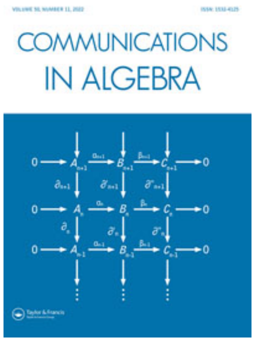


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# Weakly fully and characteristically inert socle-regular Abelian *p*-groups

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**ABSTRACT**

In regard to two recent publications in the Mediterranean J. Math. (2021) and Forum Math. (2021) related to fully and characteristically inert socle-regularity, respectively, we define and study the so-called *weakly characteristically inert socle-regular groups*. In that aspect, as a culmination of the investigations of this sort, some more global results are obtained and, moreover, some new concrete results concerning the weakly fully inert socle-regular groups, defined as in the firstly mentioned above paper, are also established. In particular, we prove that all torsion-complete groups are characteristically inert socle-regular, which encompasses an achievement from the secondly mentioned paper and completely settles the problem posed there about this class of groups.

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## 1. Introduction and fundamentals

Throughout the article, all groups under consideration, unless specified something else, are assumed to be reduced additively written Abelian $p$-groups, where $p$ is a fixed but arbitrary prime. Almost all our terminology and notations are standard and follow those from [16–18] and [25]. For instance, for any prime $p$, the symbol $G[p^n] = \{g \in G \ : \ p^n g = 0\}$ denotes the $p^n$*-socle* of the group $G$, and the symbol $p^n G = \{p^n g \ : \ g \in G\}$ denotes the *n-th power subgroup* of $G$, where $n \in \mathbb{N}$. Inductively, for any ordinal $\alpha$, $p^\alpha G = p(p^{\alpha-1}G)$ when $\alpha - 1$ exists or $p^\alpha G = \cap_{\beta<\alpha} p^\beta G$ otherwise; recall that a $p$-group $G$ is said to be *separable* if $p^\omega G = \{0\}$. We shall use for short $\mathrm{h}(x)$ to mean the *height* of the element $x$ in a given group as sometimes we shall just write $\mathrm{h}_G(x)$ if it needs to specify that the calculation is in some concrete group $G$. Also, for shortness of the exposition, we will use everywhere in the text the more compact record $p^\alpha G[p]$ which amounts to $(p^\alpha G)[p]$. Since all groups $G$ considered are reduced by the assumption alluded to above, there exists a least ordinal $\tau$ such that $p^\tau G = 0$. Likewise, as usual, the symbols $\mathrm{E}(G)$ and $\mathrm{Aut}(G)$ are reserved for the endomorphism ring and for the automorphism group of $G$, respectively.

Some years ago the second named author along with Goldsmith defined in [9] and [10], respectively, the two notions of socle-regularity and strong socle-regularity, which turned out to be

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interesting and useful concepts. The underlying idea was to circumvent the difficulties in classifying fully invariant subgroups of Abelian $p$-groups by looking at their socles rather than the full subgroup. Hence a group $G$ was said to be *socle-regular* (resp., *strongly socle-regular*) if, given any fully invariant (resp., characteristic) subgroup $H$ of $G$, there is an ordinal $\alpha$, entirely depending on $H$, having the property $H[p] = p^\alpha G[p]$. Recently, there has been a great deal of interest, arising primarily from considerations of various types of algebraic entropy, in another class of subgroups of torsion groups (and, indeed, more generally in arbitrary commutative and non-commutative groups), namely the fully inert subgroups and their related concepts (for a complete bibliography we refer the interested readers to [13, 19] and [28] as well as to [2, 3, 11, 12, 14, 20–22] and [26], respectively). Recall that the two subgroups $H$, $K$ of a group $G$ are said to be *commensurable* if the intersection $H \cap K$ has finite index simultaneously in $H$ and in $K$; this is hereafter denoted for convenience by $H \sim K$ and it is well-known to be an equivalence relation. Accordingly, a subgroup $H$ of a group $G$ is said to be *fully inert* in $G$ if the factor-group $(H + \varphi(H))/H$ is finite, that is, $H$ is commensurable with $H + \varphi(H)$, for any $\varphi \in \mathrm{E}(G)$; in particular, if $\varphi(H) \subseteq H$ for every $\varphi \in \mathrm{E}(G)$, the subgroup $H$ is known as *fully invariant*. Same applies if we replace $\mathrm{E}(G)$ by $\mathrm{Aut}(G)$ as then the subgroup $H$ is said to be *characteristically inert* and, in the particular case, *characteristic*.

It is also known, and easy to prove, that the sum and the intersection of a finite number of fully inert (resp., characteristically inert) subgroups are again fully inert (resp., characteristically inert) subgroups, and hence the socle $H[p]$ of a fully inert (resp., characteristically inert) subgroup $H$ of the group $G$ is also fully inert (resp., characteristically inert) in $G$. Fully inert (resp., characteristically inert) subgroups of an arbitrary Abelian $p$-group are, in some sense, not too far from being fully invariant (resp., characteristic) subgroups of the same group. Thus the issue of finding a natural generalization of the notion of socle-regularity arises for these special subgroups. Three natural possibilities present themselves thus:

- require for all fully inert (resp., characteristically inert) subgroups $H$ of $G$ the existence of an ordinal $\alpha$, depending on $H$, such that $H[p] = p^\alpha G[p]$.
- require for all fully inert (resp., characteristically inert) subgroups $H$ of $G$ the existence of an ordinal $\alpha$, depending on $H$, such that $H[p] \sim p^\alpha G[p]$.

A weaker alternative would be:

- require for all fully inert (resp., characteristically inert) subgroups $H$ of $G$ the existence of an ordinal $\alpha$, depending on $H$, such that $H[p] \cap p^\alpha G[p]$ is of finite index in $p^\alpha G[p]$.

We are thus able to state the following four key instruments, the first three of which already appeared in [5, 6].

**Definition 1.1.** A group $G$ is said to be *fully inert socle-regular* (resp., *characteristically inert socle-regular*) if, for all infinite fully inert (resp., characteristically inert) subgroups $H$ of $G$, there exists an ordinal $\alpha$, depending on $H$, such that $H[p] \sim p^\alpha G[p]$; alternatively $G$ is said to be *weakly fully inert socle-regular* if, for all infinite fully inert subgroups $H$ of $G$, there exists an ordinal $\alpha$, depending on $H$, such that $p^\alpha G \neq \{0\}$ and $H[p] \cap p^\alpha G[p]$ is of finite index in $p^\alpha G[p]$.

So, mimicking the above requirements, we shall say that the group $G$ is called *weakly characteristically inert socle-regular* if, for each infinite characteristically inert subgroup $H$, there exists an ordinal depending on $H$, say $\alpha$, with $p^\alpha G \neq \{0\}$ and $H[p] \cap p^\alpha G$ is of finite index in $p^\alpha G[p]$.

