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Attractors of stochastic lattice dynamical systems with a multiplicative noise and non-Lipschitz nonlinearities ☆

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ABSTRACT

In this paper we study the asymptotic behavior of solutions of a first-order stochastic lattice dynamical system with a multiplicative noise.
We do not assume any Lipschitz condition on the nonlinear term, just a continuity assumption together with growth and dissipative conditions, so that uniqueness of the Cauchy problem fails to be true.
Using the theory of multi-valued random dynamical systems we prove the existence of a random compact global attractor.

1. Introduction

This paper is devoted to the long term behavior of the following stochastic lattice differential equation

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$$\frac{du_i(t)}{dt} = \nu(u_{i-1} - 2u_i + u_{i+1}) - f_i(u_i) + \sum_{j=1}^{N} c_j u_i \circ \frac{dw_j(t)}{dt}, \quad i \in \mathbb{Z}, \tag{1.1}$$

where $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, $\mathbb{Z}$ denotes the integer set, ν is a positive constant, f_i is a continuous function satisfying a dissipative and a growth condition, $c_j \in \mathbb{R}$, for $j = 1, \ldots, N$, and w_j are mutually independent Brownian motions, where $\circ$ denotes the Stratonovich sense in the stochastic term.

Stochastic lattice differential equations arise naturally in a wide variety of applications where the spatial structure has a discrete character and uncertainties or random influences, called noises, are taken into account. These systems are used to model such systems as cellular neural networks with applications to image processing, pattern recognition, and brain science [23–26]. They are also used to model the propagation of pulses in myelinated axons where the membrane is excitable only at spatially discrete sites. In this case, u_i represents the potential at the i-th active site; see for example, [8,9,46,43,37,38]. Lattice differential equations can also be found in chemical reaction theory [30,36, 39]. Eq. (1.1) is a one-dimensional lattice system with diffusive nearest neighbor interaction, a dissipative nonlinear reaction term and a multiplicative white noise at each node. This may be the result of an environmental effect on the whole domain of the system. Also, it can appear after a spacial discretization of a parabolic stochastic differential equation.

The system with an additive noise was studied in [4,6,17,33,40,51] (see [50] for first-order retarded lattice systems as well). Also, a second-order lattice dynamical system with additive noise was studied in [49]. The case of a multiplicative noise has been considered in [15] and [33]. A sine-Gordon lattice equation with multiplicative noise has been studied in [32].

Recently, there are many works on deterministic lattice dynamical systems. For traveling waves, we refer the readers to [19,41,20,55,1,5] and the references therein. The chaotic properties of solutions for such systems have been investigated by [19] and [22,47,21,29]. In the absence of the white noise, the existence and properties of the global attractor for lattice differential equations of the type (1.1) were established in [2,7,10,42,48,52–54].

The study of global random attractors was initiated by Ruelle [44]. The fundamental theory of global random attractors for stochastic partial differential equations was developed by Crauel, Debussche, and Flandoli [27], Crauel and Flandoli [28], Flandoli and Schmalfuß [31], Imkeller and Schmalfuß [35], and others. Due to the unbounded fluctuations in the systems caused by the white noise, the concept of pullback global random attractor was introduced to capture the essential dynamics with possibly extremely wide fluctuations. This is significantly different from the deterministic case.

In the present paper, we extend the results given in [15] by proving the existence of a random global attractor for the stochastic lattice dynamical system (1.1) without assuming any Lipschitz condition of the nonlinear term f_i ensuring uniqueness of the Cauchy problem. Therefore, in order to obtain the random attractor we use the general theory of attractors for multi-valued random dynamical systems developed in [11]. Comparing with the case of uniqueness the main new difficulty which appears is the proof of the measurability of the pullback attractor.

This paper is organized as follows. In Section 2, we introduce basic concepts concerning multi-valued random dynamical systems and global random attractors. In Section 3, we show that the stochastic lattice differential equation (1.1) generates a multi-valued strict cocycle. The existence of the global random attractor is given in Section 4.

2. Multi-valued random dynamical systems

We recall now some standard definitions for set-valued non-autonomous and random dynamical systems and some results ensuring the existence of a pullback and a random global attractor for these systems.

A pair (Ω, θ) where $\theta = (\theta_t)_{t\in\mathbb{R}}$ is a flow on Ω, that is,

$$\theta : \mathbb{R} \times \Omega \to \Omega,$$

$$\theta_0 = \mathrm{id}_\Omega, \qquad \theta_{t+\tau} = \theta_t \circ \theta_\tau =: \theta_t\theta_\tau \quad \text{for } t, \tau \in \mathbb{R},$$

is called a non-autonomous perturbation.

Let $\mathcal{P} := (\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and a measurable non-autonomous flow θ:

$$\theta : \big(\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}\big) \to (\Omega, \mathcal{F}). \tag{2.1}$$

In addition, $\mathbb{P}$ is supposed to be ergodic with respect to θ, which means that every θ_t-invariant set has measure zero or one for $t \in \mathbb{R}$. Hence $\mathbb{P}$ is invariant with respect to θ_t. The quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a metric dynamical system.

If we replace in the definition of a metric dynamical system the probability space $\mathcal{P}$ by its completion $\mathcal{P}^c := (\Omega, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, the measurability property (2.1) is not true in general (see Arnold [3, Appendix A]). However, for fixed any $t \in \mathbb{R}$ we have that

$$\theta_t : (\Omega, \overline{\mathcal{F}}) \to (\Omega, \overline{\mathcal{F}})$$

is measurable.

Let $X = (X, d_X)$ be a Polish space and let $D : \omega \mapsto D(\omega) \in 2^X$ be a multi-valued mapping. The set of multi-functions $D : \omega \mapsto D(\omega) \in 2^X$ with closed and non-empty images is denoted by $C(X)$. On the other hand, $P_f(X)$ will be the set of all non-empty closed subsets of the space X. Thus, it is equivalent to write that D is in $C(X)$, or $D : \Omega \mapsto P_f(X)$.

Let $D : \omega \mapsto D(\omega)$ be a multi-valued mapping in X over $\mathcal{P}$. Such a mapping is called a random set if

$$\omega \mapsto \inf_{y \in D(\omega)} d_X(x, y)$$

is a random variable for every $x \in X$. It is well known that a mapping is a random set if and only if for every open set $\mathcal{O}$ in X the inverse image $\{\omega\colon\ D(\omega) \cap \mathcal{O} \neq \emptyset\}$ is measurable, i.e., it belongs to $\mathcal{F}$ [34, Proposition 2.1.4].

Clearly, this is also valid if we replace $\mathcal{P}$ by $\mathcal{P}^c$ and $\mathcal{F}$ by $\overline{\mathcal{F}}$. It is obvious that, if D is a random set with respect to $\mathcal{P}$, then it is also random with respect to $\mathcal{P}^c$.

We now introduce non-autonomous and random dynamical systems.

Definition 2.1. A multi-valued map $G : \mathbb{R}^+ \times \Omega \times X \to P_f(X)$ is called a multi-valued non-autonomous dynamical system (MNDS) if:

(i) $G(0, \omega, \cdot) = \mathrm{id}_X$,
(ii) $G(t + \tau, \omega, x) \subset G(t, \theta_\tau\omega, G(\tau, \omega, x))$ (cocycle property) for all $t, \tau \in \mathbb{R}^+$, $x \in X$, $\omega \in \Omega$.

It is called a strict MNDS if, moreover,

(iii) $G(t + \tau, \omega, x) = G(t, \theta_\tau\omega, G(\tau, \omega, x))$ for all $t, \tau \in \mathbb{R}^+$, $x \in X$, $\omega \in \Omega$.

An MNDS is called a multi-valued random dynamical system (MRDS) if the multi-valued mapping

$$(t, \omega, x) \mapsto G(t, \omega, x)$$

is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X)$-measurable, i.e. $\{(t, \omega, x)\colon\ G(t, \omega, x) \cap \mathcal{O} \neq \emptyset\} \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X)$ for every open set $\mathcal{O}$ of the topological space X.

We note that for any non-empty set $V \subset X$, $G(t,\omega,V)$ is defined by

$$G(t,\omega,V)=\bigcup_{x_0\in V} G(t,\omega,x_0).$$

We now formulate a general condition ensuring that an MNDS defines an MRDS.

Lemma 2.2. *(See [11, Lemma 2.5].) Let Ω be a Polish space and let $\mathcal{F}$ be the Borel σ-algebra. Suppose that $(t,\omega,x)\mapsto U(t,\omega,x)$ is upper semi-continuous. Then this mapping is measurable in the sense of Definition* 2.1.

A multi-valued mapping D is said to be negatively (positively) invariant for the MNDS G if $D(\theta_t\omega)\subset G(t,\omega,D(\omega))$ $(G(t,\omega,D(\omega))\subset D(\theta_t\omega))$ for $\omega\in\Omega$, $t\in\mathbb{R}^+$. It is called strictly invariant if it is both negatively and positively invariant.

Let $\mathcal{D}$ be a family of multi-valued mappings with values in $C(X)$. We say that a family $K\in\mathcal{D}$ is pullback D-attracting if for every $D\in\mathcal{D}$,

$$\lim_{t\to+\infty} \operatorname{dist}_X\big(G\big(t,\theta_{-t}\omega,D(\theta_{-t}\omega)\big),K(\omega)\big)=0,\quad \text{for all } \omega\in\Omega.$$

$B\in\mathcal{D}$ is said to be pullback D-absorbing if for every $D\in\mathcal{D}$ there exists $T=T(\omega,D)>0$ such that

$$G\big(t,\theta_{-t}\omega,D(\theta_{-t}\omega)\big)\subset B(\omega),\quad \text{for all } t\geqslant T. \tag{2.2}$$

We shall give now the concept of global pullback $\mathcal{D}$-attractor. We need to consider a particular set system (see [16,45]). Let $\mathcal{D}$ be a set of multi-valued mappings in $C(X)$ satisfying the inclusion closed property: if we suppose that $D\in\mathcal{D}$ and D' is a multi-valued mapping in $C(X)$ such that $D'(\omega)\subset D(\omega)$ for $\omega\in\Omega$, then $D'\in\mathcal{D}$.

Definition 2.3. A family $\mathcal{A}\in\mathcal{D}$ is said to be a global pullback $\mathcal{D}$-attractor for the MNDS G if it satisfies:

(1) $\mathcal{A}(\omega)$ is compact for any $\omega\in\Omega$;
(2) $\mathcal{A}$ is pullback $\mathcal{D}$-attracting;
(3) $\mathcal{A}$ is negatively invariant.

$\mathcal{A}$ is said to be a strict global pullback $\mathcal{D}$-attractor if the invariance property in the third item is strict.

An appropriate modification of this definition for MRDS is the following.

Definition 2.4. Suppose that G is an MRDS and suppose that the properties of Definition 2.3 are satisfied. In addition, we suppose that A is a random set with respect to $\mathcal{P}^c$. Then $\mathcal{A}$ is called a random global pullback $\mathcal{D}$-attractor.

Now we recall two general results on the existence and uniqueness of pullback and random attractors associated to MNDS and MRDS respectively, which were proved in [11].

Theorem 2.5. *Suppose that the MNDS $G(t,\omega,\cdot)$ is upper semi-continuous for $t\geqslant 0$ and $\omega\in\Omega$. Let $K\in\mathcal{D}$ be a multi-valued mapping such that the MNDS is pullback $\mathcal{D}$-asymptotically compact with respect to K, i.e. for every sequence $t_n\to+\infty$, and each $\omega\in\Omega$, every sequence $y_n\in G(t_n,\theta_{-t_n}\omega,K(\theta_{-t_n}\omega))$ is pre-compact. In addition, suppose that K is pullback $\mathcal{D}$-absorbing. Then, the set $\mathcal{A}$ given by*

$$\mathcal{A}(\omega):=\bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s} G\big(t,\theta_{-t}\omega,K(\theta_{-t}\omega)\big)} \tag{2.3}$$

is a pullback $\mathcal{D}$-attractor. Furthermore, $\mathcal{A}$ is the unique element in $\mathcal{D}$ with these properties. In addition, if G is a strict MNDS, then $\mathcal{A}$ is strictly invariant.

