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The limit of Dirichlet systems for variable monotone operators in general perforated domains

Carmen Calvo-Jurado [a], Juan Casado-Díaz [b]

[a] Departamento de Matemáticas. Escuela Politécnica, Universidad de Extremadura, Carretera de Trujillo, s/n 10071, Cáceres, Spain
[b] Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, c/Tarfia, s/n. 41012, Sevilla, Spain

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Abstract

We study the asymptotic behaviour of the solutions of nonlinear Dirichlet systems when the operators and the open sets where they are posed vary simultaneously. We obtain a representation of the limit problem and we prove that it is stable by homogenization. A corrector result is also given. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Résumé

On étudie le comportement asymptotique des solutions des systémes de Dirichlet non linéaires quand ils varient simultanement les opérateurs et les ouverts oú les problémes sont posés. On obtient une représentation du probléme limite, laquelle on montre qui est stable par homogénéisation. On donne aussi un résultat de correcteur. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

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Mots-clés : Homogénéisation ; Systémes non linéaires ; Domains perforés

Introduction

Our interest in the present paper is to study the homogenization problem

$$\begin{cases} -\operatorname{div} a_n(x, Du_n) = f_n & \text{in } \mathcal{D}'(\Omega_n, \mathbb{R}^M), \\ u_n \in W_0^{1,p}(\Omega_n, \mathbb{R}^M), \end{cases} \tag{0.1}$$

where $a_n : \Omega \times \mathcal{M}_{M\times N} \to \mathcal{M}_{M\times N}$ is a sequence of Carathéodory functions which define monotone operators in $W_0^{1,p}(\Omega, \mathbb{R}^M)$ and Ω_n is a sequence of open sets which are contained in a fixed bounded open set $\Omega \subset \mathbb{R}^N$ (no more hypotheses about Ω_n are imposed).

The homogenization of (0.1) has been studied in several papers, when Ω_n is fixed, or a_n is fixed.

When Ω_n does not vary ($\Omega_n = \Omega$ for every $n \in \mathbb{N}$), it is known (see, e.g., [16,20,21], ...) that there exists a function a which satisfies analogous conditions to a_n and does not depend of f_n or f, such that (for a subsequence) the limit problem of (0.1) is

$$\begin{cases} -\operatorname{div} a(x, Du) = f & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega, \mathbb{R}^M), \end{cases} \tag{0.2}$$

i.e., it has the same structure that (0.1).

E-mail addresses: ccalvo@unex.es (C. Calvo-Jurado), jcasado@numer.us.es (J. Casado-Díaz).

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When a_n is fixed, it has been proved in [5] (see also [2–4,6,9–13,22,23], ...) that for a subsequence, there exits a measure $\mu \in \mathcal{M}_0^p(\Omega)$ (see Notations) and a Carathéodory function $F:\Omega\times\mathbb{R}^M\to\mathbb{R}^M$ which does not depend of f_n or f, such that the limit problem of (0.1) is

$$\begin{cases} u\in W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_\mu(\Omega,\mathbb{R}^M),\\ \displaystyle\int_\Omega a(x,Du):Dv\,\mathrm{d}x+\int_\Omega F(x,u)v\,\mathrm{d}\mu=\langle f,v\rangle,\\ \forall v\in W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_\mu(\Omega,\mathbb{R}^M),\end{cases}\tag{0.3}$$

or, when μ is Radon,

$$\begin{cases} -\operatorname{div} a(x,Du)+F(x,u)\mu=f & \text{in } \mathcal{D}'(\Omega,\mathbb{R}^M),\\ u=0 & \text{in } \partial\Omega.\end{cases}\tag{0.4}$$

Thus, it contains a new term which does not appear in (0.1). However, if following G. Dal Maso and U. Mosco (see [11]), we introduce the measures $\mu_n\in\mathcal{M}_0^p(\Omega)$ as

$$\mu_n(B)=\begin{cases} +\infty & \text{if } C_p(B\cap(\Omega\backslash\Omega_n))>0,\\ 0 & \text{if } C_p(B\cap(\Omega\backslash\Omega_n))=0, \quad \forall B\subset\Omega \text{ Borel},\end{cases}$$

then (0.1) is equivalent to

$$\begin{cases} u_n\in W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_{\mu_n}(\Omega,\mathbb{R}^M),\\ \displaystyle\int_\Omega a_n(x,Du_n):Dv\,\mathrm{d}x+\int_\Omega F_n(x,u_n)v\,\mathrm{d}\mu_n=\langle f_n,v\rangle,\\ \forall v\in W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_{\mu_n}(\Omega,\mathbb{R}^M),\end{cases}\tag{0.5}$$

for a good choice of F_n (it is enough to ask $F_n(x,s)=0$ iff $s=0$). So, written (0.1) in this way, we see that (0.3) has the same structure. In fact, it is proved in [5] that if μ_n is an arbitrary sequence in $\mathcal{M}_0^p(\Omega)$ (not necessarily associated to a sequence Ω_n) and F_n satisfy analogous properties to the function F which appear in (0.3), then the limit problem of (0.5) with $a_n=a$ is still (0.3) for some F and μ (F is not exactly in the same class that F_n). Therefore, better than (0.1), let us consider in this work the homogenization of (0.5) for arbitrary a_n, F_n and μ_n. This has been realized in [15] when the operators are linear. For nonlinear equations (not systems), a previous result has been obtained by Kovalevsky in [19], where the problem is written as (0.1) for a sequence which satisfy the following hypothesis (i.e., Ω_n is not arbitrary).

There exists $\nu>0$, such that for every $u\in C_0^\infty(\Omega)$, there exists $u_n\in W_0^{1,p}(\Omega)$ which converges weakly to u in $W_0^{1,p}(\Omega)$ and it is such that

$$\limsup_{n\to\infty}\int_Q |\nabla u_n|^p\,\mathrm{d}x\leqslant \nu\int_Q\left(|\nabla u|^p+|u|^p\right)\mathrm{d}x,$$

$\forall Q\subset\Omega$ closed cube.

In the present paper we prove that if a_n and (F_n,μ_n) in (0.5) satisfy the conditions which appear in Section 2, then, for a subsequence, there exists a and (F,μ) which satisfy exactly the same conditions that a_n and (F_n,μ_n) and do not depend on f_n or f, such the limit problem of (0.5) are (0.3). The idea of the proof is to compare our problem with other ones for which the behaviour is known. The method generalize the corresponding one used by J. Casado-Díaz and A. Garroni in [5] when a_n is fixed. It can be extended to the case of pseudomonotone operators when $M=1$ by using the corresponding adaptation of the ideas used in [4].

In Section 5, we obtain a corrector result, i.e., an approach in the strong topology of $L^p(\Omega,\mathcal{M}_{M\times N})$ of the derivative of the solution u_n of (0.5). Essentially (see Theorem 5.7) we show that there exists a sequence $R_n:\Omega\times\mathbb{R}^M\to\mathcal{M}_{M\times N}$, such that if u is the weak limit in $W_0^{1,p}(\Omega,\mathbb{R}^M)$ of u_n, solutions of (0.5), and $\bar u_n$ is the solution of

$$\begin{cases} -\operatorname{div} a_n(x,D\bar u_n)=-\operatorname{div} a(x,Du) & \text{in } \Omega_n,\\ \bar u_n\in W_0^{1,p}(\Omega,\mathbb{R}^M),\end{cases}$$

then $D\bar u_n+R_n(x,u)$ is a good approach in $L^p(\Omega,\mathcal{M}_{M\times N})$ of Du_n.

From the point of view of the applications, the results exposed in the present paper can be used to study control problems for partial differential equations in which the control variables are the coefficients and the open sets in which the equations are posed. This is related with the selection of optimal materials and shapes.

