Mappings, Models, Abstraction, and Imaging: Mathematical Contributions to Modern Thinking Circa 1900

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1921, Vienna and Leipzig: the journal *Annalen der Naturphilosophie*, edited by chemist Wilhelm Ostwald,[1] publishes a hard-to-classify paper, an essay written in aphorisms about knowledge and its limits, about the world and "what cannot be spoken of," polished during the Great War by Ludwig Wittgenstein.[2] In this terse and difficult writing, barely 78 pages, one finds an attempt to definitively clarify main philosophical questions. At the core of this reflection is logic and its link with the world, a topic articulated around the notions of *Bild*, image or figure, and *Abbildung*, figuration or representation, mapping. Wittgenstein's entire attempt is unequivocally modernist in spirit, an exemplary specimen of a cultural shift intensified by the horrendous war. In those years one started to talk about *structures* in different fields, not least within mathematics, and the new architectonic efforts look for purity of lines, a purist functionalism linked (surprisingly or not) to an intense search for transcendence. This can be applied, e.g., to Wassily Kandinsky's

[1] Wilhelm Ostwald was a Nobel prize winner and tried to promote a scientific, 'monist' worldview; also famous for promoting the history of science.

[2] Ludwig Wittgenstein, "Tractatus logico-philosophicus," *Annalen der Naturphilosophie* 14 (1921): 185–262. Proposition 7 reads: "Wovon man nicht sprechen kann, […]."

[3] In what follows, to clarify my discourse, I will often employ the relevant German words. The reader should keep in mind that *Bilder* is the plural of *Bild*, from which the verb *abbilden* is derived (to represent pictorially) and the substantive *Abbildung* (representation, mapping).

English version of: José Ferreirós, "Matemáticas y pensamiento: en torno a imágenes, modelos, abstracción y figuración," in *Rondas en Sais: ensayos sobre matemáticas y cultura contemporánea*, ed. Fernando Zalamea (Bogotá: Editorial de la Universidad Nacional de Colombia, 2013), 15–40.

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M. Friedman and K. Krauthausen (eds.), *Model and Mathematics: From the 19th to the 21st Century*, Trends in the History of Science,
https://doi.org/10.1007/978-3-030-97833-4_10

canvases or the scores of Anton Webern but applies in no lesser degree to the beautiful “Tractatus logico-philosophicus.”

Let us consider more closely Wittgenstein’s pictorial or figural theory (*Bildtheorie*) of meaning. The sentences of a language are an expression of thoughts, which in turn are images, *Bilder*, of states of affairs in the world. At the basis of everything is the making of an image or *Bild* of a fact, a model of reality–figuration or representation, *abbilden* (“Tractatus,” 2.1: “Wir machen uns Bilder der Tatsachen”). Of course, Wittgenstein, and his predecessors, employ these words in a wide and somewhat abstract sense.[3] “The logical picture of the facts is the thought,” he wrote in the “Tractatus” (3.: “Das logische Bild der Tatsachen ist der Gedanke”). Even more, logic itself is, according to Wittgenstein, a picture of the world;[4] as he said in a memorable phrase, “Die Logik ist […] ein Spiegelbild der Welt,” logic is a mirror image of the world.[5]

For all of its novelty and its modernist style, the logico-philosophical work of the Viennese is also a knot that links receptively a number of proposals and developments that have already gone a long way. Few people are cited in the text, with the exceptions of Gottlob Frege, Bertrand Russell, and Heinrich Hertz. Moreover, it is well-known that the work of Hertz, a physicist, includes passages that emphasize the notions of image and representation, *Bild* and *Abbildung*, as key to understanding the function of scientific thinking and its links with reality (see section “Helmholtz and Hertz”).

The purpose of this essay will be to establish a partial genealogy of these notions, following them through an undulating path that will traverse the territories of science, mathematics, and philosophy. It should be well-known: frontiers are a work of humans, an artifice; they exist only in our social life and our imagination—regardless of how hard and consistent they may seem to be.

Generalities

Academic writers often do not pay attention to such trespassing of frontiers, like the cases I just indicated above. A source of such lack of sight is what might be called ‘disciplinary blindness’: the mathematician looks back in the search for mathematical contributions that can be recognized as relevant to the present context of mathematics. The same is true for the philosopher or the physicist. It is easy to perceive that such disciplinary blindness goes together with a certain kind of Whiggism, which is more usual among academics than in the world of art (compensated perhaps by an increase of rigor in their statements). Ideally, to understand changes in the world of thought like the ones I will discuss, a highly interdisciplinary approach is required.

[4] This thesis is polemical and very open to discussion, but we cannot enter into the question here. See: José Ferreirós, “The Road to Modern Logic – An Interpretation,” *The Bulletin of Symbolic Logic* 7 (2001): 441–84.

[5] Wittgenstein, “Tractatus,” 6.13.

Chance or not, the authors I will talk about in the following pages belong to German speaking contexts. Between 1800 and 1940 such contexts turned out to be a great breeding ground for exchanges between philosophical and scientific thought. Hybrid figures of great interest, the scientist-philosopher or the philosopher-scientist, are easy to find in that period and context—from Immanuel Kant to Albert Einstein or Hermann Weyl, to mention just some paradigmatic cases at the beginning and end of that era. In the following pages some other examples will emerge—Johann F. Herbart, Bernhard Riemann, Hermann von Helmholtz, Richard Dedekind, Hertz, and Wittgenstein.

That same richness complicates the task of tracing faithfully the maze of influence and affluence. It may be worthwhile to mention a case that I encountered early on, when I was starting to work on the history of mathematical thought. Studying Dedekind's works, highly interesting for the change of (architectonic) style in mathematics, I found many indications of the possible influence of the philosophical ideas of Gottfried W. Leibniz and Kant. A first natural impulse was to follow them in detail and trace their genealogy in particular works of those philosophers. Even today I think it is highly likely that Dedekind may have read the *Critique of Pure Reason* (*Kritik der reinen Vernunft*), the *New Essays on Human Understanding* (*Nouveaux Essais sur l'entendement humain*), etc., and for the case of Leibniz I found an elaborate argument concerning the exact how and where of those supposed influences. Yet the critical exercise of historical research led me to realize that none of my arguments was certain enough. I came to understand that Kantian and Leibnizian ideas were so widely disseminated in the German-speaking nineteenth century that they could flow in many ways, through the writings not only of philosophers (Jakob Fries, Hermann Lotze—who taught Dedekind at Göttingen) but also of mathematicians (Carl Friedrich Gauss, Riemann, William R. Hamilton, Hermann Grassmann) or scientists (Wilhelm Weber and Helmholtz). Thus, I became circumspect and I learnt a valuable lesson about direct and indirect sources of cultural transmission.

Let me return to representations or 'mappings,' what in German today are called *Abbildungen*. Dedekind coined the term in an 1888 work,[6] in the wake of the more fragmentary suggestions of Gauss and Riemann, his masters. To put it in simple terms, close to Dedekind's own, a representation or *Abbildung* φ is "a law" that correlates objects of some domain D with those of another C, the counterdomain, in such a way that to each object in D there corresponds one and only one object in C. Being d an element of D, its correlate $\varphi(d)$ is called the *image*, *Bild*, of d; analogously, one says that d is the *original* of $\varphi(d)$. Moreover, the whole set of elements in C that are images of elements of D can be called "the image of D" and denoted by φ(D); which corresponds to the whole canvas depicting an original scene.

[6] Richard Dedekind, *Was sind und was sollen die Zahlen?* [*What are numbers, and what for?*], of which the reader should employ the revised English version in *From Kant to Hilbert*, ed. William B. Ewald, vol. 2 (Oxford: Claredon Press, 1996), 787–833; see Sect. 2 on p. 799, "Representation [*Abbildung*] of a set."

The terminology is obviously inspired by painting, and Dedekind himself wrote in a letter that the mapping or representation φ is "the painter who paints" ("der abbildende Maler").[7] But it is clear that the idea can be applied to the relation between symbols in a pentagram and musical sounds, or to any other language including scientific representations.

