STRONG SOLUTIONS FOR SEMILINEAR PROBLEMS WITH ALMOST SECTORIAL OPERATORS

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Abstract. In this paper we study a semilinear parabolic problem

$$u_t + Au = f(t, u),\ t > \tau; \quad u(\tau) = u_0 \in X,$$

in a Banach space X, where $A : D(A) \subset X \to X$ is an almost sectorial operator. This problem is locally well-posed in the sense of mild solutions. By exploring properties of the semigroup of growth $1 - \alpha$ generated by $-A$, we prove that the local mild solution is actually strong solution for the equation. This is done without requiring any extra regularity for the initial condition $u_0 \in X$ and under suitable assumptions on the nonlinearity f. We apply the results for a reaction-diffusion equation in a domain with handle where the nonlinearity f satisfies a polynomial growth

$$|f(t, u) - f(t, v)| \leq C|u - v|(1 + |u|^{\rho-1} + |v|^{\rho-1})$$

and we establish values of ρ for which the problem still have strong solution.

Mathematical Subject Classification 2020: 35A01, 35D30, 35D35, 35K58
Key words and phrases: Almost sectorial operators, semigroups of growth $1 - \alpha$, semilinear problems, strong solutions.

1. Introduction

In this work, we consider the abstract semilinear problem

$$u_t + Au = f(t, u), t > \tau; \quad u(\tau) = u_0 \in X, \tag{1.1}$$

where X is a Banach space and f is a nonlinearity whose properties we specify later. For the linear part of the equation, we assume that $A : D(A) \subset X \to X$ is a closed and densely defined linear operator such that $\rho(-A)$ contains a sector $\Sigma_\varphi = \{\lambda \in \mathbb{C}; |\arg \lambda| < \varphi\}$, for some $\varphi \in (\frac{\pi}{2}, \pi)$. Moreover, there exists a constant $C > 0$ and $\alpha \in (0, 1)$ such that the resolvent of $-A$ satisfies

$$\|(\lambda I + A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{|\lambda|^\alpha + 1},\ \forall \lambda \in \Sigma_\varphi \cup \{0\}. \tag{1.2}$$

We refer to the property above as A being an *almost sectorial operator* and we say that α is the *constant of almost sectoriality.* The term "almost" comes from the fact that estimate (1.2)

[†]Research partially supported by FAPESP # 2017/09406-0 and # 2017/17502-0, Brazil.

[‡]Research partially supported by Ministerio de Ciencia Innovación y Universidades (Spain), FEDER (European Community) under grant PGC2018- 096540-B-I00, and by Junta de Andalucía (Consejería de Economía y Conocimiento) under project US-1254251 and P18-FR-4509.

[*]Research partially supported by FAPESP # 2019/26841-8, Brazil.

is satisfied for an $0 < \alpha < 1$ but never for $\alpha = 1$. We will also refer to A as $\alpha-$*almost sectorial* if we wish to highlight the constant α.

With the estimate (1.2), $-A$ does not generate a C_0-semigroup, since it fails to satisfy the classical Hille-Yosida's assumption [13, Theorem 1.5.3]. Nevertheless, $-A$ generates a type of semigroup called *semigroup of growth* $1-\alpha$. Those semigroups were first introduced by Da Prato in [7], where the growth considered was given by a positive integer n. Later on, this concept was generalized to semigroups of growth β, for any $\beta > 0$, and its properties were studied by several authors, like [11, 12, 17, 21]. The definition of this type of semigroup is presented in the sequence.

Definition 1.1. [12, Definition 1.1] *Let X be a Banach space and $\alpha \in (0, 1)$. A family $\{T(t) \in \mathcal{L}(X) : t \geq 0\}$ is a* semigroup of growth $1-\alpha$ *if*

(1) $T(0) = I$ and $T(t)T(s) = T(t+s)$, for all $t, s > 0$.
(2) There exists $\delta > 0$ such that $\|t^{1-\alpha}T(t)\|_{\mathcal{L}(X)} \leq M$, for all $0 \leq t \leq \delta$.
(3) If $T(t)x = 0$ for every $t > 0$ then $x = 0$.
(4) $X_0 = \bigcup_{t>0} T(t)[X]$ is dense X.

Condition (2) of definition above does not imply strong continuity at $t = 0$ for the semigroup of growth $1-\alpha$, as it happens for C_0-semigroups. In other words, there exists $x \in X$ such that $T(t)x \nrightarrow x$ as $t \to 0^+$. This singular behavior distinguishes semigroups of growth from the usual C_0-semigroups.

As we will see during this work, almost sectorial operators and semigroups of growth are closed related. To be precise, $\alpha-$almost sectorial operators generate semigroups of growth $1-\alpha$. Moreover, those almost sectorial operators usually emerge when we consider elliptic operators in more regular phase spaces. For instance, the minus Laplacian in a bounded domain Ω, with Dirichlet (or Neumann) boundary condition, is known to be sectorial when we consider its realization in $L^p(\Omega)$ [6, Section 1.3]. It is also sectorial if we consider it in the space of the continuous and bounded functions in Ω, see [19]. However, for more regular phase spaces, like $X = \mathcal{C}^\mu(\Omega)$, $\mu > 0$, even simple elliptic operators as the Laplacian in $1-$dimension fails to be sectorial, in fact the minus Laplacian is almost sectorial in X [10, Example 3.1.33].

Those operators also emerge when we are dealing with equations in domains with handles, that is, sets $\Omega_0 = \Omega \cup R_0$ in $\mathbb{R}^N$ formed by an open bounded subset $\Omega \subset \mathbb{R}^N$ connected to a line segment R_0 (called *handle*).The points where the line R_0 connects with Ω causes deficiencies in the resolvent estimate. We explore this example with further details in Section 5 of this work.

Due to the close relation between $\alpha-$almost sectorial operators and the generation of semigroups of growth $1-\alpha$, semilinear problems like (1.1) can be solved in terms of the semigroup that $-A$ generates. To be precise, we denote by $T_{-A}(t)$ the semigroup of growth $1-\alpha$ generated by $-A$ (which will be defined in Section 2). In [3, 14, 16] the authors proved, using a fixed point argument, the existence of *local mild solution* for (1.1), that is, a continuous function $u : (\tau, \tau+T] \to X$ that satisfies the variation of constants formula

$$u(t, \tau, u_0) = T_{-A}(t-\tau)u_0 + \int_\tau^t T_{-A}(t-s)f(s, u(s))ds, \quad t \in (\tau, \tau+T]. \tag{1.3}$$

However, differentiability of the mild solution in time variable, as well as whether or not it satisfies the equation in the usual sense, were not established. By a solution for the problem in the usual sense or *strong solution* we mean:

Definition 1.2. *A function* $u(\cdot) : (\tau, \tau + T] \to X$ *is a* strong solution *for the semilinear equation* (1.1) *if it satisfies:*

(1) $u(\cdot) \in \mathcal{C}^1((\tau, T], X)$, $u(\tau) = u_0 \in X$ *and* $u(t) \in D(A)$, *for all* $t \in (\tau, \tau + T)$.
(2) The equality $u'(t) = -Au(t) + f(t, u(t))$ *is satisfied in* X *for each* $\tau < t < \tau + T$.

Note that this definition of strong solution acknowledges that we might have discontinuity at the initial time $t = \tau$ ($u(t) \overset{t\to\tau}{\nrightarrow} u_0$) inherited by the discontinuity of $T_{-A}(t - \tau)$ at $t = \tau$.

In [8, Theorem 2], the author proved, by replaying the fixed point argument used in [3, 14], that the mild solution is also a strong solution for the problem. However, in order to achieve this, it was necessary to impose some restrictions on the initial condition u_0 and on the nonlinearity f, which were:

(C.1) The initial condition u_0 is taken in the set $\mathcal{O} = \{x \in D(A) : Ax \in \Omega\}$, where $\Omega = \{x \in X : T_{-A}(t)x \overset{t\to 0}{\longrightarrow} x\}$. In particular, $D(A^2) \subset \mathcal{O} \subset D(A)$.
(C.2) The nonlinearity $f : \mathbb{R} \times X \to X$ is locally Hölder continuous in the first variable and locally Lipschitz in the second: $\|f(t, x) - f(s, y)\|_X \le L(|t - s|^\theta + \|x - y\|_X)$.
(C.3) Moreover, when we restrict f to $D(A)$, we obtain $f : \mathbb{R} \times D(A) \to D(A)$ and it satisfies locally: $\|f(t, x) - f(t, y)\|_{D(A)} \le L\|x - y\|_{D(A)}$.

By requiring the initial conditions to be in $\mathcal{O}$, we remove every possible situation where the semigroup $T_{-A}(t)$ can present singularity and in this set it behaves like a C_0-semigroup. In this work we will not require conditions (C.1) or (C.3) to prove that the local mild solution is also strong and we will also allow other types of nonlinearities in condition (C.2), that is, nonlinearities that take elements in a more regular space $X \hookrightarrow Y$ to the less regular space Y.

We obtain the desired result by working directly with the mild formulation for the solution and proving that the following equality

$$A\left(\int_\tau^t T_{-A}(t - s)x ds\right) = x - T_{-A}(t - \tau)x, \quad x \in X, \tag{1.4}$$

to which we refer as *fundamental theorem of calculus for semigroups*, holds for semigroups of growth $1 - \alpha$.

To attend this goal, this paper is structured in the following manner: In Section 2 we present the properties of the semigroup $T_{-A}(t)$, the features that distinguish it from the C_0-semigroups and we prove (1.4). Section 3 is dedicated to study the nonautonomous linear problem

$$u_t + Au = g(t), \ t \in (\tau, T); \quad u(\tau) = u_0.$$

Assuming that $g : (\tau, T) \to X$ is Hölder continuous, we prove in Theorem 3.1 that the problem has a strong solution. In Section 4 we provide a way to reduce the semilinear case to the previous one, which allows us to obtain the existence of strong solution for the semilinear problem (Theorem 4.11). In Section 5 we apply those results to a reaction-diffusion equation in a domain with a handle. The last section, Section 6, is dedicated to a discussion between the approach we use in this work to treat semilinear parabolic problem with the usual approach with fractional power spaces and nonlinearities $f : X^\gamma \to X^\theta$.

2. Semigroups of growth $1-\alpha$

Before we introduce the concept and properties of semigroups of growth, we present some results about integral of functions in Banach spaces (also called *Bochner-integral*) that will be useful in future calculus. For a detailed discussion on the definition and properties of this integral, we recommend [18, Chapter IV, Section 1.2] or [6, Section 2.1]

2.1. Integral in Banach spaces.

In the variation of constant formula (1.3) for $u(t)$ one of the terms is an integral of a function that takes values in Banach space, that is, integrals like $\int_{t_1}^{t_2} h(t)dt$, where $h(t) \in X$.

The convergence of $\int_{t_1}^{t_2} h(t)dt$ is strictly connected with the convergence of $\int_{t_1}^{t_2} \|h(t)\|dt$: one will converge if and only if the other does. Therefore, tolls on convergence of integrals of real functions will be important, in special the ability of recognizing a *beta function* whenever it appears in the calculations. *Beta function* is the function $\mathcal{B} : (0,\infty) \times (0,\infty) \to \mathbb{R}$ given by

$$\mathcal{B}(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du$$

and a simple change of variable turns this integral into a form that shows up in the calculus frequently:

Lemma 2.1. *If $a, b > 0$ and $\tau < t$, then $\int_\tau^t (t-s)^{a-1}(s-\tau)^{b-1}ds = (t-\tau)^{a+b-1}\mathcal{B}(a,b)$.*

Another well known function involving integral is the *Gamma function*, $\Gamma : (0,\infty) \to \mathbb{R}$, given by

$$\Gamma(a) = \int_0^\infty e^{-u}u^{a-1}du.$$

Integrability properties of $h : (t_1, t_2) \to X$ are listed below and their proofs can be found in [6, Section 2.1].