We shall study in what follows only the" weakly" case as the other one was explored in detail in [5] and [6], respectively.

In this terminology our definition of weakly fully inert (resp., weakly characteristically inert) socle-regularity requires that the ordinal $\alpha$ is strictly less than the length of the group, $\alpha < \tau$.

Note also that the restriction to infinite fully inert (resp., characteristically inert) subgroups in the definition of fully inert (resp., characteristically inert) socle-regularity is not restrictive: if $H$ is finite, then $H \sim p^{\ell(G)}G = \{0\}$, where $\ell(G)$ denotes the length of $G$. However, the situation in weak fully inert (resp., weak characteristically inert) socle-regularity is rather more complicated than can be anticipated and if one allows the choice $\alpha = \ell(G)$, then every group would be weakly fully inert (resp., weakly characteristically inert) socle-regular. Therefore, our restriction to infinite fully inert (resp., characteristically inert) subgroups in both cases is quite adequate.
Likewise, note that if the length $\ell(G)$ of $G$ is limit and the relation $H[p] \cap p^\alpha G[p] \sim p^\alpha G[p]$ holds for some $\alpha < \ell(G)$, then there will exist an ordinal $\beta_0$ with $\alpha \leq \beta_0 < \ell(G)$ such that $H[p] \cap p^\beta G[p] = p^\beta G[p]$ for any $\beta \geq \beta_0$. Indeed, one has that $p^\alpha G[p] = (H[p] \cap p^\alpha G[p]) \oplus F$ for some finite $F \leq p^\alpha G[p]$, and since $\cap_{\sigma<\ell(G)} p^\sigma G = \{0\}$ such an ordinal $\beta_0$ really exists.
Clearly, a fully inert (resp., characteristically inert) socle-regular group is always weakly fully inert (resp., weakly characteristically inert) socle-regular, but we shall show via concrete examples that the converse does not hold in both cases. Moreover, it is obvious that a weakly characteristically inert socle-regular group is weakly fully inert socle-regular. About the truthfulness of the reverse, we shall present an explicit construction showing that it is impossible.

Concretely, our paper is structured thus: In the next, second section, we give some useful preliminary assertions and some concrete examples closely related to them. In the third section, we are concentrated on obtaining the main statements and on discussion of the utilized methods. We close it with a few still unsettled relevant questions.

## 2. Preliminaries and examples

The organization of our preliminary claims is as follows. First, we need the following technicality, the first part of which is a slight amendment of [6, Proposition 2.7 (iii)]:

**Lemma 2.1.**

(1) *If $G$ is a group such that $G = A \oplus B$ with a corresponding projection $\pi : G \to A$, and $C$ is characteristically inert in $G$, then $f(\pi(C)) + C \cap B \sim C \cap B$ for any $f \in \mathrm{Hom}\ (A, B)$.*

(2) *If $K$ is a direct sum of cyclic groups and $L$ is an unbounded group, then for any infinite $H \leq K[p]$ there is $f \in \mathrm{Hom}\ (K, L)$ with infinite $f(H)$.*

*Proof.*

(1) It is readily checked that the matrix $\Delta = \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix}$ represents an automorphism of $G$, so that $C + \Delta(C) = C + f(\pi(C)) \sim C$ whence $f(\pi(C)) + C \cap B \sim C \cap B$, as stated.

(2) Let $K[p] = S_0 \oplus S_1 \oplus \cdots \oplus S_n \oplus \cdots$, where $\mathrm{h}(x) = n$ for every $0 \neq x \in S_n$ and $\pi_n : K[p] \to S_n$ are the corresponding projections. Assuming now that $\pi_n(H)$ is infinite for some $n$, we may choose in $\pi_n(H)$ a countable subgroup $F = \oplus_{i\geq 1}\langle x_i \rangle$. And let $B$ be a basic subgroup of $L$ such that $B[p] = S'_0 \oplus S'_1 \oplus \cdots \oplus S'_n \oplus \cdots$, where $\mathrm{h}(y) = n$ for every $0 \neq y \in S'_n$. Since $B$ is also unbounded, for every $x_i$ there exists $0 \neq y_i \in S'_{n_i}$ with $n < n_1 < n_2 < \ldots$. However, each $x_i$ could be embedded in a cyclic direct summand, say $\langle u_i \rangle$ in $K$ of order $p^n$, as well as each $y_i$ also could be embedded in a cyclic direct summand, say $\langle v_i \rangle$ in $B$ of order $p^{n_i}$. Since $\oplus_{i\geq 1}\langle u_i \rangle$ is obviously a direct summand of $K$, there will exist $f \in \mathrm{Hom}\ (K, L)$ with $f(u_i) = p^{n_i - n} v_i$ and this will be the wanted homomorphism.
But if, however, each $\pi_n(H)$ is finite, then we keep $0 \neq x_i \in S_{n_i}$ and $0 \neq y_i \in S'_{n'_i}$ such that $n_1 < n'_1 < n_2 < n'_2 < \cdots$ and our further arguments are similar to these presented above, thus completing the proof after all. □

We next begin with a new point of view: Letting $F$ be a subgroup in $G[p]$, we shall write $\mathrm{h}_G(F)$ if $\mathrm{h}_G(x) = \mathrm{h}_G(y)$ for all $0 \neq x, y \in F$.

Suppose that $H$ is an arbitrary infinite subgroup of the reduced group $G$. Set

$$\alpha = \min\{h_G(F) \mid F \le H[p], \mid F \mid \ge \aleph_0\}$$

and write $\alpha = \min(H)$. It is clear that $\min(H)$ is not exactly determined for each subgroup $H$ even provided $H[p]$ is infinite. Notice that a slightly analogous notion was used intensively in [4].

Some useful properties, which expand those from [9, 10], hold:

- Let $H \le G$ and assume that $\alpha = \min(H)$ is determined. If $h_G(F) < \alpha$ for some $F \le H[p]$, then $F$ is necessarily finite.
  Indeed, since $h_G(F)$ is determined, it follows by the definition that $h_G(x) = h_G(y)$ for all $0 \neq x, y \in F$. So, if $F$ is infinite, then $\min(H) \le h_G(F)$ - contradiction.
  Recall that the rank $f_\alpha(G)$ of the factor-group $p^\alpha G[p]/p^{\alpha+1}G[p]$, where $\alpha$ is an arbitrary ordinal number, is called the *$\alpha$-th Ulm-Kaplansky invariant* of $G$.
- If $H \le G$ and $\alpha = \min(H)$ for some ordinal $\alpha$, then $f_\alpha(G)$ is necessarily infinite.
  Indeed, since $\min(H)$ exists, one sees that $\alpha = h_G(F)$ for some infinite $F \backslash \{0\} \subseteq p^\alpha G[p] \backslash p^{\alpha+1}G[p]$ with $F \le H[p]$, where $F \cong (F \oplus p^{\alpha+1}G[p])/p^{\alpha+1}G[p] \le p^\alpha G[p]/p^{\alpha+1}G[p]$, so that $f_\alpha(G) \ge \mathrm{rank}(F)$, as required.
- The equality $\alpha = \min(G)$ is true for some ordinal $\alpha$ if, and only if, $f_\alpha(G)$ is infinite and $f_\beta(G)$ is finite for all $\beta < \alpha$.
  In fact, the infinity of $f_\alpha(G)$ is self-evident, so if $f_\beta(G)$ were also infinite for some $\beta < \alpha$, then $\min(G) \le \beta$ - contradiction.
- If $H$ is a characteristically inert subgroup of the group $G$ and $\min(H) = n$ is a finite integer, then $H[p] \sim p^n G[p]$.

