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Longtime Dynamics of a Semilinear Lamé System

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Abstract

This paper is concerned with longtime dynamics of semilinear Lamé systems

$$\partial_t^2 u - \mu\Delta u - (\lambda + \mu)\nabla \text{div} u + \alpha\partial_t u + f(u) = b,$$

defined in bounded domains of $\mathbb{R}^3$ with Dirichlet boundary condition. Firstly, we establish the existence of finite dimensional global attractors subjected to a critical forcing $f(u)$. Writing $\lambda + \mu$ as a positive parameter ε, we discuss some physical aspects of the limit case $\varepsilon \to 0$. Then, we show the upper-semicontinuity of attractors with respect to the parameter when $\varepsilon \to 0$. To our best knowledge, the analysis of attractors for dynamics of Lamé systems has not been studied before.

Keywords System of elasticity · Global attractor · Gradient system · Upper-semicontinuity

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1 Introduction

The Lamé system is a classical model for isotropic elasticity. In three dimensions, it is given by

$$\begin{cases} \partial_t^2 u - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u = 0 & \text{in } \Omega\times\mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega\times\mathbb{R}^+, \\ u(0) = u_0, \ \ \partial_t u(0) = u_1 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, representing the elastic body in its rest configuration. Here, the vector $u = (u_1, u_2, u_3)$ denotes displacements and λ, μ are Lamé's constants with $\mu > 0$. In this model, the stress tensor is given by

$$\sigma(u)_{ij} = \lambda \mathrm{div} u\, \delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.2}$$

We refer the reader to [1,12,25,32] for modeling aspects and [9,20,30] for some applications of vector waves. Later, we discuss the physical justification of taking limit $\lambda+\mu \to 0$.

We note that the energy functional corresponding to the linear system (1.1) is given by

$$E_\ell(t) = \frac{1}{2}\int_\Omega \left(|\partial_t u|^2 + \mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div} u|^2\right) dx,$$

which is conservative since we have formally $\frac{d}{dt}E_\ell(t) = 0$. This motivated several papers on such systems where the main feature is finding suitable damping and controllers in order to get stabilization and controllability, respectively. Let us recall some related results. The exponential stabilization of Lamé systems, defined in exterior domains of $\mathbb{R}^3$ with Dirichlet boundary, was studied by Yamamoto [34]. Uniform stabilization by nonlinear boundary feedback was studied by Horn [17]. Polynomial stabilization with interior localized damping was studied by Astaburuaga and Charão [4]. By adding viscoelastic dissipation of memory type, Bchatnia and Guesmia [5] established the so-called general stability. More recently, Benaissa and Gaouar [6] studied strong stability of Lamé systems with fractional order boundary damping. With respect to controllability, we refer the reader to, for instance, [2,7, 21,23,24].

Our objective in the present article is different and goes further than considering stabilization. We are concerned with longtime dynamics of Lamé systems under nonlinear forces and frictional damping terms. Here, the above linear system (1.1) becomes

$$\begin{cases} \partial_t^2 u - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \alpha\partial_t u + f(u) = b & \text{in } \Omega\times\mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega\times\mathbb{R}^+, \\ u(0) = u_0, \ \ \partial_t u(0) = u_1 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $\alpha\partial_t u$ ($\alpha > 0$) represents a frictional dissipation, $f(u)$ stands for a nonlinear structural forcing, and $b = b(x)$ represents some external force. As far as we know, the long-time dynamics of semilinear Lamé systems (1.3) has not been studied before. We present two main results. Firstly, we establish the existence of global attractors with finite fractal-dimensional. Secondly, by taking $\lambda+\mu = \varepsilon > 0$, we study the upper semicontinuity of attractors with respect to $\varepsilon \to 0$.

In what follows we summarize the main contributions of the paper.

(i) Our first result establishes existence of global attractors for dynamics of problem (1.3) under nonlinear forces with critical growth $|f_i(u)| \approx |u|^p + |u_i|^3$, $p < 3$, $i = 1, 2, 3$. Under careful energy estimates, we show that the system is gradient and quasi-stable in the sense of [10,11]. Then we conclude that the attractors are smooth and have finite fractal dimension. See Theorem 3.1.

(ii) In Section 2.1, we discuss the physical meaning of the limit case $\lambda+\mu \to 0$ in real world applications. This arises mainly in Seismology.

(iii) Finally, setting $\varepsilon = \lambda + \mu \to 0$, we consider the ε-problem

$$\partial_t^2 u - \mu \Delta u - \varepsilon \nabla \mathrm{div} u + \alpha \partial_t u + f(u) = b,$$

depending on a parameter $\varepsilon \geq 0$. In Theorem 4.4 we show that the weak solutions of ε-problem converges to the vectorial wave equation with $\varepsilon = 0$. Then we provide all necessary analysis to prove that corresponding family of attractors $\mathcal{A}_\varepsilon$ is upper semicontinuous with respect to $\varepsilon \to 0$. This is given in a suitable phase space. See Theorem 4.5.

2 Preliminaries

2.1 Physical Aspects of $\lambda + \mu \to 0$

From the Hooke law and from the constitutive law (1.2) referring to elastic bodies, one derives the equation

$$\rho \partial_t^2 u - \mu \Delta u - (\lambda + \mu) \nabla \mathrm{div} u = \rho \mathcal{F}, \tag{2.1}$$

which may represent the displacement of vector particles for an elastic, isotropic and homogeneous body subject to external forces $\mathcal{F}$.

In Poisson [29], Timoshenko [33], Hudson [18], among others, it has been shown that Eq. (2.1) provides information about different *body waves*. In a scalar sense (P-waves), where the notation $\mathrm{div} u$ stands for fractions of volume changes from the strain tensor, it explains the behavior of compression and rarefaction in the interior of the body. From the mathematical point of view, it can be given by the identity

$$\partial_t^2 (\mathrm{div} u) - \alpha^2 \Delta (\mathrm{div} u) = \mathrm{div} \mathcal{F},$$

where $\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ represents speeds of wave propagation.

On the other hand, by considering the case $\nabla \times u$, one obtains the behavior of vector waves (S-waves) that model small rotations of lineal elements from shear forces acting within the body. In this way, the following equation arises

$$\partial_t^2 (\nabla \times u) - \beta^2 \Delta (\nabla \times u) = \nabla \times \mathcal{F},$$

where $\beta = \sqrt{\frac{\mu}{\rho}}$ means the speeds of S-wave propagation.

The analysis of the dynamics for (2.1) has shown great applications in the effect of seismic waves on various materials (e.g. harzburgite, garnet, pyroxenite, amphibolites, granite, gas sands, quartz, etc), where the propagation of the P-waves represents the change of volume in the interior of the body under compression and dilatation in the wave direction, see Fig. 1b, whereas the S-waves are cross displacements that produce vibrations in a perpendicular direction (normal to the traveling wave), see Fig. 1c.

A general existing scenario is when earthquakes generate shear waves, say S-waves, that are more effective than compression waves, say P-waves, and therefore the most damage on the body displacements is due to the “stronger” vibrations caused by S-waves. On the other hand, P-waves commonly propagate at a higher speed in relation to S-waves, reaching their highest speed, namely, the highest value for β, near the basis of the body. Thus, from

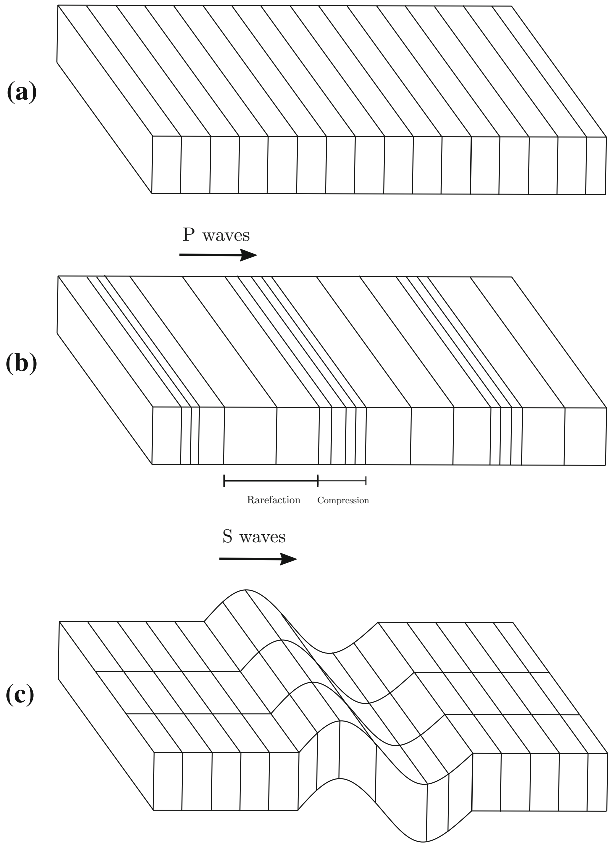

Fig. 1 In **a** we have the elastic body in a rest position. In **b** we have the effect of P-waves propagation on the material, where small contraction and dilation are produced in the same direction of the wave propagation. In **c** we exemplify the effect of transversal S-waves on the material, which are generated from the shear forces and are effective in normal directions with respect to the direction of the wave propagation

this viewpoint, it is worth mentioning that the approximation $\lambda \to -\mu$ symbolizes the approaching of the velocities with respect to S-waves in relation to P-waves.

