

# Entanglement in eight-qubit graph states

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## A B S T R A C T

Any 8-qubit graph state belongs to one of the 101 equivalence classes under local unitary operations within the Clifford group. For each of these classes we obtain a representative which requires the minimum number of controlled- $Z$  gates for its preparation, and calculate the Schmidt measure for the 8-partite split, and the Schmidt ranks for all bipartite splits. This results into an extension to 8 qubits of the classification of graph states proposed by Hein, Eisert, and Briegel [M. Hein, J. Eisert, H.J. Briegel, Phys. Rev. A 69 (2004) 062311].

## 1. Introduction

Graph states [1,2] are a type of  $n$ -qubit pure states that play several fundamental roles in quantum information theory. In quantum error-correction, the stabilizer codes which protect quantum systems from errors [3] can be realized as graph codes [4,5]. In measurement-based (or one-way) quantum computation [6], graph states are the initial resources consumed during the computation. Moreover, some graph states are universal resources for quantum computation [7]. In quantum simulation, graph states allow us to demonstrate fractional braiding statistics of anyons in an exactly solvable spin model [8]. Graph states have been used in multipartite purification schemes [9]. The Clifford group has been used for entanglement distillation protocols [10]. Graph states naturally lead to Greenberger–Horne–Zeilinger (GHZ) or all-versus-nothing proofs of Bell's theorem [11–16], which can be converted into Bell inequalities which are maximally violated by graph states [17–22]. Some specific graph states are essential for several quantum communication protocols, including entanglement-based quantum key distribution [23], teleportation [24], reduction of communication complexity [25], and secret sharing [26,27].

In addition to all these applications, graph states also play a fundamental role in the theory of entanglement. For  $n \geq 4$  qubits, there is an infinite amount of different, inequivalent classes of  $n$ -qubit pure entangled states. The graph state formalism is a use-

ful abstraction which permits a detailed (although not exhaustive) classification of  $n$ -qubit entanglement of  $n \geq 4$  qubits.

For all these reasons, a significant experimental effort is devoted to the creation and testing of graph states of an increasing number of qubits. On one hand, there are experiments of  $n$ -qubit  $n$ -photon graph states up to  $n = 6$  [28–32]. On the other hand, the combination of two techniques, hyper-entanglement (i.e., entanglement in several degrees of freedom, like polarization and linear momentum) [33–39] and the sources of 4, 5, and 6-photon entanglement using parametric down-conversion [29,40–45] allows us to create 6-qubit 4-photon graph states [46,47], 8-qubit 4-photon graph states [46], and even 10-qubit 5-photon graph states [46]. The use of 4-photon sources for preparing 8-qubit graph states is particularly suitable due to the high visibility of the resulting states.

The classification and study of the entanglement properties of graph states have been achieved, up to 7 qubits, by Hein, Eisert, and Briegel (HEB) in [1] (see also [2]). This classification has been useful to identify new two-observer all-versus-nothing proofs [16], new Bell inequalities [21,22], and has stimulated the preparation of several graph states [46]. The main purpose of this Letter is to extend the classification in [1,2] to 8-qubit graph states.

Up to 7 qubits, there are 45 classes of graph states that are not equivalent under one-qubit unitary transformations. With 8 qubits, there are 101 new classes. All these classes have been obtained by various researchers (see, e.g., [48]). The purpose here is to classify them according to several relevant physical properties for quantum information theory.

The Letter is organized as follows. In Section 2 we define qubit graph states and local complementation, which is the main clas-

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sifying tool. To establish an order between the equivalence classes we will use the criteria proposed in [1,2]. These criteria are introduced in Section 3. In Section 4 we present our results. In Section 5 we present the conclusions and point out some pending problems.

## 2. Basic concepts

### 2.1. Graph state

A  $n$ -qubit *graph state*  $|G\rangle$  is a pure state associated to a graph  $G = (V, E)$  consisting of a set  $V$  of  $n$  vertices and a set  $E$  of edges connecting some of the vertices. Each vertex represents a qubit. The graph  $G$  provides both a recipe for preparing  $|G\rangle$  and a mathematical characterization of  $|G\rangle$ .

The recipe for preparing the state is the following. First, prepare each qubit in the state  $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ . Then, for each edge connecting two qubits,  $i$  and  $j$ , apply the controlled- $Z$  gate between qubits  $i$  and  $j$ , i.e., the unitary transformation  $C_Z = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|$ .

The mathematical characterization of  $|G\rangle$  is the following. The graph state  $|G\rangle$  associated to the graph  $G$  is the only  $n$ -qubit state which fulfills

$$g_i |G\rangle = |G\rangle, \quad \text{for } i = 1, \dots, n, \quad (1)$$

where  $g_i$  are the generators of the stabilizer group of the state, defined as the set  $\{s_j\}_{j=1}^{2^n}$  of all products of the generators. Specifically,  $g_i$  is the generator operator associated to the vertex  $i$ , defined by

$$g_i := X^{(i)} \bigotimes_{j \in \mathcal{N}(i)} Z^{(j)}, \quad (2)$$

where  $\mathcal{N}(i)$  is the neighborhood of the vertex  $i$ , i.e., those vertices which are connected to  $i$ , and  $X^{(i)}$  ( $Z^{(i)}$ ) denotes the Pauli matrix  $\sigma_x$  ( $\sigma_z$ ) acting on the  $i$ th qubit.