Proposition 2.2. *If $h \in \mathcal{C}([t_1,t_2], X) \cap \mathcal{C}^1((t_1,t_2), X)$, then $h(t_2) - h(t_1) = \int_{t_1}^{t_2} h'(s)ds$.*

Proposition 2.3. *Let $A : D(A) \subset X \to X$ be a closed linear operator and $h : [\tau, t] \to X$ a continuous function with image in $D(A)$. If $Ah : [\tau, t] \to X$ is continuous, then $\int_\tau^t h(s)ds \in D(A)$ and*

$$A\int_\tau^t h(s)ds = \int_\tau^t Ah(s)ds.$$

Corollary 2.4. *Let $A : D(A) \subset X \to X$ be a closed linear operator, $h : [\tau, t) \to X$ (or $h : (\tau, t) \to X$) continuous with image in $D(A)$ and $Ah : [\tau, t) \to X$ (or $Ah : (\tau, t) \to X$) also continuous. Assume that $\int_\tau^t h(s)ds$ and $\int_\tau^t Ah(s)ds$ exist. Then, $\int_\tau^t h(s)ds \in D(A)$ and*

$$A\int_\tau^t h(s)ds = \int_\tau^t Ah(s)ds.$$

At this point, it is important to distinguish between existence of $A\int_\tau^t h(s)ds$ and existence of $\int_\tau^t Ah(s)ds$. The first can exist while the second does not. In other words, if the first term $A\int h$ exists, it does not mean that we can switch the operator with the integral, since $\int Ah$ might not exist.

At some points it will be necessary to differentiate under the integral sign and as a consequence of working directly with the differential quotient, we have the following lemma.

Lemma 2.5. *Let $f:[a,b]\times[a,b]\to X$ and $a\leq\theta_1(t)<\theta_2(t)\leq b$. Suppose that f is continuously differentiable in $[\theta_1(t),\theta_2(t)]$ for the first variable and that $\int_{\theta_1(t)}^{\theta_2(t)}\|f_t(t,\xi)\|_X d\xi$ exists. Then, we have*

$$\frac{d}{dt}\int_{\theta_1(t)}^{\theta_2(t)} f(t,\xi)d\xi=\theta_2'(t)f(t,\theta_2(t))-\theta_1'(t)f(t,\theta_1(t))+\int_{\theta_1(t)}^{\theta_2(t)} f_t(t,\xi)d\xi.$$

Remark 2.6. *If $\theta_2(t)$ is time independent, then f only needs to be differentiable in $(\theta_1,\theta_2]$. The same can be said if θ_1 is time independent or even both.*

A last result concerning estimates of integrals is a generalized version of Gronwall inequality.

Lemma 2.7. [9, p.190] *If a,b are positive real constants, $0<\alpha,\beta,\gamma$ satisfying $\beta+\gamma-1>0$ and $\alpha+\gamma-1>0$, and*

$$u(t)\leq a(t-\tau)^{\alpha-1}+b\int_\tau^t(t-s)^{\beta-1}(s-\tau)^{\gamma-1}u(s)ds,\quad t\in(\tau,T),$$

then

$$u(t)\leq a(t-\tau)^{\alpha-1}C(\beta,\alpha+\gamma-1,\beta+\gamma-1).$$

2.2. **Semigroups of growth** $1-\alpha$**.** As mentioned before, there is a connection between $\alpha-$almost sectorial operators and semigroups of growth $1-\alpha$. The resolvent estimate (1.2) for $-A$ implies that, for any $t>0$, the integral

$$T_{-A}(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda+A)^{-1}d\lambda, \tag{2.1}$$

converges, where Γ is the contour of Σ_φ, that is, $\Gamma=\{re^{-i\varphi}:r>0\}\cup\{re^{i\varphi}:r>0\}$, orientated with increasing imaginary part. This was proved in [3] alongside with several other properties that we enumerate in the sequence. As we will see, those properties allow us to conclude that $T_{-A}(t)$ given as (2.1) is a semigroup of growth $1-\alpha$.

Proposition 2.8. *Let A be an $\alpha-$almost sectorial operator and $T_{-A}(t), t>0$, the family defined in* (2.1)*, then:*

(1) The resolvent of $-A$ satisfies

$$\|A(\lambda+A)^{-1}\|_{\mathcal{L}(X)}\leq 1+C|\lambda|^{1-\alpha},\quad \forall\lambda\in\Sigma_\varphi\cup\{0\}. \tag{2.2}$$

(2) There exists $C>0$ such that

$$\|T_{-A}(t)\|_{\mathcal{L}(X)}\leq Ct^{\alpha-1},\quad \forall t>0. \tag{2.3}$$

(3) $T_{-A}(t):X\to D(A)$ and $AT_{-A}(t)$ is a bounded linear operator satisfying

$$\|AT_{-A}(t)\|_{\mathcal{L}(X)}\leq Ct^{\alpha-2},\quad \forall t>0. \tag{2.4}$$

(4) There exists $\xi > 0$ such that $T_{-A}(t)$ has an exponential decay

$$\|T_{-A}(t)\|_{\mathcal{L}(X)} \leq Ct^{\alpha-1}e^{-\xi t}, \quad \forall t > 0. \tag{2.5}$$

Proof. We only prove (2), the other results can be found in [3, Section 2]. Inequality (2.3) follows from the resolvent estimate (1.2). Indeed, parameterizing the part of Γ with positive imaginary part by $\lambda(r) = re^{i\varphi}$, $r \in [0, \infty)$ and φ is a fixed number in the interval $(\frac{\pi}{2}, \pi)$, and doing the analogous for the negative imaginary part, we have

$$\|T_{-A}(t)\|_{\mathcal{L}(X)} \leq \frac{1}{2\pi}2\int_0^\infty e^{r\cos(\varphi)t}\frac{C}{r^\alpha}dr \leq t^{\alpha-1}\frac{1}{\pi}\int_0^\infty e^{\cos(\varphi)u}\frac{C}{u^\alpha}du = Ct^{\alpha-1}.$$

□

Semigroups of growth play an important role in solving autonomous and homogeneous linear equations. The next result states that $u(t) = T_{-A}(t)u_0$ is a strong solution of the autonomous problem

$$u_t + Au = 0, \quad t > 0; \quad u(0) = u_0 \in X, \tag{2.6}$$

which allows us to conclude, from the uniqueness of the solution, that $T_{-A}(t)T_{-A}(s) = T_{-A}(t+s)$ for any $t, s > 0$. Since conditions (3) and (4) of Definition 1.1 are readily verified, we obtain that $T_{-A}(t)$ is a semigroup of growth $1 - \alpha$.

Lemma 2.9. *(*[3, Lemma 2.1 and Lemma 2.4]*) Let $T_{-A}(t)$ be the linear operator defined in* (2.1)*. The mapping $T_{-A}(t) : (0, \infty) \to \mathcal{L}(X)$ is differentiable and*

$$\frac{d}{dt}T_{-A}(t) = -AT_{-A}(t) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}(\lambda + A)^{-1}d\lambda.$$

That is, for $u_0 \in X$, $\frac{d}{dt}T_{-A}(t)u_0 + AT_{-A}(t)u_0 = 0$, for all $t > 0$, and $u(t) = T_{-A}(t)u_0$ is a strong solution of (2.6)*.*

Semigroups of growth $1 - \alpha$ are not necessarily continuous at $t = 0$. However, for elements $x \in D(A)$, the continuity of $T_{-A}(t)x$ at $t = 0$ holds, as we see next. Moreover, if $x \in D(A)^2$, then $-A$ satisfies a property (item (3) below) that resembles the definition of infinitesimal generator for C_0-semigroups.

Lemma 2.10. *Let $T_{-A}(t)$, $t > 0$, be the semigroup of growth $1 - \alpha$ obtained by $-A$.*

(1) If $x \in D(A)$, then $\|T_{-A}(t)x - x\|_X \to 0$ when $t \to 0^+$.
(2) If $x \in D(A)$, then $AT_{-A}(t)x = T_{-A}(t)Ax$.
(3) If $x \in D(A^2)$, then $\lim_{t\to 0^+} \frac{T_{-A}(t)x - x}{t} = -Ax$.
(4) If $x \in D(A^2)$, then $T_{-A}(t)x$ is continuously differentiable in $[0, \infty)$ (including $t = 0$) and

$$\frac{d}{dt}T_{-A}(t)x = \begin{cases} -AT_{-A}(t)x, & \text{if } t > 0, \\ -Ax, & \text{if } t = 0. \end{cases}$$

(5) Given any $x \in X$ and $0 < s_1 < s_2$, $T_{-A}(s_2)x - T_{-A}(s_1)x = -\int_{s_1}^{s_2} AT_{-A}(s)xds$. If $s_1 = 0$, then equality holds only for $x \in D(A^2)$.

Proof. First statement is proved in [3, Proposition 2.6]. For the second one, if $x \in D(A)$, it follows from the closedness of A that

$$AT_{-A}(t)x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}A(\lambda + A)^{-1}x d\lambda = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A)^{-1}Ax d\lambda = T_{-A}(t)Ax.$$

The proof of item (3) is given in [3, Proposition 2.7]. We then use this information to prove the fourth statement. If $x \in D(A^2)$,

$$\frac{d}{dt}T_{-A}(t)x = \begin{cases} -AT_{-A}(t)x, & t > 0, \\ -Ax, & t = 0. \end{cases}$$

The continuity for $t > 0$ is already known. To prove the continuity at $t = 0$, we note that

$$\frac{d}{dt}T_{-A}(t)x = -AT_{-A}(t)x = -T_{-A}(t)Ax \overset{t\to 0^+}{\longrightarrow} -Ax,$$

since $x \in D(A^2)$ and $Ax \in D(A)$.

Last statement follows from the fact that $(0,\infty) \ni t \mapsto \frac{d}{dt}T_{-A}(t)x = -AT_{-A}(t)x$ is continuous. If $x \in D(A^2)$, then this map is continuous including at $t = 0$. □

Another important feature for $T_{-A}(t)$ is a certain type of Hölder continuity it possesses when we consider $h \mapsto T_{-A}(t+h)$, for $t > 0$.

Lemma 2.11. *Let $T_{-A}(t)$, $t > 0$, be the semigroup of growth $1-\alpha$ generated by $-A$. Given any $0 < \mu < \alpha^2$, for $t, h > 0$, we have*

$$\|T_{-A}(t+h) - T_{-A}(t)\|_{\mathcal{L}(X)} \le Ch^\mu t^{\alpha-1-\frac{\mu}{\alpha}},$$

and $\alpha - 1 - \frac{\mu}{\alpha} \in (-1, 0)$.

Proof. Note that

$$\begin{aligned}\|T_{-A}(t+h)x - T_{-A}(t)x\|_X &= \|T_{-A}(h)T_{-A}(t)x - T_{-A}(t)x\|_X \\ &= \left\|\int_0^h \frac{d}{d\xi}T_{-A}(\xi)T_{-A}(t)x d\xi\right\|_X \le \left(\int_0^h \|T_{-A}(\xi)\|_{\mathcal{L}(X)}d\xi\right)\|AT_{-A}(t)x\|_X \\ &\le C\left(\int_0^h \xi^{\alpha-1}d\xi\right)t^{\alpha-2}\|x\|_X = Ch^\alpha t^{\alpha-2}\|x\|_X.\end{aligned}$$

A positive exponent for h appeared, but at the downside t has a power in the negative interval $(-2,-1)$, which is not suitable when convergence of integrals is being considered. However, we already know that $\|T_{-A}(t+h) - T_{-A}(t)\|_{\mathcal{L}(X)} \le (t+h)^{\alpha-1} + t^{\alpha-1} \le Ct^{\alpha-1}$.