Remark 2.6. When the property $\mathcal{A} \in \mathcal{D}$ is not satisfied (or it is not imposed in Definition 2.3), some difficulties may appear, for example, in proving the strict invariance of the attractor. This is the case when $\mathcal{D}$ is just the set of all bounded subsets of X (see [13,14]).

Theorem 2.7. *Let G be an MRDS and let the assumptions in Theorem* 2.5 *hold. Assume that $\omega \mapsto G(t, \omega, K(\omega))$ is a random set for $t \geqslant 0$ with respect to $\mathcal{P}^c$, and also that $G(t, \omega, K(\omega))$ is closed for all $t \geqslant 0$ and $\omega \in \Omega$. Then, the set $\mathcal{A}$ defined by* (2.3) *is a random set with respect to $\mathcal{P}^c$, so that it is a random global pullback $\mathcal{D}$-attractor.*

3. Stochastic lattice differential equations

We consider a stochastic lattice differential equation

$$\frac{du_i(t)}{dt} = \nu(u_{i-1} - 2u_i + u_{i+1}) - f_i(u_i) + \sum_{j=1}^{N} c_j u_i \circ \frac{dw_j(t)}{dt}, \quad i \in \mathbb{Z}, \tag{3.1}$$

where $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$, $\mathbb{Z}$ denotes the integer set, ν is a positive constant, f_i is a continuous function satisfying the assumptions below, $c_j \in \mathbb{R}$, for $j = 1, \dots, N$, and w_j are mutually independent two-sided Brownian motions on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We note that Eq. (3.1) is interpreted as a system of integral equations

$$\begin{aligned} u_i(t) = u_i(0) &+ \int_0^t \big(\nu\big(u_{i-1}(s) - 2u_i(s) + u_{i+1}(s)\big) - f_i\big(u_i(s)\big)\big)\, ds \\ &+ \int_0^t \sum_{j=1}^{N} c_j u_i(s) \circ dw_j(t), \quad i \in \mathbb{Z}, \end{aligned} \tag{3.2}$$

where the stochastic integral is understood in the sense of Stratonovich.

Assumptions on the nonlinearity f_i. Let $f_i : \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

(H1) For all $x \in \mathbb{R}$, $i \in \mathbb{Z}$,

$$f_i(x)x \geqslant \lambda x^2 - c_{0,i},$$

where $c_0 \in \ell^1, \lambda > 0$.

(H2) For all $x \in \mathbb{R}$, $i \in \mathbb{Z}$,

$$\big|f_i(x)\big| \leqslant C\big(|x|\big)|x| + c_{1,i},$$

where $c_1 \in \ell^2$, $c_{1,i} \geqslant 0$, and $C(\cdot) \geqslant 0$ is a continuous increasing function.

(H3) The maps $f_i : \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{Z}$, are continuous.

For convenience, we now formulate system (3.1) as a stochastic differential equation in ℓ^2. Denote by $\|\cdot\|$ the norm in the space ℓ^2, and by B, B^*, C_j, $j=1,\dots,N$, and A the linear operators from ℓ^2 to ℓ^2 defined as follows. For $u=(u_i)_{i\in\mathbb{Z}}\in\ell^2$,

$$(Bu)_i = u_{i+1}-u_i, \qquad \big(B^*u\big)_i = u_{i-1}-u_i, \qquad (C_ju)_i = c_ju_i$$

and

$$(Au)_i = -u_{i-1}+2u_i-u_{i+1} \quad \text{for each } i\in\mathbb{Z}.$$

Then we find that

$$A = BB^* = B^*B,$$

and

$$\big(B^*u, v\big) = (u, Bv) \quad \text{for all } u, v\in\ell^2.$$

Therefore $(Au,u)\geqslant 0$ for all $u\in\ell^2$.

Let $\tilde{f}$ be the Nemytski operator associated with f_i, that is, for $u=(u_i)_{i\in\mathbb{Z}}\in\ell^2$, let $\tilde{f}(u) = (f_i(u_i))_{i\in\mathbb{Z}}$. Then, thanks to (H1)–(H2), this operator is well defined, and we therefore have

$$\|\tilde{f}(u)\|_{\ell^2}^2 = \sum_{i\in\mathbb{Z}}\big|f_i(u_i)\big|^2 \leqslant \sum_{i\in\mathbb{Z}}\big(C\big(|u_i|\big)|u_i|+c_{1,i}\big)^2 \leqslant 2M(u)\|u\|_{\ell^2}^2+\|c_1\|_{l^2}^2, \tag{3.3}$$

where $M(u)=\max_{i\in\mathbb{Z}} C(|u_i|)$.

Similar to (3.3), one can easily see that $\tilde{f}$ also satisfies

$$\tilde{f}(u,u)\geqslant \lambda\|u\|_{\ell^2}^2-\|c_0\|_{\ell^1}, \quad \forall u\in\ell^2, \tag{3.4}$$

and that $\tilde{f}:\ell^2\to\ell^2$ is continuous and weakly continuous (see [17] for a similar proof).

The system (3.1) with initial values $u_0\equiv(u_{0,i})_{i\in\mathbb{Z}}\in\ell^2$ may be rewritten as an equation in ℓ^2 for $t\geqslant 0$ and $\omega\in\Omega$,

$$u(t) = u_0+\int_0^t\big(-\nu Au(s)-\tilde{f}\big(u(s)\big)\big)\,ds+\sum_{j=1}^N\int_0^t C_ju(s)\circ dw_j(t). \tag{3.5}$$

To prove that this stochastic equation (3.5) generates a random dynamical system, we will transform it into a random differential equation in ℓ^2. This can be done thanks to the special form of the stochastic term.

Before performing this transformation, we need to recall some properties of the Ornstein–Uhlenbeck processes. Let us start by describing a probability space $(\widetilde{\Omega},\mathcal{F},\mathbb{P})$ which will be useful for our analysis. Consider

$$\widetilde{\Omega} = \big\{\omega\in\mathcal{C}(\mathbb{R},\mathbb{R}):\ \omega(0)=0\big\} = C_0(\mathbb{R},\mathbb{R})$$

endowed with the compact open topology (see [3, Appendices A.2 and A.3]), where $\mathbb{P}$ is the corresponding Wiener measure and $\mathcal{F}$ is the Borel σ-algebra on Ω. Let

$$\theta_t\omega(\cdot) = \omega(\cdot+t)-\omega(t), \quad t\in\mathbb{R}. \tag{3.6}$$

Then $(\widetilde{\Omega},\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

Let us consider the one-dimensional stochastic differential equation

$$dz = -\alpha z\,dt + dw(t), \tag{3.7}$$

for $\alpha > 0$. This equation has a random fixed point in the sense of random dynamical systems generating a stationary solution known as the stationary Ornstein–Uhlenbeck process (see Caraballo et al. [12] for more details). In fact, we have:

Lemma 3.1. *(See Caraballo et al. [12].) There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\overline{\Omega} \in \mathcal{F}$ of $\widetilde{\Omega} = C_0(\mathbb{R}, \mathbb{R})$ of full measure such that*

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{t} = 0 \quad \textit{for } \omega \in \overline{\Omega},$$

and, for such ω, the random variable given by

$$z^*(\omega) := -\alpha \int_{-\infty}^{0} e^{\alpha\tau} \omega(\tau)\,d\tau$$

is well defined. Moreover, for $\omega \in \overline{\Omega}$, the mapping

$$(t,\omega) \mapsto z^*(\theta_t\omega) = -\alpha \int_{-\infty}^{0} e^{\alpha\tau} \theta_t\omega(\tau)\,d\tau = -\alpha \int_{-\infty}^{0} e^{\alpha\tau} \omega(t+\tau)\,d\tau + \omega(t)$$

is a stationary solution of (3.7) *with continuous trajectories. In addition, for $\omega \in \overline{\Omega}$*

$$\lim_{t\to\pm\infty} \frac{|z^*(\theta_t\omega)|}{|t|} = 0, \qquad \lim_{t\to\pm\infty} \frac{1}{t} \int_0^t z^*(\theta_\tau\omega)\,d\tau = 0,$$
$$\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t \left|z^*(\theta_\tau\omega)\right| d\tau = \mathbb{E}\left|z^*\right| < \infty. \tag{3.8}$$

Remark 3.2. We now consider θ defined in (3.6) on $\overline{\Omega}$ instead of $\widetilde{\Omega}$. This mapping possesses the same properties as the original one if we choose for $\mathcal{F}$ the trace σ-algebra with respect to $\overline{\Omega}$ denoted also by $\mathcal{F}$.

Let us consider $\alpha = 1$ and denote by z_j^* its associated Ornstein–Uhlenbeck process corresponding to (3.7) with w_j instead of w.

Then for any $j = 1, \ldots, N$ we have a stationary Ornstein–Uhlenbeck process generated by a random variable $z_j^*(\omega)$ on $\overline{\Omega}_j$ with properties formulated in Lemma 3.1 defined on the metric dynamical system $(\overline{\Omega}_j, \mathcal{F}_j, \mathbb{P}_j, \theta)$. We set

$$(\Omega, \mathcal{F}, \mathbb{P}, \theta), \tag{3.9}$$

where

$$\Omega = \overline{\Omega}_1 \times \cdots \times \overline{\Omega}_N, \qquad \mathcal{F} = \bigotimes_{i=1}^{N} \mathcal{F}_i, \qquad \mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2 \times \cdots \times \mathbb{P}_N,$$

and θ is the flow of Wiener shifts.

Now, let us note that operator C_j generates a strongly continuous semigroup (in fact, a uniformly continuous group) of operators $S_{C_j}(t)$. More precisely, $S_{C_j}(t)$ is given by

$$S_{C_j}(t)u = e^{c_j t}u, \quad \text{for } u \in \ell^2.$$

Then we denote

$$T(\omega) := S_{C_1}\big(z_1^*(\omega)\big) \circ \cdots \circ S_{C_N}\big(z_N^*(\omega)\big) = e^{\sum_{j=1}^{N} c_j z_j^*(\omega)} Id_{\ell^2},$$

which is clearly a homeomorphism in $H = \ell^2$. The inverse operator is well defined by

$$T^{-1}(\omega) := S_{C_N}\big(-z_N^*(\omega)\big) \circ \cdots \circ S_{C_1}\big(-z_1^*(\omega)\big) = e^{-\sum_{j=1}^{N} c_j z_j^*(\omega)} Id_{\ell^2}.$$

For simplicity, let us denote $\delta(\omega) = \sum_{j=1}^{N} c_j z_j^*(\omega)$. It easily follows that $\|T^{-1}(\theta_t\omega)\|$ has sub-exponential growth as $t \to \pm\infty$ for any $\omega \in \Omega$. Hence $\|T^{-1}\|$ is tempered. According to Remark 3.2 we can change our metric dynamical system with respect to $\overline{\Omega}$. However the new metric dynamical system will be denoted by the same symbols $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$.