For instance, when one considers the approach of λ to $-\mu$ on sedimentary rocks, one has atypical cases concerning bulk modulus or Poisson's ratio. This is the case when one considers e.g. $\lambda < -\frac{2\mu}{3}$ which is the case where we have negative bulk modulus or when $\lambda \sim -\mu$ which is the case where the Poisson's ratio is not defined, being $\pm\infty$ in the left or right approximation, respectively. These results seem to contradict the physical notion that we have regarding the study of thermodynamics on this type of materials, but several studies show that the compressibility of the material is closely related to the constant λ instead of approximations coming from the bulk modulus or the Poisson's ratio, see e.g. Goodway [14].

Other examples of such approximations are considered as follows. Indeed, in Moore et al. [27] the authors reveal the possibility of considering negative incremental bulk modulus on open cell foams on porous media. Also, Lakes and Wojciechowski [22] show the possibility of taking negative Poisson's ratio and bulk modulus for the same type of materials, which proves its structural stability. These are examples that show us the existence of materials (e.g., gas sands [14] and open cell foams [22,27]) that, under certain circumstances, allow us to consider the limit situation of λ to negative values. Thus, it makes sense to consider for example $\lambda \to -\mu$.

Moreover, in Ji et al. [19] the authors show that for quartz materials under a confining pressure of 600 MPa and a temperature around 650 °C, the transmission between High–Low Quartz demonstrates a significant decreasing in the speed of P-wave propagation $\left(\alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}\right)$ in relation to the perturbation of the speed of S-wave propagation $\left(\beta = \sqrt{\frac{\mu}{\rho}}\right)$. Therefore, to consider the approximation

$$\frac{\alpha}{\beta} \to 1 \quad \text{wich means} \quad \lambda \to -\mu$$

in the dynamic of seismic waves, it is equivalent to study the state of transition between High–Low Quartz in materials (say rocks) containing quartz (as for example granite, diorite, and felsic gneiss) and its behavior with respect to the wave speeds of propagation for transverse and compressible waves in the material, under proper conditions of temperature and pressure.

2.2 Assumptions

The following assumptions shall be considered throughout this paper for the functions defined on a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$.

(A1) The damping coefficient α and the Lamé coefficients λ, μ fulfill

$$\alpha, \mu > 0 \quad \text{and} \quad \lambda \in \mathbb{R} \quad \text{with} \quad \mu + \lambda \geq 0. \tag{2.2}$$

(A2) The external vector force b satisfies

$$b \in (L^2(\Omega))^3. \tag{2.3}$$

(A3) The nonlinear vector field $f = (f_1, f_2, f_3)$ is assumed to satisfy: there exist a vector field $g = (g_1, g_2, g_3) \in (C^1(\mathbb{R}^3))^3$, and functions $G \in C^2(\mathbb{R}^3)$ and $h_i \in C^2(\mathbb{R})$, $i = 1, 2, 3$, such that

$$\begin{aligned} &f_i(u_1, u_2, u_3) = g_i(u_1, u_2, u_3) + h_i(u_i), \quad i = 1, 2, 3, \\ &f_i(0) = g_i(0) = h_i(0) = 0, \quad i = 1, 2, 3, \\ &g = (g_1, g_2, g_3) = \nabla G. \end{aligned}$$

In addition, there exist constants $M, m_f \geq 0$ such that

$$f(u) \cdot u - G(u) - \sum_{i=1}^{3} \int_0^{u_i} h_i(s)ds \geq -M|u|^2 - m_f, \ \forall u \in \mathbb{R}^3, \tag{2.4}$$

$$G(u) + \sum_{i=1}^{3} \int_0^{u_i} h_i(s)ds \geq -M|u|^2 - m_f, \ \forall u \in \mathbb{R}^3, \tag{2.5}$$

with

$$0 \leq M < \frac{\mu\lambda_1}{2}, \tag{2.6}$$

where $\lambda_1 > 0$ denotes the first eigenvalue of the Laplacian operator $-\Delta$. Moreover, with respect to functions g_i and h_i, $i = 1, 2, 3$, we additionally assume:

- g fulfills the subcritical growth restriction: there exist $1 \le p < 3$ and $M_g > 0$ such that, for $i = 1, 2, 3$,

$$|\nabla g_i(u)| \le M_g(1+|u_1|^{p-1}+|u_2|^{p-1}+|u_3|^{p-1}), \ \ \forall\, u = (u_1, u_2, u_3) \in \mathbb{R}^3. \quad (2.7)$$

- For each $i = 1, 2, 3$, h_i fulfills the critical growth restriction: there exists a constant $c_h > 0$ such that

$$|h_i'(x)| \le c_h(1 + |x|^2), \quad \forall\, x \in \mathbb{R},\ i = 1, 2, 3. \quad (2.8)$$

2.3 Functional Setting

We denote the inner product in $L^2(\Omega)$ by $\langle u, v\rangle = \int_\Omega uvdx$ for $u, v \in L^2(\Omega)$. For the sake of simplicity, we use the same notation to the inner product in $(L^2(\Omega))^3$, that is, given $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in (L^2(\Omega))^3$,

$$\langle u, v\rangle := \sum_{i=1}^{3} \langle u_i, v_i\rangle .$$

Similarly, $\langle \nabla\cdot, \nabla\cdot\rangle$ stands for the inner product in $H_0^1(\Omega)$ as well as the inner product in $(H_0^1(\Omega))^3$. Thus, given $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in (L^2(\Omega))^3$,

$$\langle \nabla u, \nabla v\rangle := \sum_{i=1}^{3} \langle \nabla u_i, \nabla v_i\rangle .$$

In addition, for $p > 0$, we denote the norms in the spaces $L^p(\Omega)$ and $(L^p(\Omega))^3$ by $|\cdot|_p$ and $\|\cdot\|_p$, respectively, that is,

$$|u|_p := \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}, \quad u \in L^p(\Omega),$$

$$\|u\|_p^p := \sum_{i=1}^{3} |u_i|_p^p, \quad u = (u_1, u_2, u_3) \in (L^p(\Omega))^3.$$

In particular, for $p = 2$, one reads

$$\|u\|_2^2 = \langle u, u\rangle \ \text{ for } \ u \in (L^2(\Omega))^3 \ \text{ and } \ |u|_2^2 = \langle u, u\rangle \ \text{ for } \ u \in L^2(\Omega).$$

The elasticity operator $\mathcal{E}$, with domain $D(\mathcal{E}) := (H^2(\Omega) \cap H_0^1(\Omega))^3$, is given by

$$\mathcal{E}u = -\mu\Delta u - (\lambda + \mu)\nabla(\nabla \cdot u). \quad (2.9)$$

We consider the Hilbert space $\left((H_0^1(\Omega))^3, \langle\cdot,\cdot\rangle_e\right)$, where the inner product $\langle\cdot,\cdot\rangle_e$ is given by

$$\langle v, w\rangle_e = \mu \langle \nabla v, \nabla w\rangle + (\lambda + \mu) \langle \text{div}u, \text{div}w\rangle .$$

Remark 2.1 Under the above notations, it is easy to verify that the norms $\|\cdot\|_e^2 := \sqrt{\langle\cdot,\cdot\rangle_e}$ and $\|\nabla\cdot\|_2^2 := \sqrt{\langle\nabla\cdot, \nabla\cdot\rangle}$ are equivalent in $(H_0^1(\Omega))^3$. More precisely, one has

$$\mu\|\nabla u\|_2^2 \le \|u\|_e^2 \le a_0\|\nabla u\|_2^2, \ \ \forall\, u = (u_1, u_2, u_3) \in (H_0^1(\Omega))^3, \quad (2.10)$$

where $a_0 = \max\{\mu, 3(\lambda + \mu)\}$.

Additionally, if $u \in D(\mathcal{E})$ and $v \in (H_0^1(\Omega))^3$, then it is easy to verify that

$$\langle \mathcal{E}u, v \rangle = \langle u, v \rangle_e . \tag{2.11}$$

From (2.11), Remark 2.1 and the compact embedding of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, one sees that $\mathcal{E}$ is a positive self-adjoint operator. We denote the fractional power associated to $\mathcal{E}$ by $\mathcal{E}^r$ with domain $X^r := D(\mathcal{E}^r)$, which is endowed with the natural inner product $\langle \cdot, \cdot \rangle_r := \langle \mathcal{E}^r \cdot, \mathcal{E}^r \cdot \rangle$. In particular,

$$\begin{aligned} X^0 &= ((L_2(\Omega))^3; \langle \cdot, \cdot \rangle), \\ X^{1/2} &= \left((H_0^1(\Omega))^3; \left\langle \mathcal{E}^{1/2} \cdot, \mathcal{E}^{1/2} \cdot \right\rangle\right), \\ X^1 &= (D(\mathcal{E}); \langle \mathcal{E} \cdot, \mathcal{E} \cdot \rangle). \end{aligned}$$

Remark 2.2 From Riesz's Theorem along with density arguments and continuity, we have

$$\langle u, v \rangle_{1/2} = \langle u, v \rangle_e , \quad \forall\, u, v \in (H_0^1(\Omega))^3.$$

Finally, we define the (Hilbert) weak phase space $\mathcal{H} := X^{1/2} \times X^0$ with the usual inner product and induced norm $\| \cdot \|_{\mathcal{H}}$; and the (Hilbert) strong phase space $\mathcal{H}^1 := X^1 \times X^{1/2}$.