In order to improve the estimate, we consider the ones we already have and we interpolate them with coefficients $\frac{\mu}{\alpha}$ and $(1-\frac{\mu}{\alpha})$, $0 \le \mu \le \alpha$, resulting in

$$\|T_{-A}(t+h) - T_{-A}(t)\|_{\mathcal{L}(X)} \le Ch^\mu t^{\alpha-1-\frac{\mu}{\alpha}}.$$

The exponent of t will be in the interval $(-1, 0)$ provided that $0 < \mu < \alpha^2$. □

2.3. **Fundamental Theorem of Calculus for semigroups of growth $1-\alpha$.** If $T(t)$ is a C_0-semigroup with infinitesimal generator $-A$, an important feature of $T(t)$ is the fact that given any $x \in X$, $\int_0^t T(s)xds \in D(A)$ and

$$A\left(\int_0^t T(s)xds\right) = x - T(t)x,$$

which follows from the strong continuity of the C_0-semigroup [13, Theorem 1.2.4]. Since $-AT(t)x = \frac{d}{dt}T(t)x$, we refer to the integral above as *fundamental theorem of calculus for semigroups*. Note that the existence of the integral above does not guarantee that we can place the linear operator A inside the integral. In other words, $\int_0^t AT(t)xds$ might not exist.

In order to treat the mild solution (1.3) obtained for the semilinear problem, a similar fundamental theorem of calculus must be obtained for the semigroups of growth $1-\alpha$. The next result proves that $\int_\tau^t T_{-A}(t-s)xds \in D(A)$, for any $x \in X$, when A is almost sectorial, and a characterization of $A\left(\int_\tau^t T_{-A}(t-s)xds\right)$is obtained.

Lemma 2.12. *Let $D(A^2)$ be the domain of A^2 and assume it is a dense subset of X. Consider the linear operator $F(t,\tau) : D(A^2) \to X$, $t > \tau$, given by*

$$F(t,\tau)w = A\int_\tau^t T_{-A}(t-s)wds.$$

Then $F(t,\tau)$ is a well defined operator, it is bounded in $D(A^2)$, it satisfies

$$\|F(t,\tau)w\|_X \leq C(t-\tau)^{\alpha-1}\|w\|_X, \quad \forall w \in D(A^2),$$

and admits a bounded extension to X.

Proof. If $w \in D(A^2)$, then it follows from the results on Lemmas 2.9 and 2.10 that the following integral converges:

$$\left\|\int_\tau^t AT_{-A}(t-s)wds\right\|_X = \left\|\int_\tau^t T_{-A}(t-s)Awds\right\|_X \leq C\int_\tau^t (t-s)^{\alpha-1}ds\|Aw\|_X < \infty.$$

The existence of $\int_\tau^t AT_{-A}(t-s)wds$ and Corollary 2.4 imply that

$$F(t,\tau)w = A\int_\tau^t T_{-A}(t-s)wds = \int_\tau^t AT_{-A}(t-s)wds.$$

We prove in the sequence that there exists a constant $C > 0$ such that, for all $w \in D(A^2)$, $\|F(t,\tau)\|_X \leq C(t-\tau)^{\alpha-1}\|w\|_X$. In Proposition 2.10 it was shown that for any $w \in D(A^2)$, the function $t \mapsto T_{-A}(t)w$ is continuously differentiable in $[0,\infty)$ and

$$A\int_\tau^t T_{-A}(t-s)wds = \int_\tau^t AT_{-A}(t-s)wds = \int_\tau^t \frac{d}{ds}\left[T_{-A}(t-s)w\right]ds = w - T_{-A}(t-\tau)w.$$

In this case,

$$\begin{aligned}\|F(t,\tau)w\|_X &\leq \left\|A\int_\tau^t T_{-A}(t-s)wds\right\|_X \leq \|w - T_{-A}(t-\tau)w\|_X \\ &\leq C\left(1 + (t-\tau)^{\alpha-1}\right)\|w\|_X \leq C(t-\tau)^{\alpha-1}\|w\|_X\end{aligned}$$

and the linear operator $F(t,\tau)$ admits a bounded extension to X, which we denote the same. $\square$

The fact that $F(t,\tau)$ is bounded and admits an extension to X allows us to prove the following lemma.

Lemma 2.13. *Let $w \in X$ and A an $\alpha-$almost sectorial operator. Then $\int_\tau^t T_{-A}(t-s)wds$ belongs to $D(A)$ and*

$$A\left(\int_\tau^t T_{-A}(t-s)wds\right) = w - T_{-A}(t-\tau)w.$$

Furthermore, $\left\|A\left(\int_\tau^t T_{-A}(t-s)ds\right)\right\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-1}$.

Proof. Let (w_n) be a sequence in $D(A^2)$ such that $w_n \stackrel{n\to\infty}{\longrightarrow} w$. Since $\int_\tau^t T_{-A}(t-s)ds$ is a bounded linear operator in X,

$$\left\|\int_\tau^t T_{-A}(t-s)ds\right\|_{\mathcal{L}(X)} \leq \int_\tau^t \|T_{-A}(t-s)ds\|_{\mathcal{L}(X)} \leq C\int_\tau^t (t-s)^{\alpha-1}ds = C(t-\tau)^\alpha,$$

it follows that $\int_\tau^t T_{-A}(t-s)w_nds \stackrel{n\to\infty}{\longrightarrow} \int_\tau^t T_{-A}(t-s)wds$. The extension $F(t,\tau)$ is also a bounded linear operator and

$$A\int_\tau^t T_{-A}(t-s)w_nds = F(t,\tau)w_n \stackrel{n\to\infty}{\longrightarrow} F(t,\tau)w.$$

From closedness of A, we conclude that $\int_\tau^t T_{-A}(t-s)wds \in D(A)$ and

$$\begin{aligned} A\int_\tau^t T_{-A}(t-s)wds = F(t,\tau)w = \lim_{n\to\infty} F(t,\tau)w_n &= \lim_{n\to\infty}\left\{A\int_\tau^t T_{-A}(t-s)w_nds\right\} \\ = \lim_{n\to\infty}\{w_n - T_{-A}(t)w_n\} &= w - T_{-A}(t-\tau)w. \end{aligned}$$

□

Remark 2.14. *Note that, even though $\int_\tau^t T_{-A}(t-s)wds \in D(A)$ for any $w \in X$, it does not mean that $\int_\tau^t AT_{-A}(t-s)wds$ is defined. The second integral might not exist. We can only prove that $A\left(\int_\tau^t T_{-A}(t-s)wds\right)$, with the operator outside the integral, exists.*

Moreover, when $t \to \tau^+$, we have $\int_\tau^t T_{-A}(t-s)wds \stackrel{t\to\tau^+}{\longrightarrow} 0$, whereas $A\left(\int_\tau^t T_{-A}(t-s)wds\right)$ does not necessarily vanishes. Indeed, $A\left(\int_\tau^t T_{-A}(t-s)wds\right) = w - T_{-A}(t-\tau)w$ has the same discontinuity that the semigroup $T_{-A}(t-\tau)w$ has at $t=\tau$.

3. Strong solution for the linear nonautonomous equation

In this section we study existence of strong solution for the linear problem

$$u_t + Au = g(t), \quad \tau < t < \tau + T; \quad u(\tau) = u_0 \in X, \tag{3.1}$$

when the operator A is almost sectorial. This linear formulation will be used in the sequence to study the semilinear problem (1.1).

If $g \in L^1((\tau,\tau+T),X)$, then the function $u : (\tau,\tau+T] \to X$, given by

$$u(t) = T_{-A}(t-\tau)u_0 + \int_\tau^t T_{-A}(t-s)g(s)ds \tag{3.2}$$

is well defined and it is a *mild solution* of (3.1). If we impose further conditions on g, we can prove that this mild solution is actually a strong solution for the equation. We enunciate the theorem concerning strong solutions for the problem and we prove it throughout the section.

Theorem 3.1. *Let A, be an $\alpha-$almost sectorial operator in X with $\alpha \in (0,1)$. Also, assume $g : (\tau, \tau+T] \to X$ is a continuous function that satisfies*

$$\|g(t)-g(s)\|_X \leq C(t-s)^{\theta}(s-\tau)^{-\psi}, \quad \text{for any } \tau < s < t, \tag{3.3}$$

$$\|g(t)\|_X \leq C(t-\tau)^{-\psi}, \quad \text{for any } \tau < t, \tag{3.4}$$

where θ and ψ are positive constants satisfying $\theta > 1-\alpha$, $0 < \psi < 1$.

Then, for each $u_0 \in X$, the mild solution (3.2) is a strong solution for (3.1), that is,

(1) $u(\cdot) \in \mathcal{C}^1((\tau, \tau+T], X)$, $u(\tau) = u_0$ and $u(t) \in D(A)$, for all $\tau < t < \tau+T$.

(2) The equality $u_t(t) = -Au(t) + g(t)$, $\tau < t < \tau + T$, is satisfied in X

and the following expression for the derivative of $u(\cdot)$ holds

$$u_t(t) = -AT_{-A}(t-\tau)u_0 - A\int_{\tau}^{t} T_{-A}(t-s)[g(s)-g(t)]ds - A\int_{\tau}^{t} T_{-A}(t-s)g(t)ds + g(t).$$

Moreover, if $u_0 \in D(A)$, then $u(\cdot)$ is continuous at $t = \tau$, that is, $u(\cdot) \in \mathcal{C}([\tau, \tau+T], X) \cap \mathcal{C}^1((\tau, \tau+T], X)$.

If we tried to evaluate the derivative directly in the expression (3.2), the first term would not pose any problem, that is, $\frac{d}{dt}T_{-A}(t-\tau)u_0 = -AT_{-A}(t-\tau)u_0$. However, the expression given by the integral would be troublesome, since the expected value inside the integral is $-AT_{-A}(t-s)g(s)$ and we cannot prove convergence of the integral with such integrand (recall that $\|AT_{-A}(t-\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-2}$). We denote this term as $v(t)$, that is,

$$v(t) = \int_{\tau}^{t} T_{-A}(t-s)g(s)ds.$$

To overcome the problem mentioned above, we will consider (inspired in [9, Section 3.2]), for small $\rho > 0$, the approximations

$$[\tau+\gamma, t_0] \ni t \mapsto v_\rho(t) = \int_{\tau}^{t-\rho} T_{-A}(t-s)g(s)ds,$$

where $\gamma > 0$ is arbitrary, $t_0 \in (\tau+\gamma, \tau+T]$ and ρ is small enough such that $t-\rho > \tau+\gamma$. With this slight retreat in the domain of integration, we can prove the following result for this function.

Lemma 3.2. *Under the conditions of Theorem 3.1, the function $v_\rho : [\tau+\gamma, t_0] \to X$ is continuously differentiable in X and*

$$v_\rho'(t) = T_{-A}(\rho)g(t-\rho) - A\int_{\tau}^{t-\rho} T_A(t-s)g(s)ds. \tag{3.5}$$

Proof. This follows readily from the fact that the integrand is continuously differentiable in $(\tau, t-\rho]$ (since we are avoiding the discontinuity of the semigroup when $s = t$) and an application of Lemma 2.5. $\square$

Once we know v_ρ is differentiable, we prove:

(1) $v_\rho(\cdot)$ converges as $\rho \to 0$ to $v(\cdot)$ in $\mathcal{C}([\tau+\gamma, t_0], X)$.

(2) $v_\rho'(\cdot)$ converges as $\rho \to 0$ to $-Av(\cdot) + g(\cdot)$ in $\mathcal{C}([\tau+\gamma, t_0], X)$.

Then, differentiability of $t \mapsto v(t)$ for $t \in [\tau+\gamma, t_0]$ follows from $\mathcal{C}^1([\tau+\gamma,t_0],X)$ being a complete metric space. Moreover, $v'(t) = -Av(t) + g(t)$. From the arbitrariness of $\gamma > 0$ and t_0, we have differentiability in $(\tau, \tau+T)$.

After these two steps, Theorem 3.1 will be proved, since $u(t) = T_{-A}(t-\tau)u_0 + v(t)$ and

$$u'(t) = -AT_{-A}(t-\tau)u_0 + v'(t) = -AT_{-A}(t-\tau)u_0 - Av(t) + g(t) = -Au(t) + g(t).$$

Item (1) is easily obtained: for each $t \in [\tau+\gamma, t_0]$ we have

$$\begin{aligned}\|v_\rho(t) - v(t)\|_X &= \left\| \int_{t-\rho}^{t} T_{-A}(t-s)g(s)ds \right\|_X \leq \int_{t-\rho}^{t} C(t-s)^{\alpha-1}(s-\tau)^{-\psi}ds \\ &\leq C(t-\rho-\tau)^{-\psi}\rho^\alpha \xrightarrow{\rho\to 0} 0.\end{aligned}$$

Item (2), on the other hand, demands more attention. We first prove that $v(t) \in D(A)$.