We now argue in a heuristic informal way. Let us consider the change of variable

$$v(t) = T^{-1}(\theta_t\omega)u(t) = e^{-\delta(\theta_t\omega)}u(t), \tag{3.10}$$

where u is a solution to (3.5). Then,

$$\begin{aligned} dv(t) &= e^{-\delta(\theta_t\omega)}\, du(t) - \sum_{j=1}^{N} c_j e^{-\delta(\theta_t\omega)} u(t) \circ dz_j^*(\theta_t\omega) \\ &= e^{-\delta(\theta_t\omega)}\big(-\nu A u(t) - \tilde{f}\big(u(t)\big) + \delta(\theta_t\omega)u(t)\big)\, dt \\ &= \big(-\nu A v(t) - e^{-\delta(\theta_t\omega)}\tilde{f}\big(e^{\delta(\theta_t\omega)}v(t)\big) + \delta(\theta_t\omega)v(t)\big)\, dt. \end{aligned}$$

So we can consider the following evolution equation with random coefficients but without white noise

$$\frac{dv}{dt} = -\nu A v + \delta(\theta_t\omega)v - e^{-\delta(\theta_t\omega)}\tilde{f}\big(e^{\delta(\theta_t\omega)}v\big), \tag{3.11}$$

and initial condition $v(0) = v_0 \in H$.

From (H1) and for every $x \in \mathbb{R}$ we obtain

$$\begin{aligned} e^{-\delta(\theta_t\omega)} f_i\big(e^{\delta(\theta_t\omega)}x\big)x &\geqslant e^{-2\delta(\theta_t\omega)}\big(\lambda e^{2\delta(\theta_t\omega)}x^2 - c_{0,i}\big) \\ &= \lambda x^2 - e^{-2\delta(\theta_t\omega)}c_{0,i}. \end{aligned} \tag{3.12}$$

Now we establish the following result.

Theorem 3.3. *Let $T > 0$ and $v_0 \in H$ be fixed. Then, for every $\omega \in \Omega$, Eq.* (3.11) *admits at least a solution* $v(\cdot, \omega, v_0) \in \mathcal{C}([0, T], \ell^2)$.

Proof. For any fixed $T > 0$ and $\omega \in \Omega$, and thanks to similar arguments as those in [17, p. 169] (notice that the mapping $(t, v) \mapsto F(t, v) = -\nu A v + \delta(\theta_t\omega)v - e^{-\delta(\theta_t\omega)}\tilde{f}(e^{\delta(\theta_t\omega)}v)$ is weakly continuous), (3.11) possesses at least a local solution $v(\cdot; \omega, v_0) \in \mathcal{C}([0, T_{\max}), \ell^2)$, where $[0, T_{\max})$ is the maximal interval of existence for the solution of (3.11). We prove now that this local solution is a global one. From (3.11), (3.12) and the fact that $(Av, v) \geqslant 0$, for all $v \in \ell^2$, it follows that

$$\begin{aligned}\frac{d}{dt}\|v(t)\|^2 &= 2\big(-\nu(Av, v) - \big(e^{-\delta(\theta_t\omega)}\tilde{f}\big(e^{\delta(\theta_t\omega)}v\big), v\big) + \delta(\theta_t\omega)\|v\|^2\big)\\ &\leqslant -2\lambda\|v\|^2 + 2\delta(\theta_t\omega)\|v\|^2 + \|c_0\|_{\ell^1}e^{-2\delta(\theta_t\omega)}\\ &\leqslant \|c_0\|_{\ell^1}e^{-2\delta(\theta_t\omega)} + \big(-2\lambda + 2\delta(\theta_t\omega)\big)\|v\|^2,\end{aligned} \tag{3.13}$$

and, by Gronwall's lemma,

$$\|v(t)\|^2 \leqslant e^{-2\lambda t + 2\int_0^t \delta(\theta_s\omega)\,ds}\|v_0\|^2 + \|c_0\|_{\ell^1}\int_0^t e^{-2\delta(\theta_r\omega)}e^{\int_r^t(-2\lambda + 2\delta(\theta_s\omega))\,ds}\,dr,$$

whence

$$\begin{aligned}\|v(t)\|^2 &\leqslant e^{-2\lambda t + 2\int_0^t \delta(\theta_s\omega)\,ds}\|v_0\|^2 + \|c_0\|_{\ell^1}\int_0^t e^{-2\delta(\theta_r\omega)}e^{-2\lambda(t-r) + 2\int_r^t \delta(\theta_s\omega)\,ds}\,dr\\ &\leqslant e^{-2\lambda t + 2\int_0^t \delta(\theta_s\omega)\,ds}\|v_0\|^2 + \|c_0\|_{\ell^1}e^{-2\lambda t + \int_0^t \delta(\theta_s\omega)\,ds}\int_0^t e^{-2\delta(\theta_r\omega) + 2\lambda r - 2\int_0^r \delta(\theta_s\omega)\,ds}\,dr.\end{aligned}$$

If we denote $\beta(\omega) = \max_{t\in[0,T]}(\|c_0\|_{\ell^1}e^{-2\lambda t + \int_0^t \delta(\theta_s\omega)\,ds}\int_0^t e^{-2\delta(\theta_r\omega) + 2\lambda r - 2\int_0^r \delta(\theta_s\omega)\,ds}\,dr)$ and $\alpha(\omega) = 2\int_0^T |\delta(\theta_s\omega)|\,ds$, we then have

$$\|v(t)\|^2 \leqslant \|v_0\|^2 e^{\alpha(\omega)} + \beta(\omega), \tag{3.14}$$

which implies that the solution v is defined in $[0, T]$ (in fact in $[0, +\infty)$; see [42]). □

Now, we say that $u(\cdot) = u(\cdot, \omega, u^0)$ is a solution of (3.5) (or (3.1)) if

$$u(t) = e^{\delta(\theta_t\omega)}v\big(t, \omega, e^{-\delta(\omega)}u^0\big),$$

where $v(\cdot, \omega, e^{-\delta(\omega)}u^0)$ is a solution of (3.11) with initial value $e^{-\delta(\omega)}u^0$.

Let $\mathcal{S}(v^0, \omega)$ be the set of all solutions to (3.11) corresponding to the initial datum $v^0 \in \ell^2$ and $\omega \in \Omega$.

We define the multi-valued map $G : \mathbb{R}^+ \times \Omega \times \ell^2 \to P(\ell^2)$ as follows

$$G\big(t, \omega, u^0\big) = \big\{e^{\delta(\theta_t\omega)}v(t):\ v \in \mathcal{S}\big(e^{-\delta(\omega)}u^0, \omega\big)\big\}. \tag{3.15}$$

Arguing in a standard way (see e.g. [13,14]), it can be proved that (3.15) is a strict cocycle. Namely, the next result holds.

Lemma 3.4. *The map G defined by* (3.15) *satisfies $G(0,\omega,\cdot) = Id_{\ell^2}$ and $G(t+\tau,\omega,x) = G(t,\theta_t\omega,G(\tau,\omega,x))$, for all $t,\tau \in \mathbb{R}^+$, $x \in \ell^2$, $\omega \in \Omega$.*

4. Existence of global random attractors

In this section, we prove the existence of a global random attractor for the random lattice dynamical system generated by Eq. (3.1). As universe $\mathcal{D}$ we will consider the family of multi-valued mappings D in ℓ^2 with $D(\omega) \subset B_{\ell^2}(0,\rho(\omega))$, the closed ball with center zero and radius $\rho(\omega)$, which possesses sub-exponential growth, i.e.

$$\lim_{t\to\pm\infty} \frac{\log^+ \rho(\theta_t\omega)}{t} = 0, \quad \omega \in \Omega.$$

$\mathcal{D}$ is called the family of sub-exponentially growing multi-functions in $C(\ell^2)$. Notice that inclusion closed property of $\mathcal{D}$ also holds. Our main result is the following.

Theorem 4.1. *Assume conditions* (H1)–(H3)*. Then G is an MNDS which has a unique pullback global strictly invariant $\mathcal{D}$-attractor $\mathcal{A}(\omega)$.*

If, moreover, we assume that $\lambda > \sum_{j=1}^N |c_j|\mathbb{E}(|z_j^*|)$, then G is a multi-valued random lattice dynamical system which possesses a unique global random $\mathcal{D}$-attractor.

To prove this theorem we will use Theorems 2.5 and 2.7. In order to ensure that our strict cocycle G satisfies the assumptions in the theorem above, we will proceed in the following way. First, we will prove that there exists a pullback $\mathcal{D}$-absorbing set for G in $\mathcal{D}$. Second, we will prove that G is asymptotically compact. Next, we will check that G has closed values (hence, G is an MNDS) and that it is upper semi-continuous, obtaining thus the existence of a $\mathcal{D}$-pullback attractor. Finally, under the additional assumption $\lambda > \sum_{j=1}^N |c_j|\mathbb{E}(|z_j^*|)$ we shall prove that G is an MRDS and that the pullback $\mathcal{D}$-attractor is measurable with respect to $\mathcal{P}^c$, proving that it is a pullback random $\mathcal{D}$-attractor.

We remark that in order to obtain the existence of a pullback $\mathcal{D}$-attractor we need only conditions (H1)–(H3). Condition $\lambda > \sum_{j=1}^N |c_j|\mathbb{E}(|z_j^*|)$ is necessary only for the measurability of the pullback $\mathcal{D}$-attractor and the process G.

4.1. Existence of the pullback absorbing set for the MNDS

In the following sections we assume that conditions (H1)–(H3) hold.

We first need to prove that there exists a pullback $\mathcal{D}$-absorbing set, i.e., a set $K(\omega)$ such that, for all $B \in \mathcal{D}$ and a.e. $\omega \in \Omega$, there exists $T_{B,\omega} > 0$ such that

$$G\big(t,\theta_{-t}\omega,B(\theta_{-t}\omega)\big) \subset K(\omega), \quad \text{for all } t \geqslant T_{B,\omega}.$$

Let us start with $v(t) = v(t,\omega,e^{-\delta(\omega)}u_0)$, a solution of (3.11) for some $u_0 \in B(\theta_{-t}\omega)$. Then, by arguing as in (3.13) we obtain

$$\begin{aligned}\big\|v(t)\big\|^2 &\leqslant e^{-2\lambda t+2\int_0^t \delta(\theta_s\omega)\,ds}\|v_0\|^2 \\ &\quad + \|c_0\|_{\ell^1} e^{-2\lambda t+2\int_0^t \delta(\theta_s\omega)\,ds} \int_0^t e^{-2\delta(\theta_r\omega)+2\lambda r-2\int_0^r \delta(\theta_s\omega)\,ds}\,dr. \qquad (4.1)\end{aligned}$$

Let us now substitute ω by $\theta_{-t}\omega$ and v_0 by $e^{-\delta(\theta_{-t}\omega)}u_0$ in the expression of $v(\cdot)$. We then have that

$$
\begin{aligned}
&\left\| v\left(t, \theta_{-t}\omega, e^{-\delta(\theta_{-t}\omega)} u_0\right)\right\|^2 \\
&\quad \leqslant e^{-2\lambda t + 2\int_0^t \delta(\theta_{s-t}\omega)\, ds} \left\| e^{-\delta(\theta_{-t}\omega)} u_0 \right\|^2 \\
&\qquad + \|c_0\|_{\ell^1} e^{-2\lambda t + 2\int_0^t \delta(\theta_{s-t}\omega)\, ds} \int_0^t e^{-2\delta(\theta_{r-t}\omega) + 2\lambda r - 2\int_0^r \delta(\theta_{s-t}\omega)\, ds}\, dr \\
&\quad \leqslant e^{-2\lambda t - 2\delta(\theta_{-t}\omega) + 2\int_{-t}^0 \delta(\theta_s\omega)\, ds} \|u_0\|^2 + \|c_0\|_{\ell^1} \int_0^t e^{-2\delta(\theta_{r-t}\omega) - 2\lambda(t-r) + 2\int_r^t \delta(\theta_{s-t}\omega)\, ds}\, dr \\
&\quad \leqslant e^{-2\lambda t - \delta(\theta_{-t}\omega) + 2\int_{-t}^0 \delta(\theta_s\omega)\, ds} \|u_0\|^2 + \|c_0\|_{\ell^1} \int_{-t}^0 e^{-2\delta(\theta_r\omega) + 2\lambda r + 2\int_r^0 \delta(\theta_s\omega)\, ds}\, dr \\
&\quad \leqslant e^{-2\lambda t - 2\delta(\theta_{-t}\omega) + 2\int_{-t}^0 \delta(\theta_s\omega)\, ds} \|u_0\|^2 + \|c_0\|_{\ell^1} \int_{-\infty}^0 e^{-2\delta(\theta_r\omega) + 2\lambda r + 2\int_r^0 \delta(\theta_s\omega)\, ds}\, dr.
\end{aligned}
\tag{4.2}
$$