1. Notations

Let $M, N \in \mathbb{N}$, we denote by $\mathcal{M}_{M\times N}$ the space of $M \times N$ real matrices. The scalar product of two matrices $A, B \in \mathcal{M}_{M\times N}$ will be denoted by $A:B$. Let Ω be a bounded open subset of $\mathbb{R}^N$. For a measure μ in Ω, we denote by $L^p_\mu(\Omega, \mathbb{R}^M)$, $1 \leqslant p \leqslant +\infty$, the usual Lebesgue spaces relatives to the measure μ. If μ is the Lebesgue measure, we write $L^p(\Omega, \mathbb{R}^M)$.

We denote by $\mathcal{H}^1(\mathbb{R}^N)$ the Hardy space (see [25])

$$\mathcal{H}^1(\mathbb{R}^N) = \Big\{ f \in L^1(\mathbb{R}^N):\ \sup_{t\geqslant 0} |h_t * f| \in L^1(\mathbb{R}^N) \Big\},$$

where $h_t = (1/t^N)h(\cdot/t)$, $h \in C_0^\infty(\mathbb{R}^N)$, $h \geqslant 0$, $\operatorname{supp} h \subset B(0,1)$.

The space $\mathcal{D}(\Omega)$ is the space of C^∞ functions with compact support in Ω. Its dual is the space of distributions in Ω and it is denoted by $\mathcal{D}'(\Omega)$.

We denote by $W_0^{1,p}(\Omega, \mathbb{R}^M)$ and $W^{1,p}(\Omega, \mathbb{R}^M)$, $1 \leqslant p \leqslant +\infty$, the usual Sobolev spaces, and by $W^{-1,p'}(\Omega, \mathbb{R}^M)$, $1/p' + 1/p = 1$, $1 \leqslant p \leqslant +\infty$, the dual of $W_0^{1,p}(\Omega, \mathbb{R}^M)$. $W_c^{1,p}(\Omega, \mathbb{R}^M)$ is the subspace of functions of $W^{1,p}(\Omega, \mathbb{R}^M)$ with compact support in Ω. When $M = 1$, we omit $\mathbb{R}^M$ in these notations.

For every $A \subset \Omega$, and $p \in (1, +\infty)$, we denote by $C_p(A, \Omega)$ the C_p-capacity of A (in Ω), which is defined as the infimum of $\int_\Omega |\nabla u|^p \,dx$ over the set of the functions $u \in W_0^{1,p}(\Omega)$ such that $u \geqslant 1$ a.e. in a neighbourhood of A.

We say that a property $\mathcal{P}(x)$ holds C_p-quasi-everywhere (abbreviated as q.e.) in a set E, if there exists $N \subset E$ with $C_p(N, \Omega) = 0$ such that $\mathcal{P}(x)$ holds for all $x \in E\setminus N$.

A function $u : \Omega \to \mathbb{R}^M$ is said to be C_p-quasi-continous if for every $\varepsilon > 0$ there exists $N \subset \Omega$, with $C_p(N, \Omega) < \varepsilon$, such that the restriction of u to $\Omega\setminus N$ is continuous. It is well known that every $u \in W^{1,p}(\Omega, \mathbb{R}^M)$ has a C_p-quasi-continuous representative (see [17,18,26], ...). We always identify u with its C_p-quasi-continuous representative.

We denote by $\mathcal{M}_0^p(\Omega)$ the class of all nonnegative Borel measures which vanish on the sets of C_p-capacity zero and satisfy

$$\mu(B) = \inf\big\{\mu(A):\ A\, C_p\text{-quasi-open},\ B \subseteq A \subseteq \Omega\big\}$$

for every Borel set $B \subseteq \Omega$.

The characteristic function of $E \subset \mathbb{R}^N$ will be denoted by χ_E. For every $k > 0$, the function $T_k : \mathbb{R} \to \mathbb{R}$ is defined by

$$T_k(s) = \begin{cases} k & \text{if } s \geqslant k, \\ s & \text{if } -k \leqslant s \leqslant k, \\ -k & \text{if } s \leqslant -k. \end{cases}$$

For $s = (s_1, \dots, s_M) \in \mathbb{R}^M$, we use the notation $T_k(s)$ to mean

$$T_k(u) = \big(T_k(u_1), T_k(u_2), \dots, T_k(u_M)\big).$$

For $t, s \in \mathbb{R}$, we denote

$$t \vee s = \max\{t, s\}, \qquad t \wedge s = \min\{t, s\}.$$

Let us denote by $O_{m,n}$ (respectively O_n) a generic sequence of real numbers such that

$$\lim_{m\to\infty} \limsup_{n\to\infty} |O_{m,n}| = 0, \qquad \lim_{n\to\infty} O_n = 0.$$

Definition 1.1. Let $a_n : \Omega \times \mathcal{M}_{M\times N} \to \mathcal{M}_{M\times N}$ be a sequence of Carathéodory functions. For this sequence, we denote by $\hat{a}_n : \Omega \times \mathcal{M}_{M\times N} \to \mathcal{M}_{M\times N}$, $\widetilde{a}_n : \Omega \times \mathcal{M}_{M\times N} \times \mathcal{M}_{M\times N} \to \mathcal{M}_{M\times N}$ the functions defined by

$$\hat{a}_n(x, \xi) = a_n(x, \xi) : \xi, \quad \forall \xi \in \mathcal{M}_{M\times N}, \text{ a.e. } x \in \Omega,$$

$$\widetilde{a}_n(x, \xi_1, \xi_2) = \big(a_n(x, \xi_1) - a_n(x, \xi_2)\big) : (\xi_1 - \xi_2), \quad \forall \xi_1, \xi_2 \in \mathcal{M}_{M\times N}, \text{ a.e. } x \in \Omega.$$

We assume there exists $p \geqslant 2$, such that

(i) $a_n(x, 0) = 0$, $\quad \forall n \in \mathbb{N}$, a.e. $x \in \Omega$;

(ii) there exists a constant $\alpha > 0$ such that

$$\widetilde{a}_n(x, \xi_1, \xi_2) \geqslant \alpha |\xi_1 - \xi_2|^p, \tag{1.1}$$

$\forall n \in \mathbb{N}$, $\forall \xi_1, \xi_2 \in \mathcal{M}_{M\times N}$, a.e. $x \in \Omega$;

(iii) there exist two constants $\gamma > 0$, $\sigma \in (0, 1]$, and a function $r \in L^1(\Omega)$ such that

$$\begin{aligned}&\left|a_n(x,\xi_1) - a_n(x,\xi_2)\right| \leqslant \gamma\big(r(x) + \hat{a}_n(x,\xi_1) + \hat{a}_n(x,\xi_2)\big)^{(p-1-\sigma)/p}\widetilde{a}_n(x,\xi_1,\xi_2)^{\sigma/p},\\ &\forall n \in \mathbb{N},\ \forall \xi_1,\xi_2 \in \mathcal{M}_{M\times N},\ \text{a.e. } x \in \Omega.\end{aligned} \tag{1.2}$$

Remark 1.2. Hypotheses (i), (ii) and (iii) imply:

(iii′) there exists a constant $\gamma' > 0$ and a function $r' \in L^p(\Omega)$ such that

$$\begin{aligned}&\left|a_n(x,\xi_1) - a_n(x,\xi_2)\right| \leqslant \gamma'\big(r'(x) + |\xi_1| + |\xi_2|\big)^{p(p-1-\sigma)/(p-\sigma)}|\xi_1 - \xi_2|^{\sigma/(p-\sigma)},\\ &\forall n \in \mathbb{N},\ \forall \xi_1,\xi_2 \in \mathcal{M}_{M\times N},\ \text{a.e. } x \in \Omega.\end{aligned} \tag{1.3}$$

In particular, a_n satisfy:

(iv′) there exists a constant $\beta > 0$ and a function $h \in L^{p'}(\Omega)$ such that

$$\left|a_n(x,\xi)\right| \leqslant h(x) + \beta|\xi|^{p-1}, \quad \forall n \in \mathbb{N},\ \forall \xi \in \mathcal{M}_{M\times N},\ \text{a.e. } x \in \Omega. \tag{1.4}$$

Reciprocally, if we assume (i), (ii), (iii′), then a_n satisfy (iii) with constants $\widetilde{\gamma}$, $\widetilde{\sigma}$ and a function $\widetilde{r}$. Remark that $\widetilde{\sigma} = \sigma/(p-\sigma)$ only coincides with σ for $p = 2$ and $\sigma = 1$.