Lemma 3.3. *Under the conditions of Theorem 3.1, for any $t \in [\tau+\gamma, t_0]$, $v(t) \in D(A)$ and*

$$-Av(t) = -A\int_\tau^t T_{-A}(t-s)[g(s)-g(t)]ds - A\int_\tau^t T_{-A}(t-s)g(t)ds.$$

Proof. It follows from Lemma 2.13 that $\int_\tau^t T_{-A}(t-s)g(t)ds \in D(A)$. Furthermore, from (3.3) with $\theta > 1-\alpha$, we conclude that $\int_\tau^t AT_{-A}(t-s)[g(s)-g(t)]ds$ converges. Indeed,

$$\begin{aligned}\left\| \int_\tau^t AT_{-A}(t-s)[g(s)-g(t)]ds \right\|_X &\leq C\int_\tau^t (t-s)^{\alpha-2}(t-s)^\theta(s-\tau)^{-\psi}ds \\ &\leq C(t-\tau)^{(\alpha+\theta-1)-\psi} < \infty,\end{aligned} \tag{3.6}$$

where we used Lemma 2.1 and the fact that $\alpha+\theta-1 > 0$.

Therefore, $\int_\tau^t T_{-A}(t-s)[g(s)-g(t)]ds \in D(A)$ and from Corollary 2.4, we obtain

$$A\int_\tau^t T_{-A}(t-s)[g(s)-g(t)]ds = \int_\tau^t AT_{-A}(t-s)[g(s)-g(t)]ds.$$

□

In order to prove item (2), we must check that $v'_\rho(\cdot)$ given by (3.5) converges to $-Av(\cdot)+g(\cdot)$ which is also given by:

$$-Av(t) + g(t) = g(t) - A\int_\tau^t T_{-A}(t-s)[g(s)-g(t)]ds - A\int_\tau^t T_{-A}(t-s)g(t)ds. \tag{3.7}$$

We rearrange (3.5) in a similar way of (3.7), that is,

$$v'_\rho(t) = T_{-A}(\rho)g(t-\rho) - A\int_\tau^{t-\rho} T_{-A}(t-s)[g(s)-g(t)]ds - A\int_\tau^{t-\rho} T_{-A}(t-s)g(t)ds. \tag{3.8}$$

The second term of (3.8) converges as $\rho \to 0^+$ and it satisfies:

Lemma 3.4. *If $g : (\tau, \tau+T] \to X$ satisfies (3.3) with $\theta > 1-\alpha$, then*

$$A\int_\tau^{t-\rho} T_{-A}(t-s)[g(s)-g(t)]ds \xrightarrow{\rho\to 0^+} A\int_\tau^t T_{-A}(t-s)[g(s)-g(t)]ds.$$

Proof. This follows readily from the existence of $\int_\tau^t AT_{-A}(t-s)[g(s)-g(t)]ds$ proved in Lemma 3.3, inequality (3.6). □

For the other terms in (3.8), note that the discontinuity of the semigroup at the initial time allow situations in which

$$T_{-A}(\rho)g(t-\rho) \nrightarrow g(t) \quad \text{and} \quad T_{-A}(t-s)g(t)ds \nrightarrow A\int_\tau^t T_{-A}(t-s)g(t)ds,$$

as $\rho \to 0^+$. However, given any $0 < \rho < t-\tau$, we can rewrite $A\int_\tau^t T_{-A}(t-s)g(t)ds$ using Lemma 2.13 as

$$\begin{aligned} A\int_\tau^t T_{-A}(t-s)g(t)ds &= A\int_\tau^{t-\rho} T_{-A}(t-s)g(t)ds + A\int_{t-\rho}^t T_{-A}(t-s)g(t)ds \\ &= A\int_\tau^{t-\rho} T_{-A}(t-s)g(t)ds + g(t) - T_{-A}(\rho)g(t). \end{aligned}$$

From this, we can rewrite (3.8) to obtain

$$\begin{aligned} v_\rho'(t) &= T_{-A}(\rho)g(t-\rho) - A\int_\tau^{t-\rho} T_{-A}(t-s)[g(s)-g(t)]ds - A\int_\tau^{t-\rho} T_{-A}(t-s)g(t)ds \\ &= T_{-A}(\rho)g(t-\rho) - A\int_\tau^{t-\rho} T_{-A}(t-s)[g(s)-g(t)]ds \\ &\quad - \left[A\int_\tau^t T_{-A}(t-s)g(t)ds - g(t) + T_{-A}(\rho)g(t)\right] \\ &= g(t) - A\int_\tau^{t-\rho} T_{-A}(t-s)[g(s)-g(t)]ds - A\int_\tau^t T_{-A}(t-s)g(t)ds \\ &\quad + T_{-A}(\rho)[g(t-\rho)-g(t)]. \end{aligned}$$

First line in the last equality converges to $g(t) - Av(t)$ due to Lemma 3.4, uniformly for $t \in [\tau+\gamma, t_0]$, whereas the term in the second line of last equality vanishes as $\rho \to 0$. Indeed,

$$\|T_{-A}(\rho)[g(t-\rho)-g(t)]\|_X \leq C\rho^{\alpha-1}\rho^\theta = C\rho^{\alpha+\theta-1} \overset{\rho\to 0^+}{\longrightarrow} 0.$$

Consequently, we conclude that

$$\sup_{t\in[\tau+\gamma,T]} \left\|v_\rho'(t) - [g(t) - Av(t)]\right\|_X = \sup_{t\in[\tau+\gamma,T]} \left\|v_\rho'(t) - \left[g(t) - A\int_\tau^t T_{-A}(t-s)g(s)ds\right]\right\|_X \overset{\rho\to 0^+}{\longrightarrow} 0,$$

and Theorem 3.1 is proved.

4. Strong solution for the semilinear equation

Consider the semilinear problem

$$u_t + Au = f(t,u),\ t > \tau; \quad u(\tau) = u_0 \in X,$$

where A is an $\alpha-$almost sectorial operator with $\alpha \in (0,1)$. We assume that the nonlinearity $f(t,\cdot)$ satisfies the following conditions:

(G) There exists a Banach space Y in which X is embedded ($X \hookrightarrow Y$) and constants $C > 0$, $\rho \geq 1$, such that $f : \mathbb{R} \times X \to Y$ and, for every $u, v \in X$,

$$\|f(t,u) - f(t,v)\|_Y \leq C\|u-v\|_X \left(1 + \|u\|_X^{\rho-1} + \|v\|_X^{\rho-1}\right) \tag{4.1}$$

$$\|f(t,u)\|_Y \leq C(1 + \|u\|_X^\rho). \tag{4.2}$$

Assume also that f is Hölder continuous in time-variable, that is, there exists $C > 0$ and $\xi \in (0,1]$ such that, for all $u \in X$,

$$\|f(t,u) - f(s,u)\|_Y \leq C|t-s|^{\xi}.$$

We will refer to property (G) as the nonlinearity f having a *polynomial growth of order* ρ. A similar condition was required in [3, Section 2.2] where the authors also considered semilinear problems with almost sectorial operator. This assumption (G) allows us to understand the effects of the nonlinearity f on the elements of a Banach space X. For instance, a situation like above happens when $X = L^p(\Omega)$ and $f(u) = |u|^2$. In this case, f take elements of $L^p(\Omega)$ to the less regular space and $Y = L^{\frac{p}{2}}(\Omega)$ (Ω a bounded domain in $\mathbb{R}^N$).

We will also assume that the operator A and the space Y are related in the following sense:

(C) The operator A can be defined in Y. To be more precise, there exists $A^Y : D(A^Y) \subset Y \to Y$, such that A is the realization of A^Y in X. We assume that A^Y is almost sectorial in the Banach space Y, possibly with different exponent of almost sectoriality, $\omega \in (0,1)$. Moreover, the resolvent of A satisfies: $(\lambda + A)^{-1} : Y \to X$ (which means that $D(A^Y) \hookrightarrow X$) and there exists $\beta \in (0,1)$ such that

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(Y,X)} \leq \frac{C}{|\lambda|^{\beta}+1}, \qquad \forall \lambda \in \Sigma_{\varphi} \cup \{0\}. \tag{4.3}$$

We will refer to the property above as the *compatibility between* A *and* Y.

Remark 4.1. *To be rigorous, inequality* (4.3) *should be posed as* $\|(\lambda + A^Y)^{-1}\|_{\mathcal{L}(Y,X)} \leq \frac{C}{|\lambda|^{\beta}+1}$*, in view that* A *acts only on* X*. However, since* A *is assumed to be the realization of* A^Y *in* X*, we denote both by* A*.*

Therefore, we refer to A *acting on* X *or* A *acting on* Y *and the difference between them is expressed by the resolvent estimates:*

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{|\lambda|^{\alpha}+1}, \ \|(\lambda + A)^{-1}\|_{\mathcal{L}(Y)} \leq \frac{C}{|\lambda|^{\omega}+1} \text{ and } \|(\lambda + A)^{-1}\|_{\mathcal{L}(Y,X)} \leq \frac{C}{|\lambda|^{\beta}+1}.$$

This convention extends to the semigroup generated by $-A$*, that is, we say* $T_{-A}(t)$ *is an element of* $\mathcal{L}(X)$*,* $\mathcal{L}(Y)$ *or* $\mathcal{L}(Y,X)$*.*

To help fix the ideas above, we illustrate (C) in a simple case (a sectorial case).

Example 4.2. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded smooth domain and* $A = A_p = -\Delta_p^{\mathcal{N}} : D(\Delta_p^{\mathcal{N}}) \subset L^p(\Omega) \to L^p(\Omega)$*, where* $\Delta_p^{\mathcal{N}}$ *is the Laplacian acting in* $L^p(\Omega)$ *with Neumman boundary condition. The domain of* A_p *is* $D(\Delta_p^{\mathcal{N}}) = W_{\mathcal{N}}^{2,p} := \{u \in W^{2,p}(\Omega); \partial_n u = 0\}$ *and this operator is known to be sectorial* ($\alpha = 1$)*.*

Let $X = L^p(\Omega)$ *and suppose* $\frac{N}{2} < q \leq p$*. If we denote* $Y = L^q(\Omega)$*, then we can consider the Laplacian acting now on* Y*, that is,* $A_q : W_{\mathcal{N}}^{2,q} \subset L^q(\Omega) \to L^q(\Omega)$*. Furthermore,* $X \hookrightarrow Y$ *and* $W_{\mathcal{N}}^{2,q} \hookrightarrow L^{\infty}(\Omega) \hookrightarrow X$*, since* $q > \frac{N}{2}$*. As in Remark 4.1, we denote both* A_p *and* A_q *by* A *and* $(\lambda + A)^{-1}$ *can be seen as an operator in* $\mathcal{L}(X)$*,* $\mathcal{L}(Y)$ *or* $\mathcal{L}(Y,X)$

$$\begin{tikzcd}
W^{2,q}_{\mathcal{N}} \arrow[dr, hook] & W^{2,p}_{\mathcal{N}} \\
Y = L^q(\Omega) \arrow[u, dashed, "(\lambda+A)^{-1}"] & X = L^p(\Omega) \arrow[l, hook'] \arrow[u, dashed, "(\lambda+A)^{-1}"']
\end{tikzcd}$$

FIGURE 1. Diagram of embeds
$\hookrightarrow$: *embed* $\dashrightarrow$: *action of the operator*

Remark 4.3. *It is worth mentioning why we chose not to use the regular approach for parabolic problems (as in* [2, 5, 6, 9]*) in which one works with a scale of fractional power spaces* $\{X^\xi\}_{0\leq\xi\leq 1+\varepsilon}$ *and the growth of* f *is given as* $f:\mathbb{R}\times X^\gamma\to X^\theta$, $0\leq\gamma-\theta<1$.