### 2.2. Local complementation

For our purposes, the key point is that *local complementation* (LC) is a simple transformation which leaves the entanglement properties invariant.

Two  $n$ -qubit states,  $|\phi\rangle$  and  $|\psi\rangle$  have the same  $n$ -partite entanglement if and only if there are  $n$  one-qubit unitary transformations  $U_i$ , such that  $|\phi\rangle = \bigotimes_{i=1}^n U_i |\psi\rangle$ . If these one-qubit unitary transformations belong to the Clifford group, then both states are said to be local Clifford equivalent. The one-qubit Clifford group is generated by the Hadamard gate  $H = (|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)/\sqrt{2}$  and the phase gate  $P = |0\rangle\langle 0| + i|1\rangle\langle 1|$ .

Van den Nest, Dehaene, and De Moor found that the successive application of a transformation with a simple graphical description is sufficient to generate the complete equivalence class of graph states under local unitary operations within the Clifford group (hereafter simply referred as class or orbit) [49]. This simple transformation is LC.

On the stabilizer, LC on the qubit  $i$  induces the map  $Y^{(i)} \mapsto Z^{(i)}$ ,  $Z^{(i)} \mapsto -Y^{(i)}$  on the qubit  $i$ , and the map  $X^{(j)} \mapsto -Y^{(j)}$ ,  $Y^{(j)} \mapsto X^{(j)}$  on the qubits  $j \in \mathcal{N}(i)$  [2]. On the generators, LC on the qubit  $i$  maps the generators  $g_j^{\text{old}}$  with  $j \in \mathcal{N}(i)$  to  $g_j^{\text{new}} g_i^{\text{new}}$ .

Graphically, LC on the qubit  $i$  acts as follows: One picks out the vertex  $i$  and inverts the neighborhood  $\mathcal{N}(i)$  of  $i$ ; i.e., vertices in the neighborhood which were connected become disconnected and vice versa.

It has been shown by Van den Nest, Dehaene, and De Moor that for a particular class of qubit graph states local unitary equivalence implies local Clifford equivalence [50]. Moreover, numerical results show that local Clifford equivalence coincides with local unitary equivalence for qubit graph states associated with connected

graphs up to  $n = 7$  vertices [1,2]. It should be noted, however, that not all local unitary transformations between graph states can be represented as successive LCs. A counterexample with  $n = 27$  is described in [51].

Using LC, one can generate the orbits of all LC-inequivalent  $n$ -qubit graph states. For a small  $n$ , the number of orbits has been well known for a long time (see, e.g., [48]). Specifically, for  $n = 8$ , there are 101 LC-inequivalent classes.

## 3. Criteria for the classification

Following HEB, the criteria for ordering the classes are: (a) number of qubits, (b) minimum number of controlled- $Z$  gates needed for the preparation, (c) the Schmidt measure, and (d) the rank indexes. For instance, class No. 1 is the only one containing two-qubit graph states, class No. 2 is the only one containing three-qubit graph states [1,2]. Classes No. 3 and No. 4 both have  $n = 4$  qubits and require a minimum of  $|E| = 3$  controlled- $Z$  gates. However, class No. 3 has Schmidt measure  $E_S = 1$ , while class No. 4 has  $E_S = 2$ .

### 3.1. Minimum number of controlled- $Z$ gates for the preparation

Different members of the same LC class require a different number of controlled- $Z$  gates for their preparation starting from the state  $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$  for each qubit. The first criterion for our classification is the minimum number of controlled- $Z$  gates required for preparing one graph state *within the LC class*. This corresponds to the number of edges of the graph with the minimum number of edges within the LC class,  $|E|$ . We will provide a representative with the minimum number of edges for each LC class.

### 3.2. Schmidt measure

The *Schmidt measure* was introduced by Eisert and Briegel as a tool for quantifying the genuine multipartite entanglement of quantum systems [52] (see also [53]). Any state vector  $|\psi\rangle \in \mathcal{H}^{(1)} \otimes \dots \otimes \mathcal{H}^{(N)}$  of a composite quantum system with  $N$  components can be represented as

$$|\psi\rangle = \sum_{i=1}^R \xi_i |\psi_i^{(1)}\rangle \otimes \dots \otimes |\psi_i^{(N)}\rangle, \quad (3)$$

where  $\xi_i \in \mathbb{C}$  for  $i = 1, \dots, R$ , and  $|\psi_i^{(j)}\rangle \in \mathcal{H}^{(j)}$ , for  $j = 1, \dots, N$ . The Schmidt measure associated with a state vector  $|\psi\rangle$  is then defined as

$$E_S(|\psi\rangle) = \log_2(r), \quad (4)$$

where  $r$  is the minimal number  $R$  of terms in the sum of Eq. (3) over all linear decompositions into product states. In case of a two-component system ( $N = 2$ ), the minimal number of product terms  $r$  is given by the *Schmidt rank* of the state  $|\psi\rangle$ . Hence, the Schmidt measure could be considered a generalization of the Schmidt rank to multipartite quantum systems [see Eq. (9) below]. The Schmidt measure can be extended to mixed states by means of a convex roof extension. In this Letter, however, we will deal only with pure states.