In fact, since $\min(H) = n$, one has that $p^n G[p] = F \oplus K$, where $F$ is infinite and $h(x) = n$ for each $0 \neq x \in F$. So, $h(y) \ge n$ for each $y \in K$. We, furthermore, have $G = A \oplus B$, where $A[p] = F$. Assuming that $H[p] \nsim p^n G[p]$, we then obtain $p^n B[p] = (H[p] \cap B) \oplus L$, where $L$ is infinite. Choosing countable subgroups $\oplus_{i\ge1}\langle x_i\rangle \le F$ and $\oplus_{i\ge1}\langle y_i\rangle \le L$, we let $\oplus_{i\ge1}\langle u_i\rangle$ be such a direct summand of $A$ that $p^{n-1}u_i = x_i$ and $p^{n-1}v_i = y_i$, where $v_i \in B$. Now there will exist $\varphi \in \mathrm{Hom}\,(A, B)$ such that $\varphi(u_i) = v_i$. But then $\varphi(F) \le L$, $\varphi(F)$ is infinite and $\varphi(F) \cap H = 0$ so that $\varphi(F) + H \cap B \nsim H \cap B$. This, however, contradicts Lemma 2.1, so we get our claim.

Two more examples, which show some additional properties of this function associated with the results from the main section and that hold from our preceding considerations, are as follows:

**Example 2.2.** Let $G = \oplus_{n\in\mathbb{N}} B_n$, where $B_n \cong \mathbb{Z}(p^n)^{(\alpha_n)}$ and $\alpha_n$ are some cardinal numbers. If $H = \oplus_{n\in\mathbb{N}} H_n$ is a direct summand in $G$, where $H_n \cong \mathbb{Z}(p^n)^{(\beta_n)} \le B_n$, then $\min(H) = m - 1$ provided $\beta_m$ is infinite and, for every $k \in \mathbb{N}$ with $k < m$, the number $\beta_k$ is finite.

**Example 2.3.** Let $H \le G$. Then $\min(H) = \alpha$ if, and only if, $H \cap (p^\alpha G[p])$ is infinite and $H \cap (p^\beta G[p])$ is finite for all $\beta < \alpha$.

Thus we now come to our basic assertion. We, however, will recollect some facts needed for its proof whose details appear below. To that aim, let us notice that the infinite Ulm subgroups of weakly characteristically inert socle-regular groups retain the same property (compare with Proposition 3.8 listed below) as well as that in the bounded groups their infinite characteristically inert subgroups are always of a finite index (compare with Proposition 3.4 stated below).

**Proposition 2.4.** *The group $G$ is weakly characteristically inert socle-regular if, and only if, the subgroup $p^nG$ is weakly characteristically inert socle-regular for some positive integer $n$. In particular,* *if $A$ is a subgroup of a group $G$ and either $G/A$ is finite or $G = A \oplus B$, where $B$ is bounded, then $G$ is weakly characteristically inert socle-regular if, and only if, $A$ is weakly characteristically inert socle-regular.*

*Proof.* In view of Proposition 3.4 stated below, it is necessary to consider only the case when $G$ is unbounded.
The" **necessity"** follows directly from Proposition 3.8 listed below.
As for the" **sufficiency**", letting $H$ be an infinite characteristically inert subgroup in $G$, then since it is well known that each automorphism of $p^nG$ is induced by an automorphism of $G$, it must be that $H \cap p^nG$ is a characteristically inert subgroup in $p^nG$. Note that, if $G$ is unbounded and $H$ is infinite, then the intersection $H \cap p^nG$ is also infinite, which fact follows from Lemma 2.1. In fact, if we assume in a way of contradiction that $H \cap p^nG$ is finite, then $\pi_n(H[p])$ has to be infinite, where as usual $\pi_n : G \to B_1 \oplus \ldots \oplus B_n$ is the corresponding projection, $B = B_1 \oplus \ldots \oplus B_n \oplus \ldots$ is a basic subgroup of $G$ and $G = (B_1 \oplus \ldots \oplus B_n) \oplus (B_n^* + p^nG)$. Consulting with the already quoted Lemma 2.1, one detects that

$$f(\pi_n(H[p])) + (H[p] \cap (B_n^* + p^nG)) \sim H[p] \cap (B_n^* + p^nG),$$

where $f(\pi_n(H[p]))$ is infinite for each $f \in \mathrm{Hom}(B_1 \oplus \ldots \oplus B_n, B_n^* + p^nG)$. This is impossible substantiating that $H \cap p^nG$ is really an infinite characteristically inert subgroup in $p^nG$. But since $p^nG$ is a weakly characteristically socle-regular group, we consequently deduce that $(H \cap p^nG)[p] \cap p^{n+\alpha}G = H[p] \cap p^{n+\alpha}G \sim p^{n+\alpha}G[p]$ for some ordinal $\alpha$ (note that $n + \alpha = \alpha$ provided $\alpha$ is infinite and $p^\alpha(p^nG) = p^{n+\alpha}G$), i.e., $G$ is weakly characteristically inert socle-regular, as asserted.

Concerning the next two parts of the statement, if foremost $G/A$ is finite, then one easily writes that $G = A + F$ for some finite $F \leq G$. So, in the two supposed cases, there will exists some $i \in \mathbb{N}$ such that $p^iF = p^iB = \{0\}$, and hence $p^iG = p^iA$. Henceforth, the first part directly applies to conclude both claims. □

The following consequence is then immediate as the property of being of a" finite index" implies at once the requirements in the previous proposition, but, however, we shall provide the readers with an alternative confirmation of its validity which is relevant to [5, Theorem 4.4].

