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Partial asymptotic stability of neutral pantograph stochastic differential equations with Markovian switching

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Abstract

In this paper, we investigate the partial asymptotic stability (PAS) of neutral pantograph stochastic differential equations with Markovian switching (NPSDEwMSs). The main tools used to show the results are the Lyapunov method and the stochastic calculus techniques. We discuss a numerical example to illustrate our main results.

1 Introduction

Neutral stochastic delay differential equations with and without Markovian switching have been recently intensively investigated (see [1, 10, 11, 13, 14, 19, 20, 22], and [23]). Many systems are often subject to component repairs or failures, abrupt changes, environmental disturbances, and subsystem interconnections. The pantograph SDEs (PSDEs) have been widely used in electrodynamics and quantum mechanics. In the last decades the stability analysis of stochastic differential equations (SDEs) has received much attention (see [2, 3, 7–9, 15, 18, 25]). In general, due to the characteristics and specifications of SDEs themselves, it is difficult to obtain explicit solutions of equations. Therefore we use the Lyapunov method to study the stability and the asymptotic behavior of solutions. The almost sure polynomial and exponential stabilities were investigated by many researchers (see [2, 3], and [7–9]). The stochastic pantograph differential equations are a kind of stochastic delay differential equations (see [4, 7–9]), also called equations with proportional delay. They play an important role in industrial and mathematical problems. The NPSDEwMS are very well investigated (see [4, 25], and [17]). In [4] the authors proved the existence, uniqueness, and p-moment stability of solutions in the case $p > 0$. However, in many dynamical systems, such a stability is usually too strong to be satisfied. Therefore the notion of partial stability (PS) (see [5, 6, 12], and [16]) has been studied, and the Lyapunov method, as an important tool, has been used to investigate the PS in various practically important domains. In the literature, we did not find any result on PAS of NPSDEwMS. Using the technique of stochastic calculus and Lyapunov method, we show a new sufficient condition for the PS of a class of NPSDEwMS.

In [5] and [12] the authors investigated the PAS of the solutions of ordinary SDEs by using an appropriate Lyapunov function satisfying some specific properties. In our paper, we prove the PAS of solutions of NPSDEwMSs. In this sense, our results extend the analysis in [5] and [12] providing the neutral term and the delay in the case of the PSDE with Markovian switching.

Let us outline the framework of this paper. After preliminaries and notations (see Sect. 1), in Sect. 2, we recall some important notions and definitions. In Sect. 3, we establish the PAS for a class of NPSDEwMSs. Finally, in Sect. 4, we present a numerical example to show the applicability of our results.

2 Preliminaries and notations

Let $\{\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P}\}$ be a complete probability space with filtration $\{\mathcal{F}_s\}_{s\geq 0}$ satisfying the usual conditions, and let $W(s)$ be an m-dimensional Brownian motion defined on this probability space. Let $s \geq s_0 > 0$, let $C([qs_0, s_0]; \mathbb{R}^n) = \{\psi : [qt_0, s_0] \to \mathbb{R}^n \text{such that} \psi \text{is a continuous function}\}$ with the norm $\|\psi\| = \sup_{qs_0 \leq b \leq s_0} |\psi(b)|$, and let $|x| = \sqrt{x^T x}$ for $x \in \mathbb{R}^n$. If B is a matrix, then its trace norm is denoted by $|B| = \sqrt{\text{Trace}(B^T B)}$, and its norm is given by $\|B\| = \sup_{|x|=1} |Bx|$. Denote by $L^p_{\mathcal{F}_{s_0}}([qs_0, s_0]; \mathbb{R}^n)$ the set of all $\mathcal{F}_{s_0}$-measurable $C([qs_0, s_0]; \mathbb{R}^n)$-valued random variables $\psi = \{\psi(\theta) : qs_0 \leq \theta \leq s_0\}$ such that $E\|\psi\|^p < \infty$, where $p \in \mathbb{N}^*$.

Let $\{m(s), s \geq 0\}$ be a right-continuous Markov chain on $\{\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P}\}$ taking values in a finite state space $\bar{S} = \{1, 2, 3, \dots, N\}$, where $\Gamma = (\gamma_{jk})_{\mathbb{N}\times\mathbb{N}}$ is the generator given by

$$P\big(m(s+\varpi) = k | m(s) = j\big) = \begin{cases} \gamma_{jk}\varpi + o(\varpi) & \text{if } j \neq k, \\ 1 + \gamma_{jj}\varpi + o(\varpi) & \text{if } j = k, \end{cases}$$

for $\varpi > 0$. Here $\gamma_{jk} \geq 0$ is the transition rate from j to k if $j \neq k$, whereas

$$\gamma_{jj} = -\sum_{j\neq k} \gamma_{jk}.$$

We suppose that r and W are independent.

Consider the following NPSDEwMS:

$$\begin{aligned} &d\big(z(s) - G\big(s, z(qs), m(s)\big)\big) \\ &\quad = f\big(s, z(s), z(qs), m(s)\big)\,ds + g\big(s, z(s), z(qs), m(s)\big)\,dW(s), \quad s \geq s_0, \end{aligned} \tag{2.1}$$

with initial data $\zeta \in L^p_{\mathcal{F}_{s_0}}([qs_0, s_0]; \mathbb{R}^n)$, i.e.,

$$z(s) = \zeta(s) \quad \text{for } qs_0 \leq s \leq s_0. \tag{2.2}$$

Let $u(s) = z(s) - G(s, z(qs), m(s))$, where $G(s, z(qs), m(s)) = (G_1(s, z(qs), m(s)), G_2(s, z(qs), m(s)))^T \in \mathbb{R}^n$. We assume that

$$\begin{aligned} &f : [s_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \bar{S} \to \mathbb{R}^n, \qquad g : [s_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \bar{S} \to \mathbb{R}^{n\times m}, \\ &G : [s_0, +\infty) \times \mathbb{R}^n \times \bar{S} \to \mathbb{R}^n. \end{aligned}$$

Let $z = (z_1, z_2)^T \in \mathbb{R}^n$ be the solution of equation (2.1), where $z_1 \in \mathbb{R}^k$ and $z_2 \in \mathbb{R}^p$, and $k + p = n$.