In this fractional power approach, a good knowledge of the spaces X^ξ *as well as their embed in the* L^p-*spaces are required. To the application we have in mind, this characterization of the spaces* $X^\xi=D(A^\xi)$ *is not available, preventing us from following this way.*

On the other hand, working with the spaces X *and* Y *as in (G) and (C) allows us to treat cases where a different scale of Banach space is available, rather than the fractional power scale. We discuss this with more details in Section 6.*

Inequality (4.3) can be read as the ability of the operator $(\lambda+A)^{-1}$ to regularize elements of Y back to the space X. This regularization is transferred to the semigroup $T_{-A}(t)$. We have the following properties:

Lemma 4.4. *Let* X *and* Y *be Banach spaces as in (G) and assume* (4.3) *holds for some* $\beta\in(0,1)$. *Then, if* $T_{-A}(t)$ *is the semigroup generated by* $-A$, *we have*

$$\|T_{-A}(t)\|_{\mathcal{L}(Y,X)}\leq Ct^{\beta-1}\quad \text{and}\quad \|T_{-A}(t)\|_{\mathcal{L}(Y)}\leq Ct^{\omega-1}.$$

Proof. Both estimates are obtained in the same way as in (2.3), when we considered $\|(\lambda+A)^{-1}\|_{\mathcal{L}(X)}\leq\frac{C}{1+|\lambda|^\alpha}$. Here, we replace this resolvent estimate for one of the estimates given in Remark 4.1, according to the space we are analyzing. $\square$

4.1. **Existence of local mild solution.** We briefly comment the ideas developed in [3] that ensures the existence of local mild solution, highlighting the points that will be useful to the subsequent analysis.

Given $\tau\in\mathbb{R}$ and $u_0\in X$, the authors in [3] searched for a mild solutions in the following space

$$K(t_0,u_0)=\left\{v\in C((\tau,\tau+t_0],X);\ \sup_{t\in(\tau,\tau+t_0]}\|v(t)-T_{-A}(t-\tau)u_0\|_X\leq\mu\right\},$$

where μ is a positive constant and $t_0>0$ a suitably chosen constant. $K(t_0,u_0)$ is a Banach space with norm $\|\xi\|_K=\sup_{t\in(\tau,\tau+t_0]}\|\xi(t)-T_{-A}(t-\tau)u_0\|_X$.

Note that the condition $\|v(t)-T_{-A}(t-\tau)u_0\|_X\leq\mu$ for $t\in(\tau,\tau+t_0]$ means that we expect to find mild solution that replies the same type of discontinuity that the semigroup $T_{-A}(t-\tau)$

possesses as $t \to \tau^+$. Moreover, the functions $v(\cdot) \in K(t_0, u_0)$ satisfy

$$\begin{aligned}(t-\tau)^{1-\alpha}\|v(t)\|_X &\le (t-\tau)^{1-\alpha}\|v(t) - T_{-A}(t-\tau)u_0\|_X + (t-\tau)^{1-\alpha}\|T_{-A}(t-\tau)u_0\|_X \\ &\le t_0^{1-\alpha}\mu + C\|u_0\|_X,\end{aligned}$$

and if u_0 is in a bounded set $B \subset X$, we can obtain uniform estimates for $(t-\tau)^{1-\alpha}\|v(t)\|_X$, that is,

$$(t-\tau)^{1-\alpha}\|v(t)\|_X \le k, \text{ where } k := t_0^{1-\alpha}\mu + C \sup_{u_0 \in B} \|u_0\|_X. \tag{4.4}$$

Remark 4.5. *If $u_0 \in D(A)$, then $\|T_{-A}(t-\tau)u_0 - u_0\|_X \xrightarrow{t \to \tau^+} 0$ as a consequence of Lemma 2.10. In this case,*

$$\|v(t)\|_X \le \|v(t) - T_{-A}(t-\tau)u_0\|_X + \|T_{-A}(t-\tau)u_0\|_X \le \mu + C\|u_0\|_X = k,$$

that is, the functions are bounded for $t \in (\tau, \tau + t_0]$.

Under those conditions, mild solution was obtained as a fixed point of the contraction map (under suitable choice of t_0)

$$(Tv)(t) := T_{-A}(t-\tau)u_0 + \int_\tau^t T_{-A}(t-\tau)f(s, u(s))ds, \quad t \in (\tau, \tau + t_0],$$

defined in $K(t_0, u_0)$.

Theorem 4.6. [3, Proposition 2.11]*Let X, Y be Banach spaces with $X \hookrightarrow Y$. Suppose A is an $\alpha-$almost sectorial operator in X, satisfying the compatibility condition with Y, given in (C), with constant $\beta \in (0, 1)$. If $f : \mathbb{R} \times X \to Y$ is a nonlinearity satisfying (G) such that*

$$1 \le \rho < \frac{\beta}{1-\alpha}, \tag{4.5}$$

then, for every $u_0 \in X$, there exists $t_0 > 0$ such that the initial value problem

$$u_t + Au = f(t, u), t > \tau; \quad u(\tau) = u_0 \in X,$$

has a unique mild solution defined in $(\tau, \tau + t_0]$. This t_0 depends on u_0, but can be chosen uniformly for u_0 in bounded sets of X. Furthermore, we can extend this mild solution to a maximal interval $(\tau, \tau_M(u_0))$.

If the initial condition belongs to $D(A)$ (where the strong continuity of the semigroup takes place), Remark 4.5 implies that condition (4.5) is no longer necessary. In this case, we restate the existence result as:

Corollary 4.7. *Let X, Y be Banach spaces with $X \hookrightarrow Y$. Suppose A is an $\alpha-$almost sectorial operator in X, satisfying the compatibility condition with Y, given in (C), with constant $\beta \in (0, 1)$, and $f : \mathbb{R} \times X \to Y$ is a nonlinearity satisfying (G). Then, for every $u_0 \in D(A)$, there exists $t_0 > 0$ such that the initial value problem*

$$u_t + Au = f(t, u), t > \tau; \quad u(\tau) = u_0 \in D(A),$$

has a unique mild solution defined in $(\tau, \tau + t_0]$, which can be extended to a maximal interval $(\tau, \tau_M(u_0))$.

4.2. **Strong solution for the semilinear equation.** Under the conditions required in Theorem 4.6 for A and f, the semilinear problem has a local solution given by

$$u(t) = T_{-A}(t-\tau)u_0 + \int_\tau^t T_{-A}(t-s)f(t,u(s))ds.$$

If we define $g : (\tau, \tau+T) \to Y$ as $g(t) := f(t,u(t))$, then u also satisfies

$$u_t + Au = g(t), t \in (\tau, \tau+T); \quad u(\tau) = u_0 \in X.$$

By proving that $g(\cdot)$ is continuous and satisfies the conditions of Theorem 3.1

$$\|g(t) - g(s)\|_Y \leq C(t-s)^\theta (s-\tau)^{-\psi}, \qquad \text{for any } \tau < s < t,$$
$$\|g(t)\|_Y \leq C(t-\tau)^{-\psi}, \qquad \text{for any } \tau < t,$$

for some θ and ψ then the results on this theorem can be translated to the semilinear case.

Remark 4.8. *Whereas the mild solution is placed in the phase space X, the equation*

$$u_t + Au = g(t), t \in (\tau, \tau+T),$$

occurs in the less regular space Y, since $f(t,\cdot) : X \to Y$. Therefore, the content of Theorem 3.1 slightly changes. Since the constant of almost sectoriality of A in Y is $\omega \in (0,1)$, in order to apply the results of Theorem 3.1, the positive constants θ and ψ that we search for the estimates of g must satisfy $\theta > 1-\omega$, $0 < \psi < 1$.

In the next lemma we find values of θ and ψ for which the estimates for g hold.

Lemma 4.9. *Let X, Y be Banach spaces with $X \hookrightarrow Y$ and suppose that A is an $\alpha-$almost sectorial operator satisfying condition (C) and f a nonlinearity satisfying (G). If $1 \leq \rho < \frac{\beta}{1-\alpha}$, $u : (\tau, \tau+T) \to X$ is the mild solution of*

$$u_t(t) + Au = f(t,u(t)), t \in (\tau, \tau+T); \quad u(\tau) = u_0 \in X$$

and $g(t) = f(t,u(t))$, then, for any $0 < \mu < \alpha^2$ and

$$1 \leq \rho < \frac{\alpha - \mu}{\alpha(1-\alpha)},$$

the inequality

$$\|g(t+h) - g(t)\|_Y \leq Ch^{\min\{\xi,\mu,1-\rho(1-\alpha)\}}(t-\tau)^{\min\left\{-\frac{\mu}{\alpha}-\rho(1-\alpha),\beta-\alpha-\rho(1-\alpha)\right\}}$$

holds for $t > \tau$ and $h > 0$, where $\min\left\{-\frac{\mu}{\alpha} - \rho(1-\alpha), \beta-\alpha-\rho(1-\alpha)\right\} \in (-1, \infty)$.

Proof. From the growth condition (4.1) on f we obtain

$$\begin{aligned}
\|g(t+h) - g(t)\|_Y &= \|f(t+h, u(t+h)) - f(t,u(t))\|_Y \\
&\leq \|f(t+h,u(t+h)) - f(t,u(t+h))\|_Y + \|f(t,u(t+h)) - f(t,u(t))\|_Y \\
&\leq C|h|^\xi + C\|u(t+h) - u(t)\|_X \left(1 + \|u(t)\|_X^{\rho-1} + \|u(t+h)\|_X^{\rho-1}\right).
\end{aligned}$$

We know from (4.4) that there exists a constant k such that

$$\|u(t+h)\|_X, \|u(t)\|_X \leq k(t-\tau)^{\alpha-1}.$$

Taking this into account and selecting the factor $(t-\tau)$ with the most negative exponent (since we must control the order of discontinuity as $t\to\tau^+$), we obtain

$$\|g(t+h)-g(t)\|_Y \le C[|h|^\xi + \|u(t+h)-u(t)\|_X](t-\tau)^{-(\rho-1)(1-\alpha)},$$

which implies

$$(t-\tau)^{(\rho-1)(1-\alpha)}\|g(t+h)-g(t)\|_Y \le C[|h|^\xi + \|u(t+h)-u(t)\|_X]. \tag{4.6}$$

Let $\Psi(t)=(t-\tau)^{(\rho-1)(1-\alpha)}\|g(t+h)-g(t)\|_Y$. Then (4.6) is rewritten as

$$\Psi(t) \le C[|h|^\xi + \|u(t+h)-u(t)\|_X]. \tag{4.7}$$

We study in the sequence properties of the difference $\|u(t+h)-u(t)\|_X$ in order to obtain the desired result. Note that $\|f(t,u(t))\|_Y$ can be locally estimated:

$$\|f(t,u(t))\|_Y \le C\,(1+\|u(t)\|_X^\rho) \le C(1+(t-\tau)^{-\rho(1-\alpha)}) \le C(t-\tau)^{-\rho(1-\alpha)}. \tag{4.8}$$

From the variation of constant formula, we obtain

$$\begin{aligned} u(t+h)-u(t) &= T_{-A}(t+h-\tau)u_0 - T_{-A}(t-\tau)u_0 \\ &\quad + \left(\int_\tau^{\tau+h} + \int_{\tau+h}^{t+h}\right) T_{-A}(t+h-s)f(s,u(s))ds - \int_\tau^t T_{-A}(t-s)f(s,u(s))ds \\ &= [T_{-A}(t+h-\tau)-T_{-A}(t-\tau)]u_0 + \int_\tau^{\tau+h} T_{-A}(t+h-s)f(s,u(s))ds \\ &\quad + \int_\tau^t T_{-A}(t-s)[f(s+h,u(s+h))-f(s,u(s))]ds. \end{aligned}$$

From Lemma 2.11 and any $0<\mu<\alpha^2$,

$$\|[T_{-A}(t+h-\tau)-T_{-A}(t-\tau)]u_0\|_X \le Ch^\mu (t-\tau)^{\alpha-1-\frac{\mu}{\alpha}}.$$