**Proposition 2.5.** *If $G$ is a group with a finite index subgroup $A$, then $G$ is weakly characteristically inert socle-regular if, and only if, so is $A$.*

*Proof.* First of all, we need the following well-known general observation which may be found in, for example, [27, Lemma 16.5]: If $A$ is of finite index in $G$, one can find a direct summand, say $C$ of $G$ with $C \leq A$ such that the quotient $G/C$ is finite. We thus have the two decompositions $G = C \oplus F$ and $A = C \oplus (A \cap F)$ with $F$ finite.
To prove the necessity, assume that $G$ is characteristically inert socle-regular and suppose that $H$ is an arbitrary infinite characteristically inert subgroup of $A$. Therefore, $H \cap C$ is infinite and characteristically inert in $C$ and $H/(H \cap C) \cong (H + C)/C \leq A/C \leq G/C$ is finite. So, $H \cap C \sim H$ implying that $(H \cap C)[p] \sim H[p]$. Utilizing now Proposition 2.4, one sees that $C$ is characteristically inert socle-regular. Consequently, $(H \cap C)[p] \sim p^\beta C[p]$ for some ordinal $\beta$. But using the symmetry of the operation $\sim$, it follows that $H[p] \sim (H \cap C)[p]$ and hence, by the transitivity property of the same operation, we obtain that $H[p] \sim p^\beta C[p]$. However, $p^\beta A = p^\beta C \oplus p^\beta(A \cap F)$ whence $p^\beta A[p] \sim p^\beta C[p]$ and so, again by symmetry, $H[p] \sim p^\beta C[p] \sim p^\beta A[p]$ giving us again by transitivity that $H[p] \sim p^\beta A[p]$, as promised.
To establish the sufficiency, assume that $A$ is characteristically inert socle-regular and suppose that $X$ is an arbitrary characteristically inert subgroup of $G$. Then Proposition 2.4 again tells us that $C$ is characteristically inert socle-regular. Now, an easy check shows that $X \cap C$ is fully inert in $C$, and so $(X \cap C)[p] \sim p^\gamma C[p]$ for some $\gamma \geq 0$. But $X/(X \cap C) \cong (X + C)/C \leq G/C$ is finite,

yielding that $X \cap C \sim X$ and hence that $(X \cap C)[p] \sim X[p]$. Therefore, we deduce as above that $X[p] \sim (X \cap C)[p] \sim p^{\gamma}C[p]$ forcing $X[p] \sim p^{\gamma}C[p]$. Moreover, $p^{\gamma}G \sim p^{\gamma}C$ ensures that $p^{\gamma}G[p] \sim p^{\gamma}C[p]$ and finally $X[p] \sim p^{\gamma}G[p]$, as expected. □

We now able to exhibit the following construction. Imitating [27], here the term" *a standard group*" means that such a group has finite Ulm-Kaplansky invariants and is a basic subgroup of a given group.

**Example 2.6.** There exists an uncountable group $G$ of length $\tau = \omega \cdot 2$, that is, $p^{\tau}G = \{0\}$ and $p^{\alpha}G \neq \{0\}$ for any $\alpha < \tau$, with $G/p^{\omega}G$ a direct sum of cyclic groups, but $G$ has a countable subgroup $C$ which is characteristically (and even fully) inert in $G$ such that $C \cap p^{\alpha}G[p]$ is not commensurable with $p^{\alpha}G[p]$ for all $\alpha < \tau$.

*Proof.* Let $G_0$ be the direct sum of continuously many copies of the standard group $B$, and let $G_1$ be the usual Pierce group $P$ as constructed in [27]. Then it is easy to check by using the standard Kulikov/Fuchs criteria (see, e.g., [16–18]) that there is a group $G$ with $G/p^{\omega}G = G_0$ and $p^{\omega}G = G_1 = P$. Letting also $C$ be a basic subgroup of $P$, we then deduce that $C[p]$ is even fully inert in $G$, which fact is of a rather technical natural and so we leave it to the interested reader for an inspection, but $C[p] \cap p^{\alpha}G$ is not commensurable with $p^{\alpha}G$ for any $\alpha < \omega \cdot 2$, the length of $G$, as required. □

## 3. Main results and open questions

In this section, we will arrange our chief statements. Before doing that, we first offer the following addition in notation: Given $B$ is an arbitrary basic subgroup of the group $G$, we write $B = B_1 \oplus B_2 \oplus \ldots$, where $B_n \cong \mathbb{Z}(p^n)^{(\alpha_n)}$ for some cardinal numbers $\alpha_n$. We also let $B_n^* = B_{n+1} \oplus B_{n+2} \oplus \ldots$. Then $G = B_1 \oplus \ldots \oplus B_n \oplus G_n^*$, where $G_n^* = B_n^* + p^nG$ (see, e.g., [17, Theorem 32.4]). Notice that we will freely use these facts in the sequel without any referring.

For a convenience of the presentation, we shall distribute our chief results into five subsections as follows:

### 3.1. The separable case

We are now in a position to attack the following pivotal assertion, which curiously shows that in the separable case the notions of weak characteristically inert socle-regularity and characteristically inert socle-regularity will coincide, thus refining [5, Corollary 4.10] in a non-trivial way.

**Theorem 3.1.** *If $G$ is a separable weakly characteristically inert socle-regular group, then $G$ is characteristically inert socle-regular.*

*Proof.* Let $H$ be an infinite characteristically inert subgroup of a group $G$, and write $(H[p] \cap p^nG) + F = p^nG[p]$ for some finite subgroup $F \leq G$ and finite number $n$. Since $p^nG[p] = G_n^*[p]$, one sees that $H[p] \sim (H[p] \cap (B_1 \oplus \ldots \oplus B_n)) \oplus (H[p] \cap G_n^*)$, where $H[p] \cap G_n^* \sim G_n^*[p] = p^nG[p]$. If $\mid H[p] \cap (B_1 \oplus \ldots \oplus B_n) \mid < \aleph_0$, then $H[p] \sim p^nG[p]$ and the assertion is proved.

Assume now that $\mid H[p] \cap (B_1 \oplus \ldots \oplus B_n) \mid \geq \aleph_0$. Let $\pi_i : G \to B_i$, $i = 1, \ldots, n$, be the corresponding projections and $1 \leq m \leq n$ is the minimal natural number with $\mid \pi_m(H[p]) \mid \geq \aleph_0$. It is not too difficult to observe that in the subgroup $B_m$ it is possible to choose a countable direct summand $C$, say $B_m = C \oplus B'_m$, such that, for each $c \in C[p]$, there exists an element $x \in H$ with $\pi_m(x) = c$. As $\pi(H[p]) = C[p]$, where $\pi : G \to C$ is the corresponding projection, then by Lemma 2.1 (1) (see also [6, Proposition 2.7 (iii)]) we have that $H \sim H + f(C[p])$ for every $f \in \text{Hom}\ (C, B'_m \oplus \ldots \oplus B_n)$, and thus one derives that $H[p] \cap (B_m \oplus \ldots \oplus B_n) \sim B_m[p] \oplus \ldots \oplus B_n[p]$, i.e., $H[p] \sim p^mG[p] = B_m[p] \oplus \ldots \oplus B_n[p] \oplus G_n^*[p]$, as required. □

Note that in [5, Example 1.7] it was proved that there exists a weakly fully inert socle-regular group which is *not* fully inert socle-regular.