We will impose the following assumptions on f, g, and G:

($\mathcal{A}_1$) For each $l \in \mathbb{N}^*$, there exists $k_l > 0$ such that

$$\left|f(s,u,x,j) - f(s,\overline{u},\overline{x},j)\right|^2 \vee \left|g(s,u,x,j) - g(s,\overline{u},\overline{x},j)\right|^2 \leq k_l\left(|u-\overline{u}|^2 + |x-\overline{x}|^2\right). \tag{2.3}$$

($\mathcal{A}_2$) For all $(s,j) \in [s_0,+\infty) \times \bar{S}$ and $\varsigma, x \in \mathbb{R}^n$, there exists $\kappa_j \in (0,1)$ such that

$$\left|G(s,\varsigma,j) - G(s,x,j)\right|^2 \leq \kappa_j|\varsigma - x|^2. \tag{2.4}$$

Set $G(s,0,j) = 0$ and $\kappa = \max_{j\in\bar{S}} \kappa_j$.

Let $C^{1,2}([qs_0,+\infty) \times \mathbb{R}^n \times \bar{S};\mathbb{R}^+)$ be the set of all nonnegative functions $V(s,z,j)$ on $[qs_0,+\infty) \times \mathbb{R}^n \times \bar{S}$ that are once continuously differentiable with respect to s and twice continuously differentiable with respect to z.

For any $(s,z,v,j) \in [qs_0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times S$, $u = z - G(s,v,j)$, by the generalized Itô formula (see [18] and [24]) we have

$$V\big(s,u(s),m(s)\big) = V\big(s_0,u(s_0),m(s_0)\big) + \int_{s_0}^{s} \mathcal{L}V\big(\tau,z(\tau),z(q\tau),m(\tau)\big)\,d\tau + M(s),$$

where the stochastic process $M(s)$ and the operator $\mathcal{L}V(s,z,v,i) : [qs_0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \bar{S} \to \mathbb{R}$ are defined by

$$\begin{aligned}
M(s) &= \int_{s_0}^{s} V_z\big(\tau,u(\tau),m(\tau)\big)g\big(\tau,z(\tau),z(q\tau),m(\tau)\big)\,dW(\tau),\\
\mathcal{L}V(s,z,v,j) &= V_s(s,u,j) + V_z(s,u,j)f(s,z,v,j)\\
&\quad + \frac{1}{2}\operatorname{Trace}\big(g^T(s,z,v,j)V_{zz}(s,u,j)g(s,z,v,j)\big)\\
&\quad + \sum_{k=1}^{N} \gamma_{jk}V(s,u,k),\\
V_s &= \frac{\partial V(s,z,j)}{\partial s}, \qquad V_{zz} = \left(\frac{\partial^2 V(s,z,j)}{\partial z_j \partial z_j}\right)_{n\times n},\\
V_z &= \left(\frac{\partial V(s,z,j)}{\partial z_1},\ldots,\frac{\partial V(s,z,j)}{\partial z_n}\right).
\end{aligned}$$

($\mathcal{A}_3$) There exist functions μ_1, μ_2, μ_3, μ_4 in $\mathcal{K}$ and $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n \times \bar{S};\mathbb{R}_+)$ satisfying, for all $(s,z,v,j) \in [s_0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \bar{S}$,

(i) $\mu_1(|z_1|) \leq V(s,z,j) \leq \mu_2(|z_1|)$,

(ii) $LV(s,z,v,j) \leq -\mu_3(|z_1|) + q\mu_4(|v_1|)$.

3 Main results

We discuss the PS in probability and PAS of equation (2.1).

Definition 3.1

(i) The solution $z(s) = (z_1(s), z_2(s))$ of equation (2.1) is called PS in probability with respect to z_1 if for all $\eta > 0$ and $\lambda \in (0,1)$, there exists $\delta_0 = \delta_0(\lambda,\eta,s_0) > 0$ such that

$$P\big(\left|z_1(s)\right| < \eta, \forall s \geq s_0\big) \geq 1 - \lambda$$

whenever $\|\zeta\| < \delta_0$.

(ii) The solution $z(s) = (z_1(s), z_2(s))$ of equation (2.1) is called PAS in probability with respect to z_1 if it is stable in probability with respect to z_1 and for all $\zeta \in L^p_{\mathcal{F}_{s_0}}([qs_0, s_0]; \mathbb{R}^n)$, we have

$$P\Big(\lim_{s\to+\infty} z_1(s) = 0\Big) = 1.$$

Let $\mathcal{K}$ be the set of all continuous nondecreasing functions $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(0) = 0$ and $\mu(\nu) > 0$ for $\nu > 0$. For $H > 0$, let $S_H = \{z \in \mathbb{R}^n, |z_1| < H\}$.

Theorem 3.1 *Suppose that there exist a function* $V(s,z,j) \in C^{1,2}([s_0,+\infty) \times S_H \times S; \mathbb{R}_+)$ *and* $\mu \in \mathcal{K}$ *such that*

(i) $\mu(|z_1|) \le V(s,z,j)$ *for all* $(s,z) \in [s_0,+\infty) \times S_H$,

(ii) $\mathcal{L}V(s,z,\nu,j) \le 0$ *for all* $(s,z) \in [s_0,+\infty) \times S_H$.

Then the solution of equation (2.1) *is PS in probability with respect to* z_1.

Proof By Assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ system (2.1) has a unique global solution $z(s)$ for $s \ge s_0$ (see [17]).