Given a graph  $G = (V, E)$ , a *partition* of  $V$  is any tuple  $(A_1, \dots, A_M)$  of disjoint subsets  $A_i \subset V$ , with  $\bigcup_{i=1}^M A_i = V$ . In case  $M = 2$ , we refer to the partition as a *bipartition*, and denote it  $(A, B)$ . We will write

$$(A_1, \dots, A_N) \leq (B_1, \dots, B_M), \quad (5)$$

if  $(A_1, \dots, A_N)$  is a *finer partition* than  $(B_1, \dots, B_M)$ , which means that every  $A_i$  is contained in some  $B_j$ . The latter is then a *coarser*

partition than the former. For any graph  $G = (V, E)$ , the partitioning where  $(A_1, \dots, A_M) = V$  such that  $|A_i| = 1$ , for every  $i = 1, \dots, M$ , is referred to as the *finest partition*.

We must point out that  $E_S$  is nonincreasing under a coarse graining of the partitioning: If two components are merged in order to form a new component, then the Schmidt measure can only decrease. If we denote the Schmidt measure of a state vector  $|\psi\rangle$  evaluated with respect to a partitioning  $(A_1, \dots, A_N)$  as  $E_S^{(A_1, \dots, A_N)}(|\psi\rangle)$ , meaning that the respective Hilbert spaces are those of the grains of the partitioning, then the nonincreasing property of  $E_S$  can be expressed as

$$E_S^{(A_1, \dots, A_N)}(|\psi\rangle) \geq E_S^{(B_1, \dots, B_M)}(|\psi\rangle), \quad (6)$$

if  $(A_1, \dots, A_N) \leq (B_1, \dots, B_M)$ .

Let  $(A, B)$  be a bipartition (i.e.,  $A \cup B = V$ ;  $A \cap B = \emptyset$ ) of a graph  $G = (V, E)$ , with  $V = \{1, \dots, N\}$ , and let us denote the adjacency matrix of the graph by  $\Gamma$ , i.e., the symmetric matrix with elements

$$\Gamma_{ij} = \begin{cases} 1, & \text{if } (i, j) \in E, \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

When we are dealing with a bipartition, it is useful to label the vertices of the graph so that  $A = \{1, \dots, p\}$ ,  $B = \{p+1, \dots, N\}$ . Then, we can decompose the adjacency matrix  $\Gamma$  into submatrices  $\Gamma_A$ ,  $\Gamma_B$  (that represent edges within  $A$  and edges within  $B$ ), and  $\Gamma_{AB}$  (the  $|A| \times |B|$  off-diagonal submatrix of the adjacency matrix  $\Gamma$  that represents those edges between  $A$  and  $B$ ),

$$\begin{pmatrix} \Gamma_A & \Gamma_{AB} \\ \Gamma_{AB}^T & \Gamma_B \end{pmatrix} = \Gamma. \quad (8)$$

The Schmidt rank  $SR_A(G)$  of a graph state  $|G\rangle$  represented by the graph  $G = (V, E)$ , with respect to the bipartition  $(A, B)$ , is given by the binary rank [i.e., the rank over  $GF(2)$ ] of the submatrix  $\Gamma_{AB}$ ,

$$SR_A(G) = \text{rank}_{\mathcal{F}_2}(\Gamma_{AB}). \quad (9)$$

It follows straightforwardly from the definition that  $SR_A(G) = SR_B(G)$ , because the different bipartitions are fixed by choosing the smaller part, say  $A$ , of the bipartition  $(A, B)$ , which gives  $2^{N-1}$  bipartitions.

### 3.3. Rank indexes

While calculating the Schmidt rank with respect to all possible bipartitions of a given graph, let us count how many times a certain rank occurs in all the bipartite splits, and then classify this information according to the number of vertices in  $A$ , the smaller part of the split under consideration. There is a compact way to express this information, the so-called *rank indexes* [1,2]. The rank index for all the bipartite splits with  $p$  vertices in the smaller part  $A$  is given by the  $p$ -tuple

$$RI_p = (v_p^p, \dots, v_1^p = v_j^p \Big|_{j=p}, \quad (10)$$

where  $v_j^p$  is the number of times in which  $SR_A(G) = j$ , with  $|A| = p$ , occurs.

## 4. Procedures and results

The main results of the Letter are summarized in Fig. 2 and Table 1. In the following, we provide details on the calculations leading to these results.

### 4.1. Orbits under local complementation

We have generated all LC orbits for  $n = 8$  and calculated the number of non-isomorphic graphs in each LC orbit, denoted by  $|LC|$ . These numbers are counted up to isomorphism.

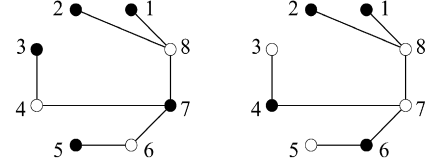


Fig. 1. The set of vertices 4, 6, and 8 is the minimal vertex cover of the graph (left). The set of vertices 3, 5, 7, and 8, is a vertex cover of the graph, but is not minimal, it has size 4 (right).