### *3.2. The general case*

We continue here with an useful claim for our further applications.

**Lemma 3.2.** *If $H$ is an infinite characteristically inert subgroup of an unbounded group $G$ and $H \nsim H \cap p^{\omega}G$, then there exists a sequence of natural numbers $n_1 < n_2 < ... < n_k < ...$ such that $H[p] \cap B_{n_k} \neq \{0\}$ for some basic subgroup $B$ of $G$.*

*Proof.* As usual, let $\pi_n : G \to B_n$ be the corresponding projection. Assuming by supposition that $H \nsim H \cap p^{\omega}G$, then either $\pi_n(H[p])$ is infinite for some $n$ or, for each natural $n$, there will exist a positive integer $m > n$ with $(\pi_1 + ... + \pi_m)H[p] > (\pi_1 + ... + \pi_n)H[p]$.
If we choose $x \in H[p] \setminus p^{\omega}G$, then one writes that $x = h_1 + ... + h_s + y$, where $0 \neq h_i \in B_{n_i}[p]$ are such that $n_1 < ... < n_s$ if $s > 1$ with $n_1 = \mathrm{h}(h_1) < ... < \mathrm{h}(h_s)$ and $y \in G^*_{n_s}[p]$. However, with the aid of [17, Corollary 27.2], the element $h_1$ can be embedded in a cyclic direct summand of order $p^{n_1}$ by saying, without loss of generality, in $B_{n_1}$. Then, it follows that $H[p] \cap B_{n_1} \neq 0$, as expected.
Next, assume that $|\ \pi_{n_1}(H[p])\ | \geq \aleph_0$ (note that this is the first possible case of two possibilities) for some $n_1$. Since $B_{n_1}$ is a direct sum of cyclic groups of the same order $p^{n_1}$, the subgroup $\pi_{n_1}(H[p])$ is a socle of some direct summand of $B_{n_1}$. Since $G = (B_1 \oplus ... \oplus B_{n_1}) \oplus G^*_{n_1}$, there will exist a homomorphism $f : B_{n_1} \to G^*_{n_1}$ with $|\ f(\pi_{n_1}(H[p]))\ | = \aleph_0$ and $f(B_{n_1}) \leq B^*_{n_1} = \oplus_{n > n_1} B_n$, where $H[p] \cap G^*_{n_1} \sim (H[p] \cap G^*_{n_1}) + f(\pi_{n_1}(H[p]))$ holds by referring to Lemma 2.1. Moreover, if $B_n$ is infinite for some $n > n_1$, then we can choose $f$ such that $f(\pi_{n_1}(H[p])) \leq B_n$ because, by what we have shown above, in this case $H \cap B_n$ is infinite. If, however, $B_n$ is finite for all $n > n_1$, then we may choose $B_{n_2}$ with $n_2 > n_1$ and $H \cap B_{n_2} \neq 0$. In fact, if $x \in (H[p] \cap G^*_{n_2}) \setminus p^{\omega}G$, then $x = h_2 + ... + h_s + y$, where $0 \neq h_i \in B_{n_i}[p]$ such that $n_2 < ... < n_s$ if $s > 2$ with $n_2 = \mathrm{h}(h_2) < ... < \mathrm{h}(h_s)$ and $y \in G^*_{n_s}[p]$. We possibly choose $B_{n_2}$ with $x \in B_{n_2}$, as needed.
Assuming now that $\pi_n(H[p])$ is finite for every $n$ (this is the second possible case), then one sees trivially that the intersection $H[p] \cap (B_1 \oplus ... \oplus B_n)$ has to be finite for all natural numbers $n$, as well. So, as above demonstrated, there exists an index $n_2 > n_1$ with $H[p] \cap B_{n_2} \neq \{0\}$, etc., repeating the same procedure, we are done. Thereby, $H[p] \cap B_{n_k} \neq \{0\}$ for some sequence of natural numbers $n_1 < n_2 < ... < n_k < ...$, as we asked for. □

We are now ready to proceed with our key instrument, which sheds some light on the property of being characteristically inert socle-regular in certain cases. With it at hand, we will successfully expand [6, Theorem 4.4, Corollary 4.5] and also completely resolve the problem posed there in the critical case for 2-groups, i.e., when $p = 2$.

**Proposition 3.3.** *The following two statements hold:*

(1) *If $G$ is an unbounded direct sum of cyclic groups, and $H$ is an infinite characteristically inert subgroup of $G$, then there exists a natural number $m > 0$ such that $H[p] \cap p^{m-1}G = p^{m-1}G[p]$ is true.*
(2) *If $G$ is an unbounded torsion-complete group and $H$ is an infinite characteristically inert subgroup of $G$, then there exists a natural number $m > 0$ such that $H[p] + F = p^{m-1}G[p]$ is true for some finite subgroup $F$ of $G$.*

*Proof.*
(1) We have $G = \oplus_{n \geq 1} B_n$, where $B_n \cong \mathbb{Z}(p^n)^{(\alpha_n)}$ for some cardinal numbers $\alpha_n$. If $H[p] \cap B_n = B_n[p]$ for all $n \geq m$ and for some $m$, then it is not too hard to check that both the equalities $B^*_m[p] = p^{m-1}G[p]$ and $p^{m-1}G[p] = H[p] \cap p^{m-1}G$ hold.

Assume for a contradiction now that $H[p] \cap B_n \neq B_n[p]$ for almost all $n \in \mathbb{N}$. Then, it is not difficult to observe that there is a sequence of natural numbers, say, $n_1 < n_2 < \ldots < n_k < \ldots$, such that $B_{n_k}[p] = (H[p] \cap B_{n_k}) \oplus B'_{n_k}$, where $B'_{n_k} \neq \{0\}$ and, for each $n_k$, in view of the choice of the subgroup $B$, there is $n_{s_k} < n_k$ with the property $H[p] \cap B_{n_{s_k}} \neq \{0\}$ – see, for completeness, Lemma 3.2 alluded to above, too. Also, the sequences $\{n_k\}$ and $\{n_{s_k}\}$ could easily be selected such that the inequalities $n_{s_k} < n_k < n_{s_{k+1}}$ hold for all $k \geq 1$. So, one writes that

$$G = \left( \bigoplus_{k \geq 1} B_{n_{s_k}} \right) \oplus A,$$

where $\oplus_{k \geq 1} B_{n_k} \leq A$.