Notice that, thanks to the properties of the Ornstein–Uhlenbeck process z_j^*, it follows that

$$
\int_{-\infty}^0 e^{-2\delta(\theta_r\omega) + 2\lambda r + 2\int_r^0 \delta(\theta_s\omega)\, ds}\, dr < +\infty.
$$

Taking into account that for any $u_0 \in B(\theta_{-t}\omega)$ it holds

$$
u(t, \theta_{-t}\omega, u_0) = e^{\delta(\omega)} v\left(t, \theta_{-t}\omega, e^{-\delta(\theta_{-t}\omega)} u_0\right),
$$

we have

$$
\begin{aligned}
\left\| u(t, \theta_{-t}\omega, u_0)\right\|^2 \leqslant{}& e^{\delta(\omega)} e^{-2\lambda t - \delta(\theta_{-t}\omega) + 2\int_{-t}^0 \delta(\theta_s\omega)\, ds} \mathrm{d}\left(B(\theta_{-t}\omega)\right)^2 \\
&+ e^{\delta(\omega)} \|c_0\|_{\ell^1} \int_{-\infty}^0 e^{-2\delta(\theta_s\omega) + \lambda s + 2\int_s^0 \delta(\theta_r\omega)\, dr}\, ds,
\end{aligned}
$$

where $\mathrm{d}(B(\theta_{-t}\omega))$ denotes the supremum of the norm of the set $B(\theta_{-t}\omega)$.

Denoting

$$
R^2(\omega) = 2e^{\delta(\omega)} \|c_0\|_{\ell^1} \int_{-\infty}^0 e^{-2\delta(\theta_s\omega) + \lambda s + 2\int_s^0 \delta(\theta_r\omega)\, dr}\, ds
$$

and noticing that

$$
\lim_{t\to+\infty} e^{\delta(\omega)} e^{-2\lambda t - \delta(\theta_{-t}\omega) + 2\int_{-t}^0 \delta(\theta_s\omega)\, ds} \mathrm{d}\left(B(\theta_{-t}\omega)\right)^2 = 0,
$$

it follows that

$$K(\omega) = B_{\ell^2}\big(0, R(\omega)\big) \tag{4.3}$$

is a pullback $\mathcal{D}$-absorbing set. We thus have the next result:

Lemma 4.2. *$K(\omega)$ defined by* (4.3) *is a pullback $\mathcal{D}$-absorbing set.*

We will now prove that $K \in \mathcal{D}$. To this end, we only have to check that

$$\lim_{t\to+\infty} e^{-\beta t} R(\theta_{-t}\omega) = 0, \quad \text{for all } \beta > 0.$$

Indeed, observe that

$$\begin{aligned}
e^{-\beta t} R^2(\theta_{-t}\omega) &= 2e^{-\beta t} e^{\delta(\theta_{-t}\omega)} \|c_0\|_{\ell^1} \int_{-\infty}^{0} e^{-2\delta(\theta_{s-t}\omega)+\lambda s+2\int_s^0 \delta(\theta_{r-t}\omega)\,dr}\, ds \\
&= 2e^{-\beta t} e^{\delta(\theta_{-t}\omega)} \|c_0\|_{\ell^1} \int_{-\infty}^{-t} e^{-2\delta(\theta_s\omega)+\lambda(s+t)t+2\int_s^{-t} \delta(\theta_r\omega)\,dr}\, ds \\
&= 2e^{-\frac{\beta}{2}t} e^{\delta(\theta_{-t}\omega)-2\int_{-t}^0 \delta(\theta_r\omega)\,dr} \|c_0\|_{\ell^1} \int_{-\infty}^{-t} e^{-2\delta(\theta_s\omega)+\lambda s+(\lambda-\frac{\beta}{2})t+2\int_s^0 \delta(\theta_r\omega)\,dr}\, ds.
\end{aligned}$$

If $\beta \geqslant 2\lambda$ the result follows directly. Let $\beta < 2\lambda$. By the properties in (3.8) we have

$$\big|-2\delta(\theta_s\omega)\big| \leqslant \frac{\beta}{4}|s|, \qquad 2\left|\int_s^0 \delta(\theta_r\omega)\,dr\right| \leqslant \frac{\beta}{4}|s|,$$

if $s \leqslant -t \leqslant -T(\beta,\omega)$. Hence,

$$\begin{aligned}
e^{-\beta t} R^2(\theta_{-t}\omega) &\leqslant 2e^{-\frac{\beta}{2}t} e^{\delta(\theta_{-t}\omega)-2\int_{-t}^0 \delta(\theta_r\omega)\,dr} \|c_0\|_{\ell^1} \int_{-\infty}^{-t} e^{(\lambda-\frac{\beta}{2})(t+s)}\, ds \\
&= 2e^{-\frac{\beta}{2}t} e^{\delta(\theta_{-t}\omega)-2\int_{-t}^0 \delta(\theta_r\omega)\,dr} \frac{\|c_0\|_{\ell^1}}{\lambda-\frac{\beta}{2}} \to 0, \quad \text{as } t \to +\infty.
\end{aligned}$$

4.2. Asymptotic compactness

In order to prove the asymptotic compactness for the MNDS G we first prove the following lemma.

Lemma 4.3. *Let $u^0(\omega) \in K(\omega)$, the absorbing set given by* (4.3)*. Then for every $\epsilon > 0$, there exist $T(\epsilon,\omega) > 0$ and $N(\epsilon,\omega) > 0$ such that any solution $u(\cdot)$ of Eq.* (3.1) *given by $u(t) = e^{\delta(\theta_t\omega)} v(t)$ with $v(\cdot) \in \mathcal{S}(u^0(\theta_{-t}\omega)e^{-\delta(\theta_{-t}\omega)}, \theta_{-t}\omega)$, satisfies*

$$\sum_{|i|\geqslant N(\epsilon,\omega)} \big|u_i\big(t,\theta_{-t}\omega,u^0(\theta_{-t}\omega)\big)\big|^2 \leqslant \epsilon, \quad \textit{for all } t \geqslant T(\epsilon,\omega).$$

Proof. Choose a smooth function ρ such that $0 \leqslant \rho(s) \leqslant 1$ for $s \in \mathbb{R}^+$, and

$$\rho(s) = 0 \quad \text{for } 0 \leqslant s \leqslant 1, \qquad \rho(s) = 1 \quad \text{for } s \geqslant 2.$$

Then there exists a constant C such that $|\rho'(s)| \leqslant C$ for $s \in \mathbb{R}^+$.

We first consider the random equation (3.11) with $v(t) = e^{-\delta(\theta_t\omega)}u(t)$. Let k be a fixed integer which will be specified later, and set $x = (x_i)_{i\in\mathbb{Z}}$ with $x_i = \rho(\frac{|i|}{k})v_i$. Then taking the inner product of Eq. (3.11) with x in ℓ^2, we get

$$\begin{aligned}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)|v_i|^2 = -2\nu(Av, x) + 2\delta(\theta_t\omega)\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)|v_i|^2 \\ - 2e^{-\delta(\theta_t\omega)}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)f_i\big(e^{\delta(\theta_t\omega)}v_i\big)v_i.\end{aligned} \tag{4.4}$$

We now estimate the terms in (4.4) as follows. First we have

$$\begin{aligned}(Av, x) &= (Bv, Bx) \\ &= \sum_{i\in\mathbb{Z}}(v_{i+1} - v_i)(x_{i+1} - x_i) \\ &= \sum_{i\in\mathbb{Z}}(v_{i+1} - v_i)\left[\left(\rho\left(\frac{|i+1|}{k}\right) - \rho\left(\frac{|i|}{k}\right)\right)v_{i+1} + \rho\left(\frac{|i|}{k}\right)(v_{i+1} - v_i)\right] \\ &= \sum_{i\in\mathbb{Z}}\left(\rho\left(\frac{|i+1|}{k}\right) - \rho\left(\frac{|i|}{k}\right)\right)(v_{i+1} - v_i)v_{i+1} + \sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)(v_{i+1} - v_i)^2 \\ &\geqslant \sum_{i\in\mathbb{Z}}\left(\rho\left(\frac{|i+1|}{k}\right) - \rho\left(\frac{|i|}{k}\right)\right)(v_{i+1} - v_i)v_{i+1}.\end{aligned}$$

By the property of the cut-off function ρ, we obtain the estimate

$$\begin{aligned}\left|\sum_{i\in\mathbb{Z}}\left(\rho\left(\frac{|i+1|}{k}\right) - \rho\left(\frac{|i|}{k}\right)\right)(v_{i+1} - v_i)v_{i+1}\right| &\leqslant \sum_{i\in\mathbb{Z}}\frac{|\rho'(\xi_i)|}{k}|v_{i+1} - v_i||v_{i+1}| \\ &\leqslant \frac{C}{k}\sum_i\big(|v_{i+1}|^2 + |v_i||v_{i+1}|\big) \leqslant \frac{2C}{k}\|v\|^2,\end{aligned}$$

which yields that

$$(Bv, Bx) \geqslant \frac{-2C\|v\|^2}{k}. \tag{4.5}$$

For the third term in (4.4), using condition (H1) we have

$$-2e^{-\delta(\theta_t\omega)}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)f_i\big(e^{\delta(\theta_t\omega)}v_i\big)v_i \leqslant 2e^{-2\delta(\theta_t\omega)}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)c_{0,i} - 2\lambda\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)|v_i|^2. \tag{4.6}$$

Then, from (4.4)–(4.6) it follows that

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)|v_i|^2+\big(2\lambda-2\delta(\theta_t\omega)\big)\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)|v_i|^2$$
$$\leqslant\frac{4\nu C}{k}\big\|v\big(t,\omega,e^{-\delta(\omega)}u_0\big)\big\|^2+2e^{-2\delta(\theta_t\omega)}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)c_{0,i}. \tag{4.7}$$

By using Gronwall's lemma, we have that for $t\geqslant T_K=T_K(\omega)$,

$$\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)\big|v_i\big(t,\omega,e^{-\delta(\omega)}u_0(\omega)\big)\big|^2$$
$$\leqslant e^{-2\lambda(t-T_K)+2\int_{T_K}^t\delta(\theta_s\omega)\,ds}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)\big|v_i\big(T_K,\omega,e^{-\delta(\omega)}u_0(\omega)\big)\big|^2$$
$$+\frac{4\nu C}{k}\int_{T_K}^{t}e^{-2\lambda(t-\tau)+2\int_\tau^t\delta(\theta_s\omega)\,ds}\big\|v\big(\tau,\omega,e^{-\delta(\omega)}u_0\big)\big\|^2\,d\tau$$
$$+\int_{T_K}^{t}e^{-2\lambda(t-\tau)+2\int_\tau^t\delta(\theta_s\omega)\,ds-2\delta(\theta_\tau\omega)}\,d\tau\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)c_{0,i}. \tag{4.8}$$