In addition, for each orbit, we have calculated a representative with the minimum number of edges  $|E|$ . As representative, we have chosen the one (or one of those) with the minimum number of edges and the minimum maximum degree (i.e., number of edges incident with a vertex). This means that the graph state associated to this graph requires the minimum number of controlled-Z gates for its preparation, and the minimum preparation depth (i.e., its preparation requires a minimum number of steps) [54]. All the representatives of each of the 101 orbits are illustrated in Fig. 2.  $|LC|$  and  $|E|$  are in Table 1.

### 4.2. Bounds to the Schmidt measure

It is a well-known fact that for any measure of multiparticle entanglement proposed so far, including the Schmidt measure  $E_S$ , the computation is exceedingly difficult for general states. In order to determine  $E_S$ , one has to show that a given decomposition in Eq. (3) with  $R$  terms is minimal. For a general state, the minimization problem involved can be a very difficult problem of numerical analysis, which scales exponentially in the number of parties  $N$  as well as in the degree of entanglement of the state itself. Nevertheless, this task becomes feasible if we restrict our attention to graph states. HEB established several upper and lower bounds for the Schmidt measure in graph theoretical terms [1,2]. These bounds make possible to determine the Schmidt measure for a large number of graphs of practical importance, because in many cases the bounds proposed are easily computable and, remarkably, the upper and lower bounds frequently coincide.

#### 4.2.1. Pauli persistency and size of the minimal vertex cover

For any graph state  $|G\rangle$ , upper bounds for its Schmidt measure  $E_S(|G\rangle)$  are the *Pauli persistency*  $PP(G)$  and the *size of the minimal vertex cover*  $VC(G)$ ,

$$E_S(|G\rangle) \leq PP(G) \leq VC(G). \quad (11)$$

The Pauli persistency is the minimal number of local Pauli measurements necessary to disentangle a graph state. Concerning this question, HEB described graphical transformation rules when local Pauli measurements are applied [1,2].

A vertex cover is a concept from graph theory: It is any subset  $V' \subseteq V$  of vertices in a graph  $G$  to which any edge of  $G$  is incident (see Fig. 1). Therefore, the minimal vertex cover of a graph is the smallest one, whose size is denoted by  $VC(G)$ . According to the graphical rules for the Pauli measurements, since each  $\sigma_z$  measurement simply deletes all edges incident to a vertex, the size of the minimal vertex cover would equal the Pauli persistency, provided that we restrict the Pauli measurements to  $\sigma_z$  measurements. Nevertheless, in graphs with many edges, i.e., very connected, a proper combination of  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$  measurements could provide a more efficient disentangling sequence, giving a better upper bound  $PP(G)$  for the Schmidt measure. See [1,2] for details.

#### 4.2.2. Maximal Schmidt rank

For any graph state  $|G\rangle$ , a lower bound for the Schmidt measure  $E_S(|G\rangle)$  is the *maximal Schmidt rank*,

$$SR_{\max}(G) \leq E_S(|G\rangle). \quad (12)$$

While calculating the Schmidt rank with respect to all possible bipartitions of a given graph  $G = (V, E)$ , if one maximizes the Schmidt rank over all bipartitions  $(A, B)$  of the graph, and takes into account the nonincreasing property of  $E_S(|G\rangle)$  [see Eq. (6)], then one obtains a lower bound for the Schmidt measure with respect to the finest partitioning. This lower bound is the maximal Schmidt rank,

$$SR_{\max}(G) := \max_{A \subseteq V} SR_A(G). \quad (13)$$

According to the definition of Schmidt rank, the maximal Schmidt rank for any state is, at most,  $\lfloor \frac{N}{2} \rfloor$ , i.e., the largest integer less than or equal to  $\frac{N}{2}$ .

#### 4.2.3. Addition and deletion of edges and vertices

As we mentioned in Section 2.2, applying the LC-rule does not change the Schmidt measure  $E_S$ . It is interesting to remark that other local changes to the graph, such as the deletion of edges or vertices, have only a limited effect on  $E_S$ . This fact is established by HEB [1] in what they call the *edge/vertex rule*: On one hand, by deleting (or adding) an edge  $e$  between two vertices of a graph  $G$  the Schmidt measure of the resulting graph  $G' = G \pm e$  can at most decrease (or increase) by 1. On the other hand, if a vertex  $v$  (including all its incident edges) is deleted, the Schmidt measure of the resulting graph  $G' = G - v$  cannot increase, and will at most decrease by one. If  $E_S(|G+e\rangle)$  denotes the Schmidt measure of the graph state corresponding to the graph  $G + e$ , then the previous rules can be summarized as

$$E_S(|G+e\rangle) \leq E_S(|G\rangle) + 1, \quad (14a)$$

$$E_S(|G-e\rangle) \geq E_S(|G\rangle) - 1, \quad (14b)$$

$$E_S(|G-v\rangle) \geq E_S(|G\rangle) - 1. \quad (14c)$$

We have used these rules in two ways: Firstly, as an internal test to check our calculations, comparing pairs of graphs connected by a sequence of addition or deletion of edges/vertices; and secondly, as a useful tool that, in some graphs, has enabled us to go from a bounded to a determined value for the Schmidt measure, once again by comparison between a problematic graph  $G$  and a resulting graph  $G'$  (typically obtained by edge or vertex deletion) of a known Schmidt measure.