Let $0 \neq y_{n_1} \in B'_{n_1}$. Assume now that already exist elements $y_{n_1}, \ldots, y_{n_t}$, $0 \neq y_{n_i} \in B'_{n_i}$, $i = 1, \ldots, t$, $t \geq 2$ with $\langle y_{n_1}, \ldots, y_{n_t} \rangle \cap H = \{0\}$. If $B'_{n_l} \leq \langle H, y_{n_1}, \ldots, y_{n_t} \rangle = H'$ for all $l \geq t+1$, then $H'[p] \cap B_n = B_n[p]$ for all $n \geq t+1$ as $p^{t-1}G[p] = H'[p] \cap p^{t-1}G$. Consequently, $H[p] \sim H[p] \cap p^{t-1}G$ since $H[p] \sim H'[p]$ and $p^{m-1}G[p] = H[p] \cap p^{m-1}G$ for some $m \geq 1$.

So, there exists $0 \neq y_{n_{t+1}} \in B'_{n_{t+1}}$ with $\langle y_{n_1}, \ldots, y_{n_{t+1}} \rangle \cap H = \{0\}$. It is thus possible to build the sequence $y_{n_1}, y_{n_2}, \ldots$ with $0 \neq y_{n_k} \in B'_{n_k}$ and $\langle y_{n_k} \mid k \geq 1 \rangle \cap H = \{0\}$. If $0 \neq x_{n_{s_k}} \in H[p] \cap B_{n_{s_k}}$, then there exists $f \in \mathrm{Hom}(\oplus_{k \geq 1} B_{n_{s_k}}, A)$ with $f(x_{n_{s_k}}) = y_{n_k}$. Furthermore, since $\langle y_{n_k} \mid k \geq 1 \rangle \leq f(H \cap (\oplus_{k \geq 1} B_{n_{s_k}}))$, one infers that $H \cap A \nsim H \cap A + f(H \cap (\oplus_{k \geq 1} B_{n_{s_k}}))$ that contradicts Lemma 2.1 (1).

(2) Since $G$ is separable, each of its countable subgroups can be embedded in some basic subgroup of $G$ by using of [17, §33, Excersize 3] as, moreover, the infinity of $H$ implies the infinity of $H[p]$. So, one may choose a basic subgroup $B \leq G$ such that $\mid B[p] \cap H \mid \geq \aleph_0$. We deduce that the automorphism $f$ of $B$, defined in the preceding point (1), is extendible to an automorphism of $G$ by consulting with [17, Theorem 69.3]. So, [6, Lemma 2.6] then applies to get that $H \cap B$ is a characteristically inert subgroup in $B$. Hence, the previous point (1) leads to $H[p] \cap p^{m-1}B = p^{m-1}B[p]$ for some $m \geq 1$.

Assume now that $(H[p] \cap p^{n-1}G) \oplus F_n = p^{n-1}G[p]$, where $\mid F_n \mid \geq \aleph_0$ for each $n \geq m$. Likewise, as we tricked in (1), we can choose $y_{n_k}$ such that $\langle y_{n_k} \mid k \geq 1 \rangle \cap H = \{0\}$ and put $\{y_{n_k}\}_{k \geq 1}$ in some basic subgroup $B'$ of $G$. Thus, it is routinely seeing that these two manipulations are possible and also that we can define the isomorphism $f : B \to B'$ (because it is well known from [17] that any two basic subgroups of a given group are isomorphic) is extendible to an automorphism of $G$ (because $G$ is by assumption torsion-complete) for which $\mid (H + f(H))/H \mid \geq \aleph_0$ – a contradiction, which completes our argumentation.

And finally, since separable weakly characteristically inert socle-regular groups are, in fact, characteristically inert socle-regular by application of Lemma 3.1 stated above, one concludes that the equality $H[p] + F = p^{m-1}G[p]$ is true for some finite subgroup $F$ of $G$ and some natural number $m$, as required. □

We now will be concerned with the bounded case, which can be somewhat argued as exhibited above. Concretely, the following holds.

**Proposition 3.4.** *Let $G$ be a bounded group such that $p^nG = \{0\}$ and $p^{n-1}G \neq \{0\}$ for some natural number $n \geq 1$, and let $H$ be an infinite characteristically inert subgroup of $G$, then $(H[p] \cap p^{m-1}G) + F = (p^{m-1}G)[p]$ is valid for some finite subgroup $F$ of $G$ with $m \leq n$.*

*Proof.* In this case, one writes that $G = B_1 \oplus \ldots \oplus B_n$. If $\pi_i(H)$ is infinite for some $i < n$, then arguing as in Proposition 3.3 one finds that $H[p] \cap p^{i+1}G \sim (p^{i+1}G)[p]$. However, if $\pi_i(H)$ is finite for all $i < n$, then $H[p] \sim B_n[p]$, where $B_n[p] = p^{n-1}G = p^{n-1}G[p]$, as required. □

As a direct consequence, we obtain the following crucial statement, which was proved in [6] in the case where $p \neq 2$ and remained left-open when $p=2$ – we thus also improve somewhat [5, Corollary 1.5].

**Theorem 3.5.** *Every torsion-complete group is characteristically inert socle-regular.*

*Proof.* It follows immediately by combining Theorem 3.1 and Propositions 3.3, 3.4. In fact, utilizing Propositions 3.3 and 3.4, we deduce that any torsion-complete group is weakly characteristically inert socle-regular. But each torsion-complete group is known to be separable, and so Theorem 3.1 gives that it is characteristically inert socle-regular, as claimed. □

Before continuing, we invoke one more observation: Let $G$ be a characteristically inert socle-regular group such that $| p^\alpha G | > \aleph_0$ for any ordinal $\alpha$ with $p^\alpha G \neq \{0\}$. Then it follows that $|H[p]| > \aleph_0$ for every infinite characteristically inert subgroup $H$ of $G$. Indeed, one checks that $H[p] \sim p^\alpha G[p]$ for some $\alpha$ and that $| p^\alpha G[p] | = | p^\alpha G |$ in view of the reduced property of $G$. So, this helps us to consider in the sequel only groups and their characteristically inert subgroups with infinite (and even uncountable) socles.

### 3.3. ULM-like theorems

We also arrive at the following statement.