The moment when that elegant idea was established marks an epoch insofar as it denotes the transition from 'classical' to 'modern,' twentieth century mathematics. Dedekind immediately applied the idea to structure-preserving mappings, what would later be called *morphisms*. This transition involved a transfiguration of mathematics, it has often been claimed that the nineteenth century was a true Renaissance for mathematics. On the one hand, Dedekind's mappings or representations link back with the concrete functions that were studied in 'classical' algebra or analysis. But he precisely abandoned the concrete element (definition by means of an analytic, explicit formula) in favor of a simplicity, generality and abstraction that is characteristically modern. Those mappings, and especially the morphisms, inaugurate the path toward new levels of conceptual abstraction that will end up in the *arrows* and *functors* of category theory. Dedekind can thus be acknowledged as a landmark in the path toward contemporary mathematics, an example being his reconceptualization of Galois theory in 1894,[8] and as such he was acknowledged by mathematicians like Emmy Noether and Emil Artin.

The coinage of the modern notion of *Abbildung* must be seen as a key moment in a transformation of *longue durée*, which has profoundly affected scientific and philosophical thought. I mean the transition from a substantial-causal conception of phenomena, characteristic of ancient and medieval thought, to a relational-functional conception that is typical of modern and contemporary thought. Substantial and causal models have been found incomplete and insufficient for the understanding of physical, human and social phenomena.[9] Simultaneously, led by physics and mathematical analysis, the idea of function came to the foreground in the context of admitted scientific models. All of science today aims to formulate relational-functional models that allow exact prediction

[7] Richard Dedekind: "Letter to Keferstein" [1890], in *From Frege to Gödel. A Source Book in Mathematical Logic 1879–1931*, ed. Jean van Heijenoort (Cambridge: Harvard University Press, 1967), 98–103, here 102.

[8] The theory of Évariste Galois, a true cornerstone of modern algebra, was born in the writings of the famous Frenchman around 1830, but Dedekind was the first to sketch the modern formulation in terms of groups of field automorphisms (in Appendix XI to the 1894 edition of *Vorlesungen über Zahlentheorie*, see: José Ferreirós, "Dedekind's Map-theoretic Period," *Philosophia Mathematica* 25, no. 3 (2017): 318–40). This very abstract version was developed by Emil Artin thirty-five years later.

[9] Note for philosophers: the idea of *causality* admits several definitions, and there is a widespread tendency to call current scientific models 'causal'—yet this linguistic usage is anachronistic when applied to the acme of causal thinking. Thus, it seems confusing to me. I employ the word 'causality' for the simple cause-effect relation that occupied Hume in his famous criticism, which Kant tried to defend by postulating it a priori as one of the categories.

of diverse kinds of phenomena, just like gravity theory (be it Newtonian or relativistic) describes interactions mediated by gravitation, like quantum theories (elementary or field-theoretic) describe the phenomena due to electromagnetism and nuclear forces.

The triumph of functional thinking, in terms of mapping or representation, surfaced during the eighteenth century and the beginning of the nineteenth, the time that today would be called—from the viewpoint of mathematics—the era of Leonhard Euler and Gauss. Since then it has only consolidated, in spite of and throughout the many transformations that have happened in science and in mathematics. I will come back to this.

The Riemann Inflexion

I will now consider some facets of the work and thought of Bernhard Riemann. This impressive intellectual figure represents a turning point in the history of mathematical and scientific thought. Of course, mathematics is collective work, and almost nothing of importance can be reduced to the contribution of a single 'genius' (a notion that is today reviled by historians, perhaps exaggeratedly). Yet some mathematicians concentrate ideas and trends in such a way that they amaze and prompt the question as to what might have been without their proposals. The work of Riemann, as one important twentieth century mathematician put it, is "full of almost cryptic messages to the future" (Lars V. Ahlfors in 1953).[10]

Relationalism and the idea of *Abbildung* play key roles in several places in Riemann's thought, but let me start here in the domain of philosophy. It should be briefly noted that Riemann studied under Gauss, as did Dedekind, who regarded his somewhat older colleague as a true master. An important characteristic of the mathematicians at Göttingen was emphasized by Gauss's collaborator, the physicist Wilhelm Weber: "With Dirichlet and Riemann, Göttingen remained the plantation of a deeply philosophical orientation in mathematical research, which it had become under Gauss."[11]

The *topos* of relations had become more and more relevant in philosophical reflections towards the period that I marked out above. It seems no coincidence that Leibniz, inventor of calculus, was a pioneer of this turn and one who insisted on relations as key for the understanding of reality. From his metaphysics, where each monad contains a (more or less partial) representation of the universe, to his scientific contributions, we see that notion appear in diverse forms. He even applied

[10] Quoted in: Samuel J. Patterson, "Reading Riemann," in *Exploring the Riemann Zeta Function: 190 Years from Riemann's Birth*, ed. Hugh Montgomery, Ashkan Nikeghbali, and Michael Th. Rassias (Cham: Springer, 2017), 265–85, here 265.

[11] From a University report, 1866; quoted in Pierre Dugac, *Richard Dedekind et les fondements des mathématiques*, (Paris: Vrin, 1976), 166: "Die Pflanzstätte der tiefer philosophischen Richtung im mathematischen Forschen, die Göttingen durch Gauß geworden, ist es unter Dirichlet und Riemann geblieben […]."

it to the matrix ideas of space and time, offering a relationalist approach that, in spite of Einstein, has never been elaborated in a totally satisfactory way. This topos followed its course with authors as central as Kant (cf. his categories of relation) and it seems to have reached a peak during the nineteenth century. In this period, a post-Kantian author who took inspiration from Leibniz, a philosopher-scientist named Herbart, went so far as to say that all our knowledge is knowledge of relations, in perception as much as in conceptual knowledge: "we live amid relations and need nothing else."[12] What there is, both in the ultimate reality postulated by metaphysics and in the contents of our mental acts,[13] is above all relations and records of relations.

Johann Friedrich Herbart is of interest because Riemann regarded him as a master in philosophical questions. But he is not among the usual canon of philosophers, partly due to the disciplinary blindness I mentioned above.[14] Herbart elaborated an interesting theory of knowledge, and here, perhaps for the first time in a philosopher, we find an overtly hypothetico-deductive epistemology, coherent with the experimental approach of the natural sciences. He insisted that our theories emerge from the gradual modification of prior ideas, from the dialectics of *experience* [*Erfahrung*] and *reflection* [*Nachdenken*], that is to say, from the reflective effort to accommodate at the level of theory the experiential data that contradict the old ideas. He rejected outright Kantian apriorism, saying that the "hypothesis" of two forms of intuition was "completely superficial, devoid of content and inadequate," since they were merely "a couple of empty infinite recipients into which the senses must pour their sensations, without any reason for the ordering and configuration."[15] As regards the categories of the understanding, which according to Kant regulate, among other things, our understanding of substances and forces, he wrote:

> the *multitude* of past mistakes [in the history of science] concerning substances and forces prove in fact that the *corresponding concepts are not fixed and determinate in the human*

[12] "Wir leben einmal in Relationen, und bedürfen nichts weiter." Johann F. Herbart, *Allgemeine Metaphysik* (Königsberg: Unzer, 1829), 415.

[13] Herbart developed a novel psychology, which figures in histories of the discipline as transitional towards scientific psychology. He is also very relevant in the history of pedagogy. See: Erhard Scholz, "Herbart's influence on Bernhard Riemann," *Historia Mathematica* 9, no. 4 (1982): 413–40.

[14] Twentieth century philosophers have projected their ideals back, tracing their own genealogy in histories that prefer to choose, as representative of the nineteenth century *Zeitgeist*, authors linked to the lines Georg W. F. Hegel–Karl Marx or Arthur Schopenhauer–Friedrich Nietzsche.

[15] "[…] aber ein paar unendliche leere Gefäße hinzustellen, in welche die Sinne ihre Empfindungen hineinschütten sollten, ohne irgend einen Grund der Anordnung und Gestaltung, das war eine völlig gehaltlose, nichtssagende, unpassende Hypothese." (Johann Friedrich Herbart, "Psychologie als Wissenschaft: neu gegründet auf Erfahrung, Metaphysik, und Mathematik. Erster, synthetischer Theil. 1824," in *Johann Friedrich Herbart's Sämtliche Werke in chronologischer Reihenfolge*, ed. Karl Kehrbach, vol. 5 (Langensalza: Hermann Beyer, 1890), 177–434, 428).