#### 4.2.4. Schmidt measure in some special types of graphs

There are some special types of graph states in which lower and upper bounds for the Schmidt measure coincide (see [1]), giving directly a determined value for  $E_S$ . Since the maximal Schmidt rank for any state can be at most  $\lfloor \frac{N}{2} \rfloor$ , and restricting ourselves to states with coincident upper and lower bounds to  $E_S$ , it is true that  $SR_{\max}(G) = E_S(|G\rangle) = PP(G) = VC(G) \leq \lfloor \frac{N}{2} \rfloor$ . This is the case for GHZ states, and states represented by trees, rings with an even number of vertices, and clusters. In our work we have used the following results concerning GHZ states and trees:

- (a) The Schmidt measure for any multipartite GHZ state is 1.
- (b) A *tree*  $T$  is a graph that has no cycles. The Schmidt measure of the corresponding graph state  $|T\rangle$  is the size of its minimal vertex cover:  $E_S(|T\rangle) = VC(T)$ .

There is another interesting kind of graphs for our purposes, the so-called *2-colorable graphs*. A graph is said to be 2-colorable when it is possible to perform a *proper 2-coloring* on it: This is a labeling  $V \rightarrow \{1, 2\}$ , such that all connected vertices are associated with a different element from  $\{1, 2\}$ , which can be identified with two colors. It is a well-known fact in graph theory that a necessary and sufficient criterion for a graph to be 2-colorable is that it does not contain any cycles of odd length. Mathematicians call these graphs

*bipartite graphs* due to the fact that the whole set of vertices can be distributed into two disjoint subsets  $A$  and  $B$ , such that no two vertices within the same subset are connected, and therefore every edge connects a vertex in  $A$  with a vertex in  $B$ .

HEB [1] provided lower and upper bounds for the Schmidt measure that could be applied to graph states represented by 2-colorable graphs:

$$\frac{1}{2} \text{rank}_{\mathcal{F}_2}(\Gamma) \leq E_S(|G\rangle) \leq \left\lfloor \frac{|V|}{2} \right\rfloor, \quad (15)$$

where  $\Gamma$  is the adjacency matrix of the 2-colorable graph. If  $\Gamma$  is invertible, then

$$E_S(|G\rangle) = \left\lfloor \frac{|V|}{2} \right\rfloor. \quad (16)$$

Besides, HEB pointed out that any graph  $G$  which is not 2-colorable can be turned into a 2-colorable one  $G'$  by simply deleting the appropriate vertices on cycles with odd length present in  $G$ . The identification of this graphical action with the effect of a  $\sigma_z$  measurement on qubits corresponding to such vertices yields new upper bounds for  $E_S(|G\rangle)$ :

$$E_S(|G\rangle) \leq E_S(|G'\rangle) + M \leq \left\lfloor \frac{|V-M|}{2} \right\rfloor + M \leq \left\lfloor \frac{|V|+M}{2} \right\rfloor, \quad (17)$$

where  $M$  is the number of removed vertices. We have used these new bounds in some graphs as a tool to check our calculations.

