BIFURCATIONS AND AGGREGATION IN LARGE SCALE SYSTEMS

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ABSTRACT

This paper deals with the following problem: assume that a qualitative analysis (behaviour modes, bifurcation points, type of atractors...) of a nonlinear dynamical system has been carried out and that afterwards this dynamical system is transformed into a large scale system through a disaggregating process of some (or all) of its variables. The problem at stake is to analyze whether the disaggregation gives rise to new behaviour modes, as a consequence of the appearance of new bifurcations in the disaggregated dynamical system. That leads us to study whether the original system and the disaggregated one are "equivalents" or whether the second one is richer in behaviours than the first one.

The paper develops general results for standard disaggregation forms. Furthermore, practical applications of the proposed methodology to urban dynamics models is included.

INTRODUCTION

Consider a dynamical system given by the equations:

$$\dot{z} = \varphi(z,q) \tag{1}$$

where $z \in R^r$ and $q \in R^t$. It is assumed that a disaggregation process is applied to (1), in such a way that every z_i ($i = 1,r$) is decomposed in parts x_j such that:

$$z_i = \sum_{\alpha}^{\beta} x_j \tag{2}$$

with

$$\alpha = \sum_{k=1}^{i-1} n_k \ , \quad \beta = \alpha + n_i$$

being n_i the number of parts in which z_i has been decomposed. The set $\{x_j | \alpha < j < \beta\}$ will be called module i, associated to z_i.

After disaggregation, the dynamical system (1) will lead to a new one, of the form:

$$\dot{x} = f(x,p) \tag{3}$$

where $x \in R^n$, $p \in R^s$ being $n = \Sigma n_i$. It should be noted that p is different from q, due to the greater richness in the description of (3) relative to (1).

The model (3) will be considered a model refinement of the model (1). The problem is to study if the system (3) will show behaviour modes not shown by (1). From a qualitative point of view that means to study if (3) will exhibit bifurcations not appearing in (1). The answer to those questions will be found through the qualitative analysis of (1) and (3). However, system (3) is a large scale system and, therefore, the qualitative analysis can be a very difficult task.

In this paper we consider only dynamical systems with point atractors; that is, we are restricted to dynamical systems with static bifurcations. For these systems we propose a method wich allows to analyze if the behaviour modes of (3) are the same as those of (1); that is, if, as a consequence of the disaggregation process, there appear bifurcations in (3) not shown by (1).

This kind of results has a lot of practical interest because the modelling process starts normally with a small dimension model (Randers 1980) which is disaggregated later on.

The paper ends with applications of the proposed method to some urban dynamics models.

BIFURCATION ANALYSIS

Considering only static bifurcations, the bifurcation analysis of the model (1) is reduced to the study of the solutions to the equation:

$$\varphi(z,q) = 0 \tag{4}$$

when the parameters q are varied, and to the stability study of each one of those solutions. The graphical representation of these solutions versus every parameter q gives rise to the bifurcation diagram of (4). These diagrams can be obtained numerically with the help of continuation methods (Kubiček 1976).

Let (z_0,q_0) be a solution of Eq. (4). If varying q around q_0 the number of solutions of (4), or just the stability of any of them, are changed, then it is said that (z_0,q_0) is a bifurcation point. These points are fundamental in the qualitative analysis of system (1), since they supply all the information needed to determine the qualitative shape of the bifurcation diagram.

For static bifurcations the bifurcation points are given by Eq. (4) and

$$\det D_z\Psi(z,q) = 0 \tag{5}$$

since in the bifurcation points an eigenvalue of jacobian matrix $D_z\Psi(z,q)$ is zero.

REDUCIBLE DYNAMICAL SYSTEMS

Consider that we are interested on the bifurcation analysis of a large scale dynamical system. This problem could be greatly simplified if we can find a subsystem of the large scale one that would "concentrate" all the bifurcations. Theorems 1 and 2 below help to cope with that problem.