Remark 1.3. The hypothesis (i) can be replaced by "$a_n(\cdot, 0)$ belongs $L^{p'}(\Omega)$". In this case, it is enough in the following to replace a_n by $\bar{a}_n$ defined by

$$\bar{a}_n(x,\xi) = a_n(x,\xi) - a_n(x,0), \quad \forall n \in \mathbb{N},\ \forall \xi \in \mathcal{M}_{M\times N},\ \text{a.e. } x \in \Omega.$$

Consider a sequence of functions $F_n : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ such that $F_n(\cdot, s)$ is μ_n-measurable for very $s \in \mathbb{R}^M$. Analogously to a_n, we define $\widehat{F}_n : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$, and $\widetilde{F}_n : \Omega \times \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M$ by

$$\widehat{F}_n(x,s) = F_n(x,s)s, \quad \forall n \in \mathbb{N},\ \forall s \in \mathbb{R}^M,\ \mu_n\text{-a.e. } x \in \Omega \quad \text{and}$$

$$\widetilde{F}_n(x,s_1,s_2) = \big(F_n(x,s_1) - F_n(x,s_2)\big)(s_1 - s_2), \quad \forall n \in \mathbb{N},\ \forall s_1,s_2 \in \mathbb{R}^M,\ \mu_n\text{-a.e. } x \in \Omega.$$

The sequence F_n is assumed to satisfy:

(A) $F_n(x,0) = 0, \quad \forall n \in \mathbb{N},\ \mu_n\text{-a.e. } x \in \Omega;$
(B) $\widetilde{F}_n(x,s_1,s_2) \geqslant \alpha|s_1 - s_2|^p, \quad \forall n \in \mathbb{N},\ \forall s_1,s_2 \in \mathbb{R}^M,\ \mu_n\text{-a.e. } x \in \Omega;$
(C) $\left|F_n(x,s_1) - F_n(x,s_2)\right| \leqslant \gamma\big[\widehat{F}_n(x,s_1) + \widehat{F}_n(x,s_2)\big]^{(p-1-\sigma)/p}\left|\widetilde{F}_n(x,s_1,s_2)\right|^{\sigma/p},$
$\forall n \in \mathbb{N},\ \forall s_1,s_2 \in \mathbb{R}^M,\ \mu_n\text{-a.e. } x \in \Omega.$

Remark 1.4. Analogously to a_n, the hypotheses (A), (B), (C) imply:

(C′) there exists a constant $\gamma' > 0$ such that

$$\begin{aligned}&\left|F_n(x,s_1) - F_n(x,s_2)\right| \leqslant \gamma'\big(|s_1| + |s_2|\big)^{p(p-1-\sigma)/(p-\sigma)}|s_1 - s_2|^{\sigma/(p-\sigma)},\\ &\forall s_1,s_2 \in \mathbb{R}^M,\ \mu_n\text{-a.e. } x \in \Omega,\ \forall n \in \mathbb{N}.\end{aligned}$$

In particular, F_n satisfies:

(D) there exists a constant $\beta \in \mathbb{R}$ such that

$$\left|F_n(x,s)\right| \leqslant \beta|s|^{p-1}, \quad \forall s \in \mathbb{R}^M,\ \mu_n\text{-a.e. } x \in \Omega,\ \forall n \in \mathbb{N}.$$

It is clear that this constant can be chosen as the same which appears in (iv′).

Reciprocally, if we assume (A), (B) and (C′), then F_n satisfy (C) for some constant $\widetilde{\gamma}$ and $\widetilde{\sigma} = \sigma/(p-\sigma)$.

Remark 1.5. Our results can be easily extended for $1 < p < 2$. In this case, (ii) and (B) must be respectively replaced by

$$\widetilde{a}_n(x,\xi_1,\xi_2) \geqslant \alpha \frac{|\xi_1-\xi_2|^p}{|\xi_1|^{2-p}+|\xi_2|^{2-p}}, \quad \forall n\in\mathbb{N},\ \forall \xi_1,\xi_2\in\mathcal{M}_{M\times N},\ \text{a.e. } x\in\Omega, \quad \text{and}$$

$$\widetilde{F}_n(x,s_1,s_2) \geqslant \alpha \frac{|s_1-s_2|^p}{|s_1|^{2-p}+|s_2|^{2-p}}, \quad \forall n\in\mathbb{N},\ \forall s_1,s_2\in\mathbb{R}^M,\ \mu_n\text{-a.e. } x\in\Omega.$$

Notation 1.6. Usually, in order to write shorter expressions, we do not specify the dependence in x of a_n and F_n. For example, we write $a_n(Du)$ to mean $a_n(x,Du(x))$ and $F_n(u)$ to mean $F_n(x,u(x))$.

We denote by C a generic constant which only depends on p,N,γ and β and can change from a line to another one.

2. Preliminary results

In order to realize the homogenization of (0.1) the idea is essentially to compare our problem with other ones for which the behaviour is known. We start this section by recalling some results related with the homogenization problem

$$\begin{cases} u_n\in W_0^{1,p}(\Omega)\cap L^p_{\mu_n}(\Omega),\\ \displaystyle\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla v\,\mathrm{d}x+\int_\Omega |u_n|^{p-2}u_n v\,\mathrm{d}\mu_n=\langle f,v\rangle,\\ \forall v\in W_0^{1,p}(\Omega)\cap L^p_{\mu_n}(\Omega), \end{cases} \tag{2.1}$$

where f is a given element in $W^{-1,p'}(\Omega)$.

Definition 2.1. For a given sequence μ_n in $\mathcal{M}_0^p(\Omega)$, we define w_n as the solution of

$$\begin{cases} w_n\in W_0^{1,p}(\Omega)\cap L^p_{\mu_n}(\Omega),\\ \displaystyle\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\nabla v\,\mathrm{d}x+\int_\Omega |w_n|^{p-2}w_n v\,\mathrm{d}\mu_n=\int_\Omega v\,\mathrm{d}x,\\ \forall v\in W_0^{1,p}(\Omega)\cap L^p_{\mu}(\Omega). \end{cases} \tag{2.2}$$

The sequence w_n has its norm bounded in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)\cap L^p_{\mu_n}(\Omega)$ and is nonnegative C_p-q.e. in Ω. Extracting a subsequence if necessary, there exists a nonnegative function $w\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, such that w_n converges weakly to w in $W_0^{1,p}(\Omega)$ and weakly-$*$ in $L^\infty(\Omega)$. Moreover the convergence is strong in $W_0^{1,q}(\Omega)$, $1\leqslant q<p$ (see [12], Theorem 6.8). It is proved in [12] that there exists a measure $\mu\in\mathcal{M}_0^p(\Omega)$ such that analogously to w_n, w satisfies

$$\begin{cases} w\in W_0^{1,p}(\Omega)\cap L^p_{\mu}(\Omega),\\ \displaystyle\int_\Omega |\nabla w|^{p-2}\nabla w\nabla v\,\mathrm{d}x+\int_\Omega |w|^{p-2}w v\,\mathrm{d}\mu=\int_\Omega v\,\mathrm{d}x,\\ \forall v\in W_0^{1,p}(\Omega)\cap L^p_{\mu}(\Omega). \end{cases} \tag{2.3}$$

Assume that the solution w_n of (2.2) converges weakly in $W^{1,p}(\Omega)$ to w and consider the measure μ defined in [12], such that (2.3) holds. The following properties about w_n, w and μ are proved in [5,12].