Let $\lambda \in (0,1)$ and $\eta > 0$ be arbitrary. We will assume that $\eta < H$. By the continuity of $V(s,z,j)$ and the fact $V(s_0, 0, m(s_0)) = 0$ we can find $\rho = \rho(\lambda, \eta, s_0) > 0$ such that

$$\frac{1}{\lambda}\sup_{z\in S_\rho}\big(V\big(s_0, z, m(s_0)\big)\big) \le \mu(\eta). \tag{3.1}$$

We can see that $\rho < \eta$. Fix an arbitrary initial condition $\zeta \in L^p_{\mathcal{F}_{s_0}}([qs_0, s_0]; \mathbb{R}^n)$ such that $\|\zeta\| < \rho$. Let ϑ be the stopping time given by

$$\vartheta = \inf_{s\ge s_0}\big\{z_1(s) \notin S_\eta\big\}.$$

By the Itô formula, for every $s \ge s_0$, we have

$$\begin{aligned}&E\big(V\big(s\wedge\vartheta, z(s\wedge\vartheta), m(s\wedge\vartheta)\big)\big)\\&\quad= E\big(V\big(s_0, z(s_0), m(s_0)\big)\big) + E\bigg(\int_{s_0}^{s\wedge\vartheta} \mathcal{L}V\big(\tau, z(\tau), \nu(\tau), m(\tau)\big)\,d\tau\bigg).\end{aligned}$$

Using (ii) and equation (3.1), we obtain that

$$E\big(V\big(s\wedge\vartheta, z(s\wedge\vartheta), m(s\wedge\vartheta)\big)\big) \le E\big(V\big(s_0, z(s_0), m(s_0)\big)\big) = \lambda\mu(\eta). \tag{3.2}$$

Notice that if $\vartheta \le s$, then

$$\big|z_1(\vartheta\wedge s)\big| = \big|z_1(\vartheta)\big| = \eta.$$

Then by (i) we have

$$E\big(V\big(s\wedge\vartheta, z(s\wedge\vartheta), m(s\wedge\vartheta)\big)\big) \ge E\big(\mathbf{1}_{\{\vartheta\le s\}}\mu\big(\big|z_1(\vartheta)\big|\big)\big) = \mu(\eta)P(\vartheta \le s). \tag{3.3}$$

Using (3.2) and (3.3), we obtain $P(\vartheta \leq s) \leq \lambda$. Letting $s \to +\infty$, we have $P(\vartheta \leq \infty) \leq \lambda$, which implies

$$P(|z_1(s)| < \eta, \forall s \geq s_0) \geq 1 - \lambda,$$

and the proof is completed. □

($\mathcal{A}_4$) There exist positive constants α_1 and p and functions μ_2, μ_3, μ_4 in $\mathcal{K}$ and $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n \times \bar{S}; \mathbb{R}_+)$ satisfying, for all $(s,z,v,j) \in [s_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \bar{S}$,

(i) $\alpha_1 |z_1|^p \leq V(s,z,j) \leq \mu_2(|z_1|)$,

(ii) $LV(s,z,v,j) \leq -\mu_3(|z_1|) + q\mu_4(|v_1|)$.

Theorem 3.2 *Suppose that assumptions ($\mathcal{A}_1$), ($\mathcal{A}_2$), and ($\mathcal{A}_4$) hold. Let μ_3 and μ_4 in $\mathcal{K}$ satisfy, for all $(s,z) \in [s_0, +\infty) \times \mathbb{R}^n$,*

$$\mu_3(|z|) \geq \mu_4(|z|), \tag{3.4}$$

where $\mu_3 - \mu_4$ is an increasing function. Then, for any initial value $\zeta \in L^p_{\mathcal{F}_{s_0}}([qs_0, s_0]; \mathbb{R}^n)$, the solution of equation (2.1) is PAS in probability with respect to z_1.

Proof We will proceed as in the proof of Theorem 3.1 in [23] with necessary changes.

By Theorem 3.1 it is easy to prove that equation (2.1) is stable in probability with respect to z_1.

Step 1. Fix $\zeta \in L^p_{\mathcal{F}_{s_0}}([qs_0, s_0]; \mathbb{R}^n)$ and $i_0 \in \bar{S}$. By the Itô formula, (i), (ii), and (3.4) we have

$$\begin{aligned}
&V(s, u(s), m(s)) \\
&\quad\leq V(s_0, u(s_0), m(s_0)) + \int_{s_0}^{s} q\mu_4(|z_1(q\tau)|)\, d\tau - \int_{s_0}^{s} \mu_3(|z_1(\tau)|)\, d\tau + M(s) \\
&\quad\leq V(s_0, u(s_0), m(s_0)) + \int_{qs_0}^{s_0} \mu_4(|z_1(\tau)|)\, d\tau - \int_{s_0}^{s} (\mu_3(|z_1(\tau)|) - \mu_4(|z_1(\tau)|))\, d\tau \\
&\qquad + M(s) \\
&\quad\leq \mu_2(|u(s_0)|) + \mu_4(\|\zeta\|)s_0(1-q) - \int_{s_0}^{s} (\mu_3(|z_1(\tau)|) - \mu_4(|z_1(\tau)|))\, d\tau + M(s),
\end{aligned} \tag{3.5}$$

where

$$M(s) = \int_{s_0}^{s} V_z(\tau, u(\tau), m(\tau)) g(\tau, z(\tau), z(q\tau), m(\tau))\, dW(\tau)$$

is a continuous local martingale with $M(s_0) = 0$ a.s. Applying Lemma 2.5 in [17] and taking $\chi = \mu_2(|u(s_0)|) + \mu_4(\|\zeta\|)s_0(1-q)$, $A(s) = 0$, $N(s) = \int_{s_0}^{s} (\mu_3(|z_1(\tau)|) - \mu_4(|z_1(\tau)|))\, d\tau$, and $M(s) = \int_{s_0}^{s} V_z(\tau, u(\tau), m(\tau)) g(\tau, z(\tau), z(q\tau), m(\tau))\, dW(\tau)$, we have

$$\limsup_{s \to +\infty} (V(s, u(s), m(s))) < \infty \quad \text{a.s.} \tag{3.6}$$

Then

$$\sup_{s_0\le s<\infty} V\big(s,u(s),m(s)\big) < \infty \quad \text{a.s.} \tag{3.7}$$

Thus using (3.4), (3.7), and (i) (in Assumption ($\mathcal{A}_4$)), we obtain

$$\sup_{s_0\le s<\infty} \big(z_1(s) - G_1\big(s,z(qs),m(s)\big)\big) < \infty. \tag{3.8}$$