Suppose we are given a large scale dynamical system:

$$\dot{u} = h(u,a)$$

that can be partitioned into the form:

$$\dot{u}_1 = h_1(u_1,u_2,a) \tag{6.1}$$

$$\dot{u}_2 = h_2(u_1,u_2,a) \tag{6.2}$$

where $u = (u_1,u_2)$. Equilibria of (6) are solutions to the equations:

$$h_1(u_1,u_2,a) = 0 \tag{7.1}$$

$$h_2(u_1,u_2,a) = 0 \tag{7.2}$$

Then, the following theorems can be stated.

Theorem 1

If Eq. (7) can be transformed into the form

$$h_1(u_1,u_2,a) = 0 \tag{8.1}$$

$$u_2 = F(a) \tag{8.2}$$

then system (6) has the same bifurcation diagram than the associated reduced system

$$\dot{u}_1 = h_1(u_1,u_2,a) \tag{9}$$

where u_2 is now a (constant) parameter, but related to parameters a by Eq. (8.2).

Theorem 2

If the hypotheses of theorem 1 are fullfilled and, furthermore, the same conditions that guarantee the stability in every branch of the bifurcation diagram of (9), can guarantee the stability of the corresponding branches in the bifurcation diagram of (6), then system (6) is reducible to (9).

These theorems will be proved and generalized in a forthcoming paper.

APPLICATION TO THE DISAGGREGATION PROCESS

Take Eqs. (3) and reorder them in such a way that the following partition could be made:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,x_2,w_1) \\ \dot{x}_2 &= f_2(x_1,x_2,w_2) \end{aligned} \tag{10}$$

where $x_1 \in R^r$ and $x_2 \in R^{n-r}$. The vector x_1 is formed by a "representative" component of every module i. The components of vector x_2 are sorted in blocks coming from the different modules obtained by disaggregation of the variables z_i .

It is convenient to transform vector (x_1, x_2) into vector (y,k), where y_i (that is, $y_i = z_i$) will be the addition of all the x_i variables belonging to module i and k_j the rate of variables x_{2j} relative to y_i. This transformation is carried out by:

$$\begin{bmatrix} y \\ \\ k \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} \tag{11}$$

where,

$$T_1 = \begin{bmatrix} I_r & 0 \\ \\ 0 & C \end{bmatrix}, \quad T_2 = \begin{bmatrix} I_r & B \\ \\ 0 & I_{n-r} \end{bmatrix} \tag{12}$$

being B an (r x n-r) matrix, with $b_{ij} = 1$ if x_{2j} belongs to module i and $b_{ij} = 0$, otherwise, and where C is a diagonal matrix with $c_{jj} = 1/y_i$ if x_{2j} belongs to module i.

Transformation (11) has an inverse, which is meaningful for studying the equilibria of (10) whether $y_i \neq 0$, or whether $y_i = 0$, and the form of the equations causes y_i disappear from the denominator. This happens when Eq. (3) has the form:

$$\dot{x}_i = f_i(x,w)x_i \tag{13}$$

After applying transformations (11) to Eq. (10) we get:

$$\begin{aligned} \dot{y} &= f_1(x_1,x_2,w)) + Bf_2(x_1,x_2,w) \\ \dot{k} &= Cf_2(x_1,x_2,w) \end{aligned} \tag{14}$$

If Eq. (3) takes the form (13) then Eq. (14.2) will take the form:

$$\dot{k}_i = f_2(x_1,x_2,w)k_i$$

In Eqs. (14) x_1 and x_2 are functions of y and k_i, and are given by Eq. (11). Reordering parameters w, Eqs. (14) can be written:

$$\begin{aligned} \dot{y} &= g_1(y,k,p_1) \\ \dot{k} &= g_2(y,k,p_2) \end{aligned} \tag{15}$$

The transformation of parameters w into (p_1,p_2) should be made in order to look for a correspondence between the variables and parameters of Eq. (16) below and the ones of Eq. (1).