2.4 Well-Posedness and Energy Estimates

Under the above assumptions and notations, we are able to state the Hadamard well-posedness of (1.3). We start by denoting

$$U = \begin{bmatrix} u \\ \partial_t u \end{bmatrix}, \quad \mathbb{E} = \begin{bmatrix} 0 & -I \\ \mathcal{E} & \alpha \end{bmatrix}, \quad \mathbb{F}(U) = \begin{bmatrix} 0 \\ f(u) \end{bmatrix}, \quad \mathbb{B}(x) = \begin{bmatrix} 0 \\ b(x) \end{bmatrix}. \tag{2.12}$$

Then, problem (1.3) is equivalent to the Cauchy problem

$$\partial_t U + \mathbb{E}U + \mathbb{F}(U) = \mathbb{B}, \quad U(0) = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \tag{2.13}$$

where $\mathbb{E} : D(\mathbb{E}) \subset \mathcal{H} \to \mathcal{H}$ with domain

$$D(\mathbb{E}) = \{(u, v) \in \mathcal{H} \mid \mathcal{E}u + \alpha v \in X^0,\ v \in X^{1/2}\} = \mathcal{H}^1.$$

Theorem 2.1 (Well-posedness) *Let us assume that* (2.2)–(2.8) *hold.*

(i) *For* $(u_0, u_1) \in \mathcal{H}$, *system* (2.13) *possesses a unique mild solution*

$$U \in C(\mathbb{R}^+; \mathcal{H}). \tag{2.14}$$

(ii) *For* $(u_0, u_1) \in \mathcal{H}^1$, *the solution is regular solution, lying in the class*

$$U \in C(\mathbb{R}^+; \mathcal{H}^1). \tag{2.15}$$

(iii) *For any* $T > 0$ *and any bounded set* $B \subset \mathcal{H}$, *there exists a constant* $C_{BT} > 0$ *such that for any two solutions* $z^i = (u^i, \partial_t u^i)$ *of* (2.13) *with initial data* $z_0^i \in B$, $i = 1, 2$, *we have*

$$\|z^1(t) - z^2(t)\|_{\mathcal{H}}^2 \le C_{BT} \|z_0^1 - z_0^2\|_{\mathcal{H}}^2, \quad \forall\, t \in [0, T].$$

Proof It is easy to check that operator $\mathbb{E}$ set in (2.12) is maximal monotone and, under the assumption (A3), $\mathbb{F}$ is a locally Lipschitz on $\mathcal{H}$. Therefore, applying the classical theory of semigroups, see e.g. [15,28], one can conclude (2.14)–(2.15) on a maximal interval $[0, T_{\max})$. In addition, the conclusion $T_{\max} = +\infty$ is a directly consequence of Proposition 1 below, which exploits the dissipativeness assumptions (2.4)–(2.6) for $f(u)$. Hence, the items (i)-(ii) holds. The continuous dependence (iii) is also obtained by using standard computations on the difference of solutions and assumptions on $f(u)$. □

In what follows we give some useful inequalities involving the energy functional. The total energy functional associated with problem (1.3) is given by

$$E(t) = \frac{1}{2}\|(u, \partial_t u)\|_{\mathcal{H}}^2 + \int_\Omega G(u)dx + \sum_{i=1}^{3}\int_\Omega\int_0^{u_i} h_i(s)dsdx - \langle b(x), u\rangle . \tag{2.16}$$

Proposition 1 *Under the hypotheses* (2.2)–(2.8)*, we have:*

(i) *the energy $E(t)$ is non-increasing with $E(t) \le E(0)$ for all $t \ge 0$;*
(ii) *there exist positive constants K_1, K_2 and K_3 such that*

$$K_2\|(u, \partial_t u)\|_{\mathcal{H}}^2 - K_3 \le E(t) \le K_1\|(u, \partial_t u)\|_{\mathcal{H}}^4 + K_3, \quad \forall t \ge 0. \tag{2.17}$$

Proof (i) Taking the multiplier u_t in problem (1.3), then a straightforward computation leads us to

$$E'(t) = -\alpha\|\partial_t u\|_2^2 \le 0, \quad \forall\, t > 0, \tag{2.18}$$

from where it readily follows the stated in item (i).
(ii) From conditions (2.2)–(2.8) and Young's inequality with $\epsilon > 0$, the expression

$$I = \int_\Omega G(u)dx + \sum_{i=1}^{3}\int_\Omega\int_0^{u_i} h_i(s)dsdx - \langle b(x), u\rangle$$

can be estimated from below and above as follows

$$\begin{aligned} I &\ge -m_f|\Omega| - \frac{\epsilon}{4}\|b\|_2^2 - \left(\frac{M}{\lambda_1\mu} + \frac{1}{\lambda_1\mu\epsilon}\right)\|(u, \partial_t u)\|_{\mathcal{H}}^2, \\ I &\le C_f|\Omega| + \frac{1}{2}\|b\|_2^2 + \frac{C_g}{\mu^{\frac{p+1}{2}}}\|(u, \partial_t u)\|_{\mathcal{H}}^{p+1} \\ &\quad + \frac{C_h}{\mu^2}\|(u, \partial_t u)\|_{\mathcal{H}}^4 + \frac{1}{2\sqrt{\lambda_1}\mu}\|(u, \partial_t u)\|_{\mathcal{H}}^2, \end{aligned}$$

where the positive generic constants depend on their index and some embedding with $H_0^1(\Omega)$, for example C_h depends on the constant c_h in (2.8) and the compact embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. From this and the definition of $E(t)$ in (2.16), we infer

$$\begin{aligned} E(t) &\le C_f|\Omega| + \frac{1}{2}\|b\|_2^2 + \frac{1}{2\sqrt{\lambda_1}\mu} + \frac{C_g}{\mu^{\frac{p+1}{2}}} \\ &\quad + \left(\frac{1}{2} + \frac{C_g}{\mu^{\frac{p+1}{2}}} + \frac{C_h}{\mu^2}\right)\|(u, \partial_t u)\|_{\mathcal{H}}^4, \\ E(t) &\ge -m_f|\Omega| - \frac{\epsilon}{4}\|b\|_2^2 + \left(\frac{1}{2} - \frac{M}{\lambda_1\mu} - \frac{1}{\lambda_1\mu\epsilon}\right)\|(u, \partial_t u)\|_{\mathcal{H}}^2. \end{aligned}$$

Therefore, from a proper choice of $\epsilon > 0$ and using condition (2.6), one can conclude the existence of positive constants K_1, K_2 and K_3 satisfying (2.17). □

Remark 2.3 We emphasizes that above constants K_1, K_2 and K_3 in (2.17) do not depend on the parameter λ.

3 Long-time Dynamics

From Theorem 2.1, one can define a dynamical system $(\mathcal{H}, S(t))$ associated with problem (1.3), where the evolution operator $S(t)$ corresponds to a non-linear C_0-semigroup (locally Lipschitz) on $\mathcal{H}$.

Our main goal in this section is to prove that $(\mathcal{H}, S(t))$ possesses a finite dimensional global attractor $\mathcal{A}$ as well as to reach its qualitative properties such as characterization and regularity. To this end, we first recall some concepts in the theory of dynamical systems, by following e.g. the references [10,11].

3.1 Some Elements of Dynamical Systems

For the sake of completeness, we recall some basic facts on dynamical systems.

- A *global attractor* for a dynamical system $(\mathcal{H}, S(t))$ is a compact set $\mathcal{A} \subset \mathcal{H}$ which is fully invariant and uniformly attracting, it means, for any bounded subset $B \subset \mathcal{H}$

$$S(t)\mathcal{A} = \mathcal{A} \text{ and } \lim_{t\to\infty} d_{\mathcal{H}}(S(t)B, \mathcal{A}) = 0.$$

- The *fractal dimension* of a compact set $B \subset \mathcal{H}$ is defined as

$$\dim_f B = \limsup_{\epsilon\to 0} \frac{ln N_\epsilon(B)}{ln(1/\epsilon)},$$

  where $N_\epsilon(B)$ is the minimal number of closed balls of radius 2ϵ necessary to cover B.
- The set of *stationary points* $\mathcal{N}$ of a dynamical system $(\mathcal{H}, S(t))$ is defined as

$$\mathcal{N} = \{V \in \mathcal{H} \mid S(t)V = V, \ \ \forall t > 0\}.$$

- A dynamical system $(\mathcal{H}, S(t))$ is called *gradient* if there exists a strict Lyapunov functional Ψ, that is, for any $z \in \mathcal{H}$, $\Psi(S(t)z)$ is decreasing with respect $t \geq 0$ and Ψ is constant on the set of stationary points $\mathcal{N}$.
- Given a set $B \subset \mathcal{H}$, its *unstable manifold* $W^u(B)$ is the set of points $z \in \mathcal{H}$ that belongs to some complete trajectory $\{y(t)\}_{t\in\mathbb{R}}$ and satisfies

$$y(0) = z \text{ and } \limsup_{t\to-\infty} \operatorname{dist}(y(t), B) = 0.$$

- *Quasi-stability.* Let X, Y be reflexive Banach spaces with compact embedding $X \overset{c}{\hookrightarrow} Y$ and $\mathcal{H} = X \times Y$. Let us suppose $(\mathcal{H}, S(t))$ is given by

$$S(t)z = (u(t), \partial_t u(t)), \ \ z = (u_0, u_1) \in \mathcal{H},$$

  where

$$u \in C(\mathbb{R}^+; X) \cap C^1(\mathbb{R}^+; Y),$$

Then, $(\mathcal{H}, S(t))$ is called *quasi-stable* on a set $B \subset \mathcal{H}$ if there exists a compact semi-norm η_X on X and non-negative scalar functions $a_1(t)$ and $a_3(t)$ locally bounded in $\mathbb{R}^+$ and $a_2(t) \in L^1(\mathbb{R}^+)$ with $\lim_{t\to\infty} a_2(t) = 0$ such that

$$\|S(t)z^1 - S(t)z^2\|_{\mathcal{H}}^2 \le a_1(t)\|z^1 - z^2\|_{\mathcal{H}}^2,$$

and

$$\|S(t)z^1 - S(t)z^2\|_{\mathcal{H}}^2 \le a_2(t)\|z^1 - z^2\|_{\mathcal{H}}^2 + a_3(t) \sup_{0\le s\le t} \left[\eta_X(u^1(s) - u^2(s))\right]^2,$$

for any $z^1, z^2 \in B$.