For $T > 0$, by Assumption ($\mathcal{A}_2$), for $s_0 \le s \le T$, we have

$$\begin{aligned}\big|z_1(s)\big| &\le \big|z_1(s) - G_1\big(s,z(qs),m(s)\big)\big| + \big|G_1\big(s,z(qs),m(s)\big)\big| \\ &\le \big|z_1(s) - G_1\big(s,z(qs),m(s)\big)\big| + k\big|z_1(qs)\big|.\end{aligned}$$

It then follows that

$$\begin{aligned}\sup_{s_0\le s\le T}\big|z_1(s)\big| &\le \kappa \sup_{s_0\le s\le T}\big|z_1(qs)\big| + \sup_{s_0\le s\le T}\big|z_1(s) - G_1\big(s,z(qs),m(s)\big)\big| \\ &\le \kappa\|\zeta\| + \kappa \sup_{s_0\le s\le T}\big|z_1(qs)\big| + \sup_{s_0\le s\le T}\big|z_1(s) - G_1\big(s,z(qs),m(s)\big)\big|.\end{aligned}$$

Thus

$$\sup_{s_0\le s\le T}\big|z_1(s)\big| \le \frac{1}{1-\kappa}\Big(\kappa\|\zeta\| + \sup_{s_0\le s\le T}\big|z_1(s) - G_1\big(s,z(qs),m(s)\big)\big|\Big).$$

Using (3.8) and letting $T \to \infty$, we have

$$\sup_{s_0\le s<\infty}\big|z_1(s)\big| \quad \text{a.s.} \tag{3.9}$$

Thus taking the expectations of both sides of (3.5) and letting $s \to +\infty$, we have

$$E\bigg(\int_{s_0}^{+\infty}\big(\mu_3\big(\big|z_1(\tau)\big|\big) - \mu_4\big(\big|z_1(\tau)\big|\big)\big)\,d\tau\bigg) < \infty. \tag{3.10}$$

This implies that

$$\int_{s_0}^{+\infty}\big(\mu_3\big(\big|z_1(\tau)\big|\big) - \mu_4\big(\big|z_1(\tau)\big|\big)\big)\,d\tau < \infty \quad \text{a.s.} \tag{3.11}$$

Step 2. Set $\mu = \mu_3 - \mu_4$ ($\mu \in C(\mathbb{R}_+,\mathbb{R}_+)$). By (3.11) we can see that (see [15])

$$\liminf_{s\to+\infty}\big(\mu\big(\big|z_1(s)\big|\big)\big) = 0 \quad \text{a.s.} \tag{3.12}$$

Now we claim that

$$\lim_{s\to+\infty}\mu\big(\big|z_1(s)\big|\big) = 0 \quad \text{a.s.} \tag{3.13}$$

If (3.13) is false, then

$$P\Big(\limsup_{s\to+\infty}\mu\big(|z_1(s)|\big)>0\Big)>0.$$

Thus there exists a positive constant λ such that

$$P(\Gamma_1)\geq 3\lambda \tag{3.14}$$

with $\Gamma_1=\{\limsup_{s\to+\infty}\mu(|z_1(s)|)>2\lambda\}$. By (3.9) and using the fact $\|\zeta\|<\infty$, we can find $h=h(\lambda)>0$ sufficiently large such that

$$P(\Gamma_2)\geq 1-\lambda, \tag{3.15}$$

where $\Gamma_2=\{\sup_{qs_0\leq s<\infty}(|z_1(s)|<h)\}$. Using (3.14) and (3.15), we have

$$P(\Gamma_1\cap\Gamma_2)\geq 2\lambda. \tag{3.16}$$

Now we define the following stopping times:

$$\begin{aligned}
&\vartheta_h=\inf\{s\geq s_0,|z_1(s)|\geq h\},\\
&\vartheta_1=\inf\{s\geq s_0,\mu\big(|z_1(s)|\big)\geq 2\lambda\},\\
&\vartheta_{2k}=\inf\{s\geq \vartheta_{2k-1},\mu\big(|z_1(s)|\big)\leq \lambda\},\quad k=1,2,3,\dots,\\
&\vartheta_{2k+1}=\inf\{s\geq \vartheta_{2k},\mu\big(|z_1(s)|\big)\geq 2\lambda\},\quad k=1,2,3,\dots.
\end{aligned}$$

By the definitions of Γ_1 and Γ_2 and (3.12) we can see that if $\omega\in\Gamma_1\cap\Gamma_2$, then

$$\vartheta_k<\infty\quad\text{and}\quad\vartheta_h=\infty\quad\forall k\in\mathbb{N}^*. \tag{3.17}$$

Since $\vartheta_{2k}<\infty$ whenever $\vartheta_{2k-1}<\infty$, by (3.10) we obtain that

$$\begin{aligned}
&\lambda\sum_{k=1}^{\infty}E\big(\mathbf{1}_{\{\vartheta_{2k-1}<\infty,\vartheta_h=\infty\}}(\vartheta_{2k}-\vartheta_{2k-1})\big)\\
&\quad\leq\sum_{k=1}^{\infty}E\bigg(\mathbf{1}_{\{\vartheta_{2k-1}<\infty,\vartheta_{2k}<\infty,\vartheta_h=\infty\}}\int_{\vartheta_{2k-1}}^{\vartheta_{2k}}\mu\big(|z_1(\tau)|\big)\,d\tau\bigg)\\
&\quad\leq E\bigg(\int_{s_0}^{+\infty}\mu\big(|z_1(\tau)|\big)\,d\tau\bigg)\\
&\quad<\infty.
\end{aligned} \tag{3.18}$$