Replace ω by $\theta_{-t}\omega$. We then estimate each term on the right-hand side of (4.8). From (4.1) with t replaced by T_K and ω by $\theta_{-t}\omega$, it follows that

$$e^{-2\lambda(t-T_K)+2\int_{T_K}^t\delta(\theta_{s-t}\omega)\,ds}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)\big|v_i\big(T_K,\theta_{-t}\omega,e^{-\delta(\theta_{-t}\omega)}u_0(\theta_{-t}\omega)\big)\big|^2$$
$$\leqslant e^{-2\lambda t+2\int_0^t\delta(\theta_{s-t}\omega)\,ds-2\delta(\theta_{-t}\omega)}\big\|u_0(\theta_{-t}\omega)\big\|^2+\|c_0\|_{\ell^1}\int_0^{T_K}e^{-2\delta(\theta_{s-t}\omega)-2\lambda(t-s)+2\int_s^t\delta(\theta_{r-t}\omega)\,dr}\,ds.$$

Thus, using (3.8), there is a $T_1(\epsilon,\omega)>T_K(\omega)$ such that if $t>T_1(\epsilon,\omega)$, then

$$e^{-2\lambda(t-T_K)+2\int_{T_K}^t\delta(\theta_{s-t}\omega)\,ds}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)\big|v_i\big(T_K,\theta_{-t}\omega,e^{-\delta(\theta_{-t}\omega)}u_0(\theta_{-t}\omega)\big)\big|^2$$
$$\leqslant\frac{1}{3}\epsilon e^{-\delta(\omega)}. \tag{4.9}$$

Next, we estimate (using again (4.1))

$$\frac{4\nu C}{k}\int_{T_K}^{t}e^{-2\lambda(t-\tau)+2\int_\tau^t\delta(\theta_{s-t}\omega)\,ds}\big\|v\big(\tau,\theta_{-t}\omega,e^{-\delta(\theta_{-t}\omega)}u_0\big)\big\|^2\,d\tau$$
$$\leqslant\frac{4\nu C}{k}\big\|u_0(\theta_{-t}\omega)\big\|(t-T_K)e^{-2\lambda t+2\int_0^t\delta(\theta_{s-t}\omega)\,ds-2\delta(\theta_{-t}\omega)}$$
$$+\frac{4\nu C}{k}\|c_0\|_{\ell^1}\int_{T_K}^{t}\Big(e^{-2\lambda(t-\tau)+2\int_\tau^t\delta(\theta_{s-t}\omega)\,ds}e^{-2\lambda\tau+2\int_0^\tau\delta(\theta_{s-t}\omega)\,ds}$$

$$
\begin{aligned}
&\times \int_0^\tau e^{-2\delta(\theta_{r-t}\omega)+2\lambda r-2\int_0^r \delta(\theta_{s-t}\omega)\,ds}\,dr\Big)\,d\tau \\
&= \frac{4\nu C}{k}\big\|u_0(\theta_{-t}\omega)\big\|(t-T_K)e^{-2\lambda t+2\int_0^t \delta(\theta_{s-t}\omega)\,ds-2\delta(\theta_{-t}\omega)} \\
&\quad + \frac{4\nu C}{k}\|c_0\|_{\ell^1}\int_{T_K}^t\int_0^\tau e^{-2\lambda(t-r)+2\int_0^t \delta(\theta_{s-t}\omega)\,ds-2\delta(\theta_{r-t}\omega)}e^{-2\int_0^r \delta(\theta_{s-t}\omega)\,ds}\,dr\,d\tau \\
&= \frac{4\nu C}{k}\big\|u_0(\theta_{-t}\omega)\big\|(t-T_K)e^{-2\lambda t+2\int_0^t \delta(\theta_{s-t}\omega)\,ds-2\delta(\theta_{-t}\omega)} \\
&\quad + \frac{4\nu C}{k}\|c_0\|_{\ell^1}\int_{T_K}^t\int_0^\tau e^{-2\lambda(t-r)+2\int_r^t \delta(\theta_{s-t}\omega)\,ds-2\delta(\theta_{r-t}\omega)}\,dr\,d\tau.
\end{aligned}
$$

Then, using (3.8), there exist $T_2(\epsilon,\omega) > T_K(\omega)$ and $N_1(\epsilon,\omega) > 0$ such that if $t > T_2(\epsilon,\omega)$ and $k > N_1(\epsilon,\omega)$, then

$$
\frac{4\nu C}{k}\int_{T_K}^t e^{-2\lambda(t-\tau)+2\int_\tau^t \delta(\theta_{s-t}\omega)\,ds}\big\|v\big(\tau,\theta_{-t}\omega,e^{-\delta(\theta_{-t}\omega)}u_0\big)\big\|^2\,d\tau \leqslant \frac{1}{3}\epsilon e^{-\delta(\omega)}. \tag{4.10}
$$

Since $c_0 \in \ell^1$, by using (3.8), there exists $N_2(\epsilon,\omega) > 0$ such that for $k > N_2(\epsilon,\omega)$

$$
\int_{T_K}^t e^{-2\lambda(t-\tau)+2\int_\tau^t \delta(\theta_s\omega)\,ds-2\delta(\theta_\tau\omega)}\,d\tau \sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)c_{0,i} \leqslant \frac{1}{3}\epsilon e^{-\delta(\omega)}. \tag{4.11}
$$

Therefore, by letting

$$
\begin{aligned}
T(\epsilon,\omega) &= \max\big\{T_1(\epsilon,\omega), T_2(\epsilon,\omega)\big\},\\
N(\epsilon,\omega) &= \max\big\{N_1(\epsilon,\omega), N_2(\epsilon,\omega)\big\},
\end{aligned}
$$

we have for $t > T(\epsilon,\omega)$ and $k > N(\epsilon,\omega)$,

$$
\sum_{|i|\geqslant 2k}\big|v_i\big(t,\theta_{-t}\omega,e^{-\delta(\theta_{-t}\omega)}u_0(\theta_{-t}\omega)\big)\big|^2 \leqslant \sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{k}\right)\big|v_i\big(t,\theta_{-t}\omega,e^{-\delta(\theta_{-t}\omega)}u_0(\theta_{-t}\omega)\big)\big|^2 \leqslant \epsilon e^{-\delta(\omega)},
$$

which, thanks to relation (3.10), implies that

$$
\sum_{|i|\geqslant N(\epsilon,\omega)}\big|u_i\big(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)\big)\big|^2 \leqslant \epsilon.
$$

This completes the proof of the lemma. □

We are now ready to show the asymptotic compactness of K.

Theorem 4.4. *For $\omega \in \Omega$, G is asymptotically compact with respect to $K(\omega)$: each sequence*

$$p^n \in G\big(t_n, \theta_{-t_n}\omega, K(\theta_{-t_n}\omega)\big)$$

with $t_n \to \infty$ has a convergent subsequence in ℓ^2.

Proof. Consider $(t_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = \infty$ and $p^n \in G(t_n, \theta_{-t_n}\omega, K(\theta_{-t_n}\omega))$. Then, there exists $x^n \in K(\theta_{-t}\omega)$ such that $p^n \in G(t_n, \theta_{-t_n}\omega, x_n)$. We will show that $\{p^n\}_{n\in\mathbb{N}}$ possesses a convergent subsequence. Since $K(\omega)$ is a bounded absorbing set, for large n, $p^n \in K(\omega)$. Thus, there exist $v \in \ell^2$ and a subsequence of $\{p^n\}_{n\in\mathbb{N}}$ (still denoted by $\{p^n\}_{n\in\mathbb{N}}$) such that

$$\big\{p^n\big\}_{n\in\mathbb{N}} \to v \quad \text{weakly in } \ell^2. \tag{4.12}$$

Next, we show that the above weak convergence is actually a strong convergence, i.e., for each $\epsilon > 0$ there is $N^*(\epsilon, \omega) > 0$ such that for $n \geqslant N^*(\epsilon, \omega)$

$$\big\|p^n - v\big\| \leqslant \epsilon.$$

Thanks to Lemma 4.3, there exist $N_1^*(\epsilon, \omega) > 0$ and $K_1(\epsilon, \omega) > 0$ such that for $n > N_1^*$

$$\sum_{|i|\geqslant K_1(\epsilon,\omega)} \big|p_i^n\big|^2 \leqslant \frac{1}{8}\epsilon^2. \tag{4.13}$$

On the other hand, since $v \in \ell^2$, there exists $K_2(\epsilon)$ such that

$$\sum_{|i|\geqslant K_2(\epsilon)} |v_i|^2 \leqslant \frac{1}{8}\epsilon^2. \tag{4.14}$$

Letting $K(\epsilon, \omega) = \max\{K_1(\epsilon, \omega), K_2(\epsilon)\}$, by the weak convergence (4.12), we have for each $|i| \leqslant K(\epsilon, \omega)$ as $n \to \infty$

$$p_i^n \to v_i,$$

which implies that there exists $N_2^*(\epsilon, \omega) > 0$ such that when $n \geqslant N_2^*(\epsilon, \omega)$,

$$\sum_{|i|\leqslant K(\epsilon)} \big|p_i^n - v_i\big|^2 \leqslant \frac{1}{2}\epsilon^2. \tag{4.15}$$

Let $N^*(\epsilon, \omega) = \max\{N_1^*(\epsilon, \omega), N_2^*(\epsilon, \omega)\}$. Then, from (4.13), (4.14) and (4.15) we obtain for $n \geqslant N^*(\epsilon, \omega)$

$$\begin{aligned}
\big\|p^n - v\big\|^2 &= \sum_{i\in\mathbb{Z}} \big|p_i^n - v_i\big|^2 \\
&= \sum_{|i|\leqslant K(\epsilon)} \big|p_i^n - v_i\big|^2 + \sum_{|i|> K(\epsilon)} \big|p_i^n - v_i\big|^2 \\
&\leqslant \frac{1}{2}\epsilon^2 + 2\sum_{|i|> K(\epsilon)} \big(\big|p_i^n\big|^2 + |v_i|^2\big) \\
&\leqslant \epsilon^2.
\end{aligned}$$

Hence, p^n converges to v strongly. □

4.3. Existence of the pullback attractor

Let us now prove other properties of the cocycle G.

Lemma 4.5. *Let v^{0n} be a sequence converging to v^0 in ℓ^2 and fix $T > 0$. Then, for any $\omega \in \Omega$ and $\epsilon > 0$, there exists $K(\epsilon, \omega)$ such that for any solution $v^n(\cdot) \in \mathcal{S}(v^{0n}, \omega)$ it follows*

$$\sum_{|i| \geqslant 2K(\epsilon,\omega)} \left|v_i^n(t)\right|^2 \leqslant \epsilon, \quad \forall t \in [0, T]. \tag{4.16}$$

Moreover, there exist $v(\cdot) \in \mathcal{S}(v^0, \omega)$ and a subsequence v^{n_k} satisfying

$$v^{n_k} \to v \quad in\ C\big([0, T], \ell^2\big). \tag{4.17}$$

Proof. For any $\epsilon > 0$ there exist $K_1(\epsilon)$, $N_1(\epsilon)$ such that

$$\sum_{i \in \mathbb{Z}} \left|v_i^{0n} - v_i^0\right|^2 < \frac{\epsilon}{4}, \quad \forall n \geqslant N_1,$$

$$\sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{K}\right)\left|v_i^0\right|^2 < \frac{\epsilon}{4}, \quad \forall K \geqslant K_1.$$

Hence,

$$\sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{K}\right)\left|v_i^{0n}\right|^2 \leqslant 2\left(\sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{K}\right)\left|v_i^{0n} - v_i^0\right|^2 + \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{K}\right)\left|v_i^0\right|^2\right) < \epsilon, \tag{4.18}$$

if $n \geqslant N_1$ and $K \geqslant K_1$. Obviously, modifying K_1 appropriately, the result holds true for all n. Also, in view of (4.1) there exists $R_0(\omega, T) > 0$ such that

$$\left\|v^n(t)\right\| \leqslant R_0(\omega, T), \quad \forall t \in [0, T],\ \forall n. \tag{4.19}$$

Using inequality (4.7) and the continuity of $t \mapsto \delta(\theta_t\omega)$, one can find $K_2(\epsilon, \omega)$ such that

$$\frac{d}{dt}\sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{K}\right)\left|v_i^n\right|^2 - 2\delta(\theta_t\omega)\sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right)\left|v_i^n\right|^2 \leqslant \epsilon, \quad \text{if } K \geqslant K_2.$$

Using Gronwall's lemma and (4.18) we obtain

$$\begin{aligned}
\sum_{|i| \geqslant 2K(\epsilon)} \left|v_i^n(t)\right|^2 &\leqslant \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{K}\right)\left|v_i^n(t)\right|^2 \\
&\leqslant \epsilon e^{2\int_0^t \delta(\theta_s\omega)\,ds} + \epsilon \int_0^t e^{2\int_r^t \delta(\theta_s\omega)\,ds}\,dr \\
&\leqslant \epsilon R_1(\omega, T),
\end{aligned} \tag{4.20}$$

if $K \geqslant \max\{K_1, K_2\}$, so that (4.16) holds.