Theorem 2.2. *The sequence w_n, the function w and the measure μ satisfy*:

(a) *The set $\{w\psi\colon\ \psi\in\mathcal{D}(\Omega)\}$ is dense in $W_0^{1,p}(\Omega)\cap L^p_\mu(\Omega)$. Moreover, the set Λ of all the functions of the form $w\sum_{i=1}^l a_i\chi_{K_i}$ where $a_i\in\mathbb{R}$ and K_i are closed subsets of Ω such that $w=0$ μ-a.e. on $K_i\cap K_j$, with $i\neq j$, is dense in $L^p_\mu(\Omega)$.*

(b) *For every Borel set $B\subset\Omega$ with $C_p(B\cap\{w=0\})>0$, we have $\mu(B)=+\infty$.*

(c) *Let $u\in W_0^{1,p}(\Omega)\cap L^p_\mu(\Omega)$ be and consider $\psi_m\in\mathcal{D}(\Omega)$ such that $w\psi_m$ converges strongly to u in $W_0^{1,p}(\Omega)\cap L^p_\mu(\Omega)$. Then*

$$\lim_{m\to\infty}\lim_{n\to\infty}\left(\int_\Omega \bigl|\nabla(w_n\psi_m-u)\bigr|^p\varphi\,\mathrm{d}x+\int_\Omega |w_n\psi_m|^p\varphi\,\mathrm{d}\mu_n\right)=\int_\Omega |u|^p\varphi\,\mathrm{d}\mu,\quad \forall\varphi\in W^{1,p}(\Omega)\cap L^\infty(\Omega). \tag{2.4}$$

(d) *Let $u_n \in W^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)$ which converges weakly in $W^{1,p}(\Omega)$ to a function u. Then*

$$\liminf_{n\to\infty}\left(\int_\Omega |\nabla u_n|^p\,dx + \int_\Omega |u_n|^p\,d\mu_n\right) \geqslant \int_\Omega |\nabla u|^p\,dx + \int_\Omega |u|^p\,d\mu, \tag{2.5}$$

$$\liminf_{n\to\infty}\left(\int_\Omega \left|\nabla(u_n-u)\right|^p\,dx + \int_\Omega |u_n|^p\,d\mu_n\right) \geqslant \int_\Omega |u|^p\,d\mu. \tag{2.6}$$

In particular, if $\|u_n\|_{L^p_{\mu_n}(\Omega)}$ is bounded, u belongs to $L^p_\mu(\Omega)$.

(e) *We consider $\varphi, \psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\varphi\psi$ belongs to $W^{1,p}_0(\Omega)$. Then, we have*

$$\lim_{n\to\infty}\left(\int_\Omega \left|\nabla(w_n\psi)\right|^p\varphi\,dx + \int_\Omega |w_n\psi|^p\varphi\,d\mu_n\right) = \int_\Omega \left|\nabla(w\psi)\right|^p\varphi\,dx + \int_\Omega |w\psi|^p\varphi\,d\mu, \tag{2.7}$$

$$\lim_{n\to\infty}\left(\int_\Omega \left|\nabla(w_n-w)\psi\right|^p\varphi\,dx + \int_\Omega |w_n\psi|^p\varphi\,d\mu_n\right) = \int_\Omega |w\psi|^p\varphi\,d\mu. \tag{2.8}$$

For every sequence $v_n \in W^{1,p}_0(\Omega) \cap L^p_{\mu_n}(\Omega)$ such that $\|v_n\|_{L^p_{\mu_n}(\Omega)}$ is bounded and converges weakly to v in $W^{1,p}(\Omega)$, we have

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega \left|\nabla(w_n\psi)\right|^{p-2}\nabla(w_n\psi)\nabla v_n\varphi\,dx + \int_\Omega |w_n\psi|^{p-2}w_n\psi v_n\varphi\,d\mu_n\\ &\quad= \int_\Omega \left|\nabla(w\psi)\right|^{p-2}\nabla(w\psi)\nabla v\varphi\,dx + \int_\Omega |w\psi|^{p-2}w\psi v\varphi\,d\mu,\end{aligned} \tag{2.9}$$

for every $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Another interesting property of w_n is given by the following proposition.

Proposition 2.3. *Let $\varphi_n \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a sequence which converges weakly in $W^{1,p}(\Omega)$, and weakly-$*$ in $L^\infty(\Omega)$ to a function $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ when n tends to infinity. If $|\nabla\varphi_n|^p$ is equiintegrable, we have*

$$\lim_{n\to\infty}\left(\int_\Omega \left|\nabla(w_n-w)\right|^p\varphi_n\,dx + \int_\Omega w_n^p\varphi_n\,d\mu_n\right) = \int_\Omega w^p\varphi\,d\mu. \tag{2.10}$$

Proof. Taking $w_n(\varphi_n-\varphi)$ as a test function in (2.2), we get

$$\int_\Omega |\nabla w_n|^p(\varphi_n-\varphi)\,dx + \int_\Omega |\nabla w_n|^{p-2}\nabla w_n\nabla(\varphi_n-\varphi)w_n\,dx + \int_\Omega w_n^p(\varphi_n-\varphi)\,d\mu_n = \int_\Omega w_n(\varphi_n-\varphi)\,dx = O_n. \tag{2.11}$$

Since $|\nabla w_n|^{p-2}\nabla w_n$ is bounded in $L^{p'}(\Omega)$ and converges in measure to $|\nabla w|^{p-2}\nabla w$, and $\nabla(\varphi_n-\varphi)$ converges weakly to zero in $L^p(\Omega)$ and its power p is equiintegrable, an easy application of the Egorov's theorem shows that the second term in (2.11) converges to zero.

On the other hand,

$$\left||\nabla w_n|^p - \left|\nabla(w_n-w)\right|^p\right| \leqslant C\left(|\nabla w_n|^{p-1} + |\nabla w|^{p-1}\right)|\nabla w|, \quad \forall n \in \mathbb{N},$$

where the right-hand side is equiintegrable and the left-hand side converges in measure to $|\nabla w|^p$. So, we deduce

$$|\nabla w_n|^p - \left|\nabla(w_n-w)\right|^p \to |\nabla w|^p \quad \text{in } L^1(\Omega).$$

So, the first term of (2.11) satisfies

$$\int_\Omega |\nabla w_n|^p(\varphi_n-\varphi)\,dx = \int_\Omega \left|\nabla(w_n-w)\right|^p(\varphi_n-\varphi)\,dx + O_n.$$

Using these estimates in (2.11) and taking into account (2.8), we conclude (2.10). □

Although it is not needed for our purpose, we recall here the role of the sequence w_n and the measure μ in the homogenization of (2.1). The following theorem has been proved in [12] (see also [10]).

Theorem 2.4. *Assume that w_n converges weakly in $W_0^{1,p}(\Omega)$ to a function w (this always holds true for a subsequence) and consider the measure μ which satisfies* (2.3)*. Then, for every sequence f_n which converges strongly in $W^{-1,p'}(\Omega)$ to a distribution f, the solution u_n of* (2.1) *converges weakly in $W_0^{1,p}(\Omega)$ and strongly in $W_0^{1,q}(\Omega)$, $1 \leqslant q < p$, to the unique solution u of*

$$\begin{cases} u \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, \mathrm{d}x + \int_\Omega |u|^{p-2} u v \, \mathrm{d}\mu = \int_\Omega f v \, \mathrm{d}x, \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega). \end{cases}$$

Moreover, if f belongs to $L^\infty(\Omega)$, we have $\nabla u_n - \nabla(w_n u) \to 0$ in $W_0^{1,p}(\Omega)$.