For the second term we use the estimates obtained in Lemma 4.4 and in (4.8),

$$\begin{aligned} \left\|\int_\tau^{\tau+h} T_{-A}(t+h-s)f(s,u(s))ds\right\|_X &\le \int_\tau^{\tau+h} \|T_{-A}(t+h-s)\|_{\mathcal{L}(Y,X)}\|f(s,u(s))\|_Y \\ &\le C\int_\tau^{\tau+h} (t+h-s)^{\beta-1}(s-\tau)^{-\rho(1-\alpha)}ds \\ &\le Ch^{1-\rho(1-\alpha)}(t-\tau)^{\beta-1}. \end{aligned}$$

Finally, for the last term, note that

$$\begin{aligned} &\left\|\int_\tau^t T_{-A}(t-s)[f(s+h,u(s+h))-f(s,u(s))]ds\right\|_X \\ &\quad = \left\|\int_\tau^t T_{-A}(t-s)(s-\tau)^{-(\rho-1)(1-\alpha)}(s-\tau)^{(\rho-1)(1-\alpha)}[g(s+h)-g(s)]ds\right\|_X \\ &\quad \le C\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds. \end{aligned}$$

We conclude that, for any $0 < \mu < \alpha^2$,

$$\begin{aligned}\|u(t+h)-u(t)\|_X &\leq Ch^{\mu}(t-\tau)^{\alpha-1-\frac{\mu}{\alpha}} + Ch^{1-\rho(1-\alpha)}(t-\tau)^{\beta-1}\\ &\quad + C\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds\\ &\leq Ch^{\min\{\mu,1-\rho(1-\alpha)\}}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\beta-1\right\}}\\ &\quad + C\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds.\end{aligned}$$

Then, from this and (4.7), we have

$$\begin{aligned}\Psi(t) &\leq C\left[h^{\xi} + h^{\min\{\mu,1-\rho(1-\alpha)\}}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\beta-1\right\}} + \int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds\right]\\ &\leq Ch^{\min\{\xi,\mu,1-\rho(1-\alpha)\}}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\beta-1\right\}} + C\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds.\end{aligned}$$

In order to apply the generalized version of Gronwall's inequality given in Lemma 2.7, the following inequalities must hold:

$$\begin{aligned}&\alpha - \frac{\mu}{\alpha} - (\rho-1)(1-\alpha) > 0 \Rightarrow \rho < \frac{\alpha-\mu}{\alpha(1-\alpha)},\\ &\beta - (\rho-1)(1-\alpha) > 0 \Rightarrow \rho < \frac{\beta}{1-\alpha} + 1\end{aligned}$$

(the second condition is already satisfied, since $\rho < \frac{\beta}{1-\alpha}$). We then obtain

$$\Psi(t) \leq Ch^{\min\{\xi,\mu,1-\rho(1-\alpha)\}}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\beta-1\right\}}.$$

Therefore,

$$\|g(t+h)-g(t)\|_Y \leq Ch^{\min\{\xi,\mu,1-\rho(1-\alpha)\}}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\beta-1\right\}}(t-\tau)^{-(\rho-1)(1-\alpha)}$$

and then,

$$\|g(t+h)-g(t)\|_Y \leq Ch^{\min\{\xi,\mu,1-\rho(1-\alpha)\}}(t-\tau)^{\min\left\{-\frac{\mu}{\alpha},\beta-\alpha\right\}}(t-\tau)^{-\rho(1-\alpha)}.$$

□

As in Corollary 4.7, if $u_0 \in D(A)$, then we do not have to worry about controlling the norm of the solution close to the initial time $t = \tau$. This simplifies the local estimates for the mild solution $u(t)$ and $f(u(t))$, that is,

$$\|u(t)\|_X \leq k \quad \text{and} \quad \|f(u(t))\|_Y \leq k, \quad \forall t \in (\tau, \tau+t_0]. \tag{4.9}$$

This allows us to restate the above result with less restrictive conditions on ρ:

Corollary 4.10. *Let X, Y be Banach spaces with $X \hookrightarrow Y$ and suppose that A is an $\alpha-$almost sectorial operator satisfying condition (C) and f a nonlinearity satisfying (G). If $u : (\tau, \tau+T) \to X$ is the mild solution of*

$$u_t(t) + Au = f(t, u(t)), t \in (\tau, \tau+T); \quad u(\tau) = u_0 \in D(A),$$

and $g(t) = f(t, u(t))$, then, for any $0 < \mu < \alpha^2$, the inequality

$$\|g(t+h) - g(t)\|_Y \leq Ch^{\min\{\xi,\mu\}}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\beta-1\right\}}$$

holds for $t > \tau$ and $h > 0$, where $\min\left\{\alpha - 1 - \frac{\mu}{\alpha}, \beta - 1\right\} \in (-1, \infty)$.

Proof. From Corollary 4.7 we obtain existence of local mild solution and from growth condition (4.1) on f we obtain

$$\|g(t+h) - g(t)\|_Y \leq C|h|^{\xi} + C\|u(t+h) - u(t)\|_X \left(1 + \|u(t)\|_X^{\rho-1} + \|u(t+h)\|_X^{\rho-1}\right).$$

It follows from (4.9) that

$$\|g(t+h) - g(t)\|_Y \leq C[|h|^{\xi} + \|u(t+h) - u(t)\|_X]. \tag{4.10}$$

Proceeding exactly as in Lemma 4.9, we obtain the following estimate for any $0 < \mu < \alpha^2$,

$$\|u(t+h) - u(t)\|_X \leq Ch^{\mu}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\beta-1\right\}} + C\int_\tau^t (t-s)^{\beta-1}\|g(s+h) - g(s)\|_Y ds.$$

Replacing it in (4.10),

$$\|g(s+h) - g(s)\|_Y \leq Ch^{\min\{\xi,\mu\}}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\beta-1\right\}} + C\int_\tau^t (t-s)^{\beta-1}\|g(s+h) - g(s)\|_Y ds.$$

From Gronwall's inequality, we obtain

$$\|g(t+h) - g(t)\|_Y \leq Ch^{\min\{\xi,\mu\}}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\beta-1\right\}}.$$

□

Using Lemma 4.9, it follows from Theorem 3.1 that for any $\tau < s < t$, given $0 < \mu < \alpha^2$, there exists $C > 0$, such that $g(t) = f(t, u(t))$ satisfies

$$\|g(t) - g(s)\|_Y \leq C(t-s)^{\theta}(s-\tau)^{-\psi},$$

where $0 < \theta < \min\{\xi, \mu, 1 - \rho(1-\alpha)\}$ and $-\psi = \min\left\{-\frac{\mu}{\alpha} - \rho(1-\alpha), \beta - \alpha - \rho(1-\alpha)\right\}$.

In order to use Theorem 3.1, note first that the equation takes place in Y, where the constant of almost sectoriality of A is ω. In this case, to conclude differentiability of the mild solution, the exponents θ and ψ must satisfy $\theta > 1 - \omega$ and $\psi > -1$. This will only be possible if

(i) $\xi > 1 - \omega$, $\mu > 1 - \omega$ and $1 - \rho(1-\alpha) > 1 - \omega$.

(ii) $-\psi > -1$.

Since $0 < \mu < \alpha^2$, there will be μ satisfying $\mu > 1 - \omega$ if

$$\alpha^2 + \omega - 1 > 0,$$

whereas the other inequality in item (i) is satisfied if

$$\rho < \frac{\omega}{1-\alpha}.$$

In order for the condition in (ii) to hold, we have already established in Lemma 4.9 that $1 \leq \rho \leq \frac{\alpha-\mu}{\alpha(1-\alpha)}$. The minimum value allowed for μ such that condition (i) holds is $1 - \omega$ and if we replace it in the expression above, we have

$$1 \leq \rho \leq \frac{\alpha + \omega - 1}{\alpha(1-\alpha)}.$$

If all the previous conditions are satisfied, then Theorem 3.1 states that the mild solution

$$u(t) = T_{-A}(t-\tau)u_0 + \int_\tau^t T_{-A}(t-s)g(s)ds$$

for the nonautonomous linear problem

$$u_t + A(t)u = g(t), \quad u(\tau) = u_0,$$

is a strong solution $u(\cdot) \in \mathcal{C}^1((\tau, \tau_M(u_0)), Y)$. But since $u(\cdot) : (\tau, \tau_M(u_0)) \to X$ is a mild solution of

$$u_t + A(t)u = f(t,u), t \in (\tau, \tau_M(u_0)); \quad u(\tau) = u_0,$$

and $g(t) = f(t, u(t))$, it follows that $u(\cdot)$ is a strong solution for the semilinear equation. Hence, the following theorem is proved.

Theorem 4.11. *Let X, Y be Banach spaces with $X \hookrightarrow Y$ and suppose that A is an $\alpha-$almost sectorial operator satisfying condition (C) and f a nonlinearity satisfying (G), with exponent of Hölder continuity $\xi > 1-\omega$. If $\alpha^2 + \omega - 1 > 0$ and ρ satisfies*

$$1 \leq \rho < \min\left\{\frac{\beta}{1-\alpha}, \frac{\alpha+\omega-1}{\alpha(1-\alpha)}, \frac{\omega}{1-\alpha}\right\},$$

then, for each $u_0 \in X$, the initial value problem

$$u_t + A(t)u = f(t, u(t)), t > \tau; \quad u(\tau) = u_0 \in X,$$

has a unique strong solution $u(\cdot) \in \mathcal{C}^1((\tau, \tau_M(u_0)), Y)$ where $(\tau, \tau_M(u_0))$ is its maximal interval.

The restrictions are simplified if the initial condition has enough regularity, as a consequence of Corollaries 4.7 and 4.10.

Corollary 4.12. *Let X, Y be Banach spaces with $X \hookrightarrow Y$ and suppose that A is an $\alpha-$almost sectorial operator satisfying condition (C) and f a nonlinearity satisfying (G), with exponent of Hölder continuity $\xi > 1-\omega$. If $\alpha^2 + \omega - 1 > 0$ then, for each $u_0 \in D(A)$, the initial value problem*

$$u_t + A(t)u = f(t, u(t)), t > \tau; \quad u(\tau) = u_0 \in D(A),$$

has a unique strong solution $u(\cdot) \in \mathcal{C}([\tau, \tau_M(u_0), Y) \cap \mathcal{C}^1((\tau, \tau_M(u_0)), Y)$ defined in its maximal interval $(\tau, \tau_M(u_0))$.

5. Application: Domain with handle

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain formed by two disjoint components: $\Omega = \Omega_L \cup \Omega_R$, $\overline{\Omega_L} \cap \overline{\Omega_R} = \emptyset$. Attached to this Ω, consider the line segment R_0 given by $R_0 = \{(r, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}; r \in (0,1)\}$. We assume that Ω and R_0 are connected by the points $(0,0) \in \mathbb{R} \times \mathbb{R}^{N-1}$ and $(1, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}$ and that there exists a cylinder centered in the line segment R_0 that only intersects Ω in its bases. Figure 2 bellow illustrate this set.

We denote $\Omega_0 = \Omega \cup R_0$ and in this domain and we consider the following system:

$$\begin{cases} w_t - div(a(x)\nabla w) + w = f(t, w), & x \in \Omega, \ t > \tau, \\ \partial_n w = 0, & x \in \partial\Omega, \\ v_t - \partial_r(a(r)\partial_r v) + v = f(t, v), & r \in R_0, \ t > \tau, \\ v(p_0) = w(p_0) \text{ and } v(p_1) = w(p_1), \end{cases} \tag{5.1}$$

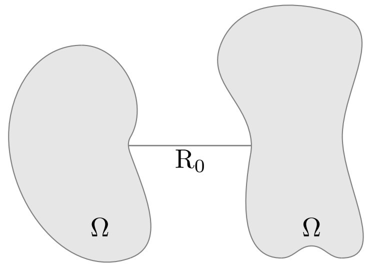

FIGURE 2. Domain with a handle

where $p_0 = (0, 0, ..., 0) \in \mathbb{R}^N$ and $p_1 = (1, 0, ..., 0) \in \mathbb{R}^N$ are the junction points between the sets Ω and R_0. We also assume that:

(A.1) $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary (C^2) formed by two disjoint components: Ω_L and Ω_R, with $p_0 \in \partial\Omega_L$ and $p_1 \in \partial\Omega_R$.