> *spirit, that they are not at all categories or innate concepts, but mutable products of a reflective thought* [Nachdenken] *stimulated by experience and altered by all kinds of opinions.*[16]

The mathematician, philosopher and scientist Riemann accepted wholeheartedly this anti-apriorist point of view, the evolutionary gradualism regarding scientific theories, and the hypothetico-deductive scheme of 'experience' and 'reflection.' Such a viewpoint, together with his noteworthy independence of thought, allowed him to advance ideas that would take decades before they seemed reasonable to other scientists. In particular, he went beyond the positivism that was dominant in his lifetime and dared to suggest that the theory of gravitation—then regarded as definitive—would be abandoned for other more adequate theories. Furthermore, he proposed extremely novel geometric ideas that would precisely establish the mathematical framework for Einstein's theory of gravitation (General relativity).

Riemann did not follow Herbart in all areas of philosophy. He rejected in particular his psychological theory and his metaphysics, to develop viewpoints that indeed turn out to emphasize relationalism even more. (E.g., Herbart established a version of the monadology, considering those immaterial, essentially active beings as substances, but Riemann replied that the supposed substantial nature of Herbart's 'monads,' the *Realen*, is contradicted by the central attributes the philosopher himself assigns to them.) Although he rejected important parts of the Herbartian conception of space and the continuum, it is beyond doubt that Herbart transmitted to him a relationalist conception of space, far from that of Kant, and that is related to the great contribution of Riemann to geometry.

Here, however, we are interested primarily in the notion of *Abbildung*, as Riemann elaborated it in his function theory, and its impact upon a pictorial or modelistic conception of scientific representation. Two themes where there seems to be a link between the author's mathematics and his philosophical ideas.

Herbart regarded mathematics as particularly close to philosophy, considering it quite possible to give a philosophical treatment of mathematics, so much so that "treated philosophically, it [mathematics] becomes itself a part of philosophy."[17] To some extent, Riemann's contributions can be regarded as a realization of such a viewpoint, a truly deep realization, quite different from what Herbart might have been able to imagine. He devoted himself to the search for basic concepts around which to restructure and reorganize whole areas of mathematics; concepts which allowed him to dig deeper into its foundations. He was convinced that mathematical research "starting from general concepts" could make a decisive contribution to scientific thinking, preventing it from "being hampered by too narrow views," so

[16] "Denn die *Mannigfaltigkeit* der Irrthümer über Substanzen und Kräfte beweist factisch, *daß die Begriffe hievon im menschlichen Geiste nicht vest stehn, daß sie keinesweges Kategorien oder angeborne Begriffe sind, sondern wandelbare Erzeugnisse eines durch die Erfahrung aufgeregten, durch allerley Meinungen umhergeworfenen, Nachdenkens*" (Johann Friedrich Herbart, "Psychologie als Wissenschaft: neu gegründet auf Erfahrung, Metaphysik, und Mathematik. Zweiter, analytischer Theil. 1825," in *Johann Friedrich Herbart's Sämtliche Werke in chronologischer Reihenfolge*, ed. Karl Kehrbach, vol. 6 (Langensalza: Hermann Beyer, 1892), 1–338, 198).

[17] See: Scholz, "Herbart's Influence on B. Riemann," 425, orig. German on 437.

that "progress in knowledge of the interdependence of things [may not be] checked by traditional prejudices."[18] So it was that he came to propose ideas related to the modern concepts of set and mapping, which in turn he tracked down and followed to the roots of scientific thinking.

As a mathematician, Riemann became famous early in life due to his contributions to 'function theory,' that is, complex analysis.[19] A few words have to be said on this topic in order to understand the links with the pictorial or modelistic conception of knowledge. Riemann thought it necessary in function theory to get away from a calculational mathematics based on formulas, opting instead for a highly conceptual approach, of the kind I have previously called abstract. The general concept that would offer the key to a new foundation of function theory was that of *analytic function*, more specifically what we call a holomorphic function.[20] This concept was defined by means of a very general characteristic property, the simple differentiability of the function around any given point (given by the Cauchy-Riemann equations). From that starting point, in order to characterize each particular function from a 'bird's eyes' view, Riemann developed a highly original approach employing geometric elements (actually topological, a discipline that he pioneered) together with elements inspired by mathematical physics. The whole approach was *conceptual*, insofar as the particular formulas employed by analysts to determine the functions (infinite series, integrals) ought to appear only at the end, as a *result* of the general abstract theory. With this, Riemann marked a turning point along the path toward twentieth century abstract mathematics.

All of that had another consequence that caught Riemann's attention: if the domain of the complex variable is conceived as a plane, the Gaussian plane, the function establishes a correspondence between two planes.[21] And holomorphic functions establish, precisely, a *conformal mapping* (*Abbildung*), that is, they "apply the minimal parts of a surface onto the other, so that the image is similar to the original in its minimal parts."[22] In other words, an infinitesimal triangle has

[18] Riemann, "On the Hypotheses which Lie at the Foundation of Geometry" (1854/1868); trans. William Kingdon Clifford (revised) in *From Kant to Hilbert*, ed. William Ewald, vol. 2 (Oxford: Claredon Press, 1996), 652–61, here 652.

[19] Based on the numbers $a + bi$, with i the famous complex unit such that $i^2 = -1$; the set of all complex numbers can be seen as a 2-dimensional geometric system, the complex plane.

[20] 'Holomorphic' was a term employed by Charles Briot and Jean-Claude Bouquet, students of Cauchy, in their *Théorie des fonctions doublement périodiques* (Paris, 1859); it stems from Greek ὅλος (*holos*), 'whole,' and μορφή (*morphē*), 'form' or 'appearance.'

[21] If the function $w = f(z)$ is continuous, says Riemann in his dissertation, "it will be possible to conceive that dependence of magnitude w from z as a mapping [*Abbildung*] of the plane A onto plane B;" in German: "Man wird sich also diese Abhängigkeit der Grösse w von z vorstellen können als eine Abbildung der Ebene A auf der Ebene B." (Bernhard Riemann, "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse," in *Bernhard Riemann's Gesammelte mathematische Werke und wissenschaftlicher Nachlass*, 2nd ed., ed. Heinrich Weber (Leipzig: Teubner, 1876), 3–47, here 5).

[22] This is a phrase from the title of a paper by Gauss published in 1825, which inspired his disciple; see: ibid., "Ueber diesen Gegenstand sehe man: 'Allgemeine Auflösung der Aufgabe:

as its image, in the other plane, another triangle with the same angles and proportional sides. We find here, in a geometric and highly pictorial version, the ideas of mapping (*Abbildung*), original and image (*Bild*)—the function is the *pictor*, who paints and conforms. There is an 'isomorphism' between the complex variable and its holomorphic correlate, the forms are preserved. Related to this topic, Riemann proposed and proved, though not with sufficient rigor, his celebrated *mapping theorem* (that there is a conformal mapping between any simply connected domain and the unit disk), one of the most important results in complex analysis.

Seen from a higher standpoint, *mutatis mutandis*, something analogous occurs with any structure-preserving function or mapping, even if the correlation may eliminate all traces of geometric form. Even so, the very existence of a continuous functional relation will establish profound structural analogies between one and the other domain. The idea is very philosophical, but without a doubt it occupied Riemann and inspired some of his epistemological considerations. It also became dear to Dedekind, encouraging him to amplify the pictorial terminology of *Bild* and *Abbildung* and use it for any function or mapping in general.