Fix now $t \in [0, T]$. In view of (4.19), passing to a subsequence, we can state that $v^n(t) \to w$ weakly in ℓ^2. Then, for any $\sigma > 0$, there exist $N_2(\sigma)$ and $K_3(\sigma)$ such that

$$\begin{aligned}\left\|v^n(t)-w\right\|_{\ell^2}^2 &\leqslant \sum_{|i|\leqslant K_3(\sigma)}\left|v_i^n(t)-w_i\right|^2+\sum_{|i|>K_3(\sigma)}\left|v_i^n(t)-w_i\right|^2\\ &\leqslant \sum_{|i|\leqslant K_3(\sigma)}\left|v_i^n(t)-w_i\right|^2+2\sum_{|i|>K_3(\sigma)}\left|v_i^n(t)\right|^2+2\sum_{|i|>K_3(\sigma)}\left|w_i(t)\right|^2\\ &<\sigma,\end{aligned}$$

if $n \geqslant N_2$. Hence, $v^n(t) \to w$ strongly in ℓ^2. It follows that the sequence $v^n(t)$ is pre-compact for any t. Now, using (3.3) and that $\|Av^n(t)\|_{\ell^2} \leqslant 4\|v^n(t)\|_{\ell^2}$ over (3.11), we have that

$$\begin{aligned}\left\|\frac{dv^n(t)}{dt}\right\|_{\ell^2} &\leqslant 4\nu\left\|v^n(t)\right\|_{\ell^2}+\delta(\theta_t\omega)\left\|v^n(t)\right\|_{\ell^2}\\ &\quad+e^{-\delta(\theta_t\omega)}\left(2M\left(v^n(t)\right)\left\|v^n(t)\right\|_{\ell^2}+\|c_1\|_{\ell^2}\right).\end{aligned}$$

By (4.19),

$$\left\|\frac{d}{dt}v^n(t)\right\|_{\ell^2} \leqslant R_2(\omega,T), \quad \forall n\in\mathbb{N},\ \forall t\in[0,T],$$

proving that the sequence v^n is equi-continuous. The Ascoli–Arzelà theorem implies the existence of a subsequence v^{n_k} converging in $C([0,T],\ell^2)$ to some function $v(\cdot)$. It is then easy to show that v is a solution of (3.11). Also, it is clear that $v(0)=v^0$. □

Lemma 4.5 implies several consequences.

Corollary 4.6. *For any $\omega\in\Omega$ and $t\geqslant 0$ the graph of the map $u^0\mapsto G(t,\omega,u^0)$ is closed. Hence, G possesses closed values.*

Proof. For $p^n\in G(t,\omega,u_n^0)$ there is $v_n(\cdot)\in\mathcal{S}(e^{-\delta(\omega)}u_n^0,\omega)$ such that $p^n=e^{\delta(\theta_t\omega)}v_n(t)$. Assume that $p^n\to p$ and $u_n^0\to u^0$. Applying Lemma 4.5 we obtain, passing to a subsequence, that $v_n\to v$ in $C([0,t],\ell^2)$, where $v(\cdot)\in\mathcal{S}(e^{-\delta(\omega)}u^0,\omega)$. Therefore, $p=e^{\delta(\theta_t\omega)}v(t)\in G(t,\omega,u^0)$. □

Corollary 4.7. *G is a strict MNDS.*

Proof. It follows from Lemma 3.4 and Corollary 4.6. □

Corollary 4.8. *For any $\omega\in\Omega$ and $t\geqslant 0$, the map $G(t,\omega,\cdot)$ has compact values.*

Proof. Let $p^n\in G(t,\omega,u^0)$ be an arbitrary sequence. Take $v_n(\cdot)\in\mathcal{S}(e^{-\delta(\omega)}u^0,\omega)$ such that $p^n=e^{\delta(\theta_t\omega)}v_n(t)$. By Lemma 4.5 there exists $v(\cdot)\in\mathcal{S}(e^{-\delta(\omega)}u^0,\omega)$ satisfying $v_{n_k}\to v$ in $C([0,t],\ell^2)$ for some subsequence. Hence, $p^{n_k}=e^{\delta(\theta_t\omega)}v_{n_k}(t)\to e^{\delta(\theta_t\omega)}v(t)\in G(t,\omega,u^0)$ in ℓ^2. □

Proposition 4.9. *For any $\omega\in\Omega$ and $t\geqslant 0$, the map $u^0\mapsto G(t,\omega,u^0)$ is upper semi-continuous.*

Proof. Suppose the opposite. Then there exist u^0, $t>0$, a neighborhood $\mathcal{O}$ of $G(t,\omega,u^0)$ and sequences $u_n^0\to u^0$, $p^n\in G(t,\omega,u_n^0)$ such that $\xi^n\notin\mathcal{O}$. Let $p^n=e^{\delta(\theta_t\omega)}v_n(t)$, where $v_n(\cdot)\in\mathcal{S}(e^{-\delta(\omega)}u_n^0,\omega)$. By Lemma 4.5 we obtain that, up to a subsequence, $v_n\to v$ in $C([0,t],\ell^2)$, where $v(\cdot)\in\mathcal{S}(e^{-\delta(\omega)}u^0,\omega)$. Thus, $p^n\to e^{\delta(\theta_t\omega)}v(t)\in G(t,\omega,u^0)$ in ℓ^2, which is a contradiction. □

Finally, we have the following result, which proves the first part of Theorem 4.1.

Theorem 4.10. *The MNDS G possesses a unique pullback global strictly invariant $\mathcal{D}$-attractor $\mathcal{A}(\omega)$, defined by* (2.3)*, where $K(\omega)$ is the set given in* (4.3)*.*

Proof. This theorem follows from Theorem 2.5 using Corollary 4.7, Lemma 4.2, Theorem 4.4 and Proposition 4.9. □

4.4. The random attractor

In order to ensure that $\mathcal{A}(\omega)$ is a random pullback $\mathcal{D}$-attractor we need to check that it is a random set, i.e., its measurability. For this aim we need to obtain some properties concerning the map $\omega \mapsto G(t,\omega,K(\omega))$, where $K(\omega) \in \mathcal{D}$ is the pullback $\mathcal{D}$-absorbing set given in Lemma 4.2. Also, we have to prove that G is an MRDS.

Assume further that

$$\lambda > \sum_{j=1}^{N} |c_j| \mathbb{E}\big(\big|z_j^*\big|\big). \tag{4.21}$$

For $M \in \mathbb{N}$, we consider the sets

$$\Omega_M := \left\{ \begin{array}{l} \omega \in \Omega\colon\ \big|\delta(\theta_t\omega)\big| \leqslant \varepsilon|t|, \quad \displaystyle\int_0^t \big|\delta(\theta_s\omega)\big|\,ds \leqslant \left(\sum_{j=1}^{N} |c_j|\mathbb{E}\big(\big|z_j^*\big|\big) + \varepsilon\right)|t|, \\ \big|\omega_j(t)\big| \leqslant \varepsilon|t|, \quad \forall j,\ \text{for } |t| \geqslant M \end{array} \right\}, \tag{4.22}$$

where

$$0 < 2\varepsilon < \lambda - \sum_{j=1}^{N} |c_j| \mathbb{E}\big(\big|z_j^*\big|\big).$$

These sets are well defined in view of Lemma 3.1 and condition (4.21).

Lemma 4.11. $\Omega = \bigcup_M \Omega_M$.

Proof. Let us show that for any $\omega \in \Omega$ there exists M such that $\omega \in \Omega_M$, which will imply that $\Omega = \bigcup_M \Omega_M$.

Since $\lim_{t\to\pm\infty} \frac{|z_j^*(\theta_t\omega)|}{t} = 0$ and $\lim_{t\to\pm\infty} \frac{|\omega_j(t)|}{t} = 0$, there exists $M(\varepsilon)$ such that

$$\big|z_j^*(\theta_t\omega)\big| \leqslant \frac{\varepsilon}{\sum_{j=1}^{N} |c_j|}|t|,$$
$$\big|\omega_j(t)\big| \leqslant \varepsilon|t|, \quad \forall j,\ \text{if } |t| \geqslant M,$$

and

$$\big|\delta(\theta_t\omega)\big| \leqslant \sum_{j=1}^{N} |c_j|\big|z_j^*(\theta_t\omega)\big| \leqslant \varepsilon|t|, \quad \text{if } |t| \geqslant M.$$

By the property $\lim_{t\to\pm\infty}\frac{\int_0^t |z_j^*(\theta_s\omega)|\,ds}{t}=\mathbb{E}(|z_j^*|)$ arguing in a similar way we have

$$\int_0^t \left|\delta(\theta_s\omega)\right| ds \leqslant \left(\sum_{j=1}^N |c_j|\mathbb{E}\left(\left|z_j^*\right|\right)+\varepsilon\right)|t|, \quad \text{for } |t|\geqslant M. \qquad \square$$

We need the continuity of the map $\mathbb{R}\times\Omega_M \ni (t,\omega)\mapsto \delta(\theta_t\omega)$, which implies that $\Omega_M\in\mathcal{F}$.

Lemma 4.12. *The set Ω_M is closed, hence, $\Omega_M\in\mathcal{F}$ and it is a Polish space. The map $\mathbb{R}\times\ \Omega_M\ni(t,\omega)\mapsto \delta(\theta_t\omega)$ is continuous.*

Proof. Let $t_n\to t_0$, $\omega_n\to\omega_0$, where $\omega_n\in\Omega_M$ and $\omega_0\in\Omega$. Then

$$\begin{aligned}\left|\delta(\theta_{t_n}\omega_n)-\delta(\theta_{t_0}\omega_0)\right| &= \left|\sum_{j=1}^N c_j\left(\int_{-\infty}^0 e^s\theta_{t_n}\omega_{nj}(s)\,ds-\int_{-\infty}^0 e^s\theta_{t_0}\omega_{0j}(s)\,ds\right)\right| \\ &\leqslant \sum_{j=1}^N |c_j|\int_{-\infty}^0 e^s\left|\theta_{t_n}\omega_{nj}(s)-\theta_{t_0}\omega_{0j}(s)\right| ds \\ &\leqslant \sum_{j=1}^N |c_j|\left(\int_{-\infty}^0 e^s\left|\omega_{nj}(s+t_n)-\omega_{0j}(s+t_0)\right| ds+\left|\omega_{nj}(t_n)-\omega_{0j}(t_0)\right|\right).\end{aligned}$$