Let us now give some results related with the homogenization problem

$$\begin{cases} -\operatorname{div} a_n(\nabla u_n) = f_n \quad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^M), \\ u_n \in W_0^{1,p}(\Omega, \mathbb{R}^M), \end{cases} \tag{2.12}$$

where f_n is in $W^{-1,p'}(\Omega, \mathbb{R}^M)$ and a_n satisfy (i)–(iii) of the previous section.

The homogenization of (2.12) is given by the following theorem (see [16,20,21], ...).

Theorem 2.5. *There exits a subsequence of a_n, still denoted by a_n, and a Carathéodory function $a : \Omega \times \mathcal{M}_{M\times N} \to \mathcal{M}_{M\times N}$ such that for every sequence f_n which converges strongly in $W^{-1,p'}(\Omega)$ to a distribution f, the solution u_n of* (2.12) *converges weakly in $W_0^{1,p}(\Omega, \mathbb{R}^M)$ to the unique solution u of*

$$\begin{cases} -\operatorname{div} a(\nabla u) = f \quad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^M), \\ u \in W_0^{1,p}(\Omega, \mathbb{R}^M). \end{cases}$$

Analogously to a_n, the function a satisfies (i), (ii) *and* (iii)*, for the same constants α, γ, σ and the same function r.*

The next theorem will be frequently used.

Lemma 2.6. *Assume that $f_n \in W^{-1,p'}(\Omega, \mathbb{R}^M)$ converges strongly in $W^{-1,p'}(\Omega, \mathbb{R}^M)$ to a distribution f and let u_n be the solution of* (2.12)*. Then, the sequence $|\nabla u_n|^p \chi_K$ is equiintegrable for every $K \subset \Omega$, compact.*

The proof of Lemma 2.6 is based on a simple application of $\mathcal{H}^1$ regularity and the following result due to R. Coiffman, P.L. Lions, Y. Meyer and S. Semmes (see [8]).

Theorem 2.7. *There exists a constant $C > 0$ with the following property: if $A \in L^p(\mathbb{R}^N, \mathbb{R}^N)$, $B \in L^{p'}(\mathbb{R}^N, \mathbb{R}^N)$, $1 < p < \infty$, are such that* $\operatorname{div}(A) = 0$ *and* $\operatorname{curl}(B) = 0$ *in $\mathcal{D}'(\mathbb{R}^N)$, then AB belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^N)$ and satisfies*

$$\|AB\|_{\mathcal{H}^1(\mathbb{R}^N)^N} \leqslant C\|A\|_{L^p(\mathbb{R}^N)} \cdot \|B\|_{L^{p'}(\mathbb{R}^N)}.$$

Proof of Lemma 2.6. Assume first that f belongs to $L^{p'}(\Omega, \mathbb{R}^M)$. Since the solution v_n of

$$\begin{cases} -\operatorname{div} a_n(Dv_n) = f \quad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^M), \\ v_n \in W_0^{1,p}(\Omega, \mathbb{R}^M), \end{cases}$$

satisfies that $v_n - u_n$ converges strongly to zero in $W_0^{1,p}(\Omega, \mathbb{R}^M)$, we can assume $f_n = f$ for every $n \in \mathbb{N}$.

We consider $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geqslant 0$. Since $\operatorname{div}(a_n(Du_n)\varphi)$ is bounded in $L^{p'}(\mathbb{R}^N, \mathbb{R}^M)$, there exists a sequence

$$\psi_n \in W^{1,p'}(\mathbb{R}^N, \mathcal{M}_{M\times N}) \quad \text{such that}$$

$$\operatorname{div} \psi_n = \operatorname{div}(a_n(Du_n)\varphi) \quad \text{and} \quad \|\psi_n\|_{W^{1,p'}(\mathbb{R}^N, \mathcal{M}_{M\times N})} \leqslant C\|\operatorname{div}(a_n(Du_n)\varphi)\|_{L^{p'}(\mathbb{R}^N, \mathbb{R}^M)}.$$

Taking in Theorem 2.7 A_n and B_n respectively the ith row of $a_n(Du_n)\varphi - \psi_n$ and Du_n $1 \leqslant i \leqslant n$ (u_n is assumed to be zero outside of Ω), we then deduce that $[a_n(Du_n)\varphi - \psi_n] : Du_n$ is bounded in $\mathcal{H}^1$. On the other hand, by the Sobolev's imbedding theorem, $\psi_n Du_n$ is bounded in $L^r(\mathbb{R}^N, \mathcal{M}_{M\times N})$ for some $r > 1$ and then in $\mathcal{H}^1$. Thus, the sequence $\varsigma_n = a_n(Du_n) : Du_n\varphi$ is bounded in $\mathcal{H}^1$. Since it is nonnegative, we conclude that $\varsigma_n \log \varsigma_n$ is bounded in $L^1(\mathbb{R}^N)$ (see [7,24] and [25]). By (i) and (ii) we deduce that $|\nabla u_n|^p\varphi$ is equiintegrable for every $\varphi \in \mathcal{D}(\Omega)$.

Using that every $f \in W^{-1,p'}(\Omega, \mathbb{R}^M)$ is the limit in $W^{-1,p'}(\Omega, \mathbb{R}^M)$ of a sequence $f_n \in L^{p'}(\Omega, \mathbb{R}^M)$, it is easy to extend the result to the case f in $W^{-1,p'}(\Omega, \mathbb{R}^M)$. □

3. Estimates and a first representation of the limit problem

Extracting a sequence if necessary, we assume that μ_n satisfies there exist $w \in W_0^{1,p}(\Omega)$ and $\mu \in \mathcal{M}_0^p(\Omega)$ such that the solution w_n of (2.2) converges weakly in $W_0^{1,p}(\Omega)$ to a function w and (2.3) holds. Moreover, we assume there exists a in the conditions of Theorem 2.5.

In the following, we consider sequences of distributions $f_n \in W^{-1,p'}(\Omega, \mathbb{R}^M)$, of functions $u_n \in W_0^{1,p}(\Omega, \mathbb{R}^M)$, a distribution $f \in W^{-1,p'}(\Omega, \mathbb{R}^M)$ and a function $u \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_\mu(\Omega, \mathbb{R}^M)$ such that

$$f_n \to f \quad \text{in } W^{-1,p'}\big(\Omega, \mathbb{R}^M\big), \tag{3.1}$$

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}\big(\Omega, \mathbb{R}^M\big), \tag{3.2}$$

$$\begin{cases} u_n \in W_0^{1,p}\big(\Omega, \mathbb{R}^M\big) \cap L^p_{\mu_n}\big(\Omega, \mathbb{R}^M\big), \\ \displaystyle\int_\Omega a_n(Du_n) : Dv\,\mathrm{d}x + \int_\Omega F_n(u_n)v\,\mathrm{d}\mu_n = \langle f_n, v\rangle, \\ \forall v \in W_0^{1,p}\big(\Omega, \mathbb{R}^M\big) \cap L^p_{\mu_n}\big(\Omega, \mathbb{R}^M\big). \end{cases} \tag{3.3}$$

Remark 3.1. Using u_n as a test function in this equation, we easily deduce that $\|u_n\|_{L^p_{\mu_n}(\Omega)}$ is bounded, then by Theorem 2.2(d), the function u is in $L^p_\mu(\Omega)$.