Proposition 2 [11, Corollary 7.5.7] *Let $(\mathcal{H}, S(t))$ be a gradient asymptotically smooth dynamical system. Additionally, if its Lyapunov function $\Psi(x)$ is bounded from above on any bounded subset of $\mathcal{H}$, the set $\Psi_R = \{x \in \mathcal{H} : \Psi(x) \le R\}$ is bounded for every R and the set $\mathcal{N}$ of stationary points of $(\mathcal{H}, S(t))$ is bounded, then $(\mathcal{H}, S(t))$ possesses a compact global attractor characterized by $\mathcal{A} = W^u(\mathcal{N})$.*

Proposition 3 [11, Proposition 7.9.4] *Let us assume that the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on every bounded forward invariant set $B \subset \mathcal{H}$. Then, $(\mathcal{H}, S(t))$ is asymptotically smooth.*

Proposition 4 [11, Theorem 7.9.6] *Let $(\mathcal{H}, S(t))$ a quasi-stable dynamical system. If $(\mathcal{H}, S(t))$ possesses a compact global attractor $\mathcal{A}$ and is quasi-stable on $\mathcal{A}$, hen the attractor $\mathcal{A}$ has a finite fractal dimension* $\dim_f \mathcal{A} < \infty$.

3.2 Main Result and Proofs

We are now in condition to state and prove the main result concerning global attractors associated with problem (1.3). It reads as follows.

Theorem 3.1 *Under the assumptions (2.2)–(2.8), we have:*

(i) *The dynamical system $(\mathcal{H}, S(t))$ corresponding to problem (1.3) has a unique global attractor $\mathcal{A}$ with finite fractal dimension $\dim_f \mathcal{A} < \infty$, and is characterized by the unstable manifold $\mathcal{A} = W^u(\mathcal{N})$ emanating from the set of stationary points $\mathcal{N}$ of $(\mathcal{H}, S(t))$.*
(ii) *Moreover, if $h_i = 0$, $i = 1, 2, 3$, then $\mathcal{A}$ is bounded in the strong phase space $\mathcal{H}^1$. In particular, any full trajectory $\{(u(t), \partial_t u(t)), t \in \mathbb{R}\}$ that belongs to $\mathcal{A}$ has the following regularity properties*

$$\partial_t u \in L^\infty(\mathbb{R}; (H_0^1(\Omega))^3) \cap C(\mathbb{R}; (L^2(\Omega))^3), \quad \partial_t^2 u \in L^\infty(\mathbb{R}; (L^2(\Omega))^3), \tag{3.1}$$

and there exists $R > 0$ such that

$$\|(\partial_t u(t), \partial_t^2 u(t))\|_{\mathcal{H}}^2 \le R^2, \tag{3.2}$$

where R does not depend on λ.

The proof of Theorem 3.1 will be concluded at the end of this section as a consequence of some technical results provided in the sequel.

3.2.1 Gradient Property

Lemma 3.2 *Under the assumptions of Theorem 3.1, let us define the functional*

$$\begin{aligned}\Psi : \mathcal{H} &\to \mathbb{R}\\ z &\mapsto \Psi(z) := \Psi(u, v)\end{aligned}$$

given by

$$\Psi(u, v) = \frac{1}{2}\|(u, v)\|_{\mathcal{H}}^2 + \int_\Omega G(u)dx + \sum_{i=1}^{3}\int_\Omega\int_0^{u_i} h_i(s)dsdx - \langle b(x), u\rangle . \tag{3.3}$$

Then:

1. Ψ *is a strict Lyapunov functional;*
2. $\Psi(z) \to \infty$ *if and only if* $\|z\|_{\mathcal{H}} \to \infty$*;*
3. $\mathcal{N}$ *is bounded on* $\mathcal{H}$

As a consequence, the dynamical system $(\mathcal{H}, S(t))$ *associated with problem* (1.3) *is a gradient system.*

Proof Let fix $z_0 \in \mathcal{H}$ and recall that $\mathcal{N}$ is the set of stationary points of $(\mathcal{H}, S(t))$. Also, from (2.16) one sees that $\Psi(u(t), \partial_t u(t)) = E(u(t), \partial_t u(t)) := E(t)$. Then, we infer:

- From (2.18), it is clear that $\Psi(S(t)z_0)$ is decreasing with respect to time and from (2.17), $\Psi(z) = \Psi(S(0)z) \to \infty$ if and only if $||z||_{\mathcal{H}} \to \infty$.
- Let us consider the stationary problem:

$$\begin{cases}\mathcal{E}u + f(u) = b(x) & \text{in} \quad \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases} \tag{3.4}$$

  Thus, a simple computation shows that $\mathcal{N}$ is given by

$$\mathcal{N} = \{(u, 0) \in \mathcal{H} \mid u \text{ is the solution of } (3.4)\} .$$

  In addition, from (2.18) it is easy to prove that Ψ is constant on $\mathcal{N}$. Finally, multiplying (3.4) by u, integrating on Ω and using (2.4) and (2.5), we obtain that for any $\epsilon > 0$

$$\left(1 - \frac{2M}{\lambda_1\mu} - \frac{1}{4\lambda_1\mu\epsilon}\right)\|u\|_e^2 \le 2m_f|\Omega| + \epsilon\|b\|_2^2, \tag{3.5}$$

  from where (along with (2.6)) we conclude that $\mathcal{N}$ is bounded on $\mathcal{H}$, for $\epsilon > 0$ properly chosen.

Therefore, the items 1–3 are proved. □

3.2.2 Quasi-Stability Property

Proposition 5 (Stabilizability Estimate) *Under the assumptions of Theorem 3.1, let us consider a bounded subset* $B \subset \mathcal{H}$ *and two weak solutions* $z^1 = (u^1, \partial_t u^1)$ *and* $z^2 = (u^2, \partial_t u^2)$ *of problem* (1.3) *with initial data* $z_0^1 = (u_0^1, u_1^1)$, $z_0^2 = (u_0^2, u_1^2) \in B$*. Then,*

$$\|z^1(t) - z^2(t)\|_{\mathcal{H}}^2 \le a_2(t)\|z_0^1 - z_0^2\|_{\mathcal{H}}^2 + c(t)\sup_{0\le s\le t}\|u^1(s) - u^2(s)\|_{p_0}^2, \tag{3.6}$$

where $p_0 = \max\{4, \frac{6}{4-p}\} < 6$, $a_2 \in L^1(\mathbb{R}^+)$ *with* $\lim_{t\to\infty} a_2(t) = 0$ *and* $c(t)$ *is a locally bounded function.*

Proof The estimate (3.6) is one of the main cores of the present article. Its proof is quite technical and long, and for this reason we are going to proceed in several steps as follows.
Step 1. Setting the difference problem and functionals. Let us denote $w = u^1 - u^2$ with $u^i = (u_1^i, u_2^i, u_3^i)$, $i = 1, 2$. Then, a simple computation shows that w is a solution (in the weak and strong sense) of the following problem

$$\begin{cases} \partial_t^2 w + \mathcal{E}w + \alpha\partial_t w + f(u^1) - f(u^2) = 0 \text{ in } \quad \Omega \times \mathbb{R}^+, \\ w = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega \times \mathbb{R}^+, \\ w(x,0) = u_0^1(x) - u_0^2(x), \ x \in \Omega, \\ \partial_t w(x,0) = u_1^1(x) - u_1^2(x), \ x \in \Omega. \end{cases} \tag{3.7}$$

The energy associated with system (3.7) is given by

$$\Xi(t) := \frac{1}{2}\|(w, \partial_t w)\|_{\mathcal{H}}^2 = \frac{1}{2}\|z^1(t) - z^2(t)\|_{\mathcal{H}}^2, \quad t \geq 0. \tag{3.8}$$

We also set the functional

$$\chi(t) = \langle w, \partial_t w \rangle \,,$$

and the perturbed energy functional

$$\Upsilon(t) = \epsilon_1 \Xi(t) + \epsilon_2 \chi(t),$$

where the constants $\epsilon_1, \epsilon_2 > 0$ will be chosen later.
Step 2. Equivalence. There exist constants $C_1, C_2 > 0$ such that

$$C_2 \Xi(t) \leq \Upsilon(t) \leq C_1 \Xi(t). \tag{3.9}$$

Indeed, the inequalities in (3.9) follow by taking $K' = \max\{\frac{c_p^2}{\mu}, 1\}$, with $c_p > 0$ coming from the Poincaré inequality and μ from (2.10), $\epsilon_1 > \epsilon_2 K'$, $C_2 = \epsilon_1 - \epsilon_2 K'$ and $C_1 = \epsilon_1 + \epsilon_2 K'$.
Step 3. Estimate for Ξ'. Given $\xi > 0$, there exists a constant $C(\xi, B) > 0$, which depends on ξ and B, such that

$$\Xi'(t) \leq -\alpha\|\partial_t w\|_2^2 + C(\xi, B)\|w\|_{\frac{6}{4-p}}^2 + \xi\|\partial_t w\|_2^2 + I, \tag{3.10}$$

where we set

$$I := \sum_{i=1}^{3} \left\langle h_i(u_i^2) - h_i(u_i^1), \partial_t w_i \right\rangle. \tag{3.11}$$

In fact, we first observe that deriving $\Xi(t)$ and using (3.7), we get

$$\Xi'(t) = -\,\alpha\|\partial_t w\|_2^2 - \left\langle g(u^1) - g(u^2), \partial_t w \right\rangle + I.$$

Since

$$|\left\langle g(u^1) - g(u^2), \partial_t w \right\rangle| \leq \sum_{i=1}^{3} \int_\Omega M_g \left\{ 1 + \sum_{i=1}^{3} |u_i^1|^{p-1} + \sum_{i=1}^{3} |u_i^2|^{p-1} \right\} |w||\partial_t w_i| dx,$$

then applying Hölder's inequality, we obtain

$$|\left\langle g(u^1) - g(u^2), \partial_t w \right\rangle| \leq \sum_{i=1}^{3} \tilde{C}_f \|w\|_{\frac{6}{4-p}} \|\partial_t w_i\|_2, \tag{3.12}$$

where

$$\tilde{C}_f = M_f \left\{ |\Omega|^{\frac{p-1}{6}} + \sum_{i=1}^{3} \|u_i^1\|_6^{p-1} + \sum_{i=1}^{3} \|u_i^2\|_6^{p-1} \right\} \le C(B) < \infty,$$

is a constant depending on B. Therefore, the estimate (3.10) follows from Young's inequality with $\xi > 0$.