In view of the definition of Ω_M for $|\tau|\geqslant M$ we have $|\omega_{nj}(\tau)|\leqslant\varepsilon|\tau|$. Also, by $\omega_n\to\omega_0$ it is clear that $|\omega_{0j}(\tau)|\leqslant\varepsilon|\tau|$ if $|\tau|\geqslant M$. Therefore, $|\omega_{nj}(\tau)-\omega_{0j}(\overline{\tau})|\leqslant|\omega_{nj}(\tau)|+|\omega_{0j}(\overline{\tau})|\leqslant\varepsilon(|\tau|+|\overline{\tau}|)$, if $|t|\geqslant M$. Thus, for any $\beta>0$, there exists $T(\beta)>t_0$ (and then $T(\beta)>t_n$ also) such that

$$\sum_{j=1}^N |c_j|\int_{-\infty}^{-T} e^s\left|\omega_{nj}(s+t_n)-\omega_{0j}(s+t_0)\right| ds\leqslant 2\sum_{j=1}^N|c_j|\int_{-\infty}^{-T} e^s 2\varepsilon\left(|s|+|\bar{t}|\right) ds<\frac{\beta}{3},$$

where $|t_0|<|\bar{t}|$ (and thus, again, we can assume $|t_n|<|\bar{t}|$). Thus, if we take $n_0(\beta,T(\beta))$ such that

$$\sum_{j=1}^N|c_j|\left|\omega_{nj}(s+t_n)-\omega_{0j}(s+t_0)\right|<\frac{\beta}{3},$$

$$\sum_{j=1}^N|c_j|\left|\omega_{nj}(t_n)-\omega_{0j}(t_0)\right|<\frac{\beta}{3},$$

for any $s\in[-T,0]$ and $n\geqslant n_0$, we then obtain

$$\left|\delta(\theta_{t_n}\omega_n)-\delta(\theta_{t_0}\omega_0)\right|<\beta, \quad \text{if } n\geqslant n_0.$$

We have proved that $\delta(\theta_{t_n}\omega_n)\to\delta(\theta_{t_0}\omega_0)$, so that ω_0 satisfies the properties in (4.22), and then $\omega_0\in\Omega_M$. Thus, we have proved that Ω_M is closed (hence, $\Omega_M\in\mathcal{F}$) and also that the map $\mathbb{R}\times\Omega_M\ni(t,\omega)\to\delta(\theta_t\omega)$ is continuous.

Finally, as a subspace of Ω, the space Ω_M is separable and metrizable. Since Ω_M is closed and Ω is complete, Ω_M is also complete, and then a Polish space. $\square$

Let $\mathcal{F}_{\Omega_M}$ be the trace σ-algebra of $\mathcal{F}$ with respect to Ω_M and let $B_{\Omega_M}(a,r)$, $a \in \Omega_M$, $r > 0$, be a ball in Ω_N. These balls can be generated by $B_\Omega(a,r) \cap \Omega_M$ where $B_\Omega(a,r)$ is a ball in Ω. The same is true for all open sets in Ω_M. Hence $\mathcal{F}_{\Omega_M}$ is just the Borel σ-algebra of Ω_M. Moreover, since $\Omega_M \in \mathcal{F}$ we have $\mathcal{F}_{\Omega_M} \subset \mathcal{F}$.

Let us define

$$\mathbb{P}_{\Omega_M}(A) := \mathbb{P}(A), \quad \text{for } A \in \mathcal{F}_{\Omega_M},$$

that is, $\mathbb{P}_{\Omega_M}$ is just the restriction of $\mathbb{P}$ to $\mathcal{F}_{\Omega_M}$. Also, let $\overline{\mathcal{F}}_{\Omega_M}$ be the completion of $\mathcal{F}_{\Omega_M}$ with respect to $\mathbb{P}_{\Omega_M}$.

The following facts can be proved as in [11].

(1) $\mathbb{P}_{\Omega_N}$ is a finite measure on $(\Omega_N, \mathcal{F}_{\Omega_N})$.
(2) If $A \in \overline{\mathcal{F}}_{\Omega_M}$, then $A \in \overline{\mathcal{F}}$.

Now we establish the continuity of the random radius $R(\omega)$ given in Lemma 4.2 over Ω_M. We recall that

$$R^2(\omega) = 2e^{\delta(\omega)} \|c_0\|_{\ell^1} \int_{-\infty}^{0} e^{-2\delta(\theta_s\omega)+2\lambda s+2\int_s^0 \delta(\theta_r\omega)\,dr}\,ds.$$

Lemma 4.13. *The map $\omega \mapsto R(\omega)$ is continuous on Ω_M.*

Proof. Let $\omega_n \to \omega_0$ in Ω_M. By Lemma 4.12 we have

$$e^{-2\delta(\theta_s\omega_n)} \to e^{-2\delta(\theta_s\omega_0)} \quad \text{as } n \to \infty.$$

The convergence $\delta(\theta_{t_n}\omega_n) \to \delta(\theta_{t_0}\omega_n)$ and the continuity of $t \mapsto \delta(\theta_t\omega_n)$ imply that $\delta(\theta_t\omega_n)$ converges to $\delta(\theta_t\omega_0)$ uniformly with respect to t in a finite interval, so that $|\delta(\theta_r\omega_n)|$ is uniformly bounded on any finite interval $[a,b]$. Thus, by $\delta(\theta_r\omega_n) \to \delta(\theta_r\omega_0)$ and the existence of some L such that $|\delta(\theta_r\omega_n)| \leqslant L_s$, for all n and $r \in [s,0]$, Lebesgue's theorem implies that

$$\int_s^0 \delta(\theta_r\omega_n)\,dr \to \int_s^0 \delta(\theta_r\omega_0)\,dr.$$

Hence,

$$e^{-2\delta(\theta_s\omega_n)+2\lambda s+2\int_s^0 \delta(\theta_r\omega_n)\,dr} \to e^{-2\delta(\theta_s\omega_0)+2\lambda s+2\int_s^0 \delta(\theta_r\omega_0)\,dr}, \quad \text{for any } s \leqslant 0,$$

as $n \to \infty$. On the other hand, by (4.22) we obtain the majorant

$$e^{-2\delta(\theta_s\omega_n)+2\lambda s+2\int_s^0 \delta(\theta_r\omega_n)\,dr} \leqslant e^{2(\lambda-2\varepsilon-\sum_{j=1}^N |c_j|\mathbb{E}(|z_j^*|))s}, \quad \text{for } s \leqslant -M.$$

Then by Lebesgue's theorem and condition (4.21) we have

$$\int_{-\infty}^{-T} e^{-2\delta(\theta_s\omega_n)+2\lambda s+2\int_s^0 \delta(\theta_r\omega_n)\,dr}\,ds \to \int_{-\infty}^{-T} e^{-2\delta(\theta_s\omega_0)+2\lambda s+2\int_s^0 \delta(\theta_r\omega_0)\,dr}\,ds.$$

On the other hand, by $|\delta(\theta_r\omega_n)| \leqslant L_T$, for all n and $r \in [-T,0]$, we get

$$e^{-2\delta(\theta_s\omega_n)+2\lambda s+2\int_s^0 \delta(\theta_r\omega_n)\,dr} \leqslant e^{2L_T+2TL_T}, \quad \text{for } s \in [-T,0].$$

Hence, again by Lebesgue's theorem

$$\int_{-T}^{0} e^{-2\delta(\theta_s\omega_n)+2\lambda s+2\int_s^0 \delta(\theta_r\omega_n)\,dr}\,ds \to \int_{-T}^{0} e^{-2\delta(\theta_s\omega_0)+2\lambda s+2\int_s^0 \delta(\theta_r\omega_0)\,dr}\,ds.$$

Since by Lemma 4.12 we have $e^{\delta(\omega_n)} \to e^{\delta(\omega_0)}$, the continuity of $\omega \mapsto R(\omega)$ follows. □

Concerning Ω_M we can obtain stronger properties for the cocycle G.

Lemma 4.14. *The map $\mathbb{R}^+ \times \Omega_M \times \ell^2 \ni (t,\omega,u^0) \mapsto G(t,\omega,u^0)$ is upper semi-continuous.*

Proof. If this is not true, then there exist u^0, $t_0 > 0$, $\omega_0 \in \Omega_M$, a neighborhood $\mathcal{O}$ of $G(t_0,\omega_0,u^0)$ and sequences $t_n \to t_0$, $\omega_n \to \omega_0$ in Ω_M, $u_n^0 \to u^0$ in ℓ^2, $\xi^n \in G(t_n,\omega_n,u_n^0)$ such that $\xi^n \notin \mathcal{O}$. We shall prove that, up to a subsequence, $\xi^n \to \xi \in G(t_0,\omega_0,u^0)$, which is a contradiction.

Let $v^n(\cdot) \in \mathcal{S}(v^{0n},\omega_n)$ be such that $\xi^n = e^{\delta(\theta_{t_n}\omega_n)}v^n(t_n)$, where $v^{0n} = e^{-\delta(\omega_n)}u^{0n}$. By Lemma 4.12 we have that $\delta(\theta_{t_n}\omega_n) \to \delta(\theta_{t_0}\omega_0)$ and $v^{0n} \to v^0 = e^{-\delta(\omega_n)}u^0$.

Due to these properties and (4.1), arguing as in Lemma 4.13 it follows that (4.18) and (4.19) hold, where $T > t_n$, $T > t_0$, and R_0 is a common constant for any ω_n.

Lemma 4.12 and the continuity of $t \mapsto \delta(\theta_t\omega_0)$ imply that $\delta(\theta.\omega_n) \to \delta(\theta.\omega_0)$ in $C([0,T])$. Then, using inequalities (4.7) and (4.19), we can find $K(\epsilon)$ and $\alpha > 0$ such that

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}} \rho\left(\frac{|i|}{K}\right)\left|v_i^n\right|^2 \leqslant \alpha \sum_{i\in\mathbb{Z}} \rho\left(\frac{|i|}{k}\right)\left|v_i^n\right|^2 + \epsilon, \quad \text{if } K \geqslant K(\epsilon).$$

Using Gronwall's lemma and (4.18) we obtain

$$\begin{aligned}\sum_{|i|\geqslant 2K(\epsilon)} \left|v_i^n(t)\right|^2 \leqslant \sum_{i\in\mathbb{Z}} \rho\left(\frac{|i|}{K}\right)\left|v_i^n(t)\right|^2 &\leqslant \epsilon e^{\alpha t} + \epsilon \int_0^t e^{\alpha(t-r)}\,dr \\ &\leqslant \epsilon\left(e^{\alpha T} + \frac{e^{\alpha T}-1}{\alpha}\right).\end{aligned} \tag{4.23}$$

As in the proof of Lemma 4.5 we obtain that the sequence $v^n(t)$ is pre-compact for any t. By (3.3), (4.19) and the fact that $\delta(\theta.\omega_n) \to \delta(\theta.\omega_0)$ in $C([0,T])$, we obtain

$$\left\|-\nu A v^n(t) + \delta(\theta_t\omega_n)v^n(t) - e^{-\delta(\theta_t\omega_n)}\tilde{f}\left(e^{\delta(\theta_t\omega_n)}v^n(t)\right)\right\|_{\ell^2}^2 \leqslant C_0, \quad \forall n \in \mathbb{N},\ \forall t \in [0,T],$$

so that

$$\left\|\frac{d}{dt}v^n(t)\right\|_{\ell^2} \leqslant C_1, \tag{4.24}$$

which proves that the sequence v^n is equi-continuous. The Ascoli–Arzelà theorem implies then the existence of a subsequence v^{n_k} converging to some function $v(\cdot)$ in $C([0,T],\ell^2)$.

From the continuity of the maps $\tilde{f}:\ell^2\to\ell^2$ and $(t,\omega)\mapsto\delta(\theta_t\omega)$ it is easy to show that $v(\cdot)$ is a solution of (3.11). Also, it is clear that $v(0)=v^0$.

It follows that $\xi^{n_k}\to\xi=e^{\theta_{t_0}\omega_0}v(t_0)\in G(t_0,\omega_0,u^0)$, which is a contradiction. □

Now, the following result is a consequence of Lemma 2.2.