We can now go on to see how the idea finds application in Riemann's epistemology. The hypothetico-deductive method of Riemann is based on the notion of truth, understood along classical lines (following Aristotle) as the correspondence with facts. Thus, he writes:

> I. When is our conception of the world true?
> 'When the connection between our mental representations [*Zusammenhang unserer Vorstellungen*] corresponds with the connection between things.'
> II. How can the connection between things [*Zusammenhang der Dinge*] be established?
> 'Starting from the connections between phenomena [*Erscheinungen*].'[23]

Even though the ideas expressed above may seem well known, at this level too Riemann manages to introduce some innovative views that have interesting counterparts in the twentieth century. In a Herbartian vein, he indicates that the interesting correspondence is not between simple elements in our conceptual systems and the simple real elements but between their *interrelations*. The relations between elements in our image of the world must faithfully reflect the relations between things. Commenting upon no. I he writes:

'Die Theile einer gegebenen Fläche so abzubilden, dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird, von C. F. Gauss. [...].'" A conformal mapping preserves angles and forms around each point, but it may deform figures and affect their size.

[23] Bernhard Riemann, "Fragmente philosophischen Inhalts," in *Bernhard Riemann's Gesammelte mathematische Werke und wissenschaftlicher Nachlass*, 2nd ed., ed. Heinrich Weber (Leipzig: Teubner, 1892), 507–38, here 523. "I. Wann ist unsere Auffassung der Welt wahr? 'Wenn der Zusammenhang unserer Vorstellungen dem Zusammenhang der Dinge entspricht.' […] II. Woraus soll der Zusammenhang der Dinge gefunden werden? 'Aus dem Zusammenhange der Erscheinungen.'"

> The elements in our image of the world [*Bild von der Welt*] are wholly different from the corresponding elements of the represented real. They are something in us, the real elements are something outside us. But the relations [*Verbindungen*] between the elements in the image and in the represented [*Elementen im Bilde und im Abgebildeten*] must coincide if the image is to be true. The truth of the image is independent of its degree of finesse, it does not depend on whether the image-elements represent greater or lesser amounts of the real. But their relations must correspond, in the image one ought not assume an immediate action [*unmittelbare Wirkung*] between two elements, when in reality there is only a mediated one. If so, the image would be false and need rectification.[24]

Riemann's explanations on this point are highly reminiscent of the well-known *Bildtheorie* of thought and language proposed by Hertz, or also that of Wittgenstein at the beginning of his *Tractatus.* Both employ the very Riemannian terminology of 'Bild' and 'Abbildung' (just quoted), which was first published in 1876. It would be interesting to know whether Wittgenstein, besides being influenced by Hertz, could have read with some care Riemann's *Gesammelte mathematische Werke* (1876, 1st ed.). Probably not. What seems highly unlikely is that Hertz did *not* read them: among other indications, in his *The Principles of Mechanics* (*Die Prinzipien der Mechanik*, 1894) Hertz introduces a "geometry of systems of points" ("Geometrie der Punktsysteme") that is a form of *n*-dimensional Riemannian geometry.[25]

Reflections in Science and Mathematics … and New Flashes

The highly pictorial terminology of image and representation, *Bild* and *Abbildung*, produces at the same time an inversion—a semantic displacement which distances

[24] Riemann, "Fragmente," 523: "Die Elemente unseres Bildes von der Welt sind von den entsprechenden Elementen des abgebildeten Realen gänzlich verschieden. Sie sind etwas in uns; die Elemente des Realen etwas ausser uns. Aber die Verbindungen zwischen den Elementen im Bilde und im Abgebildeten müssen übereinstimmen, wenn das Bild wahr sein soll. Die Wahrheit des Bildes ist unabhängig von dem Grade der Feinheit des Bildes; sie hängt nicht davon ab, ob die Elemente des Bildes grössere oder kleinere Mengen des Realen repräsentiren. Aber die Verbindungen müssen einander entsprechen; es darf nicht im Bilde eine unmittelbare Wirkung zweier Elemente auf einander angenommen werden, wo in der Wirklichkeit nur eine mittelbare stattfindet."

[25] See: Jesper Lützen, "Geometrising Configurations. Heinrich Hertz and his Mathematical Precursors," in *The Symbolic Universe: Geometry and Physics 1890–1930*, ed. Jeremy Gray (Oxford: Oxford University Press, 1999), 25–46, also for qualifications (p. 39–40) concerning Hertz's conception of geometry, which seems to have been rather Kantian and not Riemannian. Heinrich Hertz, *Die Prinzipien der Mechanik* (Leipzig: J. A. Barth, 1894), 36: "Die Sammlung und Ordnung aller hier auftretenden Beziehungen gehört in die Geometrie der Punktsysteme und die Entwickelung dieser Geometrie hat eigenen mathematischen Reiz; wir verfolgen dieselbe aber nur soweit als es der augenblickliche Zweck der physikalischen Anwendung erfordert. Da ein System von *n* Punkten eine *3n*fache Mannigfaltigkeit der Bewegung darbietet, welche aber durch die Zusammenhänge des Systems auch auf jede beliebige Zahl vermindert werden kann, so entstehen viele Analogien mit der Geometrie eines mehrdimensionalen Raumes, welche zum Teil so weit gehen, daß dieselben Sätze und Bezeichnungen hier und dort Bedeutung haben können."

it from any naïve conception. In fact, Riemann and Dedekind, following a form of Leibnizian and Herbartian relationalism, insist upon the idea that the image representing some real element does not need to bear resemblance to it. The form that is preserved in adequate representations is not the form of the elements but only of the relations between them. Thus, the suggestive terminology, seemingly so close to familiar ideas, contains a new idea that is far from common sense: the symbolic, not iconic nature of scientific images,[26] the conventionality of models and scientific representations, will reappear in several scientific achievements of the time, which in turn will promote it.

Two historical experiences of great epistemological depth came together in the nineteenth century: in mathematics the move beyond the Euclidean geometric framework, in physics the abandonment of mechanistic images of the world. Both times we find some German-speaking authors centrally involved, whom I will invoke soon, i.e., Riemann, Helmholtz, Hertz (together, of course, with others of various nationalities, Gauss, Nikolai Lobachevsky, János Bolyai, Michael Faraday, James C. Maxwell, Ernst Mach, Hendrik A. Lorentz, Henri Poincaré, Einstein). A third achievement not entirely unrelated to these two, which I can only mention here, is a deeper reflection on sensations and perceptual experience upon new experimental bases, in the hands of Helmholtz, Mach and others.

If there is a mathematical or scientific sign that we regard as prototypically iconic, this is the geometric figures, the "triangles, circles" and other "characters" which Galileo Galilei deemed elements of the language in which the great book of nature is written.[27] The geometrization of the physical image of the world, due to Johannes Kepler and Galilei, seemed thus to offer a direct route to the inner side of things, an intimate familiarity with real forms. Precisely for this reason, perhaps the most pronounced change in scientific epistemology, during the nineteenth century, was the abandonment of such naïve views. In the terms we have established, one can say briefly and concisely: the key was the very novel idea that *geometric images of physical reality are not iconic but symbolic*. At least when moving away from the middle ranges of experience, characteristic of our bodies and common experience, and going towards the very small (atomic) or the very large (cosmic range).

The problem was implicated already in the idea of a straight line, so hard to make conceptually precise since the essential point in the behavior of straight lines—expressed in the parallel postulate—is not a local property but a global

[26] I follow here the terminology of Charles Sanders Peirce, who employs 'sign' as a generic term, under which icons, indices and symbols fall; the icon, unlike the symbol, bears resemblance with the represented. The reader should know that Riemann's work antedates Peirce, and also Dedekind's is independent.