We also define $\bar{u}_n \in W_0^{1,p}(\Omega, \mathbb{R}^M)$ as the solution of

$$\begin{cases} -\operatorname{div} a_n(D\bar{u}_n) = -\operatorname{div} a(Du) \quad \text{in } \mathcal{D}'\big(\Omega, \mathbb{R}^M\big), \\ \bar{u}_n \in W_0^{1,p}\big(\Omega, \mathbb{R}^M\big). \end{cases} \tag{3.4}$$

Our aim in the present section is to obtain some estimates about $D(u_n - \bar{u}_n)$, which we will need later in order to obtain the problem satisfied by u.

Remark 3.2. By Theorem 2.5, $\bar{u}_n$ converges weakly to u in $W_0^{1,p}(\Omega, \mathbb{R}^M)$ and by Lemma 2.6, $|D\bar{u}_n|^p\chi_K$ is equiintegrable for every compact set $K \subset \Omega$.

Let us now obtain an estimate for $|D\bar{u}_n|^p$.

Proposition 3.3. *The sequence $\bar{u}_n$ satisfies*

$$\lim_{n\to\infty} \int_{\{u=0\}\cap K} |D\bar{u}_n|^p\,\mathrm{d}x = 0, \quad \forall K \subset \Omega \text{ compact.} \tag{3.5}$$

Proof. For $\varepsilon > 0$ and $K \subset \Omega$ compact, we consider $\varphi \in W_c^{1,p}(\Omega, \mathbb{R}^M) \cap L^\infty(\Omega, \mathbb{R}^M)$, $0 \leqslant \varphi \leqslant 1$, C_p-q.e. in Ω, such that

$$\varphi = \begin{cases} 1 & C_p\text{-q.e. in } \{u = 0\} \cap K, \\ 0 & C_p\text{-q.e. in } \{u > \varepsilon/2\}. \end{cases} \tag{3.6}$$

Then, we take $(T_k(\bar{u}_n) - T_k(u))\varphi \in W_0^{1,p}(\Omega,\mathbb{R}^M)$, $k>0$, as a test function in (3.4). This gives

$$\int_\Omega a_n(D\bar{u}_n) : D\big(T_k(\bar{u}_n) - T_k(u)\big)\varphi\,\mathrm{d}x + \int_\Omega a_n(D\bar{u}_n) : \big[\big(T_k(\bar{u}_n) - T_k(u)\big)\otimes\nabla\varphi\big]\,\mathrm{d}x = \big\langle -\operatorname{div} a(Du), \big(T_k(\bar{u}_n) - T_k(u)\big)\varphi\big\rangle. \tag{3.7}$$

By the weak convergence in $W_0^{1,p}(\Omega,\mathbb{R}^M)$ of $(T_k(\bar{u}_n) - T_k(u))\varphi$ to zero and the Rellich–Kondrachov's theorem, we have

$$\big\langle -\operatorname{div} a(Du), \big(T_k(\bar{u}_n) - T_k(u)\big)\varphi\big\rangle = O_n, \quad \forall k\in\mathbb{N},$$
$$\int_\Omega a_n(D\bar{u}_n) : \big[\big(T_k(\bar{u}_n) - T_k(u)\big)\otimes\nabla\varphi\big]\,\mathrm{d}x = O_n, \quad \forall k\in\mathbb{N}.$$

Therefore, (3.7) gives

$$\int_\Omega a_n(D\bar{u}_n) : D\big(T_k(\bar{u}_n) - T_k(u)\big)\varphi\,\mathrm{d}x = O_n, \quad \forall k\in\mathbb{N}. \tag{3.8}$$

On the other hand, we have

$$\int_\Omega a_n(D\bar{u}_n) D\big(T_k(\bar{u}_n) - T_k(u)\big)\varphi\,\mathrm{d}x = \int_\Omega a_n(D\bar{u}_n) : D(\bar{u}_n - u)\varphi\,\mathrm{d}x + \int_\Omega a_n(D\bar{u}_n) : D\big(T_k(\bar{u}_n) - \bar{u}_n\big)\varphi\,\mathrm{d}x + \int_\Omega a_n(D\bar{u}_n) D\big(u - T_k(u)\big)\varphi\,\mathrm{d}x. \tag{3.9}$$

Using $DT_k(\bar{u}_n) = D\bar{u}_n$ in $\{|\bar{u}_n|_\infty < k\}$, we have

$$\left|\int_\Omega a_n(D\bar{u}_n) : D\big(T_k(\bar{u}_n) - \bar{u}_n\big)\varphi\,\mathrm{d}x\right| \leqslant C\left(\int_{\{|\bar{u}_n|_\infty > k\}} \big|a_n(D\bar{u}_n)\big|^{p'}\varphi\,\mathrm{d}x\right)^{\frac{1}{p'}}\left(\int_{\{|\bar{u}_n|_\infty > k\}} |D\bar{u}_n|^p\varphi\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

By the equintegrability of $|D\bar{u}_n|^p\varphi$ and the Rellich–Kondrachov's compactness theorem we get

$$\int_\Omega a_n(D\bar{u}_n) : D\big(T_k(\bar{u}_n) - \bar{u}_n\big)\varphi\,\mathrm{d}x = O_{k,n}.$$

Analogously,

$$\int_\Omega a_n(D\bar{u}_n) : D\big(T_k(\bar{u}) - \bar{u}\big)\varphi\,\mathrm{d}x = O_{k,n}.$$

Returning to (3.8), we deduce

$$\int_\Omega a_n(D\bar{u}_n) : D\bar{u}_n\varphi\,\mathrm{d}x = \int_\Omega a_n(D\bar{u}_n) : Du\varphi\,\mathrm{d}x + O_n, \tag{3.10}$$

which implies

$$\alpha\int_{\{u=0\}\cap K} |D\bar{u}_n|^p\,\mathrm{d}x \leqslant \int_{\{0<|u|<\varepsilon\}} \big|a_n(D\bar{u}_n)\big||Du|\,\mathrm{d}x + O_n \leqslant \left(\int_\Omega \big|a_n(D\bar{u}_n)\big|^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}}\left(\int_{\{0<|u|<\varepsilon\}} |Du|^p\,\mathrm{d}x\right)^{\frac{1}{p}} + O_n, \quad \forall\varepsilon>0. \tag{3.11}$$

Since

$$\lim_{\varepsilon\to 0}\int_{\{0<|u|<\varepsilon\}} |Du|^p\,\mathrm{d}x = 0,$$

we deduce (3.5). □

Let us now study how close Du_n is to $D\bar{u}_n$. We start by showing (see [1,5,14]):

Lemma 3.4. *The sequences u_n and $\bar{u}_n$ satisfy*

$$u_n - \bar{u}_n \to 0 \text{ in } W_0^{1,q}(\Omega, \mathbb{R}^M), \quad 1 \leqslant q < p. \tag{3.12}$$

Proof. Let $\varepsilon > 0$ be given. For every $n \in \mathbb{N}$, we consider $\varepsilon_n \in (0, \varepsilon)$ which we shall fix later and $\Phi_{\varepsilon_n} \in \mathcal{D}(\mathbb{R}^M)$ such that

$$\Phi_{\varepsilon_n}(y) = \begin{cases} 1 & \text{if } |y| \leqslant \varepsilon_n, \\ 0 & \text{if } |y| > 2\varepsilon_n, \end{cases}$$