Step 4. Estimate for χ'. There exists a constant $C(B) > 0$ depending on B such that

$$\chi'(t) \le -\Xi(t) - \frac{1}{2}\|w\|_e^2 + \frac{\alpha}{2}\|w\|_2^2 + C(B)\|w\|_4^2 + \frac{3+\alpha}{2}\|\partial_t w\|_2^2. \tag{3.13}$$

Indeed, multiplying $(3.7)_1$ by w and integrating on Ω, we obtain

$$\begin{aligned}\chi'(t) = &- \Xi(t) - \frac{1}{2}\|w\|_e^2 + \frac{\alpha}{2}\|w\|_2^2 + \frac{\alpha+3}{2}\|\partial_t w\|_2^2 \\ &- \langle g(u^1) - g(u^2), w\rangle - \sum_{i=1}^{3} \langle h_i(u_i^1) - h_i(u_i^2), w_i\rangle .\end{aligned}$$

Now, noting that

$$\left|\langle g(u^1) - g(u^2), w\rangle\right| \le 3 \left\{ |\Omega|^{p-1} + \sum_{i=1}^{3} \|u_i^1\|_{p+1}^{p-1} + \sum_{i=1}^{3} \|u_i^2\|_{p+1}^{p-1} \right\} \|w\|_{p+1}^2 \le \tilde{C}_B \|w\|_{p+1}^2,$$

and

$$\left|\sum_{i=1}^{3} \langle h_i(u_i^1) - h_i(u_i^2), w_i\rangle\right| \le \sum_{i=1}^{3} (|\Omega|^2 + \|u_i^1\|_4^2 + \|u_i^2\|_4^2)\|w_i\|_4^2 \le C_B\|w\|_4^2,$$

where the constants $\tilde{C}_B, C_B > 0$ depend only on B, then the estimate (3.13) follows.

Step 5. Estimate for Υ. There exists a constant $C_3 > 0$ depending on B such that

$$\Upsilon(t) \le e^{-\frac{\epsilon_2 t}{C_1}} \Upsilon(0) + C_3 \int_0^t e^{-\frac{\epsilon_2}{C_1}(t-s)} \|w(s)\|_{p_0}^2 ds + \epsilon_1 e^{-\frac{\epsilon_2 t}{C_1}} J, \tag{3.14}$$

where $C_1 > 0$ comes from (3.9) and we set

$$J := \int_0^t e^{\frac{\epsilon_2 s}{C_1}} I ds = \sum_{i=1}^{3} \int_0^t e^{\frac{\epsilon_2 s}{C_1}} \langle h_i(u_i^2(x,s)) - h_i(u_i^1(x,s)), \partial_t w_i(x,s)\rangle ds. \tag{3.15}$$

First, we note that from (3.10) and (3.13), one has

$$\begin{aligned}\Upsilon'(t) \le &- \epsilon_2 \Xi(t) + \frac{\alpha\epsilon_2}{2}\|w\|_2^2 + \epsilon_2 C(B)\|w\|_4^2 + \epsilon_1 C(\xi, B)\|w\|_{\frac{6}{4-p}}^2 \\ &+ \epsilon_1 I + \left(\frac{3\epsilon_2 + \alpha\epsilon_2}{2} + \epsilon_1\xi - \alpha\epsilon_1\right)\|\partial_t w\|_2^2.\end{aligned}$$

We now choose $\epsilon_1, \epsilon_2, \xi > 0$ small enough such that

$$\epsilon_2 K' < \epsilon_1 \quad \text{and} \quad \frac{3\epsilon_2 + \alpha\epsilon_2}{2} + \epsilon_1\xi < \alpha\epsilon_1.$$

It is worth mentioning that $\epsilon_1, \epsilon_2, \xi > 0$ do not depend on λ. Thus, from this choice, setting $p_0 = \max\{\frac{6}{4-p}, 4\}$ and using (3.9), there exists a constant $C_3 = C(B) > 0$ such that

$$\Upsilon'(t) \le -\frac{\epsilon_2}{C_1}\Upsilon(t) + C_3\|w\|_{p_0}^2 + \epsilon_1 I,$$

from where it follows the estimate (3.14) with J given in (3.15).

Remark 3.1 Since the choices for ϵ_1, ϵ_2 do not depend on λ, then $C_3 > 0$ is a constant that does not depend on λ as well.

Step 6. Estimate for J. There exist constants $\gamma_0 > 0$ and $C_4 > 0$ depending on B such that

$$J \leq C_4 e^{\gamma_0 t} \sup_{0<s<t} \|w\|_4^2 + C_4 \int_0^t (\|\partial_t u^1(s)\|_2 + \|\partial_t u^2(s)\|_2) e^{\gamma_0 s} \Upsilon(s) ds. \tag{3.16}$$

Firstly, in view of the assumption (2.8), for any constant $\gamma > 0$ and each $i = 1, 2, 3$, there exists a constant $K_i' > 0$ such that

$$\begin{aligned}
&\int_0^t e^{\gamma s} \left\langle h_i(u_i^2(s)) - h_i(u_i^1(s)), \partial_t w_i(s) \right\rangle ds \\
&\quad \leq K_i' e^{\gamma t} \sup_{0<s<t} \|w_i(s)\|_4^2 + K_i' \int_0^t \left(\|\partial_t u_i^1(s)\|_2 + \|\partial_t u_i^2(s)\|_2\right) e^{\gamma s} \|\nabla w_i(s)\|_2^2 ds.
\end{aligned} \tag{3.17}$$

Indeed, the justification of (3.17) follows by taking similar arguments as in [8, Lemma 4.9]. For the sake of completeness, we present a short proof of such an inequality. Note that

$$\begin{aligned}
&\int_0^t e^{\gamma s} \left\langle h_i(u_i^2(s)) - h_i(u_i^1(s)), \partial_t w_i(s) \right\rangle ds \\
&= \frac{1}{2} \int_0^t e^{\gamma s} \int_\Omega \frac{d}{ds} |w_i|^2 \int_0^1 h_i'(u_i^2 + \lambda(u_i^1 - u_i^2)) d\lambda dx ds \\
&= \frac{e^{\gamma s}}{2} \int_\Omega \int_0^1 h_i'(u_i^2(s) + \lambda(u_i^1(s) - u_i^2(s))) d\lambda |w_i(s)|^2 dx \Big|_0^t \\
&\quad - \frac{1}{2} \int_0^t \int_\Omega \frac{d}{ds} \left(e^{\gamma s} \int_0^1 h_i'(u_i^2(s) + \lambda(u_i^1(s) - u_i^2(s))) d\lambda \right) |w_i(s)|^2 dx ds \\
&\leq K_i' e^{\gamma t} \sup_{0<s<t} \|w_i(s)\|_4^2 \\
&\quad - \frac{1}{2} \int_0^t e^{\gamma s} \int_\Omega \int_0^1 h_i''(u_i^2 + \lambda(u_i^1 - u_i^2))(\partial_t u_i^2 + \lambda(\partial_t u_i^1 - \partial_t u_i^2)) d\lambda |w_i(s)|^2 dx ds.
\end{aligned}$$

For the last term we use the fact that $h_i \in C^2(\mathbb{R})$, condition (2.8), Hölder's inequality, and the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$.

Therefore, from (3.15) and (3.17) it prompt follows

$$J \leq \sum_{i=1}^3 K_i' \sup_{0<s<t} \|w(s)\|_4^2 + \max\{K_i'\} \int_0^t e^{\gamma_0 s} (\|\partial_t u^1(s)\|_2 + \|\partial_t u^2(s)\|_2) \|\nabla w(s)\|_2^2 ds,$$

for $\gamma_0 = \frac{\epsilon_2}{C_1} > 0$. Additionally, taking $C_4 = \max\{K_1' + K_2' + K_3', \frac{2\max\{K_i'\}}{\mu C_2}\} > 0$ and noting that

$$\|\nabla w(s)\|_2^2 \leq \frac{1}{\mu} \|w\|_e^2 \leq \frac{2}{\mu} \Xi(s) \leq \frac{2}{\mu C_2} \Upsilon(s),$$

then estimate (3.16) follows as desired.

Remark 3.2 We emphasize that constants γ_0 and C_4 do not depend on λ.