Corollary 4.15. *The map $(t,\omega,u^0)\mapsto G(t,\omega,u^0)$ is $\mathcal{B}(\mathbb{R}^+)\otimes\mathcal{F}_{\Omega_M}\otimes\mathcal{B}(\ell^2)$-measurable.*

We need now some properties of the map $\Omega_M\ni\omega\mapsto G(t,\omega,K(\omega))$. For this aim we shall use the following auxiliary lemma.

Lemma 4.16. *Let $v^{0n}\to v^0$ weakly in ℓ^2, $\omega_n\to\omega_0$ in Ω_M and fix $T>0$. Then there exist $v(\cdot)\in\mathcal{S}(v^0,\omega_0)$ and a subsequence $v^{n_k}\in\mathcal{S}(v^{0n_k},\omega_{n_k})$ such that*

$$v^{n_k}\to v\quad\text{weakly in } L^2\big(0,T;\ell^2\big),$$
$$v^{n_k}(t)\to v(t)\quad\text{weakly in } \ell^2 \text{ for all } t\in[0,T].$$

Proof. In view of (4.1) and arguing as in Lemma 4.13 there exists $R_0>0$ such that

$$\big\|v^n(t)\big\|_{\ell^2}\leqslant R_0,\quad\forall t\in[0,T],\ \forall n.$$

Arguing now as in Lemma 4.14 we have

$$\left\|\frac{d}{dt}v^n(t)\right\|_{\ell^2}\leqslant C_0. \tag{4.25}$$

Hence, there exist $v,\chi\in L^2(0,T;\ell^2)$ such that, up to a subsequence,

$$v^n\to v,\qquad \frac{d}{dt}v^n\to\frac{d}{dt}v,\qquad F^{\omega_n}\big(\cdot,v^n(\cdot)\big)\to\chi\quad\text{weakly in } L^2\big(0,T;\ell^2\big), \tag{4.26}$$

where $F^{\omega_n}(t,v^n(t))=-\nu Av^n(t)+\delta(\theta_t\omega_n)v^n(t)-e^{-\delta(\theta_t\omega_n)}\tilde{f}(e^{\delta(\theta_t\omega_n)}v^n(t))$.

Let $\varphi(\cdot)\in\mathcal{C}^1([0,T])$ be a function such that $\varphi(T)=0$ and $\varphi(0)=1$ and let $\xi\in\ell^2$. Then

$$\frac{d}{dt}\big(v^n,\xi\varphi\big)=\big(F^{\omega_n}\big(t,v^n(t)\big),\xi\big)\varphi(t)+\big(v^n(t),\xi\big)\varphi'(t).$$

Integrating over $(0,T)$ and using (4.26) we have

$$\begin{aligned}0&=\big(v^n(0),\xi\big)+\int_0^T\big(F^{\omega_n}\big(t,v^n(t)\big),\xi\big)\varphi(t)\,dt+\int_0^T\big(v^n(t),\xi\big)\varphi'(t)\,dt\\&\to\big(v^0,\xi\big)+\int_0^T\big(\chi(t),\xi\big)\varphi(t)\,dt+\int_0^T\big(v(t),\xi\big)\varphi'(t)\,dt=0.\end{aligned}\tag{4.27}$$

On the other hand, since $v(t)$ is absolutely continuous, we have

$$\frac{d}{dt}(v,\xi\varphi)=\left(\frac{dv}{dt},\xi\right)\varphi(t)+\big(v(t),\xi\big)\varphi'(t)$$
$$=\big(\chi(t),\xi\big)\varphi(t)+\big(v(t),\xi\big)\varphi'(t)\quad \text{for a.a. } t\in(0,T),$$

so

$$0=\big(v(0),\xi\big)+\int_0^T\big(\chi(t),\xi\big)\varphi(t)\,dt+\int_0^T\big(v(t),\xi\big)\varphi'(t)\,dt. \tag{4.28}$$

Hence, (4.27), (4.28) imply that $v(0)=v^0$.

Therefore, for any $\xi\in\ell^2$ we have

$$\big(v^n(t),\xi\big)=\big(v^{0n},\xi\big)+\int_0^t\left(\frac{dv^n}{d\tau},\xi\right)d\tau\to\big(v^0,\xi\big)+\int_0^t\left(\frac{dv}{d\tau},\xi\right)d\tau=\big(v(t),\xi\big),$$

where the last equality follows from $v,\frac{dv}{dt}\in L^2(0,T;\ell^2)$ and $v(0)=v^0$. Hence,

$$v^n(t)\to v(t)\quad \text{weakly in } \ell^2 \text{ for all } t\in[0,T].$$

It follows from the weakly continuity of the map $\tilde f:\ell^2\to\ell^2$ and the continuity of $(t,\omega)\mapsto\delta(\theta_t\omega)$ that

$$F^{\omega_n}\big(t,v^n(t)\big)\to F^{\omega_0}\big(t,v(t)\big)\quad \text{weakly in } \ell^2 \text{ for all } t\in[0,T].$$

Also by (3.3) we obtain

$$\big|\big(F^{\omega_n}\big(t,v^n(t)\big),\xi\big)\big|\leqslant C_1\|\xi\|_{\ell^2},\quad \forall n\in\mathbb{N},\ \forall t\in[0,T],$$

and then Lebesgue's theorem gives

$$\big(F^{\omega_n}\big(t,v^n(t)\big),\xi\big)\to\big(F^{\omega_0}\big(t,v(t)\big),\xi\big)\quad \text{in } L^1(0,T) \text{ for all } \xi\in\ell^2.$$

Thus,

$$F^{\omega_0}\big(\cdot,v(\cdot)\big)=\chi,$$

and

$$v(t)=v^0+\int_0^t\frac{dv}{d\tau}\,d\tau=v^0+\int_0^t F^{\omega_0}\big(\tau,v(\tau)\big)\,d\tau,\quad \text{for all } t\in[0,T],$$

which implies that $v(\cdot)\in\mathcal{S}(v^0,\omega_0)$. $\square$

Lemma 4.17. *The map $\Omega_M\ni\omega\mapsto G(t,\omega,K(\omega))$ is $\overline{\mathcal{F}}_{\Omega_M}$-measurable for any $t\geqslant 0$. Also, $G(t,\omega,K(\omega))$ is closed for all $t\geqslant 0$, $\omega\in\Omega_M$.*

Proof. It is well known [18, Chapter III] that $D(\omega)$ is a random set with respect to $\mathcal{P}^c$ if and only if the graph of $D(\omega)$, given by

$$\mathrm{Gr}(D) := \{(\omega, x) \in \Omega_M \times \ell^2 \colon x \in D(\omega)\},$$

belongs to $\overline{\mathcal{F}}_{\Omega_M} \otimes \mathcal{B}(\ell^2)$. Thus, in order to prove the first statement it is sufficient to show that the graph of the map $\omega \mapsto G(t, \omega, K(\omega))$ belongs to $\overline{\mathcal{F}}_{\Omega_M} \otimes \mathcal{B}(\ell^2)$, and this is true if the graph is closed.

Let $\omega \to \omega_0$ in Ω_N and $\xi^n \to \xi$ in ℓ^2, where $\xi^n \in G(t, \omega_n, u^{0n})$ and $u^{0n} \in K(\omega_n)$. We have to show that $\xi \in G(t, \omega_0, K(\omega_0))$. Take $v_n(\cdot) \in \mathcal{S}(e^{-\delta(\omega_n)} u^{0n}, \omega_n)$ such that $\xi^n = e^{\delta(\theta_t \omega_n)} v_n(t)$.

By the definition of $K(\omega)$ we have that $v^{n0} = e^{-\delta(\omega_n)} u^{0n}$ satisfies $\|v^{0n}\|_{\ell^2} \leqslant e^{-\delta(\omega_n)} R(\omega_n)$. Then by Lemmas 4.12, 4.13 we obtain, passing to a subsequence, that $v^{n0} \to v^0$ weakly in ℓ^2, where $\|v^0\|_{\ell^2} \leqslant e^{-\delta(\omega_0)} R(\omega_0)$, and $u^{0n} \to u^0 = e^{\delta(\omega_0)} v^0 \in K(\omega_0)$ weakly in ℓ^2.

In view of Lemma 4.16 there exist $v(\cdot) \in \mathcal{S}(v^0, \omega_0)$ and a subsequence such that

$$v^n(t) \to v(t) \quad \text{weakly in } \ell^2 \text{ for all } t \in [0, T].$$

Thus, $\xi^n = e^{\delta(\theta_t \omega_n)} v_n(t) \to e^{\delta(\theta_t \omega_0)} v(t)$ weakly in ℓ^2, so that $\xi = e^{\delta(\theta_t \omega_0)} v(t) \in G(t, \omega_0, u_0) \subset G(t, \omega_0, K(\omega_0))$.

As the graph is closed, it is obvious that $G(t, \omega, K(\omega))$ is closed for all $t \geqslant 0$, $\omega \in \Omega_M$. $\square$

The following result, together with Theorem 4.10, proves completely Theorem 4.1.

Theorem 4.18. *The MNDS G is an MRDS. Also, the pullback $\mathcal{D}$-attractor $\mathcal{A}(\omega)$ given in Theorem* 4.10 *is a random set with respect to $\overline{\mathcal{F}}$, and then it is the unique random global pullback $\mathcal{D}$-attractor for G.*

Proof. Let us prove that G is an MRDS. As G is an MNDS, it remains to show that the map $(t, \omega, x) \mapsto G(t, \omega, x)$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(\ell^2)$-measurable. Let $\mathcal{O}$ be an open set of ℓ^2. Then, by Corollary 4.15, we have that the set

$$A_{M,\mathcal{O}} := \{(t, \omega, x) \in \mathbb{R}^+ \times \Omega_M \times \ell^2 \colon G(t, \omega, x) \cap \mathcal{O} \neq \emptyset\}$$

belongs to $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}_{\Omega_M} \otimes \mathcal{B}(\ell^2)$, so that $A_{M,\mathcal{O}} \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(\ell^2)$. Hence

$$\begin{aligned} &\{(t, \omega, x) \in \mathbb{R}^+ \times \Omega \times \ell^2 \colon G(t, \omega, x) \cap \mathcal{O} \neq \emptyset\} \\ &\quad = \bigcup_{N=1}^{\infty} \{(t, \omega, x) \in \mathbb{R}^+ \times \Omega_M \times \ell^2 \colon G(t, \omega, x) \cap \mathcal{O} \neq \emptyset\} \\ &\quad = \bigcup_{N=1}^{\infty} A_{N,\mathcal{O}} \in \mathcal{B}\big(\mathbb{R}^+\big) \otimes \mathcal{F} \otimes \mathcal{B}\big(\ell^2\big), \end{aligned}$$

and then G is an MRDS.

Furthermore, in view of Lemma 4.17, the map $\Omega_M \ni \omega \mapsto G(t, \omega, K(\omega))$ is $\overline{\mathcal{F}}_{\Omega_M}$-measurable for any $t \geqslant 0$. Hence, for a fixed $t \geqslant 0$, the set

$$C_{N,\mathcal{O}} := \{\omega \in \Omega_M \colon G\big(t, \omega, K(\omega)\big) \cap \mathcal{O} \neq \emptyset\}$$

belongs to $\overline{\mathcal{F}}_{\Omega_M}$, and then

$$\{\omega \in \Omega \colon G\big(t, \omega, K(\omega)\big) \cap \mathcal{O} \neq \emptyset\} = \bigcup_{N=1}^{\infty} C_{M,\mathcal{O}} \in \overline{\mathcal{F}}.$$

In view of Theorem 2.7 the pullback $\mathcal{D}$-attractor $\mathcal{A}(\omega)$ is a random set with respect to $\overline{\mathcal{F}}$, and then it is the unique random global pullback $\mathcal{D}$-attractor for G. $\square$

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