In fact, by assumption $(\mathcal{A}_1)$ there exists $k_h>0$ such that

$$\big|g(s,z,v,j)\big|^2\vee\big|f(s,z,v,j)\big|^2\leq k_h$$

whenever $(s,j) \in [s_0,+\infty) \times \bar{S}$ and $|z| \vee |\nu| \leq h$. Using the Hölder and Doob martingale inequalities, we have that for $k = 1,2,3,\dots$ and $T > 0$,

$$\begin{aligned}
&E\Big(\mathbf{1}_{\{\vartheta_h\wedge\vartheta_{2k-1}<\infty\}} \sup_{s_0\leq s\leq T}\big|z_1\big(\vartheta_h \wedge (\vartheta_{2k-1}+s)\big) - z_1(\vartheta_h\wedge\vartheta_{2k-1})\big|^2\Big) \\
&\leq 2E\bigg(\mathbf{1}_{\{\vartheta_h\wedge\vartheta_{2k-1}<\infty\}} \sup_{s_0\leq s\leq T}\bigg|\int_{\vartheta_h\wedge\vartheta_{2k-1}}^{\vartheta_h\wedge(\vartheta_{2k-1}+s)} f\big(\tau,z(\tau),z(q\tau),m(\tau)\big)\,d\tau\bigg|^2\bigg) \\
&\quad + 2E\bigg(\mathbf{1}_{\{\vartheta_h\wedge\vartheta_{2k-1}<\infty\}} \sup_{s_0\leq s\leq T}\bigg|\int_{\vartheta_h\wedge\vartheta_{2k-1}}^{\vartheta_h\wedge(\vartheta_{2k-1}+s)} g\big(\tau,z(\tau),z(q\tau),m(\tau)\big)\,dW(\tau)\bigg|^2\bigg) \\
&\leq 2TE\bigg(\mathbf{1}_{\{\vartheta_h\wedge\vartheta_{2k-1}<\infty\}} \int_{\vartheta_h\wedge\vartheta_{2k-1}}^{\vartheta_h\wedge(\vartheta_{2k-1}+T)} \big|f\big(\tau,z(\tau),z(q\tau),m(\tau)\big)\big|^2\,d\tau\bigg) \\
&\quad + 8E\bigg(\mathbf{1}_{\{\vartheta_h\wedge\vartheta_{2k-1}<\infty\}} \int_{\vartheta_h\wedge\vartheta_{2k-1}}^{\vartheta_h\wedge(\vartheta_{2k-1}+T)} \big|g\big(\tau,z(\tau),z(q\tau),m(\tau)\big)\big|^2\,d\tau\bigg) \\
&\leq 2k_h T(T+4). \qquad (3.19)
\end{aligned}$$

We know that if μ is a continuous function in $\mathbb{R}^n$, then it is uniformly continuous in $\overline{B}_h = \{z \in \mathbb{R}^n : |z| \leq h\}$. Thus we can choose sufficiently small $\varphi = \varphi(\lambda) > 0$ such that

$$\big|\mu(z) - \mu(\nu)\big| < \frac{\lambda}{2} \quad \text{whenever } z,\nu \in \overline{B_h}, |z-\nu| < \varphi. \qquad (3.20)$$

Set $T = T(\lambda,\varphi,h) > 0$ sufficiently small such that $\frac{2k_h T(T+4)}{\varphi^2} < \lambda$. By (3.19) we have

$$P\Big(\{\vartheta_h\wedge\vartheta_{2k-1}<\infty\} \cap \Big\{\sup_{s_0\leq s\leq T}\big|z_1\big(\vartheta_h\wedge(\vartheta_{2k-1}+s)\big) - z_1(\vartheta_h\wedge\vartheta_{2k-1})\big| \geq \varphi\Big\}\Big) < \lambda.$$

We can see that

$$\{\vartheta_h = \infty, \vartheta_{2k-1} < \infty\} = \{\vartheta_h\wedge\vartheta_{2k-1} < \infty, \vartheta_h = \infty\} \subset \{\vartheta_h\wedge\vartheta_{2k-1} < \infty\}.$$

Then we obtain

$$P\Big(\{\vartheta_{2k-1}<\infty, \vartheta_h = \infty\} \cap \Big\{\sup_{s_0\leq s\leq T}\big|z_1(\vartheta_{2k-1}+s) - z_1(\vartheta_{2k-1})\big| \geq \varphi\Big\}\Big) < \lambda.$$

Using (3.16) and (3.17), we deduce

$$\begin{aligned}
&P\Big(\{\vartheta_{2k-1}<\infty, \vartheta_h = \infty\} \cap \Big\{\sup_{s_0\leq s\leq T}\big|z_1(\vartheta_{2k-1}+s) - z_1(\vartheta_{2k-1})\big| < \varphi\Big\}\Big) \\
&= P\big(\{\vartheta_{2k-1}<\infty, \vartheta_h = \infty\}\big) \\
&\quad - P\Big(\{\vartheta_{2k-1}<\infty, \vartheta_h = \infty\} \cap \Big\{\sup_{s_0\leq s\leq T}\big|z_1(\vartheta_{2k-1}+s) - z_1(\vartheta_{2k-1})\big| \geq \varphi\Big\}\Big) \\
&> 2\lambda - \lambda = \lambda.
\end{aligned}$$

Therefore by (3.20) we have

$$P\Big(\{\vartheta_{2k-1}<\infty, \vartheta_h = \infty\} \cap \Big\{\sup_{s_0\leq s\leq T}\big|\mu\big(z_1(\vartheta_{2k-1}+s)\big) - \mu\big(z_1(\vartheta_{2k-1})\big)\big| < \lambda\Big\}\Big) > \lambda. \qquad (3.21)$$

Set $\overline{M}_k = \{\sup_{s_0\le s\le T}|\mu(z_1(\vartheta_{2k-1}+s))-\mu(z_1(\vartheta_{2k-1}))|<\lambda\}$. Notice that if $\omega \in \{\vartheta_{2k-1}<\infty, \vartheta_h = \infty\}\cap \overline{M}_k$, then

$$\vartheta_{2k}(\omega)-\vartheta_{2k-1}(\omega)\ge T.$$

By (3.18) and (3.21) we can derive that

$$\begin{aligned}\infty &> \lambda \sum_{k=1}^{\infty} E\big(\mathbf{1}_{\{\vartheta_{2k-1}<\infty,\vartheta_h=\infty\}}(\vartheta_{2k}-\vartheta_{2k-1})\big)\\ &\ge \lambda \sum_{k=1}^{\infty} E\big(\mathbf{1}_{\{\vartheta_{2k-1}<\infty,\vartheta_h=\infty\}\cap\overline{M}_k}(\vartheta_{2k}-\vartheta_{2k-1})\big)\\ &\ge \lambda T\sum_{k=1}^{\infty} P\big(\{\vartheta_{2k-1}<\infty,\vartheta_h=\infty\}\cap\overline{M}_k\big)\\ &\ge \lambda T\sum_{k=1}^{\infty}\lambda=\infty,\end{aligned}$$

which is impossible. Then (3.13) holds.