Step 7. Conclusion of the proof. We are finally in position to complete the proof of (3.6). Indeed, from (3.14) and (3.16), there exists a constant $C_5 > 0$ depending on B, but independently of λ, such that

$$e^{\gamma_0 t}\Upsilon(t) \leq C_5\Upsilon(0) + C_5 e^{\gamma_0 t} \sup_{0<s<t} \|w\|_{p_0}^2 + C_5 \int_0^t (\|\partial_t u^1(s)\|_2 + \|\partial_t u^2(s)\|_2) e^{\gamma_0 s}\Upsilon(s)ds,$$

and applying Gronwall's inequality, one gets

$$\Upsilon(t) \leq C_5 \left\{ e^{-\gamma_0 t}\Upsilon(0) + \sup_{0<s<t} \|w(s)\|_{p_0}^2 \right\} e^{\left(C_5 e^{-\gamma_0 t} \int_0^t (\|\partial_t u^1(s)\|_2 + \|\partial_t u^2(s)\|_2) e^{\gamma_0 s} ds \right)}. \quad (3.18)$$

Now, from (2.17) and (2.18), and also in view of Remark 2.3, we have

$$\int_0^t \|\partial_t u(s)\|_2^2 ds = -\frac{1}{\alpha}\int_0^t E'(s)ds \leq \frac{2|E(0)|}{\alpha} \leq Q, \quad u := u^1, u^2,$$

where $Q > 0$ is a constant depending on B and f, but independent of λ. Thus, using Hölder and Young's inequalities, we obtain

$$e^{-\gamma_0 t} \int_0^t (\|\partial_t u^1(s)\|_2 + \|\partial_t u^2(s)\|_2) e^{\gamma_0 s} ds \leq 2\sqrt{Q}\sqrt{t} \leq \epsilon t + \frac{2Q}{\epsilon},$$

for any $t > 0$ and $\epsilon > 0$. Replacing the latter estimate in (3.18), we arrive at

$$\Upsilon(t) \leq C_5 e^{(\epsilon C_5 t + \frac{2C_5 Q}{\epsilon})} \left\{ e^{-\gamma_0 t}\Upsilon(0) + \sup_{0<s<t} \|w(s)\|_{p_0}^2 \right\}.$$

Taking $\epsilon = \frac{\gamma_0}{2C_4}$ and using (3.9), we have

$$\Xi(t) \leq \frac{C_1 C_4 e^{\frac{Q}{\gamma_0}}}{C_2} e^{\frac{-\gamma_0}{2}t}\Xi(0) + \frac{C_4 e^{\frac{Q}{\gamma_0}}}{C_2} e^{\frac{\gamma_0}{2}t} \sup_{0<s<t} \|w(s)\|_{p_0}^2. \quad (3.19)$$

Finally, regarding the definition of $\Xi(t)$, $t \geq 0$, in (3.8) and setting

$$a_2(t) := \frac{C_1 C_4 e^{\frac{Q}{\gamma_0}}}{C_2} e^{\frac{-\gamma_0}{2}t} \quad \text{and} \quad a_3(t) := \frac{2C_4 e^{\frac{Q}{\gamma_0}}}{C_2} e^{\frac{\gamma_0}{2}t}, \quad (3.20)$$

then (3.19) leads to (3.6) as desired.

The proof of Proposition 5 is therefore concluded. □

Corollary 1 (Quasi-stability) *Under the assumptions of Theorem 3.1, the dynamical system $(\mathcal{H}, S(t))$ associated with problem (1.3) is quasi-stable on any bounded set $B \subset \mathcal{H}$.*

Proof It is a direct consequence of Theorem 2.1 - (iii) and Proposition 5 by noting the semi-norm given by $n_{H_0^1}(u^1 - u^2) = \|u^1 - u^2\|_{p_0}$ is compact. □

3.2.3 Conclusion of the Proof of Theorem 3.1

(i) From Proposition 3 and Corollary 1, the dynamical system $(\mathcal{H}, S(t))$ related to problem (1.3) is asymptotically smooth. Therefore, using Lemma 3.2 and Propositions 2 and 4, the conclusion of Theorem 3.1 - (i) is complete.

(ii) In case $h_i = 0$, $i = 1, 2, 3$, then going back to (3.11), one sees that $I = 0$ and, consequently, from (3.15) one gets $J = 0$. Thus, (3.14) reduces to

$$\Upsilon(t) \leq e^{-\frac{\epsilon_2 t}{C_1}}\Upsilon(0) + \frac{C_3 C_1}{\epsilon_2} \sup_{0\leq s\leq t} \|w(s)\|^2_{p_0'}\left(1 - e^{-\frac{\epsilon_2}{C_1}t}\right),$$

$p_0' = \max\{\frac{6}{4-p}, p+1\}$. In this way, one reaches (3.19) (respec. (3.6)) with

$$a_3(t) := \frac{C_3 C_1}{\epsilon_2}\left(1 - e^{-\frac{\epsilon_2}{C_1}t}\right),$$

instead of $a_3(t)$ given in (3.20). Thus, $c_\infty = \sup_{t\in\mathbb{R}^+} a_3(t) < \infty$, and from [11, Theorem 7.9.8], the regularity properties (3.1)–(3.2) are ensured, that is, the conclusion of Theorem 3.1-(ii) is complete.

Therefore, the proof of Theorem 3.1 is ended.

4 Upper Semicontinuity

Along this section ε denotes a real number in [0, 1] and assume $\lambda + \mu = \varepsilon$. Thus, problem (1.3) can be rewritten as follows

$$\begin{cases} \partial_t^2 u - \mu\Delta u - \varepsilon\nabla \text{div} u + \alpha\partial_t u + f(u) = b & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0) = u_0, \ \ \partial_t u(0) = u_1 & \text{in } \Omega, \end{cases} \tag{4.1}$$

In this way, instead of operator (2.9), we write the ε-operator

$$\mathcal{E}_\varepsilon u := -\mu\Delta u - \varepsilon\nabla\text{div}(u), \quad \text{for} \quad u = (u_1, u_2, u_3).$$

Hereafter, we denote by P_ε the ε-problem (4.1) and, in view of Theorem 3.1, we also denote by $\mathcal{A}_\varepsilon$ the compact finite dimensional global attractor of its associated dynamical system. The energy corresponding to P_ε is still given by (2.16) and denoted here as $E_\varepsilon(t)$.

Using the same notation as in Section 2, we define the inner-product

$$\langle u, v\rangle_\varepsilon = \mu\langle\nabla u, \nabla v\rangle + \varepsilon\langle \text{div} u, \text{div} v\rangle .$$

Then, the norm $\|\cdot\|_\varepsilon = \sqrt{\langle u, v\rangle_\varepsilon}$ satisfies that

$$\mu\|\nabla\cdot\|_2^2 \leq \|\cdot\|_\varepsilon^2 \leq \max\{\mu, 3\}\|\nabla\cdot\|_2^2. \tag{4.2}$$

Additionally, let us denote by

$$\mathcal{H}_\varepsilon = ((H_0^1(\Omega))^3, \|\cdot\|_\varepsilon) \times ((L^2(\Omega))^3, \|\cdot\|_2)$$

the space of weak solutions associated to P_ε, and

$$\mathcal{H}_0^1 = (D(-\Delta), \|\mu\Delta\cdot\|_2) \times ((H_0^1(\Omega))^3, \|\mu\nabla\cdot\|_2)$$

the space of strong solutions associated to P_0.

Analogously, we denote by $(\mathcal{H}_\varepsilon, S_\varepsilon(t))$ the dynamical system associated with P_ε, and by $\mathcal{N}_\varepsilon$, its corresponding set of stationary solutions. The existence of a global attractor $\mathcal{A}_\varepsilon$ as well as its properties are ensured by Theorem 3.1.

In this section, our main goal is to study the upper semicontinuity of attractors $\mathcal{A}_\varepsilon$ with respect to the parameter $\varepsilon \to 0$. More precisely, our main results are presented in Theorems 4.4 and 4.5. To this end, we need to prove the following two properties: the existence of an absorbing set which does not depend on ε and the convergence in some sense of the solutions of P_ε when $\varepsilon \to 0$.

For the existence of an absorbing set we need the following result which is a direct consequence of [11, Remark 7.5.8].

Theorem 4.1 *Under the conditions* (2.2)–(2.8)*, the following inequality holds true for the attractor* $\mathcal{A}$ *in Theorem* 3.1 *and* Ψ *given in Lemma* 3.2*:*

$$\sup\{\Psi(u, \partial_t u) : (u, \partial_t u) \in \mathcal{A}\} \leq \sup\{\Psi(u, 0) : (u, 0) \in \mathcal{N}\}.$$

With respect to the convergence of solutions, it is important to note that the phase space $\mathcal{H}_\varepsilon$ changes when $\varepsilon \to 0$. So, the convergence of the solutions of P_ε is *singular* in the same sense proposed in [26].

