_p + 197h^2 {h_p}^2 + 100h{h_p}^3 + 20{h_p}^4 \right)}{\left(h + h_p \right)^2 h^2}$$
$$-\frac{e_{15} \left(15h^3 h_p + 10h^2 {h_p}^2 \right)}{\left(h + h_p \right)^2 h^2}$$
$$\lambda_2 = \frac{4}{3} \Xi_{11} h_p, \quad \lambda_3 = 2e_{31} h_p\,, \quad \lambda_4 = \frac{16\Xi_{33}}{h_p}$$

Analytical solution

Consider a simply supported FG rectangular plat with piezoelectric layers bonded to its surface. The Navier method is adopted for solution of obtained governing equations. The boundary conditions for simply supported plate are taken as below:

At edges x=0 and x=a: v_0=0,w_b=0,w_s=0, $M^b{}_x$=0, $M^s{}_x$=0, N_x=0, ϕ=0.

(14)

At edges y=0 and y=b: u_0=0,w_b=0,w_s=0, $M^b{}_y$=0, $M^s{}_y$=0, N_y=0, ϕ =0.

Using following infinite Fourier series for independent variables:

$$u_0 = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} u_{0,mn}\, e^{i\omega t} \cos(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b})$$
$$v_0 = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} v_{0,mn}\, e^{i\omega t} \sin(\frac{m\pi x}{a}) \cos(\frac{n\pi y}{b})$$
$$w_b = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} w_{b,mn} e^{i\omega t} \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \tag{15}$$
$$w_s = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} w_{s,mn}\, e^{i\omega t} \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b})$$

also the electrostatic potential can be approximated as following double Fourier expansions:

$$\varphi(x,y,t) = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \varphi_{mn}\, e^{i\omega t} \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \tag{16}$$

where ω is natural frequency. Substituting Eqs. (15) and (16) into Eq. (9), natural frequency can be obtained from the below Eigen-value equations:

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} \end{bmatrix} \begin{Bmatrix} u_{0,mn} \\ v_{0,mn} \\ w_{b,mn} \\ w_{s,mn} \\ \varphi_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{17}$$

Setting the determinant of the coefficient matrix equal to zero, the natural frequencies of the plate with piezoelectric layer resting on elastic foundation can be obtained.

Numerical results and discussion

To verifying accuracy of present theory, several numerical examples are solved and results are compared with other theories, Also effects of piezoelectric thickness and elastic parameters are investigated. Material properties used in examples are listed in Table 1.

Table 1. Material property

(1)

Property	Core plate			
	Ti-6Al-4V	Aluminum oxide	Al	alumina
E (GPa)	105.7	320.24	70	380
υ	0.2981	0.260	0.3	0.3
ρ(kg m^{-3})	4429	3750	2707	3800

Property	Piezoelectric layer	
	G-1195 N	PZT-4
E (GPa)	63.0	-
υ	0.3	-
C_{11}(GPa)	-	132
C_{12}(GPa)	-	71
C_{33}(GPa)	-	115
C_{13}(GPa)	-	73
C_{55}(GPa)	-	26
e_{31}(cm^{-2})	44.37	-4.1
e_{33}(cm^{-2})	50.18	14.1
e_{15}(cm^{-2})	14.13	10.5
Ξ_{11}(nFm^{-1})	15.30	7.124
Ξ_{33}(nFm^{-1})	15.00	5.841
ρ(kg m^{-3})	7600	7500

Table 2 presented non dimensional natural frequencies of square Al/Al_2O_3 FG (aluminum as metal and alumina as ceramic) plate with different piezoelectric (*PZT*-4) thickness and rested on elastic foundation ($\bar{K}_w = \bar{K}_s$ =100). As it is seen in Table 2, the obtained results are in good agreement with third order shear deformation plate theory with five unknown functions. It is clear that as piezoelectric thickness goes to zero, the frequency of the hybrid plate approaches that of the homogeneous plate.

$$\varpi = \omega h \left(\frac{\rho_m}{E_m} \right)^{0.5},\ \bar{K}_w = \frac{K_w b^4}{D_m},\ \bar{K}_s = \frac{K_s b^2}{D_m}$$
$$D_m = E_m \frac{h^3}{12\left(1 - \nu^2\right)} \tag{18}$$

Table 2. Non-dimensional natural frequencies (ϖ) of square FG plate with piezoelectric layers. (h/a=0.05)

h_p/h	theory	Power law index				
		0	0.5	1	2	5
10^{-1}	Present	0.0360	0.0341	0.0333	0.0329	0.0328
10^{-2}	Present	0.0404	0.0385	0.0377	0.0373	0.0376
10^{-4}	present	0.0411	0.0392	0.0384	0.0381	0.0383
0	present	0.0411	0.0392	0.0384	0.0381	0.0383
0	Ref [9]	0.0411	0.0395	0.0388	0.0386	0.0388

Effects of elastic parameters on natural frequencies of a simply supported square FG plate attached with G-1195N piezoelectric layers are investigated in Tables 3 and 4. Ti-6Al-4V and aluminum oxide are selected as the metal and ceramic parts of the FG core. The side and thickness of core plate are 400 and 5mm and the thickness of each piezoelectric layer is 0.1mm. Increasing value of foundation parameters tend to increase the natural frequency.

Table 3. Natural frequencies (Hz) of square FG plate with piezoelectric layers.

$\bar{K}_s$	Theory	Power law index				
		0	0.5	1	5	100
10^3	present	1731.6	1778.1	1802.1	1853.6	1881.2
10^2	Present	564.44	589.33	600.74	626.38	644.72
10	Present	224.99	256.67	268.72	296.65	321.44
0	Present	144.39	186.03	200.34	232.77	261.95
0	Ref [7]	145.35	186.26	200.57	233.04	262.68
0	Ref [12]	144.25	185.45	198.92	230.46	259.35

Table 4. Natural frequencies (Hz) of square FG plate with piezoelectric layers.

$\bar{K}_w$	Theory	Power law index				
		0	0.5	1	5	100
10^3	Present	414.35	439.35	450.14	474.87	494.40
10^2	Present	189.56	224.61	237.45	267.04	293.59
10	Present	149.52	190.25	204.35	236.42	265.28
0	Present	144.39	186.04	200.34	232.77	261.95
0	Ref [7]	145.35	186.26	200.57	233.04	262.68
0	Ref [12]	144.25	185.45	198.92	230.46	259.35

Conclusions

In this study employing the four-variable refined plate theory, analytical solutions for free vibration of FG plate integrated with piezoelectric layers and rested on elastic foundation were presented. The equations of motion were obtained using Hamilton's principle and in order to solve these equations, the Naveir solution was adopted. To verify the accuracy of the present theory, some comparisons between obtained results and already published ones were made. It was observed that in comparison to other theories, the present formulation gave more accurate results in predicting natural frequencies of FG plate integrated with piezoelectric layers and rested on elastic foundation. It should be noted that the present theory involves only four unknown functions and also does not need the shear correction factor. Also effects of piezoelectric thickness and foundation parameters were investigated. It observed that increasing value of foundation parameters tend to increase the natural frequency.

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