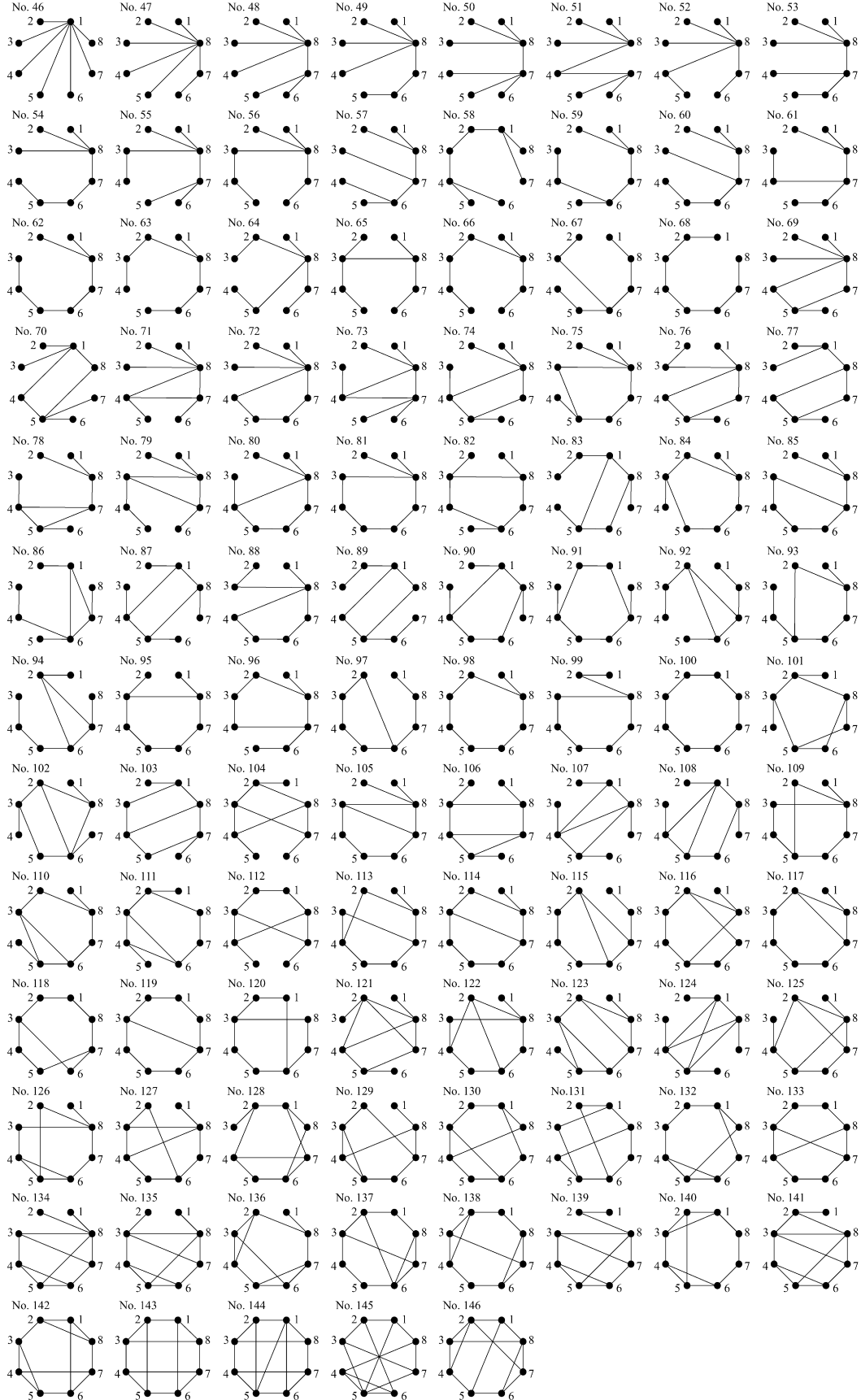
## 5. Conclusions, open problems, and future developments

To sum it all up, we have extended to 8 qubits the classification of the entanglement of graph states proposed in [1] for  $n < 8$  qubits. Notice that for  $n=8$  we have 101 classes, while for  $n < 8$  there are only 45 classes. For each of these classes we obtain a representative which requires the minimum controlled-Z gates for its preparation (see Fig. 2), and calculate the Schmidt measure for the 8-partite split (which measures the genuine 8-party entanglement of the class), and the Schmidt ranks for all bipartite splits (see Table 1).

This classification will help us to obtain new all-versus-nothing proofs of Bell's theorem [16] and new Bell inequalities. Specifically, any 8-qubit graph state belonging to a class with a representative with 7 edges (i.e., a tree) has a specific type of Bell inequality [22]. More generally, it will help us to investigate the nonlocality (i.e., the non-simulability of the predictions of quantum mechanics by means of non-local hidden variable models) of graph states [21].

Extending the classification in [1] a further step sheds some light on the limitations of the method of classification. The criteria used in [1] to order the classes (see Section 3) already failed to distinguish all classes in  $n=7$ . For instance, classes No. 40, No. 42, and No. 43 in [1,2] have the same number of qubits, require the same minimum number of controlled-Z gates for the preparation, and have the same Schmidt measure and rank indexes. The same problem occurs between classes No. 110 and No. 111, between classes No. 113 and No. 114, and between classes No. 116 and No. 117 in our classification (see Table 1). Following [1,2], we have placed the class with lower  $|LC|$  in the first place. However, this solution is not satisfactory, since  $|LC|$  is not related to the entanglement properties of the class. On the other hand, Van den Nest, Dehaene, and De Moor have proposed a finite set of invariants that characterizes all classes [55]. However, this set has more than  $2 \times 10^{36}$  invariants already for  $n=7$ . The problem of obtaining a minimum set of invariants capable of distinguishing all classes with  $n \leq 8$  qubits will be addressed elsewhere [56].

Another weak point in the method is that the precise value of  $E_S$  is still unknown for some classes. The good news is that, for most of these classes, the value might be fixed if we knew the



**Fig. 2.** Graphs associated to the 101 classes on 8-qubit graph states inequivalent under local complementation and graph isomorphism. We have chosen as representative of the class the one (or one of those) with minimum number of edges and minimum maximum degree (i.e., number of edges incident with a vertex), which means that it requires the minimum number of controlled-Z gates in the preparation and minimum preparation depth.



**Table 1**  
Entanglement of the 101 classes of 8-qubit graph states. No. is the number of the class; it is assigned attending to  $|E|$ ,  $E_S$ , and  $Rl_j$ ; the numbering starts at the point in which the one in Refs. [1,2] ends.  $|LC|$  is the number of nonisomorphic elements of the class.  $|E|$  is the number of edges of those representatives with the minimum number of edges.  $E_S$  is the Schmidt measure (or its lower and upper bounds).  $Rl_j$  is the rank index with  $j$  qubits in the smaller bipartition (i.e., the number of bipartite splits in which a certain rank occurs; ranks appear in decreasing order from left to right). 2-col indicates whether or not a 2-colorable representative exists.