$0 \leqslant \psi_{\varepsilon_n} \leqslant 1$ in $\mathbb{R}^N$ and $|\nabla \psi_{\varepsilon_n}| \leqslant C/\varepsilon_n$. Then for ψ_{ε_n} defined by $\psi_{\varepsilon_n}(y) = \Phi_{\varepsilon_n}(y)y$, we take

$$\psi_{\varepsilon_n}(u_n - \bar{u}_n)w_n \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_{\mu_n}(\Omega, \mathbb{R}^M)$$

as a test function in the difference of (3.3) and (3.4), and $\psi_{\varepsilon_n}(u_n) \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_{\mu_n}(\Omega, \mathbb{R}^M)$ as a test function in (3.3). Adding, we get

$$\begin{aligned} &\int_\Omega \big[a_n(Du_n) - a_n(D\bar{u}_n)\big] : D\big[\psi_{\varepsilon_n}(u_n - \bar{u}_n)\big] w_n \,\mathrm{d}x + \int_\Omega \big[a_n(Du_n) - a_n(D\bar{u}_n)\big] : \big[\psi_{\varepsilon_n}(u_n - \bar{u}_n) \otimes \nabla w_n\big] \,\mathrm{d}x \\ &\quad + \int_\Omega a_n(Du_n) : D\psi_{\varepsilon_n}(u_n) \,\mathrm{d}x + \int_\Omega F_n(u_n)\big(\psi_{\varepsilon_n}(u_n - \bar{u}_n)w_n + \psi_{\varepsilon_n}(u_n)\big) \,\mathrm{d}\mu_n \\ &\quad = \big\langle f_n, \psi_{\varepsilon_n}(u_n - \bar{u}_n)w_n + \psi_{\varepsilon_n}(u_n)\big\rangle - \int_\Omega a(Du)D\big[\psi_{\varepsilon_n}(u_n - \bar{u}_n)w_n\big] \,\mathrm{d}x. \end{aligned} \tag{3.13}$$

From (A) and (B), we have

$$\int_\Omega F_n(u_n)\psi_{\varepsilon_n}(u_n) \,\mathrm{d}\mu_n \geqslant 0.$$

By (iii′), (D) and since u_n and w_n are bounded in $W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_{\mu_n}(\Omega, \mathbb{R}^M)$, we deduce there exists a constant $C_1 > 0$ such that

$$\left| \int_\Omega \big[a_n(Du_n) - a_n(D\bar{u}_n)\big] : \big[\psi_{\varepsilon_n}(u_n - \bar{u}_n) \otimes \nabla w_n\big] \,\mathrm{d}x \right| \leqslant C_1 \varepsilon, \qquad \left| \int_\Omega F_n(u_n) w_n \psi_{\varepsilon_n}(u_n - \bar{u}_n) \,\mathrm{d}\mu_n \right| \leqslant C_1 \varepsilon. \tag{3.14}$$

For the second member of (3.13), we have

$$\big\langle f_n, \psi_{\varepsilon_n}(u_n - \bar{u}_n)w_n \big\rangle = O_n, \tag{3.15}$$

$$\int_\Omega a(Du)D\big[\psi_{\varepsilon_n}(u_n - \bar{u}_n)w_n\big] \,\mathrm{d}x = O_n. \tag{3.16}$$

So, by (ii), (iii′), (iv′), w_n bounded in $L^\infty(\Omega)$ and the properties of ψ_{ε_n}, we get

$$\begin{aligned} &\int_{\{|u_n - \bar{u}_n| < \varepsilon_n\}} \big|D(u_n - \bar{u}_n)\big|^p w_n \,\mathrm{d}x + \int_{\{|u_n| < \varepsilon_n\}} |Du_n|^p \,\mathrm{d}x \\ &\quad \leqslant C \int_{\{\varepsilon_n \leqslant |u_n - \bar{u}_n| \leqslant 2\varepsilon_n\}} \big(r'(x) + |D\bar{u}_n| + |Du_n|\big)^{\frac{p(p-1-\sigma)}{p-\sigma}} \big|D(u_n - \bar{u}_n)\big|^{\frac{p}{p-\sigma}} \,\mathrm{d}x \\ &\qquad + C \int_{\varepsilon_n \leqslant |u_n| \leqslant 2\varepsilon_n} \big(h(x) + |Du_n|^{p-1}\big)|Du_n| \,\mathrm{d}x + \big\langle f_n, \psi_{\varepsilon_n}(u_n)\big\rangle + 2C_1\varepsilon + O_n. \end{aligned} \tag{3.17}$$

Now, since u_n and $\bar{u}_n$ are bounded in $W_0^{1,p}(\Omega, \mathbb{R}^M)$, there exists a constant $M > 0$, such that

$$\int_\Omega \big(r'(x) + |D\bar{u}_n| + |Du_n|\big)^{\frac{p(p-1-\sigma)}{p-\sigma}} \big|D(u_n - \bar{u}_n)\big|^{\frac{p}{p-\sigma}} \,\mathrm{d}x + \int_\Omega \big(h(x) + |Du_n|^{p-1}\big)|Du_n| \,\mathrm{d}x \leqslant M.$$

For $K\in\mathbb{N}$, $\delta>0$, we have

$$\sum_{k=1}^{K}\Bigg(\int\limits_{\{2^{k-1}\delta\leqslant|u_n-\bar u_n|\leqslant 2^k\delta\}}\big(r'(x)+|D\bar u_n|+|Du_n|\big)^{\frac{p(p-1-\sigma)}{p-\sigma}}\big|D(u_n-\bar u_n)\big|^{\frac{p}{p-\sigma}}\,\mathrm{d}x + \int\limits_{2^{k-1}\delta\leqslant|u_n|\leqslant 2^k\delta}\big(h(x)+|Du_n|^{p-1}\big)|Du_n|\,\mathrm{d}x\Bigg)\leqslant M.$$

So, for every $n\in\mathbb{N}$, there exits $k(n)\in\{1,\ldots,K\}$ such that

$$\int\limits_{\{2^{k(n)-1}\delta\leqslant|u_n-\bar u_n|\leqslant 2^{k(n)}\delta\}}\big(r'(x)+|D\bar u_n|+|Du_n|\big)^{\frac{p(p-1-\sigma)}{p-\sigma}}\big|D(u_n-\bar u_n)\big|^{\frac{p}{p-\sigma}}\,\mathrm{d}x + \int\limits_{2^{k(n)-1}\leqslant|u_n|\leqslant 2^{k(n)}}\big(h(x)+|Du_n|^{p-1}\big)|Du_n|\,\mathrm{d}x\leqslant\frac{M}{K}.$$

Taking δ and K such that $\varepsilon=2^K\delta$ and then $\varepsilon_n=2^{k(n)-1}\delta$, we deduce from (3.17):

$$\int\limits_{\{|u_n-\bar u_n|<\delta\}}\big|D(u_n-\bar u_n)\big|^p w_n\,\mathrm{d}x+\int\limits_{\{|u_n|<\delta\}}|Du_n|^p\,\mathrm{d}x\leqslant C\frac{M}{K}+C_1 2^{K+1}\delta+\big\langle f_n,\psi_{\varepsilon_n}(u_n)\big\rangle+O_n.$$

Let us pass to the limit in this inequality, for this purpose, since $\psi_{\varepsilon_n}(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ by a constant which does not depend on K nor in δ and $|\psi_{\varepsilon_n}(u_n)|\leqslant 2\varepsilon_n$, we can assume (it is true for a subsequence) that there exists $u^*_{K,\delta}\in W_0^{1,p}(\Omega)$ such that $\psi_{\varepsilon_n}(u_n)$ converges weakly to $u^*_{K,\delta}$ in $W_0^{1,p}(\Omega)$. Moreover, $u^*_{k,\delta}$ is bounded in $W_0^{1,p}(\Omega)$ and satisfies $|u^*_{K,\delta}|\leqslant 2^K\delta$. So for every $K>0$, $u^*_{K,\delta}$ converges weakly to zero in $W_0^{1,p}(\Omega)$, when δ tends to zero. Thus, taking the limit, first in n, then in δ and then in K, implies

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\Bigg(\int\limits_{\{|u_n-\bar u_n|<\delta\}}\big|D(u_n-\bar u_n)\big|^p w_n\,\mathrm{d}x+\int\limits_{\{|u_n|<\delta\}}|Du_n|^p\,\mathrm{d}x\Bigg)=0. \tag{3.18}$$