[27] Galileo Galilei, *Il Saggiatore* (1623), in *Opere*, vol. VI, 232. Yet their relation with the designated is far from trivial, since geometric icons are operated upon, not as they are depicted but as conceived; thus they may lack empirically given referents (see: José Ferreirós, "Ancient Greek Mathematics: A Role for Diagrams," Chap. 5 in *Mathematical Knowledge and the Interplay of Practices* (Princeton: Princeton University Press, 2015), 112–52).

one, which involves the infinitely large. Gauss already saw this problem and realized that it was an empirical matter to determine whether it is Euclid's geometry, or perhaps that of Lobachevsky, that most adequately symbolizes physical relations in the largest scales. Riemann was his successor here, and at the same time he left Gauss astonished with the richness of geometric possibilities exposed with the introduction of Riemannian manifolds, i.e., with spaces of variable curvature (tensor), in the famous lecture "On the Hypotheses which Lie at the Foundation of Geometry" ("Über die Hypothesen, welche der Geometrie zu Grunde liegen," 1854, published 1868). Within this framework, *geodesics* (lines of minimal length) play the role of straight lines, and if the space curvature is equal at all points, we come down to more intuitive geometries like those of Euclid and Lobachevsky (and the so-called 'elliptical' geometry of Riemann). Eventually, in the limit case when curvature is always zero, we find Euclidean geometry, and geodesics coincide with the straight lines of our intuition. The dialectics between local and global, which has played such a central role in mathematical thought ever since, was a fundamental issue for Riemann—maybe the first mathematician to take this step. Given that I cannot enter into this topic any further, I refer the reader to Riemann's mathematical works.[28]

The question of the local or extremely small was crucial for nineteenth century science, since the core of mathematics was formed by real and complex analysis, developed around infinitesimal calculus, and at the core of physics one found laws expressed by differential equations.[29] Once more, Riemann saw this very clearly, and went as far as to say that "truly elementary laws can only occur in the infinitely small, only for points in space and time."[30] At the atomic scale "the concepts of a solid body and of a ray of light" cease to be valid, but they had been the basis for our usual metric and geometric determinations, and for that reason the geometry valid in the "infinitely small" may not be Euclidean. And so it should be assumed "if we can thereby obtain a simpler explanation of phenomena."[31] The issue was of utmost relevance for physical theory then, since the central decades of the nineteenth century saw great efforts to find a correct theory of electromagnetism. This theory, linked with the name of Maxwell, became the greatest step forward in physics after Newton and before the quantum theories.

[28] English edition: Bernhard Riemann, *Collected papers*, trans. Roger Baker, Charles Christenson, and Henry Orde (Heber City: Kendrick Press 2004); German edition: *Bernhard Riemann's Gesammelte mathematische Werke und wissenschaftlicher Nachlass*, 2nd ed., ed. Heinrich Weber (Leipzig: Teubner, 1892). Many titles deal with such questions, among them e.g. the elegant book *Poetry of the Universe* by Robert Osserman (New York: Anchor Books, 1995). See also: Detlef Laugwitz, *Bernhard Riemann 1826–1866* (Basel: Birkhäuser, 1996).

[29] Partial differential equations like the wave equation, those of Laplace and Hamilton in classical mechanics, the electromagnetic equations of Ampère and later Maxwell's, etc.

[30] Cited in Thomas Archibald, "Riemann and the Theory of Electrical Phenomena: Nobili's Rings," *Centaurus* 34, no. 3 (1991): 247–71, here 269.

[31] Riemann, "On the Hypotheses," 661.

There developed in this field a complex dialectics between the requirement of intelligibility and exploitation of the new theoretical freedom implied in the pictorial-abstract conception of models, a freedom made explicit in the new mathematical tools. Riemann too, like Faraday before and Maxwell after him, was in favor of field theories instead of distance-action theories. They tried to avoid something that was hardly compatible with physical intuition, namely “that one body may act upon another at a distance through a vacuum without the mediation of anything else”—an idea that Newton himself adopted only as a useful expedient, but which he regarded as “so great an absurdity” as to be unacceptable to a person “who has in philosophical matters any competent faculty of thinking.”[32] Riemann and Maxwell put in its place the notion of local action between ethereal points, in accordance with some partial differential equations. The celebrated four equations of Maxwell (which do not correspond to what he actually published) enjoyed unprecedented success not only in the explanation of electricity, magnetism and their interrelations but also because they reduced light waves to an electromagnetic phenomenon, and they predicted the existence of radio waves. Interestingly, Riemann assisted Weber and Rudolf Kohlrausch in some experiments where the measurement of an electromagnetic constant led to a result very close to the speed of light. Subsequently, he speculated on its theoretical significance, writing a paper for the Göttingen Scientific Society where he communicated his discovery of a connection between electricity and light.[33]

As well known, it was precisely Helmholtz’s disciple Hertz who managed in 1888 to build an apparatus that emitted and detected electromagnetic waves, setting the basis for a technology that dominates our lives today. Like Maxwell and Helmholtz, like Isaac Newton himself, Hertz too was both a great theoretician and a great experimenter. We shall see in a moment that he followed Riemann’s path in adopting the abstract-pictorial conception of scientific models. But now it will be interesting to consider another side of his work.

In the search for a theory, Maxwell employed relatively simple mechanical models to back his equations. It was the age of mechanicism, the belief in the unlimited power of attractive and repulsive mechanical actions to explain all phenomena, from planetary movements and light to nerve stimuli. Maxwell himself was aware of the fact that his imaginary mechanisms could not account fully for the actions described in the equations he was handling. In effect the problem was endemic: the ether hypothesis adopted in wave theories of light (from Augustin J. Fresnel onwards) had proven to be intractable—if the medium was mechanical, luminiferous ether seemed to enjoy inconsistent properties.

Hertz inherited this problem through Helmholtz’s work on electromagnetism, and within a few years he came to the conviction that Maxwell’s theory was right

[32] Third letter to Bentley, mentioned by Riemann (see: Riemann, “Fragmente,” 534).

[33] Bernhard Riemann, “Ein Beitrag zur Electrodynamik (1858),” in *Bernhard Riemann’s Gesammelte mathematische Werke und wissenschaftlicher Nachlass*, ed. Heinrich Weber (Leipzig: Teubner, 1876), 270–75, here 270.

and had put an end to the dominance of Newtonian action at a distance, inaugurating the reign of physical fields. Actually, Hertz became convinced of the need to go one step further, a small step from the technical point of view, but huge as it involved a big shift towards abstraction, with obvious epistemological implications. He abandoned the hope to find mechanistic interpretations of the field models and expressed the idea in eloquent terms: "To the question 'what is Maxwell's theory?' I know of no shorter and more concrete answer than the following: Maxwell's theory is the system of Maxwellian equations." "[… They must be] considered as hypothetical assumptions, so that their probability depends upon the great number of natural laws which they encompass."[34]

With these words he was taking note of the new status attained by physical theory around 1900; mechanicism was abandoned, the age of *Anschauung* or visualizable models based on mechanics had ended, the era of mathematical abstraction was beginning.[35] Hertz himself acknowledged that, as a result, the theory acquires a very "abstract and colourless" appearance.[36]

The situation can be regarded, with certain qualifications, as highly Pythagorean. Weyl has remarked how attempts to provide physical equations with an immediate physical sense, through the means of a suggestive language, do not manage to go farther than more or less fortunate metaphors, analogies which are always a bit unsatisfactory, vague, imprecise, superficial. Hence, the common experience that physical theory can only be adequately understood by those who have command of its mathematical structure (together with the way it is interpreted and applied in paradigmatic laboratory cases, in real and also imaginary experiments).

Helmholtz and Hertz

Let me go back to our story about the pictorial approach (*Bildtheorie*) to scientific theorizing. In the case of Wittgenstein, Hertz seems to have been the crucial influence, although we know that other authors (like Ludwig Boltzmann) may have played a role. Some experts have indicated that Hertz inherited his conception

[34] Heinrich Hertz, *Electric Waves*, trans. Daniel Evan Jones (New York: Dover, 1962), 21, 139; German: "Auf die Frage 'Was ist die Maxwell'sche Theorie' wüsste ich also keine kürzere und bestimmtere Antwort als diese: Die Maxwell'sche Theorie ist das System der Maxwell'schen Gleichungen." "[Es erscheint folgerichtig, dieselben] als eine hypothetische Annahme zu betrachten und ihre Wahrscheinlichkeit auf der sehr grossen Zahl an Gesetzmässigkeiten beruhen zu lassen, welche sie zusammenfassen." (Heinrich Hertz, *Untersuchungen über die Ausbreitung der elektrischen Kraft* (Leipzig: Barth, 1892), 23, 148).

[35] A great popular exposition of this topic, which explains it without a single mathematical formula, can be found in Albert Einstein and Leopold Infeld, *The Evolution of Physics* (Cambridge: Cambridge University Press, 1938).

[36] Hertz, *Electric Waves*, 28.

partially from the polymath Helmholtz, his master at the Friedrich-Wilhelms-Universität Berlin.[37] However, the truth may be different, and the influence of Riemann on Hertz seems quite likely.