No.	$ LC $	$ E $	$E_S$	$Rl_4$	$Rl_3$	$Rl_2$	2-col	No.	$ LC $	$ E $	$E_S$	$Rl_4$	$Rl_3$	$Rl_2$	2-col
46	2	7	1	(0,0,0,35)	(0,0,56)	(0,28)	yes	97	154	8	4	(8,22,4,1)	(42,13,1)	(27,1)	no
47	6	7	2	(0,0,20,15)	(0,30,26)	(12,16)	yes	98	542	8	4	(8,22,5,0)	(42,14,0)	(27,1)	no
48	6	7	2	(0,0,30,5)	(0,45,11)	(15,13)	yes	99	300	8	4	(12,16,7,0)	(44,11,1)	(27,1)	yes
49	16	7	2	(0,0,30,5)	(0,45,11)	(17,11)	yes	100	214	8	4	(14,17,4,0)	(48,8,0)	(28,0)	yes
50	4	7	2	(0,0,34,1)	(0,48,8)	(16,12)	yes	101	14	9	3	(0,20,15,0)	(24,32,0)	(25,3)	no
51	16	7	2	(0,0,34,1)	(0,51,5)	(19,9)	yes	102	66	9	3	(0,28,7,0)	(32,24,0)	(25,3)	no
52	10	7	3	(0,12,16,7)	(16,28,12)	(20,8)	yes	103	66	9	3	(0,30,5,0)	(36,20,0)	(26,2)	yes
53	16	7	3	(0,12,22,1)	(16,34,6)	(20,8)	yes	104	6	9	3	(0,32,0,3)	(32,24,0)	(24,4)	yes
54	44	7	3	(0,12,22,1)	(16,35,5)	(21,7)	yes	105	57	9	3 < 4	(0,30,5,0)	(36,19,1)	(25,3)	no
55	16	7	3	(0,18,14,3)	(18,33,5)	(21,7)	yes	106	28	9	4	(8,18,9,0)	(38,18,0)	(25,3)	no
56	44	7	3	(0,18,14,3)	(22,29,5)	(23,5)	yes	107	17	9	4	(8,20,6,1)	(32,24,0)	(24,4)	no
57	10	7	3	(0,18,15,2)	(18,36,2)	(21,7)	yes	108	72	9	4	(8,20,7,0)	(36,20,0)	(25,3)	no
58	25	7	3	(0,18,16,1)	(18,36,2)	(22,6)	yes	109	87	9	4	(8,20,7,0)	(40,16,0)	(27,1)	no
59	44	7	3	(0,18,16,1)	(22,31,3)	(23,5)	yes	110	114	9	4	(8,22,5,0)	(40,16,0)	(26,2)	no
60	44	7	3	(0,24,9,2)	(24,30,2)	(23,5)	yes	111	372	9	4	(8,22,5,0)	(40,16,0)	(26,2)	no
61	26	7	3	(0,24,10,1)	(28,25,3)	(23,5)	yes	112	70	9	4	(8,24,2,1)	(40,16,0)	(26,2)	no
62	120	7	3	(0,24,10,1)	(28,26,2)	(24,4)	yes	113	264	9	4	(8,24,3,0)	(44,12,0)	(27,1)	no
63	66	7	3	(0,26,7,2)	(30,24,2)	(25,3)	yes	114	542	9	4	(8,24,3,0)	(44,12,0)	(27,1)	no
64	14	7	4	(8,12,12,3)	(32,18,6)	(24,4)	yes	115	156	9	4	(12,18,5,0)	(46,9,1)	(27,1)	no
65	25	7	4	(8,12,14,1)	(32,20,4)	(24,4)	yes	116	174	9	4	(12,20,3,0)	(46,10,0)	(27,1)	no
66	120	7	4	(8,14,12,1)	(34,19,3)	(25,3)	yes	117	542	9	4	(12,20,3,0)	(46,10,0)	(27,1)	no
67	72	7	4	(8,16,10,1)	(36,17,3)	(25,3)	yes	118	262	9	4	(12,20,3,0)	(48,8,0)	(28,0)	no
68	172	7	4	(8,18,8,1)	(38,16,2)	(26,2)	yes	119	802	9	4	(14,19,2,0)	(50,6,0)	(28,0)	no
69	10	8	2	(0,0,34,1)	(0,52,4)	(20,8)	yes	120	117	9	4	(16,16,3,0)	(50,6,0)	(28,0)	yes
70	10	8	2	(0,0,35,0)	(0,54,2)	(21,7)	yes	121	10	10	3	(0,32,2,1)	(32,24,0)	(24,4)	no
71	10	8	3	(0,12,22,1)	(16,36,4)	(20,8)	no	122	37	10	3	(0,32,3,0)	(40,16,0)	(27,1)	yes
72	21	8	3	(0,12,22,1)	(16,36,4)	(22,6)	no	123	36	10	4	(8,22,5,0)	(44,12,0)	(26,2)	no
73	10	8	3	(0,18,17,0)	(18,36,2)	(21,7)	no	124	7	10	4	(8,24,0,3)	(32,24,0)	(24,4)	no
74	44	8	3	(0,18,17,0)	(22,32,2)	(23,5)	yes	125	103	10	4	(8,24,3,0)	(42,14,0)	(26,2)	no
75	66	8	3	(0,18,17,0)	(22,33,1)	(24,4)	no	126	46	10	4	(8,24,3,0)	(44,12,0)	(27,1)	no
76	26	8	3	(0,20,14,1)	(24,30,2)	(24,4)	yes	127	170	10	4	(8,26,1,0)	(46,10,0)	(27,1)	no
77	26	8	3	(0,24,10,1)	(24,31,1)	(23,5)	yes	128	74	10	4	(12,20,3,0)	(46,10,0)	(27,1)	yes
78	28	8	3	(0,24,10,1)	(28,27,1)	(23,5)	no	129	340	10	4	(12,22,1,0)	(48,8,0)	(27,1)	no
79	44	8	3	(0,24,11,0)	(28,26,2)	(23,5)	no	130	254	10	4	(12,22,1,0)	(50,6,0)	(28,0)	no
80	132	8	3	(0,24,11,0)	(28,27,1)	(24,4)	no	131	433	10	4	(14,21,0,0)	(52,4,0)	(28,0)	no
81	114	8	3	(0,24,11,0)	(30,25,1)	(25,3)	yes	132	476	10	4	(16,18,1,0)	(52,4,0)	(28,0)	no
82	72	8	3	(0,26,9,0)	(30,26,0)	(25,3)	no	133	28	10	4 < 5	(12,22,0,1)	(48,8,0)	(28,0)	no
83	72	8	3	(0,28,6,1)	(32,23,1)	(25,3)	yes	134	9	11	3 < 4	(0,30,5,0)	(40,15,1)	(25,3)	no
84	198	8	3	(0,28,7,0)	(34,22,0)	(26,2)	yes	135	39	11	4	(8,26,1,0)	(44,12,0)	(26,2)	no
85	56	8	3 < 4	(0,30,4,1)	(34,21,1)	(25,3)	no	136	46	11	4	(12,20,3,0)	(50,6,0)	(27,1)	no
86	28	8	4	(8,16,10,1)	(32,22,2)	(24,4)	no	137	208	11	4 < 5	(16,18,1,0)	(52,4,0)	(28,0)	no
87	10	8	4	(8,16,10,1)	(32,24,0)	(24,4)	yes	138	298	11	4 < 5	(18,17,0,0)	(54,2,0)	(28,0)	no
88	56	8	4	(8,16,10,1)	(36,18,2)	(26,2)	no	139	24	11	4 < 5	(20,10,5,0)	(50,5,1)	(27,1)	no
89	66	8	4	(8,16,11,0)	(36,18,2)	(25,3)	yes	140	267	11	4 < 5	(20,14,1,0)	(54,2,0)	(28,0)	no
90	72	8	4	(8,18,9,0)	(34,22,0)	(25,3)	no	141	4	12	4	(28,0,7,0)	(56,0,0)	(28,0)	no
91	63	8	4	(8,18,9,0)	(36,20,0)	(26,2)	yes	142	22	12	4 < 5	(14,21,0,0)	(56,0,0)	(28,0)	no
92	66	8	4	(8,18,9,0)	(38,16,2)	(25,3)	no	143	46	12	4	(20,12,3,0)	(50,6,0)	(27,1)	yes
93	176	8	4	(8,18,9,0)	(38,17,1)	(26,2)	no	144	28	13	4	(28,4,3,0)	(56,0,0)	(28,0)	no
94	76	8	4	(8,20,6,1)	(36,19,1)	(25,3)	no	145	7	13	4 < 5	(16,18,1,0)	(56,0,0)	(28,0)	no
95	194	8	4	(8,20,7,0)	(38,18,0)	(26,2)	yes	146	51	13	4 < 5	(24,10,1,0)	(56,0,0)	(28,0)	no
96	352	8	4	(8,20,7,0)	(40,15,1)	(26,2)	no								