Step 3. By (3.9) and (3.13) there is $\Omega_0\subset\Omega$ with $P(\Omega_0)=1$ such that for all $\omega\in\Omega_0$,

$$\lim_{s\to+\infty}\mu\big(\big|z_1(s,\omega)\big|\big)=0,\quad \text{and}\quad \sup_{s_0\le s\le\infty}\big|z_1(s,\omega)\big|<\infty. \tag{3.22}$$

Now we must show that

$$\lim_{s\to+\infty} z_1(s,\omega)=0\quad \forall\omega\in\Omega_0. \tag{3.23}$$

If we suppose that (3.23) is false, then there is $\hat\omega\in\Omega_0$ such that $\lim_{s\to+\infty}\sup|z_1(s,\hat\omega)|>0$. Thus there exist subsequences $\{z_1(s_k,\hat\omega)\}_{k\ge0}$ of $\{z_1(s,\hat\omega)\}_{s\ge s_0}$ satisfying $|z_1(s_k,\hat\omega)|>\bar\alpha$ for some $\bar\alpha>0$ and all $k\ge0$. Since $\{z_1(s_k,\hat\omega)\}_{k\ge0}$ is bounded, we can find an increasing subsequence $\{\hat s_k\}_{k\ge0}$ such that $\{z_1(\hat s_k,\omega)\}_{k\ge0}$ converges to some $\bar z\in\mathbb{R}^n$ such that $|\bar z|>\bar\alpha$. Therefore $\mu(|\bar z|)=\lim_{k\to\infty}\mu(|z_1(s_k,\omega)|)>0$. However, by (3.22) we have $\mu(|\bar z|)=0$, a contradiction.

Consequently, the solution of system (2.1) is asymptotically stable in probability with respect to z_1. □

4 Asymptotic instability of NPSDEwMS

We will state a theorem about the asymptotic instability with respect to all variables of NPSDEwMS.

Definition 4.1 The solution $z(s)=(z_1(s),z_2(s))$ of equation (2.1) is called asymptotically unstable in probability if it is unstable in probability or for all $\zeta\in L^p_{\mathcal{F}_{s_0}}([qs_0,s_0];\mathbb{R}^n)$,

$$P\Big(\lim_{s\to+\infty} z_1(s)\ne0\Big)=1.$$

Theorem 4.1 *Suppose that there exist a function* $V\in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^n\times\bar S;\mathbb{R}_+)$ *and* μ_1, μ_2, μ_3, *and* μ_4 *in* $\mathcal{K}$ *such that for all* $(s,z,v,j)\in[s_0,+\infty)\times\mathbb{R}^n\times\mathbb{R}^n\times\bar S$,

(i) $\mu_1(|z|) \le V(s,z,j) \le \mu_2(|z|)$,

(ii) $\mathcal{L}V(s,z,\nu,j) \ge -\mu_3(|z|) + q\mu_4(|\nu|)$.

Then for any initial value $\zeta \in L^p_{\mathcal{F}_{s_0}}([qs_0,s_0];\mathbb{R}^n)$, *the solution of equation* (2.1) *is asymptotically unstable in probability.*

Proof The proof is similar to that of Theorem 4.3 in [6]. □

5 Example and numerical solution

We now give a numerical example to illustrate the application of our results.

Let $W(s)$ be a three-dimensional Brownian motion. Let $m(s)$ be a right-continuous Markov chain taking values in $\bar{S} = \{1,2,3\}$ with $\Gamma = (\gamma_{jk})_{1\le j,k\le 3}$ given by

$$\Gamma = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Moreover, we assume that $W(s)$ and $m(s)$ are independent. Consider the following NPS-DEwMS:

$$\begin{cases} d(z_1(s) - G(s,z_1(qs),m(s))) \\ \quad = f_1(s,z(s),z(qs),m(s))\,ds + g_1(s,z(s),z(qs),m(s))\,dW_1(s), \\ d(z_2(s) - G(s,z_2(qs),m(s))) \\ \quad = f_2(s,z(s),z(qs),m(s))\,ds + g_2(s,z(s),z(qs),m(s))\,dW_2(s), \\ d(z_3(s) - G(s,z_3(qs),m(s))) \\ \quad = f_3(s,z(s),z(qs),m(s))\,ds + g_3(s,z(s),z(qs),m(s))\,dW_3(s), \end{cases} \tag{5.1}$$

with initial data $\zeta(s)$. Moreover, for $(s,z,\nu,j) \in [s_0,+\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \bar{S}$, let

$$G(s,z,j) = \begin{cases} \frac{1}{5}z & \text{if } j=1, \\ \frac{1}{6}z & \text{if } j=2, \\ \frac{1}{9}z & \text{if } j=3, \end{cases} \qquad f_1(s,z,\nu,j) = \begin{cases} -(z_1 + \frac{1}{5}\nu_1) & \text{if } j=1, \\ -(z_1 + \frac{1}{6}\nu_1) & \text{if } j=2, \\ -(z_1 + \frac{1}{9}\nu_1) & \text{if } j=3, \end{cases}$$

$$f_2(s,z,\nu,j) = \begin{cases} -\frac{1}{3}(z_1 - \frac{1}{5}\nu_1)^2(z_2 - \frac{1}{5}\nu_2) & \text{if } j=1, \\ -\frac{1}{3}(z_1 - \frac{1}{6}\nu_1)^2(z_2 - \frac{1}{6}\nu_2) & \text{if } j=2, \\ -\frac{1}{3}(z_1 - \frac{1}{9}\nu_1)^2(z_2 - \frac{1}{9}\nu_2) & \text{if } j=3, \end{cases}$$