**Proposition 3.6.** *Let $G$ be a reduced group such that $G/p^\omega G$ is a direct sum of cyclic groups and let $G = G_1 \oplus G_2$, where both $G_1$, $G_2$ are unbounded. Then $G$ is weakly characteristically inert socle-regular if $p^\omega G$ is weakly characteristically inert socle-regular.*

*Proof.* If $p^\omega G[p]$ is finite, then one verifies that $H \cap p^\omega G[p] \sim p^\omega G[p]$, where $H$ is an infinite characteristically inert subgroup of $G$. So, let now $p^\omega G[p]$ be infinite. We shall prove in this case that $H \cap p^\omega G[p]$ is also infinite. To do that, assume the opposite that $H \cap p^\omega G[p]$ is finite. Since $p^\omega G[p] = p^\omega G_1[p] \oplus p^\omega G_2[p]$, $G$ is reduced and $G_1$, $G_2$ are both unbounded, one derives that $G_i/p^\omega G_i[p]$, where $i=1, 2$, are also both unbounded.
Let $\pi_i : G \to G_i$, $i=1, 2$, be the corresponding projections. For concreteness, assume that $\pi_1(H[p])$ is infinite. If $f : G_1 \to G_1/p^\omega G_1$ is the canonical epimorphism, then $f(\pi_1(H[p]))$ is infinite being a direct sum of cyclic groups. So, with the aid of Lemma 2.1 (2), one inspects that there will exist $\varphi \in \mathrm{Hom}\ (G_1, G_2)$ with the property that $\varphi(f(\pi_1(H[p])))$ is an infinite subgroup of $p^\omega G_2[p]$ and that its subgroup is embedding in $H \cap p^\omega G_2[p]$, as required. □

The next construction is somewhat helpful to understand the behavior of fully inert subgroups by showing manifestly that the case where the infinite characteristically inert subgroup is at most countable is rather difficult, thus contrasting with the assertion in the previous Proposition 3.6.

**Example 3.7.** There exists a group $G$ with a basic subgroup $B$ such that $H_1 \leq B$ and $H_2 \leq p^\omega G$ for some two infinite fully inert subgroups $H_1$ and $H_2$ of $G$.

*Proof.* Let $P$ be a separable group such that the equality $\mathrm{E}(P) = J_p \cdot 1_{\mathrm{E}(P)} \oplus \mathrm{E}_s(P)$ holds, and let $G$ be a group such that $p^\omega G$ is countable and elementary with $G/p^\omega G \cong P$ (see, e.g., [27] and [23]). Then, by virtue of [23, Lemma 4.4], one writes that $\mathrm{E}(G) = J_p \cdot 1_{\mathrm{E}(G)} \oplus \mathrm{E}_s(G)$. We remember that here $J_p$ and $\mathrm{E}_s(P)$ stand for the ring of $p$-adic integers and for the ideal of $\mathrm{E}(P)$ consisting only of small endomorphisms of (the standard group) $P$, respectively; similarly for $\mathrm{E}_s(G)$ associated with the group $G$.
Let us now $H_1 = F[p]$, where $F$ is a pure subgroup in $B$ such that $f_n(F)$ is finite for all $n$, and let $H_2$ be an infinite subgroup in $p^\omega G$. An easy inspection shows that each $f \in \mathrm{E}(G)$ is the form $f = \pi + \varphi$, where $\pi \in J_p$, $\varphi \in \mathrm{E}_s(G)$. Consequently, $f(H_1) + H_1 = \pi(H_1) + \varphi(H_1) + H_1$, where $\pi(H_1) \leq H_1$

and $\varphi(H_1)$ are finite, because of the finiteness of $f_n(H_1)$ and $f(H_2) = \pi(H_2) \leq H_2$. Furthermore, by a routine check, the subgroup $H_1$ is fully inert and the subgroup $H_2$ is fully invariant, as asserted. □

The following statement is somewhat an Ulm-like type affirmation for the newly defined classes of groups from the introductory section.

**Proposition 3.8.**

(1) *Given $G$ is a weakly characteristically inert socle-regular group, then for any ordinal $\alpha$, if $p^\alpha G$ is infinite, the subgroup $p^\alpha G$ is also weakly characteristically inert socle-regular.*

(2) *Given $G$ is a weakly fully inert socle-regular group, then for any ordinal $\alpha$, if $p^\alpha G$ is infinite, the subgroup $p^\alpha G$ is also weakly fully inert socle-regular.*

*Proof.*

(1) Let $H$ be an infinite characteristically inert subgroup of $p^\alpha G$. Since $p^\alpha G$ is characteristic in $G$, the subgroup $H$ is also characteristically inert in $G$. Thus there is an ordinal $\sigma$ with $p^\sigma G \neq \{0\}$ such that

$$(H \cap p^\sigma G)[p] \sim p^\sigma G[p].$$

But then

$$(H \cap p^\sigma G)[p] \cap p^\alpha G \sim p^\sigma G[p] \cap p^\alpha G,$$

so that $(H \cap p^\beta G)[p] \sim p^\beta G[p]$, where $\beta = \max\{\alpha, \sigma\}$. Note also that $p^\beta G \neq \{0\}$, because both the inequalities $p^\sigma G \neq \{0\}$ and $p^\alpha G \neq \{0\}$ hold. Writing $\beta = \alpha + \lambda$ for some $\lambda$, we have that

$$(H \cap p^\lambda(p^\alpha G))[p] \sim p^\lambda(p^\alpha G)[p]$$

and hence $p^\alpha G$ is weakly characteristically inert socle-regular, as claimed.

(2) It can be proved as (1) by a way of similarity, so omit the details by leaving them to the interested readers. □

### 3.4. Groups of special type

It is well know that (see [1]) the group $G$ is called of *type A* if $\mathrm{Aut}(G) \restriction p^\omega G = \mathrm{Aut}(p^\omega G)$. By a reason of symmetry, we will say now that the group $G$ is of *type E* provided $\mathrm{E}(G) \restriction p^\omega G = \mathrm{E}(p^\omega G)$. Here the abbreviation $\restriction p^\omega G$ stands for those automorphisms (resp., endomorphisms) of $G$ which act on $p^\omega G$.

We will obtain some new results of this branch by adapting the structure of the first Ulm subgroup $p^\omega G$ to this aim (for some other analogous approaches of using $p^\omega G$, we refer to [7, 8] and [24] as well). So, we pose the next incidental requirement: We shall say that this concrete subgroup $p^\omega G$ is *continually infinite* if it is infinite itself and, moreover, the intersection with the socle of any infinite characteristically inert subgroup of $G$ continues to be infinite too. A visible example of such subgroups is the following one:

**Example 3.9.** Suppose $B = p^\omega G$ is a bounded group and $G/p^\omega G$ is a direct sum of cyclic groups. If $H \leq G$ is infinite fully inert and $H \cap p^\omega G$ is finite, then $H = H' \oplus F$ for some finite subgroup $F$ with $H \cap p^\omega G \leq F$. Then the socle $H[p] \sim H'[p]$ is also infinite.