(A.2) $a \in \mathcal{C}^1(\overline{\Omega_0}, \mathbb{R}^+)$ and has its image in a closed interval $[a_0, a_1] \subset (0, \infty)$.

(A.3) The nonlinearity f is continuously differentiable, $f \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and satisfies a *polynomial growth condition* for the second variable, that is,

$$|f_\xi(t, \xi)| \leq C(1 + |\xi|^{\rho-1}), \text{ for some } \rho \geq 1. \tag{5.2}$$

Remark 5.1. *We will use x for the variable that takes values in Ω, r for the variable that takes values in R_0 and $t, s, \tau \in \mathbb{R}$ for a given instant of time. Note that r has the form $(z, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}$, with $z \in [0, 1]$. Therefore, at some points, we will consider r as an element in the interval $[0, 1]$ and treat $v_r(t, r)$ as the derivative of v in the real variable $r \in [0, 1]$.*

The phase space is given by $U_p^0 = L^p(\Omega) \times L^p(0, 1)$, with norm $\|(w, v)\|_{U_p^0} = \|w\|_{L^p(\Omega)} + \|v\|_{L^p(0,1)}$. In this case, $(U_p^0, \|\cdot\|_{U_p^0})$ is Banach and equation (5.1) originates the following abstract problem:

$$(w, v)_t + A_0(w, v) = F_0(t, (w, v)), t > \tau; \quad (w, v)(\tau) = (w_0, v_0) \in U_p^0, \tag{5.3}$$

where $A_0 : D(A_0) \subset U_p^0 \to U_p^0$ is the linear operator given by

$$D(A_0) = \left\{(w, v) \in W^{2,p}(\Omega) \times W^{2,p}(0, 1) \colon \partial_n w = 0 \text{ in } \partial\Omega \text{ and } v(p_i) = w(p_i), i = 1, 2\right\}, \tag{5.4}$$

$$A_0(w, v) = (-div(a(x)\nabla w) + w, -\partial_r(a(r)\partial_r v) + v), \quad \text{for } (w, v) \in D, \tag{5.5}$$

and the nonlinearity F_0 is given by

$$F_0(t, (w, v))(x) = \begin{cases} f(t, w(x)), & x \in \Omega, \\ f(t, v(r)), & r \in R_0. \end{cases} \tag{5.6}$$

Remark 5.2. *The condition imposed on p_0 and p_1 in (5.4) only makes sense if $w \in C(\overline{\Omega})$. Therefore, the restriction on $p > \frac{N}{2}$ must be required at this point, which ensures that $W^{2,p}(\Omega) \hookrightarrow C(\overline{\Omega})$* [1, Theorem 5.4].

In [3, Proposition 3.1] several properties of A_0 are presented, including its almost sectoriality.

Proposition 5.3. *The linear operator A_0 satisfies:*

(1) A_0 is a closed and densely defined linear operator.
(2) A_0 has compact resolvent and the semigroup $T_{-A_0}(s)$ is compact.
(3) There exists $\varphi \in \left(\frac{\pi}{2}, \pi\right)$ and $C > 0$ such that $\Sigma_\varphi \subset \rho(-A_0)$ and, for $\frac{N}{2} < q \leq p$, $\lambda \in \Sigma_\varphi \cup \{0\}$, we have

$$\|(\lambda + A_0)^{-1}\|_{\mathcal{L}(U_q^0, U_p^0)} \leq \frac{C}{|\lambda|^\beta + 1},$$

for each $0 < \beta < 1 - \frac{N}{2q} - \frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)$. In particular, the case $p = q$ yields

$$\|(\lambda + A_0)^{-1}\|_{\mathcal{L}(U_p^0)} \leq \frac{C}{|\lambda|^\alpha + 1},$$

for $0 < \alpha < 1 - \frac{N}{2p} < 1$.
(4) The spectrum of A_0 consists entirely of real and positive isolated eigenvalues.

Remark 5.4. *The operator A_0 given in* (5.5) *differs from the operator considered in* [3]*, which is $A_0(w, v) = (-\Delta + I, -\frac{d^2}{dr^2} + I)$. Despite the difference, the proof of each statement above is exactly the same as the ones presented there, since it only depends on the sectoriality of the operator $-\Delta + I$ in Ω, on the sectoriality of $-\frac{d^2}{dr^2} + I$ (with Dirichlet boundary condition) in R_0, and on Sobolev embeddings.*

Even tough the linear operators on Ω and on R_0 are sectorial, the condition at p_0 and p_1 imposes restriction on the estimate of the resolvent that culminates with $A_0(w, v) = (-\Delta + I, -\frac{d^2}{dr^2} + I)$ being almost sectorial (it is expressed in (3.14) of [3]*).*

We turn our attention to the nonlinearity f. The growth condition (5.2) and the mean value theorem imply the existence of a constant $C > 0$ such that

$$|f(t, \xi) - f(t, \psi)| \leq C|\xi - \psi|(1 + |\xi|^{\rho-1} + |\psi|^{\rho-1}) \quad \text{ and } \quad |f(t, \xi)| \leq C(1 + |\xi|^\rho).$$

Moreover, $t \mapsto f(t, \cdot)$ is locally Lipschitz, that is, $|f(t, \cdot) - f(s, \cdot)| \leq C|t - s|$. Those properties of f reflect on the operator F_0:

Lemma 5.5. *Let F_0 be the nonlinearity defined in* (5.6) *and suppose* (5.2) *is satisfied. Then F_0 take elements of U_p^0 to elements in U_q^0, that is, $F : U_p^0 \to U_q^0$, where $q = \frac{p}{\rho}$. Furthermore, for each $(w, v) \in U_p^0$, we have*

$$\begin{aligned}
&\|F_0(w, v) - F_0(\tilde{w}, \tilde{v})\|_{U_q^0} \leq C\|(w, v) - (\tilde{w}, \tilde{v})\|_{U_p^0}(1 + \|(w, v)\|_{U_p^0}^{\rho-1} + \|(\tilde{w}, \tilde{v})\|_{U_p^0}^{\rho-1}),\\
&\|F_0(t, (w, v))\|_{U_q^0} \leq C(1 + \|(w, v)\|_{U_p^0}^\rho),\\
&\|F_0(t, (w, v)) - F_0(s, (w, v))\|_{U_q^0} \leq C|t - s|.
\end{aligned}$$

Proof. We only verify the first inequality. The second follows in a similar way. Note that

$$\begin{aligned}
&\|F_0(t, (w, v)) - F_0(t, (\tilde{w}, \tilde{v}))\|_{U_q^0}\\
&\quad = \left[\int_\Omega |f(t, w(x)) - f(t, \tilde{w}(x))|^q dx\right]^{\frac{1}{q}} + \left[\int_0^1 |f(t, v(s)) - f(t, \tilde{v}(s))|^q ds\right]^{\frac{1}{q}}.
\end{aligned}$$

We consider the integrals separately. Firstly, we have

$$\begin{aligned}
\int_\Omega |f(t,w(x)) - f(t,\tilde{w}(x))|^q dx &\le \int_\Omega C^q |w(x)-\tilde{w}(x)|^q (1+|w(x)|^{q(\rho-1)} + |\tilde{w}(x)|^{q(\rho-1)})dx \\
&\le C\left(\int_\Omega |w-\tilde{w}|^p\right)^{\frac{q}{p}} \left(\int_\Omega \left[1+|w|^{q(\rho-1)}+|\tilde{w}|^{q(\rho-1)}\right]^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} \\
&\le C\|w-\tilde{w}\|^q_{L^p(\Omega)} \left(1+\int_\Omega |w|^{q(\rho-1)\frac{p}{p-q}} + |\tilde{w}|^{q(\rho-1)\frac{p}{p-q}}\right)^{\frac{p-q}{p}},
\end{aligned}$$

and we used that $q(\rho-1)\frac{p}{p-q} = p$. Therefore

$$\begin{aligned}
\left[\int_\Omega |f(t,w(x)) - f(t,\tilde{w}(x))|^q dx\right]^q &\le C\|w-\tilde{w}\|_{L^p(\Omega)} \left(1+\|w\|^p_{L^p(\Omega)} + \|\tilde{w}\|^p_{L^p(\Omega)}\right)^{\frac{p-q}{pq}} \\
&\le C\|w-\tilde{w}\|_{L^p(\Omega)} \left(1+\|w\|^{\rho-1}_{L^p(\Omega)} + \|\tilde{w}\|^{\rho-1}_{L^p(\Omega)}\right).
\end{aligned}$$

For the second term, we obtain

$$\left[\int_0^1 |f(t,v(s)) - f(t,\tilde{v}(s))|^q ds\right]^{\frac{1}{q}} \le C\|v-\tilde{v}\|_{L^p(0,1)}(1+\|v\|^{\rho-1}_{L^p(0,1)} + \|\tilde{v}\|^{\rho-1}_{L^p(0,1)}).$$

Using the above inequalities,

$$\begin{aligned}
&\|F_0(t,(w,v)) - F_0(t,(\tilde{w},\tilde{v}))\|_{U^0_q} \\
&\quad \le C(\|w-\tilde{w}\|_{L^p(\Omega)} + \|v-\tilde{v}\|_{L^p(0,1)})(1+\|w\|^{\rho-1}_{L^p(\Omega)} + \|v\|^{\rho-1}_{L^p(0,1)} + \|\tilde{w}\|^{\rho-1}_{L^p(\Omega)} + \|\tilde{v}\|^{\rho-1}_{L^p(0,1)}) \\
&\quad \le C\|(w,v)-(\tilde{w},\tilde{v})\|_{U^0_p}(1+\|(w,v)\|^{\rho-1}_{U^0_p} + \|(\tilde{w},\tilde{v})\|^{\rho-1}_{U^0_p}).
\end{aligned}$$

□

5.1. **Functional setting.** As a starting point, in order for the problem to be well defined (see Remark 5.2), we require $p > \frac{N}{2}$.

The phase space in which the initial data will be taken is $X = U^0_p$. In this space, the operator A_0 is $\alpha-$almost sectorial with α being any real number satisfying

$$0 < \alpha < 1 - \frac{N}{2p} =: \alpha^+.$$

The nonlinearity F_0, which is known to have a growth of order ρ, will take elements of U^0_p and decrease its regularity to an element of U^0_q, where $q = \frac{p}{\rho}$.

We denote $Y = U^0_q$. Assume for now that $1 \le \rho < \rho_0$ is such that $q = \frac{p}{\rho} > \frac{N}{2}$ (later on we will calculate the range for which this situation can occur). Proposition 5.3 states that A_0 is $\omega-$almost sectorial in $Y = U^0_q$ with ω in the interval

$$0 < \omega < 1 - \frac{N}{2q} =: \omega^+.$$

The connection between those two spaces X, Y and the operator A_0 is then established one more time via Proposition 5.3, which ensures the existence of a constant β in the interval

$$0 < \beta < 1 - \frac{N}{2q} - \frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right) =: \beta^+$$

such that the resolvent of $-A_0$ satisfies $\|(\lambda + A_0)^{-1}\|_{\mathcal{L}(U_q^0, U_p^0)} \leq \frac{C}{|\lambda|^{\beta}+1}$. This means that the operator A_0^{-1} (or any $(\lambda + A_0)^{-1}$ for $\lambda \in \Sigma_\varphi$) take elements in the less regular space Y back to X.

5.2. **Local well-posedness.** To establish existence of local mild solution for general initial condition $(w_0, v_0) \in U_p^0$, it is necessary to ensure that condition $1 \leq \rho < \frac{\beta}{1-\alpha}$ (see Theorem 4.6) holds. The next lemma provides conditions on p and q such that this is satisfied.