$$f_3(s,z,\nu,j) = \begin{cases} -2(z_3 + \frac{1}{5}\nu_3) & \text{if } j=1, \\ -2(z_3 + \frac{1}{6}\nu_3) & \text{if } j=2, \\ -\frac{11}{2}(z_3 + \frac{1}{9}\nu_3) & \text{if } j=3, \end{cases} \qquad g_1(s,z,\nu,j) = \begin{cases} \frac{1}{\sqrt{5}}\nu_2 & \text{if } j=1, \\ \frac{1}{\sqrt{6}}\nu_2 & \text{if } j=2, \\ \frac{1}{3}\nu_2 & \text{if } j=3, \end{cases}$$

$$g_2(s,z,\nu,j) = \begin{cases} \sqrt{\frac{2}{3}}(z_1 - \frac{1}{5}\nu_1)(z_2 - \frac{1}{5}\nu_2) & \text{if } j=1, \\ \sqrt{\frac{2}{3}}(z_1 - \frac{1}{6}\nu_1)(z_2 - \frac{1}{6}\nu_2) & \text{if } j=2, \\ \sqrt{\frac{2}{3}}(z_1 - \frac{1}{9}\nu_1)(z_2 - \frac{1}{9}\nu_2) & \text{if } j=3, \end{cases} \qquad g_3(s,z,\nu,j) = \begin{cases} \frac{2}{\sqrt{5}}\nu_3 & \text{if } j=1, \\ \sqrt{\frac{2}{3}}\nu_3 & \text{if } j=2, \\ \frac{2}{3}\nu_3 & \text{if } j=3. \end{cases}$$

Let $V(s,z,j) = z_1^2 + z_2^2 + z_3^2$ for $j \in \bar{S}$. Then for $j = 1$, we have

$$\begin{aligned}
\mathcal{L}V(s,z,\nu,1) &= -2\left(z_1^2 - \frac{1}{25}\nu_1^2\right) + \frac{1}{5}\nu_2^2 - 4\left(z_3^2 - \frac{1}{25}\nu_3^2\right) + \frac{4}{5}\nu_3^2 \\
&= -2z_1^2 + \frac{2}{25}\nu_1^2 - 4z_3^2 + \frac{24}{25}\nu_3^2 + \frac{1}{5}\nu_2^2 \\
&\geq -4\left(z_1^2 + z_2^2 + z_3^2\right) + \frac{2}{25}\left(\nu_1^2 + \nu_2^2 + \nu_3^2\right) \\
&= -4|z|^2 + \frac{2}{25}|\nu|^2.
\end{aligned}$$

For $j = 2$, it follows that

$$\begin{aligned}
\mathcal{L}V(s,z,\nu,2) &= -2\left(z_1^2 - \frac{1}{36}\nu_1^2\right) + \frac{1}{6}\nu_2^2 - 4\left(z_3^2 - \frac{1}{36}\nu_3^2\right) + \frac{2}{3}\nu_3^2 \\
&= -2z_1^2 + \frac{1}{18}\nu_1^2 + \frac{1}{6}\nu_2^2 - 4z_3^2 + \frac{7}{9}\nu_3^2 \\
&\geq -4\left(z_1^2 + z_2^2 + z_3^2\right) + \frac{1}{18}\left(\nu_1^2 + \nu_2^2 + \nu_3^2\right) \\
&= -4|z|^2 + \frac{1}{18}|\nu|^2.
\end{aligned}$$

For $j = 3$, we deduce

$$\begin{aligned}
\mathcal{L}V(s,z,\nu,3) &= -2\left(z_1^2 - \frac{1}{81}\nu_1^2\right) + \frac{1}{9}\nu_2^2 - 11\left(z_3^2 - \frac{1}{81}\nu_3^2\right) + \frac{4}{9}\nu_3^2 \\
&= -2z_1^2 + \frac{2}{81}\nu_1^2 + \frac{1}{9}\nu_2^2 - 11z_3^2 + \frac{47}{81}\nu_3^2 \\
&\geq -11\left(z_1^2 + z_2^2 + z_3^2\right) + \frac{2}{81}\left(\nu_1^2 + \nu_2^2 + \nu_3^2\right) \\
&= -11|z|^2 + \frac{2}{81}|\nu|^2.
\end{aligned}$$

Thus for $j \in \bar{S}$, we obtain

$$\mathcal{L}V(s,z,\nu,3) \geq -11|z|^2 + \frac{2}{81}|\nu|^2. \tag{5.2}$$

Therefore by Theorem 4.1, system (5.1) is asymptotically unstable with respect to all variables.

For $j \in \bar{S}$, we define V_1 by

$$V_1(s,z,j) = \begin{cases} z_3^2 & \text{if } j = 1,2, \\ \frac{1}{2}z_3^2 & \text{if } j = 3. \end{cases}$$

For $j = 1$, we have

$$\begin{aligned}
\mathcal{L}V_1(s,z,\nu,1) &= -4\left(z_3^2 - \frac{1}{25}\nu_3^2\right) + \frac{4}{5}\nu_3^2 - \frac{1}{2}\left(z_3 - \frac{1}{5}\nu_3\right)^2 \\
&= -4z_3^2 + \frac{24}{25}\nu_3^2 - \frac{1}{2}z_3^2 + \frac{1}{5}z_3\nu_3 - \frac{1}{50}\nu_3^2
\end{aligned}$$

$$\begin{aligned}
&= -\frac{9}{2}z_3^2 + \frac{47}{50}v_3^2 + \frac{1}{5}z_3 v_3 \\
&\le -\frac{9}{2}z_3^2 + \frac{47}{50}v_3^2 + \frac{z_3^2}{100} + v_3^2 \\
&= -\frac{449}{100}z_3^2 + \frac{97}{50}v_3^2 \\
&= -4.49z_3^2 + 1.94v_3^2.
\end{aligned}$$