Curiously, Riemann and Helmholtz form a couple that affords much insight into the deep tendencies of scientific thinking in the nineteenth century. Philosophically both were heirs to reactions toward Kantian epistemology, they enjoyed deep knowledge of physics and mathematics, and their conceptions pointed in different directions, but both would be of great influence in the twentieth century.

Helmholtz represents a kind of scientific neo-Kantianism, oriented towards empiricism, and since decades he has been hailed as a predecessor of logical empiricism. Riemann was a Herbartian and thus radically against Kantian apriorism, but he was no less scientific; his tendency was to emphasize the hypothetical nature of any scientific theorizing, its character as a 'symbolic construction' of the world. To put it in simplistic terms: if Helmholtz was a predecessor of the Vienna Circle, Riemann opened the way towards positions such as Karl Popper's, or also Weyl's (despite matters of emphasis and nuance).

The divergence was already quite clear in the reactions of the one scientist to the core contributions of the other. One of them is found in the last article Riemann was preparing, a reply to Helmholtz's great book *The Theory of the Sensations of Tone as a Physiological Basis for Musical Theory* (*Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik*, 1863).[38] Another, inversely, took place after the publication of Riemann's epoch-making "On the Hypotheses which Lie at the Foundations of Geometry" ("Ueber die Hypothesen, welche der Geometrie zu Grunde liegen," 1868). Helmholtz was quick to reply in the same year, and his paper was named "On the Facts which lie at the Foundations of Geometry" ("Ueber die Thatsachen, die der Geometrie zum Grunde liegen," 1868).[39] *Tatsachen* versus *Hypothesen—facts* that would speak for geometries of constant curvature, against the daring *hypotheses* that opened up the world of Riemannian manifolds. And yet Helmholtz's supposed facts would be contradicted by a new empirically supported theory, Einstein's general relativity, in which Riemann's geometries of varying curvature featured centrally.

[37] See: Gregor Schiemann "The Loss of World in the Image: Origin and Development of the Concept of Image in the Thought of Hermann von Helmholtz and Heinrich Hertz," in *Heinrich Hertz: Classical Physicist, Modern Philosopher*, ed. Davis Baird, Richard I. G. Hughes, and Alfred Nordmann (Dordrecht: Kluwer, 1998), 25–38, and Michael Heidelberger, "From Helmholtz's Philosophy of Science to Hertz's Picture-Theory," in *Heinrich Hertz: Classical Physicist, Modern Philosopher*, ed. Davis Baird, Richard I. G. Hughes, and Alfred Nordmann (Dordrecht: Kluwer, 1998), 9–24.

[38] See: Bernhard Riemann, "Mechanik des Ohres," in *Bernhard Riemann's Gesammelte mathematische Werke und wissenschaftlicher Nachlass*, ed. Heinrich Weber (Leipzig: Teubner, 1876), 316–28. See also: Hermann von Helmholtz, *Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik* (Braunschweig: Vieweg und Sohn, 1863).

[39] Hermann von Helmholtz, "Ueber die Thatsachen, die der Geometrie zum Grunde liegen," *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen* 9 (1868): 193–221.

Those who think that Hertz's *Bildtheorie* stems from his master cite some passages from an important conference that Helmholtz gave in 1878, "The Facts in Perception" ("Die Tatsachen in der Wahrnehmung"). He insists upon an idea that, no doubt, is linked with the topics we have been revisiting:

> Our sensations are indeed effects produced in our organs by external causes: and how such an effect expresses itself naturally depends quite essentially upon the kind of apparatus upon which the effect is produced. Inasmuch as the quality of our sensation gives us a report of what is peculiar to the external influence by which it is excited, it may count as a *sign* [*Zeichen*] of it, but not as an *image* [*Abbild*]. For from an image one requires some kind of alikeness with the object of which it is an image—from a statue alikeness of form, from a drawing alikeness of perspective projection in the visual field, from a painting alikeness of colors as well. But a sign need not have any kind of similarity at all with what it is the sign of. The relation between the two of them is restricted to the fact that like objects exerting an influence under like circumstances evoke like signs, and that therefore unlike signs always correspond to unlike influences. […] Since like things are indicated in our world of sensations by like signs, an equally regular sequence will also correspond in the domain of our sensations to the sequence of like effects by law of nature upon like causes.[40]

We find here an emphasis on the idea of functional correspondence between stimulus and sensation, and at the same time insistence on the non-resemblance between the elements, i.e., the 'conventional' or dissimilar nature of secondary qualities.[41] But Helmholtz chose his terminology in such a manner that he rejects the pictorial conception—they can be regarded as signs (*Zeichen*), he says, but not as images, not as *Abbilder*. And it is so because he insists (to explain it this way) on the old

[40] Hermann von Helmholtz, "The Facts in Perception," in *From Kant to Hilbert*, ed. William Ewald, vol. 2 (Oxford: Claredon Press 1996), 689–727, here 695–96; German: "Unsere Empfindungen sind eben Wirkungen, welche durch äussere Ursachen in unseren Organen hervorgebracht werden, und wie eine solche Wirkung sich äussert, hängt natürlich ganz wesentlich von der Art des Apparats ab, auf den gewirkt wird. Insofern die Qualität unserer Empfindung uns von der Eigenthümlichkeit der äusseren Einwirkung, durch welche sie erregt ist, eine Nachricht giebt, kann sie als ein *Zeichen* derselben gelten, aber nicht als ein *Abbild*. Denn vom Bild verlangt man irgend eine Art der Gleichheit mit dem abgebildeten Gegenstande, von einer Statue Gleichheit der Form, von einer Zeichnung Gleichheit der perspectivischen Projection im Gesichtsfelde, von einem Gemälde auch noch Gleichheit der Farben. Ein Zeichen aber braucht gar keine Art der Aehnlichkeit mit dem zu haben, dessen Zeichen es ist. Die Beziehung zwischen beiden beschränkt sich darauf, dass das gleiche Object, unter gleichen Umständen zur Einwirkung kommend, das gleiche Zeichen hervorruft, und dass also ungleiche Zeichen immer ungleicher Einwirkung entsprechen. […] Da Gleiches in unserer Empfindungswelt durch gleiche Zeichen angezeigt wird, so wird der naturgesetzlichen Folge gleicher Wirkungen auf gleiche Ursachen, auch eine ebenso regelmässige Folge im Gebiete unserer Empfindungen entsprechen." (Hermann von Helmholtz, *Die Tatsachen in der Wahrnehmung* (Darmstadt: Wissenschaftliche Buchgesellschaft, 1959), 18–19).

[41] This old theme was discussed by Galileo and Descartes, but it can be found already in Greek philosophy. Democritus says: "By convention sweet and by convention bitter, by convention hot, by convention cold, by convention color; but in reality atoms and void. [And the senses reply:] Unhappy mind! You obtain your beliefs from us, and you want to end us? Our fall will be your ruin." (Hermann Diels and Walther Kranz, *Die Fragmente der Vorsokratiker*, vol. 2 (Dublin: Weidmann, 1972–74), fragment 125).

style of figurative painting, not imagining the turn towards abstraction that will come. Specialists thus have to talk about Helmholtz's *Zeichentheorie*, which is not the same as the *Bildtheorie* we find in Riemann and Hertz.

Riemann's option is the opposite one, more forward looking, which is what Hertz will follow. And the point is that this new opposition between Riemann and Helmholtz seems to me quite coherent with the tension between the hypothetico-deductivism of the former and (the relative) empiricism of the latter. And it is also interesting to realize that Hertz did not share his Berlin master's tendency to present everything as derived from 'facts,' insisting instead on the hypothetical character of physical principles. One should make clear that Riemann's epistemological fragments became public in 1876, with the first edition of his *Werke*, and the related ideas of Dedekind were published in a purely logico-mathematical context in 1888. All of it years before the key text by Hertz (1894).