value for the 5-qubit ring cluster state, which is the first graph state in the classification for which the value of  $E_S$  is unknown [1,2]. Unfortunately, we have not made any progress in calculating  $E_S$  for the 5-qubit ring cluster state.

Table 1 shows that there are no 8-qubit graph states with rank indexes  $Rl_p = [v_j^p]_{j=p}^1$  with  $v_j^p \neq 0$  if  $j = p$ , and  $v_j^p = 0$  if  $j < p$ , i.e., with maximal rank with respect to all bipartite splits, i.e., such that entanglement is symmetrically distributed between all parties. These states are robust against disentanglement by a few measurements. Neither there are 7-qubit graph states with this property [1,2]. This makes more interesting the fact that there is a single 5-qubit and a single 6-qubit graph state with this property [1,2].

### Acknowledgements

The authors thank H.J. Briegel, O. Gühne, M. Grassl, M. Hein, C. Santana, and M. Van den Nest, for their help. This work has

benefited from the use of the program *nauty* [57] for computing automorphism groups of graphs. A.C., A.J.L., and P.M. acknowledge support from projects No. P06-FQM-02243, No. FIS2008-05596, and No. PAI-FQM-0239. J.R.P. acknowledges support from projects No. P06-FQM-01649, No. MTM2008-05866-C03-01, and No. PAI-FQM-0164.

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