For $j = 2$, we derive

$$\begin{aligned}
\mathcal{L}V_1(s,z,v,2) &= -4\left(z_3^2 - \frac{1}{36}v_3^2\right) + \frac{2}{3}v_3^2 - \frac{1}{2}\left(z_3 - \frac{1}{6}v_3\right)^2 \\
&= -4z_3^2 + \frac{7}{9}v_3^2 - \frac{1}{2}z_3^2 + \frac{1}{6}z_3 v_3 - \frac{1}{72}v_3^2 \\
&= -\frac{9}{2}z_3^2 + \frac{55}{72}v_3^2 + \frac{1}{6}z_3 v_3 \\
&\le -\frac{9}{2}z_3^2 + \frac{55}{72}v_3^2 + \frac{z_3^2}{144} + v_3^2 \\
&= -\frac{648}{144}z_3^2 + \frac{127}{72}v_3^2 \\
&= -4.5z_3^2 + 1.76v_3^2.
\end{aligned}$$

For $j = 3$, we deduce

$$\begin{aligned}
\mathcal{L}V_1(s,z,v,3) &= -\frac{11}{2}\left(z_3^2 - \frac{1}{81}v_3^2\right) + \frac{2}{9}v_3^2 + \left(z_3 - \frac{1}{9}v_3\right)^2 \\
&= -\frac{11}{2}z_3^2 + \frac{11}{162}v_3^2 + \frac{2}{9}v_3^2 + z_3^2 - \frac{2}{9}z_3 v_3 + \frac{1}{81}v_3^2 \\
&\le -\frac{9}{2}z_3^2 + \frac{49}{162}v_3^2 + \frac{z_3^2}{9} + \frac{v_3^2}{9} \\
&= -\frac{79}{18}z_3^2 + \frac{67}{162}v_3^2 \\
&= -4.38z_3^2 + 0.41v_3^2.
\end{aligned}$$

Then for $j \in \bar{S}$, it follows that

$$\mathcal{L}V_1(s,z,v,j) \le -4.38z_3^2 + (0.5)(3.88)v_3^2.$$

Consequently, by Theorem 3.2 system (5.1) is asymptotically stable with respect to z_3 with $\mu_1(|z_3|) = 4.38z_3^2$ and $\mu_2(|v_3|) = 3.88v_3^2$.

For system (5.1), we conduct a simulation using the Euler–Maruyama scheme with step size 0.001, $q = 0.35$, $s_0 = 1$, and the linear initial function $\zeta(s) = (s, -s, s-1)$ for $0.35 \le s \le 1$. Next, we provide the simulations for system (5.1). In Fig. 1, we show the stability of the component z_3 by simulation of its trajectories. In Fig. 2, we illustrate the instability of the components z_1 and z_2.

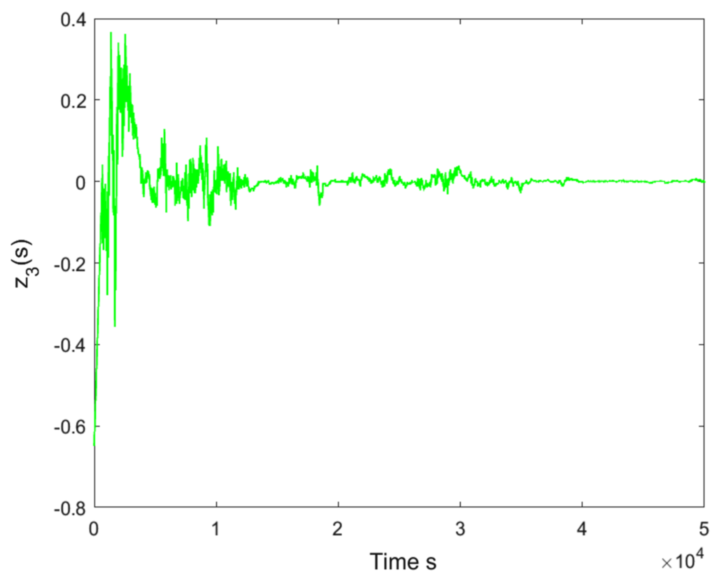

Figure 1 Simulations of the trajectory of $z_3(s)$ in system (5.1) with $\zeta_3(s) = s - 1$ for $s \in [0.35, 5 \times 10^4]$

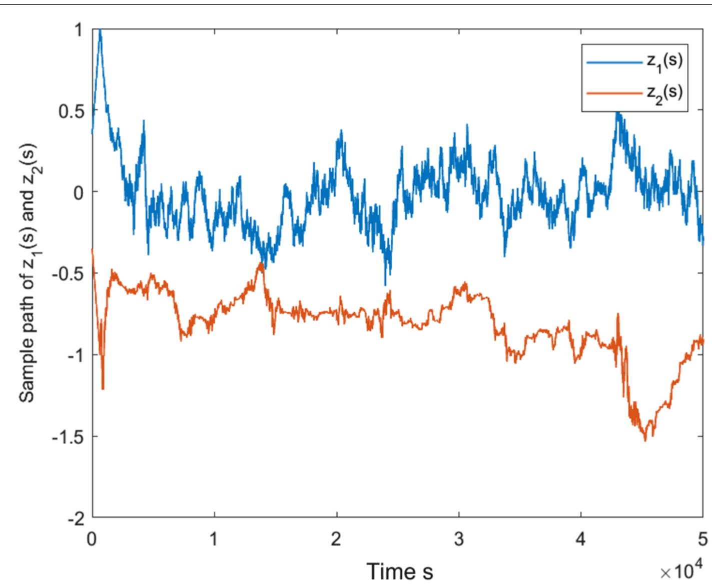

Figure 2 Simulations of the trajectories of the components $z_1(s)$ and $z_2(s)$ with $\zeta_1(s) = s$ and $\zeta_2(s) = -s$ on $[0.35, 5 \times 10^4]$

The simulation results clearly show that the trajectories of the corresponding stochastic system converge asymptotically to the equilibrium state for any given initial values, thus verifying the effectiveness of theoretical results.

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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LM and TC carried out the problem and gave the instructions while writing the paper. LM and TC deduced the mathematical computation and theorems involved and wrote the manuscript. MR has done the numerical simulations and wrote the manuscript. All authors read and approved the final manuscript.

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