If the hypoteses of theorems 1 and 2 are fullfilled, then the dynamical system (15) is reduced to:

$$\dot{y} = g_1(y,k,p_1) \tag{16}$$

where k is now a constant parameter.

SOME SPECIAL CASES

Previous results can be applied to two kinds of disaggregation, which are widely used.

a) Linear disaggregation.

Suppose that functions f_{2i} appearing in Eq. (10) are linear functions of variables x_i beloging to the same module as x_{2i}. Then the disaggegation is called linear. In such a case, functions g_2 of (15) do not depend on y, due to the form of transformation (11).

Indeed, transformation (11) can be considered as an application of two successive transformations. The first one transforms (x_1,x_2) into (y,x_2) through matrix T_2. The second one, transforms (y,x_2) into (y,k) by means of matrix T_1. Function f_{2i} is transformed into f_{2i} through T_2. This last function f_{2i} is linear in y_i and in variables x_{2j}, where the latter belong to module i.

Through T_1 functions g_{2j} take the form $g_{2j} = f_{2j}/y_i$. Since f_{2j} is linear in y_i and x_{2j}, expressions f_{2j}/y_i only depend on k.

If apart from it being a linear disaggregation, matrix $D_k g_2$ in (15) is stable then the above theorems can be applied and the disaggregation does not add new bifurcations (new behaviour modes). In next section an example of this case will be presented.

b) Disaggregation with kernel

This disaggregation occurs when the non linearities that appear in f_1 and f_2 have as the only argument the variables y_i (that is, the additions of all the variables x_j belonging to module i) and, furthermore, when, after transformation (11), theorems 1 and 2 can be applied.

APPLICATIONS TO URBAN DYNAMICS

In urban dynamics (Alfeld and Graham, 1976) the evolution of the housing, or of the business structures, is described by a model of the form

$$\dot{z} = z(q_1\tau(hz)-q_2) \tag{17}$$

If the case of the housing evolution is considered, then z stands for housing, q_1 for the rate of housing demolition and $q_1\tau(hz)$ for the rate of housing construction. Function $\tau(hz)$ represents the housing-land multiplier and its shape is show in Fig. 1.

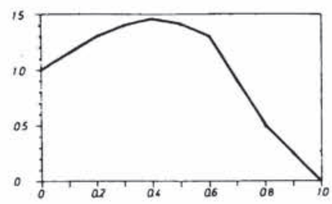

Fig. 1

The qualitative analysis of this model can be found elsewhere (Aracil, 1981), and some related material in (Aracil, 1984).

Model (17) describes the housing evolution. However, a disaggregation of the housing sector, taking into account the connection between housing units and the socioeconomic status of their occupants, can lead to a model refinement. This is done in (Alfeld and Graham, chap. 9) where the following model is proposed as a disaggregation of (17).

$$\begin{aligned} \dot{x}_1 &= n_2(x_1+n_2x_2)\tau(x_1+x_2+x_3)-n_5x_1 \\ \dot{x}_2 &= n_5x_1-x_2 \\ \dot{x}_3 &= n_6x_2-x_3 \end{aligned} \tag{18}$$

where the total number of houses z has been disaggregated into variables x_1,x_2,x_3 corresponding to upper income, middle income and lower income houses. The disaggregation from (17) to (18) is of the same type as the one from (1) to (3).

A transformation of type (11) can be applied to this model, giving:

$$\begin{aligned} \dot{y} &= n_2(1-k_2-k_3+n_2k_2)\tau(y)y-n_7k_3y \\ \dot{k}_2 &= n_5(1-k_2-k_3)-n_6k_2 \\ \dot{k}_3 &= n_6k_2-n_7k_3 \end{aligned} \tag{19}$$

It should be noticed that the disaggregation is of linear type. System (19) is reducible provided that matrix $D_k g_2$ is stable. In this case, we have:

$$D_k g_2 = \begin{bmatrix} -n_5 - n_6 & n_5 \\ n_6 & -n_7 \end{bmatrix}$$

whose stability is guaranted provided that $n_i > 0$.