Lemma 4.2 *Under the conditions* (2.2)–(2.8)*, there exists a bounded absorbing set* $\mathcal{B}$ *for* $(\mathcal{H}_\varepsilon, S_\varepsilon(t))$*, that does not depend on* ε*.*

Proof Denoting by Ψ_ε the Lyapunov functional defined on $\mathcal{H}_\varepsilon$, then from (2.17), Remarks 2.3 and Theorem 4.1,

$$\begin{aligned}\sup_{z\in\mathcal{A}_\varepsilon} \|z\|^2_{\mathcal{H}_\varepsilon} &\leq \frac{\sup_{z\in\mathcal{A}_\varepsilon} \Psi_\varepsilon(z) + K_3}{K_2}\\ &\leq \frac{\sup_{z\in\mathcal{N}_\varepsilon} \Psi_\varepsilon(z) + K_3}{K_2}\\ &\leq \frac{K_1 \sup_{z\in\mathcal{N}_\varepsilon} \|z\|^4_{\mathcal{H}_\varepsilon} + 2K_3}{K_2}.\end{aligned}$$

Thus, from (3.5) there exists a constant R_1 which does not depend on ε such that

$$\sup_{z\in\mathcal{A}_\varepsilon} \|z\|^2_{\mathcal{H}_\varepsilon} \leq R_1^2, \quad \forall \varepsilon \in [0, 1].$$

Let us define $\mathcal{B} = \left\{z \in \mathcal{H}_0 : \|z\|^2_{\mathcal{H}_0} \leq R_1 + 1\right\}$, then from (4.2) $A_\varepsilon \subset \mathcal{B}$, for all ε. □

Lemma 4.3 *Let* B *be a bounded subset in* $\mathcal{H}_0$ *and* $\{z_\varepsilon = (u_\varepsilon, v_\varepsilon)\}_\varepsilon \subset B$ *a family of initial data related to each* P_ε *with solutions* $\{S_\varepsilon(t)z_\varepsilon\}_\varepsilon$*. Then there exists a constant* $\hat{C}$ *that does not depend on* t, ε *such that*

$$E_\varepsilon(t) \leq \hat{C} \quad \textit{and} \quad \|S_\varepsilon(t)z_\varepsilon\|_{\mathcal{H}_\varepsilon} \leq \hat{C}, \quad \forall\, \varepsilon, t > 0.$$

Proof From (2.17), (4.2), Remark 2.3 and the fact that for each ε, E_ε is decreasing, we have

$$\begin{aligned}K_2\|S_\varepsilon(t)z_\varepsilon\|_{\mathcal{H}_\varepsilon} - K_3 &\leq E_\varepsilon(t) \leq E_\varepsilon(0)\\ &\leq K_1(\|u_\varepsilon\|^2_\varepsilon + \|v_\varepsilon\|^2_2)^4 + K_3\\ &\leq K_1\hat{K}(B, \mu) + K_3\end{aligned}$$

where $\hat{K}(B, \mu)$ is a constant which depends only on B and μ, and K_1, K_2 and K_3 do not depend on t and ε. □

Now we are in position to state and prove the main results of this section.

Theorem 4.4 (Singular limit) *Under the assumptions of* (2.2)–(2.8). *Given a sequence* $\{\varepsilon_n\}$ *of positive numbers, let* $(u_n(t), \partial_t u_n(t))$ *be the weak solution to* P_{ε_n} *with initial data* $(v_0, v_1) \in \mathcal{H}_0$. *Then if* $\varepsilon_n \to 0$ *when* $n \to \infty$, *there exist a weak solution* $(u(t), \partial_t u(t))$ *of* P_0 *with the same initial data, such that for any* $T > 0$:

$$u_n \overset{*}{\rightharpoonup} u \ in \ L^\infty(0, T; (H_0^1(\Omega))^3),$$
$$\partial_t u_n \overset{*}{\rightharpoonup} \partial_t u \ in \ L^\infty(0, T; (L^2(\Omega))^3).$$

Proof Using Lemma 4.3 for $B = \{(v_0, v_1)\}$ and Eq. (4.2), for some constant K,

$$\|(u_n(t), \partial_t u_n(t))\|_{\mathcal{H}_0} \le K. \tag{4.3}$$

Then, we have for any $T > 0$,

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0, T; (H_0^1(\Omega))^3),$$
$$\partial_t u_n \overset{*}{\rightharpoonup} \partial_t u \text{ in } L^\infty(0, T; (L^2(\Omega))^3).$$

Fixing n and multiplying P_{ε_n} by a function $\phi \in (H_0^1(\Omega))^3$, we get

$$\begin{aligned} &\frac{d}{dt}\langle \partial_t u_n, \phi\rangle + \mu\langle \nabla u_n, \nabla\phi\rangle + \varepsilon_n\langle \mathrm{div} u_n, \mathrm{div}\phi\rangle \\ &\quad + \alpha\langle \partial_t u_n, \phi\rangle + \langle f(u_n), \phi\rangle = \langle b, \phi\rangle . \end{aligned} \tag{4.4}$$

It is clear that

$$\langle \nabla u_n, \nabla\phi\rangle \underset{n\to\infty}{\longrightarrow} \langle \nabla u, \nabla\phi\rangle ,$$
$$\langle \partial_t u_n, \phi\rangle \underset{n\to\infty}{\longrightarrow} \langle \partial_t u, \phi\rangle ,$$
$$\varepsilon_n\langle \mathrm{div} u_n, \mathrm{div}\phi\rangle \underset{n\to\infty}{\longrightarrow} 0 \quad \text{from (4.3)}.$$

Additionally, we have

$$\begin{aligned} \langle f(u_n) - f(u), \phi\rangle &\le \sum_{i=1}^{3}\int_\Omega |f_i(u_n) - f_i(u)||\phi_i| \\ &\le \sum_{i=1}^{3}\int_\Omega M_g(1 + \sum_{j=1}^{3}|u_n^j|^{p-1} + |u^j|^{p-1})|u_n - u||\phi_i|. \end{aligned}$$

Then proceeding analogously to (3.14)–(3.15) and using Simon's compactness theorem [31] we have that

$$\|u_n - u\|_2, \|u_n - u\|_{\frac{6}{4-p}} \to 0,$$

which implies

$$\langle f(u_n), \phi\rangle \underset{n\to\infty}{\longrightarrow} \langle f(u), \phi\rangle .$$

Therefore (4.4) converges to

$$\frac{d}{dt}\langle \partial_t u, \phi\rangle + \mu\langle \nabla u, \nabla\phi\rangle + \alpha\langle \partial_t u, \phi\rangle + \langle f(u), \phi\rangle = \langle b, \phi\rangle , \tag{4.5}$$

which means that $(u, \partial_t u)$ is a weak solution of P_0 and $u(0) = u_0$.

Finally we multiply Eqs. (4.4) and (4.5) by a test function $\psi \in H^1([0,T])$ such that $\psi(0) = 1$, $\psi(T) = 0$ and integrating on $[0, T]$, we obtain for all $\phi \in (H_0^1(\Omega))^3$,

$$\int_0^T \frac{d}{dt}\langle \partial_t u_n, \phi\rangle\, \psi dt + \mu \int_0^T \langle \nabla u_n, \nabla \phi\rangle\, \psi dt + \varepsilon_n \lambda_0 \int_0^T \langle \mathrm{div} u_n, \mathrm{div}\phi\rangle\, \psi dt$$
$$\alpha \int_0^T \langle \partial_t u_n, \phi\rangle\, \psi dt + \int_0^T \langle f(u_n), \phi\rangle\, \psi dt = \int_0^T \langle b, \phi\rangle\, \psi dt,$$
$$\int_0^T \frac{d}{dt}\left\langle \partial_t u, \phi\right\rangle \psi dt + \mu \int_0^T \left\langle \nabla u, \nabla \phi\right\rangle \psi dt + \alpha \int_0^T \langle \partial_t u, \phi\rangle\, \psi dt + \int_0^T \langle f(u), \phi\rangle\, \psi dt$$
$$= \int_0^T \langle b, \phi\rangle\, \psi dt.$$

Solving the integrals and taking $n \to \infty$,

$$-\langle v_1, \phi\rangle - \int_0^T \langle \partial_t u, \phi\rangle \frac{d}{dt}\psi dt + \mu \int_0^T \langle \nabla u, \nabla\phi\rangle\, \psi dt + \alpha \int_0^T \langle \partial_t u, \phi\rangle\, \psi dt$$
$$+ \int_0^T \langle f(u), \phi\rangle\, \psi dt = \int_0^T \langle b, \phi\rangle\, \psi dt,$$
$$-\langle \partial_t u(0), \phi\rangle - \int_0^T \langle \partial_t u, \phi\rangle \frac{d}{dt}\psi dt + \mu \int_0^T \left\langle \nabla u, \nabla\phi\right\rangle \psi dt + \alpha \int_0^T \langle \partial_t u, \phi\rangle\, \psi dt$$
$$+ \int_0^T \langle f(u), \phi\rangle\, \psi dt = \int_0^T \langle b, \phi\rangle\, \psi dt.$$

Therefore, $\partial_t u(0) = v_1$, which ends the proof. □

Remark 4.1 From Theorem 4.4 and its proof, it is worth making two comments as follows:

- the limit in the previous theorem is singular in the sense that $\mathcal{H}_\varepsilon$ is not the same for ε varying the range $[0, 1]$;
- the space $\mathcal{H}_\varepsilon$ for weak solutions is defined by means of $(H_0^1(\Omega))^3 \times (L^2)^3$, where $H_0^1(\Omega)$ is provided with the norm $\|\cdot\|_\varepsilon$. Therefore, it makes sense to consider (v_0, v_1) as initial data to any P_ε.

Theorem 4.5 (Upper semicontinuity) *Under the assumptions (2.2)–(2.8), the family of attractors $\{\mathcal{A}_\varepsilon\}$ is upper semicontinuous with respect to $\varepsilon \to 0$. More precisely,*

$$\lim_{\varepsilon\to 0} d_{\mathcal{H}_0}(\hat{\imath}_\varepsilon(\mathcal{A}_\varepsilon), \mathcal{A}_0) = 0,$$

where $d_{\mathcal{H}_0}$ denotes Hausdorff semi-distance and $\hat{\imath}_\varepsilon : \mathcal{H}_\varepsilon \to \mathcal{H}_0$ is the identity map.