I will now quote the celebrated passage of Hertz's introduction to *The Principles of Mechanics* (1894), where he speaks of theories and scientific hypotheses as images, *Bilder*:

> The most direct, and in a sense the most important, problem which our conscious knowledge of nature should enable us to solve is the anticipation of future events, so that we may arrange our present affairs in accordance with such anticipation. As a basis for the solution of this problem we always make use of our knowledge of events which have already occurred, obtained by chance observation or by prearranged experiment. In endeavoring thus to draw inferences as to the future from the past, we always adopt the following process. We form for ourselves inner images [*innere Scheinbilder*] or symbols of external objects; and the form which we give them is such that the necessary consequents of the images [*Bilder*] in thought [*denknotwendigen Folgen*] are always the images of the necessary consequents in nature [*naturnotwendigen Folgen*] of the things pictured [*abgebildeten Gegenstände*]. In order that this requirement may be satisfied, there must be a certain conformity between nature and our thought. Experience teaches us that the requirement can be satisfied, and hence that such a conformity does in fact exist.[42]

[42] Heinrich Hertz, *The Principles of Mechanics, Presented in a New Form* (New York: Dover, 1956), 1; German: "Es ist die nächste und in gewissem Sinne wichtigste Aufgabe unserer bewußten Naturerkenntnis, daß sie uns befähige, zukünftige Erfahrungen vorauszusehen, um nach dieser Voraussicht unser gegenwärtiges Handeln einrichten zu können. Als Grundlage für die Lösung jener Aufgabe der Erkenntnis benutzen wir unter allen Umständen vorangegangene Erfahrungen, gewonnen durch zufällige Beobachtungen oder durch absichtlichen Versuch. Das Verfahren aber, dessen wir uns zur Ableitung des Zukünftigen aus dem Vergangenen und damit zur Erlangung der erstrebten Voraussicht stets bedienen, ist dieses: wir machen uns innere Scheinbilder oder Symbole der äußeren Gegenstände, und zwar machen wir sie von solcher Art, daß die denknotwendigen Folgen der Bilder stets wieder die Bilder seien von den denknotwendigen Folgen der abgebildeten Gegenstände. Damit diese Forderung überhaupt erfüllbar sei, müssen gewisse Übereinstimmungen vorhanden sein zwischen der Natur und unserem Geiste. Die Erfahrung lehrt uns, daß die Forderung erfüllbar ist und daß also solche Übereinstimmungen in der That bestehen." (Heinrich Hertz, *Die Prinzipien der Mechanik in neuem Zusammenhange dargestellt* (Leipzig: Barth, 1894), 1).

The key sentence, as one can see, combines some terminology close to Helmholtz (symbols) with Riemann's pregnant terminology, which has become famous while its origin is forgotten. Maybe it is difficult to grasp on a first read what Hertz is expressing: "the necessary consequents of the images in thought are always images of necessary natural consequents," but it is an elementary mathematical idea, typical of mappings and analyzed by Dedekind in full precision. Call a and b the two natural things that are one consequence of the other, and call $f(a)$ and $f(b)$ their correlated images. As Hertz states, the key condition is: If $f(a)$ follows from $f(b)$ in the image, then a causes b in nature.[43] (R).

He continues insisting that the images he is talking about are our mental representations of things [*Vorstellungen*], and that they are in conformity with things "in a single important respect," namely that they satisfy requirement (R). There need not exist any other kind of conformity and in effect we are not in a position to know whether our mental representations may exhibit any other kind of conformity. It is the same idea Riemann insisted on.

In fact, things are even more complex, as scientists and philosophers of science would discover in the twentieth century.[44] The relations (be they causal or not) that are ascertained by empirical means, "by chance observation or by prearranged experiment," are not precisely the relations between the elements that conform to abstract theory. The elements in this abstract theory, its objects (assumptions) and principles, form a complex whole. Considering this *holos*, the whole, often in combination with one or more auxiliary theories, one establishes particular *models* for certain kinds of concrete systems. These models act as mediators, and it is them that we use to infer, by logical or mathematical or computational means, some predictions that concern the relations one ought to expect to hold among natural phenomena. The validity of requirement (R)—in Hertz's or maybe better in Riemann's version—for these predictions derived from models does not quite imply conformity between the relations of high theory and the relations between natural things.

Many scientists, in particular theoretical physicists, have a strong tendency to think that their theories are even more real than phenomena themselves. This tendency stems in the last analysis from a somewhat primitive epistemology, from a lack of acknowledgment of the subtle relations between the theories (that they handle so well in practice), the models subsequently established, and the data obtained from their experimental colleagues. It is the theoretician's *hybris* and a form of residual Platonism (ideas being more real than appearances), paradoxically

[43] Riemann, as you will remember, thought it this way: If $f(a)$ is related with $f(b)$ in the image, then a is related with b in nature; and if a is not related immediately with b, then $f(a)$ will not be related immediately with $f(b)$.

[44] The locus classicus for methodological issues related to holism is Pierre Duhem's work, *The Aim and Structure of Physical Theory* (Princeton: Princeton University Press, 1954; orig. 1906); also relevant is Willard Van Orman Quine, "Two Dogmas of Empiricism," *Philosophical Review* 60, no. 1 (1951): 20–43. But the topic reappears in many other twentieth century authors, like Thomas Kuhn for instance.

the outcome of a too pragmatic, hence, naïve orientation. The subtle equilibrium that scientists like Hertz or Maxwell managed to attain, probably thanks to their mastery of both theoretical and experimental practices, and their acquaintance with conceptual and philosophical difficulties, seems to have been lost in our age of hyper-specialized scientists.

Longue Durée

In 1910, Ernst Cassirer published *Substanzbegriff und Funktionsbegriff*, in simplified translation *Substance and Function*, an important book which aimed to account for a fundamental transformation in the logic of scientific thinking, involving all fields of knowledge and especially the natural sciences. Chapter 1 explains the general idea, before proceeding to trace its course through several sciences, and begins by talking about the Aristotelian way of representing world phenomena, focusing on "generic concepts" (basically for classification) whose metaphysical counterpart are substances;[45] to end up emphasizing the new "logic of the mathematical concept of function." Cassirer wrote:

> 'Every mathematical function represents a universal law, which, by virtue of the successive values which the variable can assume, contains within itself all the particular cases for which it holds.' If, however, this is once recognized, a completely new field of investigation is opened for logic. In opposition to the logic of the generic concept, which, as we saw, represents the point of view and influence of the concept of substance, there now appears the *logic of the mathematical concept of function*. However, the field of application of this form of logic is not confined to mathematics alone. On the contrary, it extends over into the field of the knowledge of nature; for the concept of function constitutes the general schema and model according to which the modern concept of nature has been molded in its progressive historical development.[46]

[45] Ernst Cassirer, *Substance and Function* (New York: Dover, 1980), 8. The comment (ibid., 8) is noteworthy that all determinations of being were subordinated to the substances they inhere in, and this affected in particular the Aristotelian category of relation, "forced into a dependent and subordinate position;" relations can only be modifications of substances, they cannot alter their real "nature."

[46] Cassirer, *Substance and Function*, 20–21; German: "'[...] Denn jede Funktion stellt ein allgemeines Gesetz dar, das vermöge der successiven Werte, welche die Variable annehmen kann, zugleich alle einzelnen Fälle, für die es gilt, unter sich begreift.' Wird dies aber einmal anerkannt, so eröffnet sich damit zugleich für die Logik ein völlig neues Gebiet der Untersuchung. Der Logik des Gattungsbegriffs, die, wie wir sahen, unter dem Gesichtspunkt und der Herrschaft des Substanzbegriffs steht, tritt jetzt die *Logik des mathematischen Funktionsbegriffs* gegenüber. Das Anwendungsgebiet dieser Form der Logik aber kann nicht im Gebiet der Mathematik allein gesucht werden. Vielmehr greift hier das Problem sogleich auf das Gebiet der *Naturerkenntnis* über: denn der Funktionsbegriff enthält in sich zugleich das allgemeine Schema und das Vorbild, nach welchem der moderne Naturbegriff in seiner fortschreitenden geschichtlichen Entwicklung sich gestaltet hat." (Ernst Cassirer, *Substanzbegriff und Funktionsbegriff. Untersuchung über die Grundfragen der Erkenntniskritik* (Berlin: Bruno Cassirer, 1910), 27).