Lemma 5.6. *Let $\frac{N}{2} < q \leq p$ and $\rho = \frac{p}{q}$. There exist $0 < \beta < 1 - \frac{N}{2q} - \frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)$ and $0 < \alpha < 1 - \frac{N}{2p}$ such that $1 \leq \rho < \frac{\beta}{1-\alpha}$ if and only if, for fixed $p > N$, we have*

$$\frac{p(2N+1)}{2p+1} < q \leq p. \tag{5.7}$$

Proof. It is enough to obtain a condition on q such that $\frac{p}{q} = \rho < \frac{\beta^+}{1-\alpha^+}$. This inequality will be satisfied if $\frac{p}{q} < \frac{1-\frac{N}{2q}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}{1-\left(1-\frac{N}{2p}\right)} \Leftrightarrow q > \dfrac{p(2N+1)}{2p+1}$. Also, the condition $\frac{p(2N+1)}{2p+1} < q \leq p$ will only make sense if $p > N$.

□

Inequality (5.7) allows us to calculate the largest growth F_0 can have so that the problem is still locally well-posed (in the sense of mild solution). This largest value is given by

$$\rho_c = \frac{p}{\frac{p(2N+1)}{2p+1}} = \frac{2p+1}{2N+1}.$$

Remark 5.7. *As illustrated in the previous calculus, any lower bound $l = l(p)$ for q creates a restriction of the type $l(p) < q \leq p$, which generates a critical value for ρ given by $\frac{p}{l(p)}$.*

The above results allow us to write an existence result for the equation being considered.

Proposition 5.8. *Assume that $p > N$ and $\max\left\{\frac{N}{2}, \frac{p(2N+1)}{2p+1}\right\} < q \leq p$, $X = U_p^0$, $Y = U_q^0$, $a : \overline{\Omega_0} \to \mathbb{R}^+$ satisfies (A.2) and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies (A.3). Then (5.3) has a local mild solution $(w, v)(\cdot) : (\tau, \tau_M(w_0, v_0)) \to U_p^0$ given by*

$$(w, v)(t) = U_0(t, \tau)(w_0, v_0) + \int_\tau^t U_0(t, s) F_0(s, (w, v)(s)) ds.$$

5.3. **Strong solution for the semilinear equation.** To ensure regularity of the mild solution, we must check the remaining conditions posed in Theorem 4.11, which are

$$\alpha^2 + \omega - 1 > 0 \quad \text{and} \quad 1 \leq \rho < \min\left\{\frac{\alpha + \omega - 1}{\alpha(1-\alpha)}, \frac{\omega}{1-\alpha}\right\}.$$

Note that the first inequality provides a lower bound for the values of q:

$$\alpha^2 + w - 1 > 0 \Rightarrow q > \frac{2p^2 N}{(2p-N)^2}$$

and this is obtained by replacing α^+, ω^+ in the relation and some manipulation (as it was done in Lemma 5.6). Second inequality provides two lower bounds for q (recall that $\rho = \frac{p}{q}$):

$$\rho < \frac{\alpha + \omega - 1}{\alpha(1-\alpha)} \Rightarrow q > \frac{N(N-4p)}{2(N-2p)} \quad \text{and} \quad \rho < \frac{w}{1-\alpha} \Rightarrow q > N.$$

Therefore, in order for the mild solution to be regular, all the lower bounds established for q above, added to (5.7) obtained for local well-posedness, must be satisfied, that is,

$$\max\left\{\frac{p(2N+1)}{2p+1}, \frac{2p^2N}{(2p-N)^2}, \frac{N(N-4p)}{2(N-2p)}, N\right\} < q \leq p.$$

Therefore, we have the following result that ensures existence of regular solution.

Proposition 5.9. *Assume* $\max\left\{\frac{p(2N+1)}{2p+1}, \frac{2p^2N}{(2p-N)^2}, \frac{N(N-4p)}{2(N-2p)}, N\right\} < q \leq p$, $X = U_p^0$, $Y = U_q^0$, $a : \overline{\Omega_0} \to \mathbb{R}^+$ *satisfies (A.2) and* $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *satisfies (A.3). Then* (5.3) *has a strong* $Y-$*solution* $(w,v)(\cdot) : (\tau, \tau_M(w_0, v_0))$ *given by*

$$(w,v)(t) = U_0(t,\tau)(w_0,v_0) + \int_\tau^t U_0(t,s)F_0(s,(w,v)(s))ds.$$

Taking into account Remark 5.7, the lower bounds for q on Proposition 5.9 provide restrictions on the growth ρ for F_0, each comes from dividing p by a lower bound of q.

$$\rho_{c_1} = \frac{2p+1}{(2N+1)}, \quad \rho_{c_2} = \frac{(2p-N)^2}{2pN}, \quad \rho_{c_3} = \frac{2p(N-2p)}{N(N-4p)}, \quad \rho_{c_4} = \frac{p}{N}.$$

Note that it is only required to know N in order to establish values of p and ρ ($\rho = \frac{p}{q}$) for which the problem can be solved. For instance, if $N = 3$ Figure 3 bellow provides regions for which, given a certain p, the problem can be locally solved as long as (p,ρ) belongs to the shaded region. The figure on the right-side states those values (p,ρ) for which the mild solution is also strong. Despite the presence of four different restrictions (ρ_{c_1}, ρ_{c_2}, ρ_{c_3} and ρ_{c_4}), only two of them restrict the most the shaded region. The other two are located above this region.

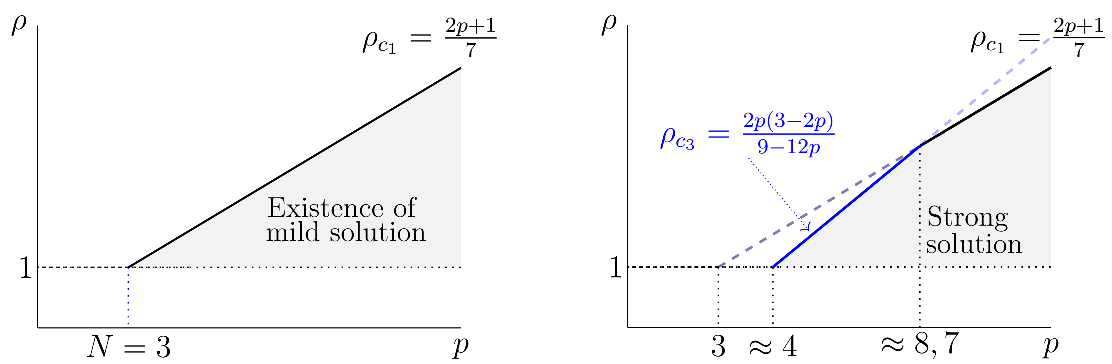

FIGURE 3. Mild and strong solution when $N = 3$

5.4. **Initial condition in $D(A)$.** For initial conditions in $D(A)$, Corollary 4.12 establishes that the only necessary condition for existence of strong solution is $\alpha^2 + \omega - 1 > 0$, which implies

$$q > \frac{2p^2N}{(2p-N)^2} \quad \text{and} \quad \rho < \frac{(2p-N)^2}{2pN}.$$

Moreover, in order for the boundary condition of A_0 action on U_q^0 to make sense, we must have $q > \frac{N}{2}$ ($W^{2,q}(\Omega) \hookrightarrow C(\overline{\Omega})$). We also have to take into account this lower bound for q, which establishes a second critical value for ρ given by $\rho < \frac{2p}{N}$. In Proposition 5.9 we did not bother to write this condition since $q > N$ was more restrictive.

For this case, we have the following existence result on strong solutions

Proposition 5.10. *Assume* $\max\left\{\frac{2p^2N}{(2p-N)^2}, \frac{N}{2}\right\} < q \leq p$, $X = U_p^0$, $Y = U_q^0$, $a : \overline{\Omega_0} \to \mathbb{R}^+$ *satisfies (A.2) and* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies (A.3). Then, given any* $(w_0, v_0) \in D(A)$, (5.3) *has a strong* $Y-$*solution* $(w, v) \in \mathcal{C}([\tau, \tau_M(w_0, v_0)), Y) \cap \mathcal{C}^1((\tau, \tau_M(w_0, v_0)), Y)$ *given by*

$$(w, v)(t) = U_0(t, \tau)(w_0, v_0) + \int_\tau^t U_0(t, s)F_0(s, (w, v)(s))ds.$$

6. Remarks and discussion

As mentioned in Remark (4.3), when dealing with semilinear parabolic problems as the one considered in this work, it is usual to approach it with the theory of fractional powers of operators. We briefly comment how to connect the results obtained in this work with the case where the growth of the nonlinearity involves fractional powers.

Suppose $A : D(A) \subset Z \to Z$ is an almost sectorial operators (with constant $\phi \in (0, 1)$) in the Banach space Z. In [15, 20] a functional calculus for almost sectorial operators was developed and fractional powers of almost sectorial operators were defined. We can obtain an associated scale of fractional power spaces $Z^\xi = D(A^\xi)$ in the same sense that we do for sectorial operators.

However, the deficiency in the resolvent allows us to define those powers only on the interval $1 - \phi < \xi < 1$ and the Momentum Inequality only holds for $1 - \phi < \xi < \phi$ (see [4, p. 24]). Those restrictions reflect on the semilinear problem, as we see next.

Let $f : Z^\gamma \to Z^\theta$, with $1 - \phi < \theta < \gamma < 1$ and assume $1 - \phi < \gamma - \theta < \phi^2$. Suppose also that f has a growth given by $\rho \geq 1$. Under those conditions, Theorem 3.1 in [4] proves the existence of mild solutions for

$$u_t + Au = f(u), t > \tau; \quad u(\tau) = u_0 \in Z^\gamma$$

(actually they study a singularly nonautonomous case where A is time dependent, that is, $A(t)$ is a family of almost sectorial operators. We consider $A(t) = A$ to our purpose).

Comparing with terminology used in this work, we set $X = Z^\gamma$ and $Y = Z^\theta$. Using the almost sectoriality of A, the characterization of the resolvent $(\lambda + A)^{-1}$ as the Laplace transform of the semigroup $T_{-A}(t)$, which also holds for almost sectorial operators (see [12, Lemma 3.1]), and the Momentum Inequality [4, Proposition 2.1], we have

$$\begin{aligned}
\|(\lambda + A)^{-1}\|_{\mathcal{L}(Y,X)} &= \|(\lambda + A)^{-1}\|_{\mathcal{L}(Z^\theta,Z^\gamma)} = \|A^\gamma(\lambda + A)^{-1}A^{-\theta}\|_{\mathcal{L}(Z)} \\
&\leq \left\| A^{\gamma-\theta} \int_0^\infty e^{-\lambda s} T_{-A}(s) ds \right\|_{\mathcal{L}(Z)} \leq C \int_0^\infty e^{-\lambda s} s^{-1+\phi-\frac{(\gamma-\theta)}{\phi}} ds \\
&\leq C \int_0^\infty e^{-u} u^{-1+\phi-\frac{(\gamma-\theta)}{\phi}} \left(\frac{1}{\lambda}\right)^{-1+\phi-\frac{(\gamma-\theta)}{\phi}} \frac{1}{\lambda} du = C \frac{1}{\lambda^{\phi-\frac{(\gamma-\theta)}{\phi}}} \Gamma\left(\phi - \tfrac{(\gamma-\theta)}{\phi}\right),
\end{aligned}$$

since $\gamma - \theta < \phi^2$.

Therefore, the constant β in condition (C), in this case, would be $\beta = \phi - \frac{(\gamma-\theta)}{\phi}$. Furthermore, if $\gamma = \theta$ then we can see that

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} = \|(\lambda + A)^{-1}\|_{\mathcal{L}(Y)} = \|(\lambda + A)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{1}{\lambda^\phi},$$

that is, the constant of sectoriality in X, Y and Z are all the same: $\alpha = \omega = \phi$.

Theorem 4.6 states that local solvability for the problem is guaranteed if

$$\rho < \frac{\beta}{1-\alpha} = \frac{\phi - \frac{(\gamma-\theta)}{\phi}}{1-\phi}.$$

Inequality above is exactly the largest value for ρ established in Theorem 3.1 of [4]. Using the relations above, we can extend the results obtained in this work to the setting where fractional power features.

Acknowledgments

The authors would like to thank the anonymous referee for his comments and suggestions which improved an earlier version of this work.

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