Proof The proof is done by contradiction arguments and follows similar lines as presented e.g. in [13,16,26].

Let us assume, for some $\epsilon > 0$, that

$$\sup_{y\in\mathcal{A}_\varepsilon} \inf_{z\in\mathcal{A}_0} \|\hat{\imath}_\varepsilon(y) - z\|_{\mathcal{H}_0} \geq \epsilon.$$

Since for any ε, $\mathcal{A}_\varepsilon$ is compact, there exists a sequence $\{y_n^0\}_n$ such that $y_n^0 \in \mathcal{A}_{\varepsilon_n}$ and

$$\inf_{z\in\mathcal{A}_0} \|\hat{\imath}_{\varepsilon_n}(y_n^0) - z\|_{\mathcal{H}_0} \geq \epsilon.$$

Let $y_n(t) = (u_n(t), \partial_t u_n(t))$ be a full trajectory in $\mathcal{A}_{\varepsilon_n}$ such that $y_n(0) = y_n^0$. From Lemma 4.2,

$$\|y_n(t)\|_{\mathcal{H}_0} \leq R_1 + 1. \tag{4.6}$$

Also, from Theorem 3.1, for each $n \in \mathbb{N}$, there exists $R_2^{\varepsilon_n} > 0$ such that

$$\|\partial_t y_n(t)\|_{\mathcal{H}_0} \leq \|\partial_t y_n(t)\|_{\mathcal{H}_{\varepsilon_n}} \leq R_2^{\varepsilon_n}.$$

Additionally, from (4.2), we obtain the existence of $R_2 > 0$, that does not depend on ε_n for all n, such that

$$\|\partial_t y_n(t)\|_{\mathcal{H}_0} \leq \|\partial_t y_n(t)\|_{\mathcal{H}_{\varepsilon_n}} \leq R_2, \quad \forall\, t, n.$$

In this way, one sees that

$$\mathcal{E}_{\varepsilon_n} u = -\alpha \partial_t u - f(u) - \partial_{tt} u + h \in (L^2(\Omega))^3.$$

Thus, multiplying this identity by $\mathcal{E}_{\varepsilon_n} u$, integrating and using Hölder's inequality, there exists $R_3 > 0$, not depending on ε_n, such that

$$\|\mathcal{E}_{\varepsilon_n} u(t)\|_2 \leq R_3, \quad \forall\, t, n,$$

from where it follows that

$$\begin{aligned} (y_n) &\text{ is bounded on } L^\infty(\mathbb{R}, \mathcal{H}_0), \\ (\partial_t y_n) &\text{ is bounded on } L^\infty(\mathbb{R}, \mathcal{H}_0). \end{aligned}$$

Using Simon's Theorem of compactness for the spaces $\mathcal{H}_0^1 \overset{c}{\hookrightarrow} \mathcal{H}_0 \hookrightarrow \mathcal{H}_0$, we have that for any $T > 0$, there exists a subsequence $\{y_{n_l}\}$ and $y \in C([-T, T], \mathcal{H}_0)$ such that

$$\lim_{l \to \infty} \sup_{t \in [-T, T]} \|y_{n_l}(t) - y(t)\|_{\mathcal{H}_0} = 0.$$

In particular,

$$\lim_{l \to \infty} \|\hat{\imath}_{\varepsilon_{n_l}}(y_{n_l}^0) - y(0)\|_{\mathcal{H}_0} = 0.$$

In order to get the desired contradiction, it remains to prove $y(0) \in \mathcal{A}_0$. In fact, since $\{y_{n_l}^0\}_l$ is bounded on $\mathcal{H}_0$, we can process as in the proof of Theorem 4.4, and prove that y is a solution of P_0 for time varying $t \in [-T, T]$ with initial data $y(0)$. Since $T > 0$ is arbitrary and (4.6) holds true, then $y(t)$ is a bounded full trajectory of P_0. This implies that $y(0) \in \mathcal{A}_0$. The proof Theorem 4.5 is complete. □

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References

1. Achenbach, J.: Wave Propagation in Elastic Solids. North-Holland, Amsterdam (1973)

2. Alabau, F., Komornik, V.: Boundary observability, controllability, and stabilization of linear elastodynamic systems. SIAM J. Control Optim. **37**, 521–542 (1999)
3. Arrieta, J., Carvalho, A.N., Hale, J.K.: A damped hyperbolic equation with critical exponent. Commun. Partial Differ. Equ. **17**, 841–866 (1992)
4. Astaburuaga, M.A., Charão, R.C.: Stabilization of the total energy for a system of elasticity with localized dissipation. Differ. Integral Equ. **15**, 1357–1376 (2002)
5. Bchatnia, A., Guesmia, A.: Well-posedness and asymptotic stability for the Lamé system with infinite memories in a bounded domain. Math. Control Relat. Fields **4**, 451–463 (2014)
6. Benaissa, A., Gaouar, S.: Asymptotic stability for the Lamé system with fractional boundary damping. Comput. Math. Appl. **77**, 1331–1346 (2019)
7. Belishev, M.I., Lasiecka, I.: The dynamical Lame system: regularity of solutions, boundary controllability and boundary data continuation. ESAIM: Control Optim. Calc. Var. **18**, 143–167 (2002)
8. Cavalcanti, M.M., Fatori, L.H., Ma, T.F.: Attractors for wave equations with degenerate memory. J. Differ. Equ. **260**, 56–83 (2016)
9. Cerveny, V.: Seismic Ray Theory. Cambridge University Press, Cambridge (2001)
10. Chueshov, I.: Dynamics of Quasi-Stable Dissipative Systems, Universitext. Springer, Cham (2015)
11. Chueshov, I., Lasiecka, I.: Von Karman evolution equations: well-posedness and long-time dynamics. In: Springer Monographs in Mathematics. Springer, New York (2010)
12. Ciarlet, P.G.: Mathematical Elasticity, Three-Dimensional Elasticity, vol. I. North-Holland, Amsterdam (1988)
13. Geredeli, P.G., Lasiecka, I.: Asymptotic analysis and upper semicontinuity with respect to rotational inertia of attractors to von Karman plates with geometrically localized dissipation and critical nonlinearity. Nonlinear Anal. Theory Methods Appl. **91**, 72–92 (2013)
14. Goodway, B.: AVO and Lamé constants for rock parameterization and fluid detection. CSEG Rec. **26**, 30–60 (2001)
15. Hale, J.K.: Asymptotic Behavior of Dissipative Systems. American Mathematical Society, Providence (2010)
16. Hale, J.K., Raugel, G.: Upper semicontinuity of the attractor for a singularly perturbed hyperbolic equation. J. Differ. Equ. **73**, 197–214 (1988)
17. Horn, M.A.: Stabilization of the dynamic system of elasticity by nonlinear boundary feedback. In: Hoffmann, K.-H., Leugering, G., Troltzsch, F. (eds.) Optimal Control of Partial Differential Equations, International Conference in Chemnitz, Germany, April 20–25, 1998. Springer, Basel (1999)
18. Hudson, J.: The Excitation and Propagation of Elastic Waves. Cambridge University Press, Cambridge (1984)
19. Ji, S., Sun, S., Wang, Q., Marcotte, D.: Lamé parameters of common rocks in the Earth's crust and upper mantle. J. Geophys. Res. **115** (2010) article B06314
20. Kline, M., Kay, I.: Electromagnetic Theory and Geometrical Optics. Interscience, New York (1965)
21. Lagnese, J.: Boundary stabilization of linear elastodynamic systems. SIAM J. Control Optim **21**, 968–984 (1983)
22. Lakes, R., Wojciechowski, K.W.: Negative compressibility, negative Poisson's ratio, and stability. Phys. Stat. Sol. (B) **245**, 545–551 (2008)
23. Lions, J.L.: Contrôlabilité Exacte, Perturbations et Stabilization de Systèmes Distribués, Tome 1. Masson, Paris (1988)
24. Liu, W.-J., Krstić, M.: Strong stabilization of the system of linear elasticity by a Dirichlet boundary feedback. IMA J. Appl. Math. **65**, 109–121 (2000)
25. Love, A. E. H.: A Treatise on Mathematical Theory of Elasticity. Cambridge (1892)
26. Ma, T.F., Monteiro, R.N.: Singular limit and long-time dynamics of Bresse systems. SIAM J. Math. Anal. **49**(4), 2468–2495 (2017)
27. Moore, B., Jaglinski, T., Stone, D.S., Lakes, R.S.: Negative incremental bulk modulus in foams. Philos. Mag. Lett. **86**, 651–659 (2006)
28. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations, vol. 44. Springer, Berlin (2012)
29. Poisson, S.D.: Mémoire sur l'équilibre et le mouvement des corps élastiques. Mémoires de l'Académie Royal des Sciences de l'Institut de France **VII**(I), 357–570 (1829)
30. Pujol, J.: Elastic Wave Propagation and Generation in Seismology. Cambridge University Press, Cambridge (2003)
31. Simon, J.: Compact sets in the space $L^p(O, T; B)$. Annali di Matematica **146**, 65–96 (1986)
32. Teodorescu, P.P.: Treatise on Classical Elasticity, Theory and Related Problems. Springer, Dordrecht (2013)
33. Timoshenko, S.P.: History of the Strength of Materials. McGraw-Hill, New York (1953)

34. Yamamoto, K.: Exponential energy decay of solutions of elastic wave equations with the Dirichlet condition. Math. Scand. **65**, 206–220 (1989)