Now, letting $\pi : G \to G/p^\omega G$ be the canonical epimorphism, one detects that $\pi \restriction H'[p]$ is an injection. Since $G/p^\omega G$ is a direct sum of cyclic groups, we find that for the infinite subsocle $K \leq \pi(H'[p])$ there exists a homomorphism $f : G/p^\omega G \to p^\omega G$ such that $f \restriction K$ is an injection, too. And since $f\pi H[p] \sim H \cap p^\omega G$, we deduce that $H[p] \cap p^\omega G$ is infinite which contradicts the assumption that $H \cap p^\omega G$ is finite. This finishes the arguments after all.

Besides, it is not too difficult to construct a group $G$ with infinite $p^{\omega}G$, containing an infinite fully inert subgroup $H$, such that $H \cap p^{\omega}G = \{0\}$. In fact, choose $p^{\omega}G$ to be a countably infinite elementary group with $\mathrm{E}(G) = J_p \cdot 1_{\mathrm{E}(G)} \oplus \mathrm{E}_s(G)$. Set $B = \oplus_{n \geq 1} B_n$, where all $B_n$ are direct sums of the groups $\mathbb{Z}(p^n)$. Given $H \leq G[p]$ with $H = \oplus_{n \geq 1} H_n$, where each $H_n$ is finite in $B_n[p]$. Thus, for every $\varphi \in \mathrm{E}(G)$, the image $\varphi(H)$ is finite too, i.e., we have at once that $H$ is fully inert and $H \cap p^{\omega}G = \{0\}$, as expected.
We have now all the ingredients necessary to proceed by proving the next two statements which somewhat show that the reversibility in Proposition 3.8 is possible in some special concrete situations.

**Proposition 3.10.** *Let $G$ be a group of the type A. If $p^{\omega}G$ is continually infinite, then $G$ is weakly characteristically inert socle-regular if, and only if, $p^{\omega}G$ is weakly characteristically inert socle-regular.*

*Proof.* It follows from Proposition 3.8 (1) that if $G$ is an arbitrary weakly characteristically inert socle-regular group, then for every ordinal $\alpha$ either $p^{\alpha}G$ is finite or is weakly characteristically inert socle-regular for each ordinal $\alpha$, as required.
Conversely, suppose that $H$ is infinite and characteristically inert in $G$. Since $G$ is of type A, the intersection $H[p] \cap p^{\omega}G$ is also characteristically inert in $p^{\omega}G$ and so, applying the assumption, it is infinite as well. Consequently, $H[p] \cap p^{\omega}G$ is of finite index in $p^{\omega}G[p]$, as expected, which finishes of proof. □

Note that the following assertion does not appear in [5], so it could be of some usefulness to discover the structure of the class of weakly fully inert socle-regular groups.

**Proposition 3.11.** *Let $G$ be a group of the type E. If $p^{\omega}G$ is continually infinite, then $G$ is weakly fully inert socle-regular if, and only if, $p^{\omega}G$ is weakly fully inert socle-regular group.*

*Proof.* It follows from Proposition 3.8 (2) that if $G$ is an arbitrary weakly fully inert socle-regular group, then either $p^{\alpha}G$ is finite or is weakly fully inert socle-regular for every ordinal $\alpha$, as required.
Conversely, suppose that $H$ is infinite and fully inert in $G$. Since $G$ is of type E, the intersection $H[p] \cap p^{\omega}G$ is also fully inert in $p^{\omega}G$ and thus, employing the assumption, it is infinite too. Consequently, $H[p] \cap p^{\omega}G$ is of finite index in $p^{\omega}G[p]$, which completes the proof. □

### *3.5. Square of groups*

It was proved in [6, Theorem 4.6] that a group $G$ is fully inert socle-regular if, and only if, the square $G \oplus G$ is characteristically inert socle-regular. We shall now slightly extend this necessary and sufficient condition to the following one, which also improve [5, Proposition 3.1] as follows:

**Proposition 3.12.** *The group $G$ is weakly fully inert socle-regular if, and only if, the square $G \oplus G$ is weakly characteristically inert socle-regular.*

*Proof.* Suppose that $G$ is weakly fully inert socle-regular and $C$ is characteristically inert in $K = G_1 \oplus G_2$, where $G_1 \cong G \cong G_2$. A well-known result of Kaplansky (see, e.g., [25]) then tells us that every endomorphism of $K$ is the sum of 3 automorphisms of $K$, and so it follows with this at hand that $C$ is fully inert in $K$. Writing now $C_1 = C \cap G_1, C_2 = C \cap G_2$, it then follows that $C \sim C_1 \oplus C_2$ and thus that $C_1$, $C_2$ are fully inert in $G_1$, $G_2$ respectively. Furthermore, if $\varphi : G_1 \to G_2$, $\psi : G_2 \to G_1$ are two arbitrary homomorphisms, then the relations $\varphi(C_1) + C_2 \sim C_2$ and $\psi(C_2) + C_1 \sim C_1$ are immediately valid.

Since all of the $G_i$'s are weakly fully inert socle-regular, there are ordinals $\alpha$, $\beta$ such that $(C_1)[p] \cap p^\alpha G_1 \sim p^\alpha G_1[p]$ and $(C_2)[p] \cap p^\beta G_2 \sim p^\beta G_2[p]$. Now choosing $\varphi, \psi$ to be the isomorphisms identifying $G_1$ and $G_2$, we see that $C_1 \sim C_1 + C_2 \sim C_2$; in other words $p^\alpha G_1[p] \sim p^\beta G[p] \sim p^\alpha G_2[p]$ and thus $C[p] \sim p^\gamma G_1[p] \oplus p^\gamma G_2[p]$, where $\gamma = \max\{\alpha, \beta\}$. Therefore, $C[p] \sim p^\gamma K[p]$ and $K$ is then weakly characteristically inert socle-regular, as needed.
Conversely, suppose that $G \oplus G$ is weakly characteristically socle-regular and let $F$ be an arbitrary fully inert subgroup of $G$. Then $F \oplus F$ is fully inert in $G \oplus G$, and hence $F \oplus F$ is certainly characteristically inert in $G \oplus G$. Thus there is an ordinal $\beta$ such that $(F[p] \oplus F[p]) \cap p^\beta(G \oplus G) \sim p^\beta G[p] \oplus p^\beta G[p]$. It now follows easily from standard properties of commensurability that $F[p] \cap p^\beta G \sim p^\beta G[p]$. Since $F$ was arbitrarily chosen, the group $G$ is weakly characteristically inert socle-regular, as asserted. □

We finish our work with the following two questions of some interest and importance. The first one immediately arises in connection with Theorem 3.1 and the comment given after it (see also [5, Example 1.7]):

**Problem 3.13.** Does there exist an inseparable weakly characteristically inert socle-regular group which is *not* characteristically inert socle-regular?

**Problem 3.14.** Decide whether or *not* every generally $p^\alpha$ torsion-complete Abelian group, as defined in [15], is still characteristically inert socle-regular for each ordinal $\alpha$ strictly less than the length of the group.

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