Consequently, system (19) is reducible to:

$$\dot y = n_2(1 - k_2 - k_3 + n_2 k_2)\tau(y)y - n_7 k_3 y \quad (20)$$

It should be noticed that (20) is equivalent to (17) so that a correspondence between the parameters of (18) and those of (17) can be stated in the form:

$$\begin{aligned} q_1 &\longleftrightarrow n_2(1 - k_2 - k_3 + n_2 k_2) \\ q_2 &\longleftrightarrow n_7 k_3 \end{aligned} \quad (21)$$

where k_2 and k_3 can be expressed as functions of parameters n_i from equilibrium equations of system (19).

The disaggregation process has not supplied new bifurcations, but it has given a more detailed way of computing the parameters of the aggregated model.

If now expresion (17) models the evolution of business structures, then in (Alfeld and Graham 1976, chap.8) a different form of disaggregation is considered. Taking into account the aging and obsolescence of the business structures a disaggregated model of the following form is proposed

$$\begin{aligned} \dot x_1 &= x_1(n_1\tau(h(x_1 + x_2 + x_3)) - n_4) \quad (22) \\ \dot x_2 &= n_4 x_1 + x_2(n_6\tau(h(x_1 + x_2 + x_3)) - n_7 - n_8) \\ \dot x_3 &= n_7 x_2 + x_3(n_9\tau(h(x_1 + x_2 + x_3)) - n_{10}) \end{aligned}$$

where now z in Eq. (17) stands for the number of business, and x_2, x_1 and x_3 in Eq. (22) for new business, mature business, and deteriorating business.

Reordering (22) into the form (x_1, x_2, x_3) and applying transformation (11), we obtain:

$$\dot y = (k_1 n_1 + (1 - k_1 - k_3)n_6 + k_3 n_9)\tau(hy)y - (n_8(1 - k_1 - k_3) + n_{10} k_3)y \quad (23.1)$$

$$\dot k_1 = (n_1\tau(hy) - n_4)k_1 \quad (23.2)$$

$$\dot k_3 = n_7(1 - k_1 - k_3) + k_3(n_9\tau(hy) - n_{10}) \quad (23.3)$$

Taking into account the equilibria of (23), the values of k_i at the equilibrium point can be obtained from parameters n_i through the equations:

$$(k_1 n_1 + (1 - k_1 - k_3)n_6 + k_3 n_9)n_4/n_1 - (n_8(1 - k_1 - k_3) + n_{10} k) = 0 \quad (24.1)$$

$$n_7(1 - k_1 - k_3) + k_3(n_9 n_4/n_1 - n_{10}) = 0 \quad (24.2)$$

If we are given n_i then Eq. (24) are linear systems for k_i, and, therefore, there is a function $u_2 = F(a)$ exists as required by theorem 1.

Eq. (23.1) can be written:

$$\dot y = b_1\tau(hy) - b_2 y \quad (25)$$

The condition for an equilibrium $y \neq 0$ of equation (25) to be stable is $\tau'(hy) < 0$. This condition guarantees the stability of the corresponding branch in the bifurcation diagram of (23), with $y \neq 0$ and $k_i \neq 0$, since the jacobian matrix of this last system works out to be:

$$\begin{bmatrix} c_1\tau'(hy) & c_4 & -c_5 \\ c_2\tau'(hy) & 0 & 0 \\ c_3\tau'(hy) & -c_6 & -c_7 \end{bmatrix}$$

where all parameters c_i are positive if n_i are within the range of values meaningful to the model. The condition for (26) to be stable is $\tau'(hy) < 0$ as it is easily shown.

Comparing Eq. (23.1) with Eq. (17) it is easy to deduce the grouping of parameters from Eq. (22) which are equivalent to the parameters from Eq. (17).

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