Cassirer's idea seems basically correct and highly perceptive. On the one hand, the evolution of scientific research over several centuries, from Copernicus's *De Revolutionibus* (1543) onward, involved a constant distancing from the conception of phenomena in terms of substance and accident, to the point that the substantialist conception—if not entirely superseded—was increasingly marginalized, less and less visible in the details of scientific theories and models. On the other hand, it is also true that the impact of functional thinking on logic itself has been decisive: modern logic emerged from the effort to understand the logical relations put forward in mathematics, and it is well known that Frege's logic was founded on the abandonment of the subject-predicate scheme (correlate of the notions of substance and attribute, or substance and accident) in favor of the function-argument scheme, which Frege came to regard as fundamentally logical in nature (and which became the basis for his metaphysics of functions and objects).[47]

Another side of the question, which Cassirer did not research into, but which was a topic for contemporaries like Mach and Carnap, was the dissolution of causal thinking in its primeval form. This aspect has been less reflected upon and less incorporated in our cultural consciousness, to the point that it would not be strange if the reader is surprised to read about it. The truth is that, to paraphrase Cassirer, the concepts of relation and function have given form to the general scheme and pattern according to which the contemporary conception of nature has been molded, especially since around 1850. The old and familiar scheme of cause and effect has died out. To be sure, it remains familiar and intuitive because it is the way agents like us represent the possibilities of action in their immediate environment (manipulative causation). Of course, there are many particular circumstances in which it makes sense to ask for causes, and we still do so in scientific practice, but the question of the cause is essentially pragmatic, context-dependent, and does not point toward a complete explanation.

If you ask, why is the sun eclipsed, the answer may be 'because of the moon,' and if you ask why the blood moves, we will answer 'thanks to the heart pumping;' or, why does the orbit of Uranus present some deviations, 'because of the force imparted by Neptune.' But these are only partial indications, contextually relevant given the information previously available, which do not account for the phenomenon in anything like a complete explanation. Causal explanations are very often demanded in practice,[48] but a causal explanation will never be a full explanation of the phenomenon—for which we would resort instead to a

[47] Aristotelian propositions express substantial or accidental attributions (some men are small, all humans are mortal), but they are unable to analyze elementary mathematical inferences; here is a related example given by Leibniz, of an inference which escapes Aristotle: Mary was the mother of Jesus of Nazareth, therefore Mary was the mother of God. One could go much deeper into this topic: the logic of quantifiers $\forall x$, $\exists x$ emerged in relation with critical reflections upon the foundations of analysis, a theory which focuses on the study of functions (mappings); see: José Ferreirós, "The Road to Modern Logic," *Bulletin of Symbolic Logic* 7, no. 4 (2001): 441–84.

[48] The pragmatic theory of explanation has been defended by Bas van Fraassen, *The Scientific Image* (Oxford: Clarendon Press, 1980).

relational-functional model. E.g., a three-body model complying with Newton's equations for the system sun-earth-moon, or a complex model of the autonomous nervous system and the movements of the heart and blood, and so on.

The ultimate implications of these deep changes may not yet be clear. By giving primacy to relations and functions, the question arises to what extent the *relata* must antecede relations, a topic in between science and metaphysics. Is the world made of 'bricks' (atoms, particles) or is it fundamentally made of relations? It has been philosophers, above all, who dared to suggest the radical priority of relations over *relata*.[49] Common sense still recommends the opposite idea that *relata*, the objects entering into relations, are prior, no matter how much the sciences belie the notion of permanent objects (in a strong sense) and promote the thesis of the processual and relational character of a reality whose principles are expressed through functional thinking. Maybe this antinomy, which I cannot further develop here, holds the key to a solution of many perplexities in physical theory.

Other Reflections

There are some other related reflections and bright sparks that I could comment upon here. Indeed, the topics I have revisited suggest connections that, more than once, may leave us perplexed. I am thinking above all about how to understand the links between the rise of a pictorial-abstract approach in science and mathematics, and the contemporary (albeit somewhat delayed) emergence of several forms of non-figurative painting. A flat and somewhat reductionistic perspective on the relations between art and science would insist, perhaps, in how photographic techniques emptied the meaning of traditional forms of painting, forcing painters to leave aside realistic representation and to reconsider work on the canvas in a much more autonomous way. This may well be true, but it seems only one factor among several (the function is in several variables). The developments I have reviewed suggest deeper and more philosophical lines of connection, perhaps a bit subterranean but certainly powerful.

Paul Cézanne was a contemporary of Dedekind and Hertz whose exploration of the inner space of the canvas, his free speculation about the reciprocal relations between forms and colors,—which great masters of the twentieth century would consider pioneering and paradigmatic—has many analogies with the architectonic exploration of structural possibilities that guided the most avant-garde

[49] Examples can be found in the neo-Kantians Paul Natorp and Hermann Cohen, in Herbart (see above), but apparently also in a physicist like Wilhelm Eduard Weber; see: M. Norton Wise, "German Concepts of Force, Energy, and the Electromagnetic Ether: 1845–1880," in *Conceptions of Ether: Studies in the history of ether theories 1740–1900*, ed. Geoffrey N. Cantor and Michael J. S. Hodge (Cambridge: Cambridge University Press, 1981), 269–307.

mathematicians of his time.[50] In the same period, the sculptures of Auguste Rodin problematize the limits of the image, speculating with the relations between frame and figure, between matter and sculptural form, opening up a network of more complex relations, more autonomous and problematic than in the previous tradition. The final decades of the nineteenth century, and the early ones of the twentieth, saw the transition from the space-frame characteristic of figurative painting (analogous to the ambient space of Cartesian and Newtonian physics and geometry) to the space-network characteristic of the avant-garde (analogous in turn to the relativistic geometry of Riemann and Einstein). There is a surprising parallelism between the evolution of geometries linked with non-Euclidean theories and the abstract explorations in the world of art, immediately afterwards.[51] It can hardly be mere coincidence.

One could go deeper into the contributions of Einstein, the works of Kandinsky, the writings of Wittgenstein, or the new and multiple structures emerging in experimental music around the same years. But limits of space, and of course my own limits, prevent me from continuing down these paths.

Just one last remark. Contemporaneous with the process I have been studying, there was quite a lot of interest among mathematicians in models (actual, 3D models) and model making. They were employed as tools for education, highly recommended by Klein, Alexander Brill and others (plaster models thus made their way to leading universities around the world), but they were also artefacts for research.[52] Indeed, the notions of *Bild* and *Abbild*, and the relation to *Vorstellung*, are often mentioned by those mathematicians, when they literally speak about material models. Was there any close relation? One would tend to say, at first sight, that the two processes are parallel and disconnected: one is the Riemannian line of abstract thinking about *Abbildung*, another the Kleinian line of emphasis on actual hands-on experience and 3D models. Yet this is not quite convincing, if only because Klein regarded himself as a conceptual thinker and direct heir to Riemann. Moreover, one can find intermediary cases, like the 'models' for the non-Euclidean plane in Euclidean space (Minding, Beltrami, etc.). The question certainly deserves further attention, but I leave it open here.

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[50] Even some analogies in life and mental orientations, so to say, between Paul Cézanne and Dedekind are salient and noteworthy. See my forthcoming paper: José Ferreirós, "Paradise Recovered? Some Thoughts on *Mengenlehre* and Modernism," in *Science as Cultural Practice, Vol. II: Modernism in the Sciences, ca. 1900–1940*, ed. Moritz Epple and Falk Müller (Berlin: Akademie Verlag, forthcoming).

[51] On this topic see: Capi Corrales Rodrigáñez, "Dallo spacio come contenitore allo spazio come rete," in *Matematica e Cultura 2000*, ed. Michele Emmer (Milano: Springer Italia 2000), 123–38. "Local–Global in Mathematics and Painting," in *The Visual Mind*, ed. Michele Emmer, vol. 2 (Cambridge, MA: The MIT Press, 2005), 273–94.

[52] David Rowe, "Mathematical Models as Artefacts for Research: Felix Klein and the Case of Kummer Surfaces," *Mathematische Semesterberichte* 60, no. 1 (2013): 1–24.