Let $\rho,\delta>0$ be, two parameters devoted to converge to zero, and consider $\varphi_\rho\in\mathcal{D}(\Omega)$, $0\leqslant\varphi_\rho\leqslant 1$ in Ω which pointwise converges to 1 in Ω. For $q\in[1,\ p)$, we get

$$\begin{aligned}
&\int\limits_{\Omega}\big|D(u_n-\bar u_n)\big|^q\,\mathrm{d}x\\
&\quad=\int\limits_{\{|u_n-\bar u_n|<\delta\}\cap\{\rho\leqslant w\}}\big|D(u_n-\bar u_n)\big|^q\left(\frac{w_n}{w}\right)^{q/p}\mathrm{d}x+\int\limits_{\{|u_n|<\delta\}\cap\{w=0\}}\big|D(u_n-\bar u_n)\big|^q\varphi_\rho\,\mathrm{d}x\\
&\qquad+\int\limits_{\Omega}\big|D(u_n-\bar u_n)\big|^q\left(1-\left(\frac{w_n}{w}\right)^{\frac{q}{p}}\chi_{\{\rho\leqslant w\}\cap\{|u_n-\bar u_n|<\delta\}}-\varphi_\rho\chi_{\{|u_n|<\delta\}\cap\{w=0\}}\right)\mathrm{d}x\\
&\quad\leqslant\frac{1}{\rho^{\frac{q}{p}}}\Bigg(\int\limits_{\{|u_n-\bar u_n|<\delta\}}\big|D(u_n-\bar u_n)\big|^p w_n\,\mathrm{d}x\Bigg)^{\frac{q}{p}}|\Omega|^{\frac{p-q}{q}}+C\Bigg(\int\limits_{\{|u_n|<\delta\}\cap\{w=0\}}\big|D(u_n-\bar u_n)\big|^p\varphi_\rho\,\mathrm{d}x\Bigg)^{\frac{q}{p}}|\Omega|^{\frac{p-q}{p}}\\
&\qquad+\Bigg(\int\limits_{\Omega}\big|D(u_n-\bar u_n)\big|^p\,\mathrm{d}x\Bigg)^{\frac{q}{p}}\Bigg(\int\limits_{\Omega}\left(1-\left(\frac{w_n}{w}\right)^{\frac{q}{p}}\chi_{\{\rho\leqslant w\}\cap\{|u_n-\bar u_n|<\delta\}}-\varphi_\rho\chi_{\{w=0\}\cap\{|u_n|<\delta\}}\right)^{\frac{p}{p-q}}\mathrm{d}x\Bigg)^{\frac{p-q}{q}}.
\end{aligned}$$

Since $\{w=0\}$ is contained in $\{u=0\}$ (this is consequence of Theorem 2.2), (3.5) and (3.18), taking the limit in the above inequality first when n tends to infinity, then when δ tends to zero and then when ρ tends to zero, we conclude that $u_n-\bar u_n$ converges strongly to zero in $W^{1,q}(\Omega)$. □

Corollary 3.5. *The sequence $a_n(Du_n)$ satisfies*

$$a_n(Du_n)\rightharpoonup a(Du)\quad\text{in } L^{p'}\big(\Omega,\mathbb{R}^M\big). \tag{3.19}$$

Proof. By (iii′) and (3.12), $a_n(Du_n) - a_n(D\bar{u}_n)$ converges to zero in $L^r(\Omega, \mathbb{R}^M)$, $1 \leqslant r < p'$. Since $a_n(Du_n)$ is bounded in $L^{p'}(\Omega, \mathbb{R}^M)$ and $a_n(D\bar{u}_n)$ converges weakly to $a(Du)$ in $L^{p'}(\Omega, \mathbb{R}^M)$, we conclude (3.19). □

The following lemma replaces Lemma 6.6 in [5] (see also Lemma 2.5 in [2]) and permits to obtain a first representation of the limit problem of (3.3).

Lemma 3.6. *For every $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geqslant 0$ C_p-q.e. in Ω, we have*

$$\limsup_{n\to\infty}\left(\int_\Omega \left|D(u_n - \bar{u}_n)\right|^p \varphi\,\mathrm{d}x + \int_\Omega |u_n|^p \varphi\,\mathrm{d}\mu_n\right) \leqslant C \int_\Omega |u|^p \varphi\,\mathrm{d}\mu. \tag{3.20}$$

Proof. Let w_n and w be respectively the solutions of problems (2.2) and (2.3). For every $n, m \in \mathbb{N}$, we define

$$w_{n,m} = \frac{w_n}{w \vee 1/m}.$$

By Theorem 2.2(a), it is easy to show that there exits $\psi_m \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^\infty(\Omega, \mathbb{R}^M)$ which is zero C_p-q.e. in $\{w < 1/m\}$ and such that $w\psi_m$, converges strongly to u in $W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_\mu(\Omega, \mathbb{R}^M)$.

For every $n, m \in \mathbb{N}$, we define $\bar{u}_{n,m}$ as the solution of the problem

$$\begin{cases} -\operatorname{div} a_n(D\bar{u}_{n,m}) = -\operatorname{div} a\big(D(w\psi_m)\big) & \text{in } W^{-1,p'}\big(\Omega, \mathbb{R}^M\big), \\ \bar{u}_{n,m} \in W_0^{1,p}\big(\Omega, \mathbb{R}^M\big). \end{cases} \tag{3.21}$$

By Lemma 2.6 and since $w\psi_m$ converges strongly in $W_0^{1,p}(\Omega)$, we easily have that for every compact set $K \subset \Omega$, $|\nabla \bar{u}_{n,m}|^p \chi_K$ is equiintegrable (in n and m).

For $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geqslant 0$ C_p-q.e. in Ω, we take

$$\big[u_n - w_{n,m}T_m(\bar{u}_{n,m})\big]\varphi \in W_0^{1,p}\big(\Omega, \mathbb{R}^M\big) \cap L^p_{\mu_n}\big(\Omega, \mathbb{R}^M\big)$$

as a test function in (3.3). This gives

$$\begin{aligned} &\int_\Omega a_n(Du_n) : D\big[u_n - w_{n,m}T_m(\bar{u}_{n,m})\big]\varphi\,\mathrm{d}x + \int_\Omega a_n(Du_n) : \big(\big[u_n - w_{n,m}T_m(\bar{u}_{n,m})\big] \otimes \nabla\varphi\big)\,\mathrm{d}x \\ &\quad + \int_\Omega F_n(u_n)\big[u_n - w_{n,m}T_m(\bar{u}_{n,m})\big]\varphi\,\mathrm{d}\mu_n = \big\langle f_n, \big[u_n - w_{n,m}T_m(\bar{u}_{n,m})\big]\varphi\big\rangle, \end{aligned} \tag{3.22}$$

where using that $[u_n - w_{n,m}T_m(\bar{u}_{n,m})]$ converges weakly in $W_0^{1,p}(\Omega, \mathbb{R}^M)$ to zero when n and then m tends to infinity, it is easy to see that the second and fourth terms are equal to $O_{m,n}$. Thus, we have

$$\int_\Omega a_n(Du_n) : D\big[u_n - w_{n,m}T_m(\bar{u}_{n,m})\big]\varphi\,\mathrm{d}x + \int_\Omega F_n(u_n)\big[u_n - w_{n,m}T_m(\bar{u}_{n,m})\big]\varphi\,\mathrm{d}\mu_n = O_{m,n}. \tag{3.23}$$

This implies

$$
\begin{aligned}
&\int_\Omega a_n(Du_n):D(u_n-\bar u_n)\varphi\,\mathrm{d}x+\int_\Omega F_n(u_n)u_n\varphi\,\mathrm{d}\mu_n\\
&\quad=\int_\Omega a_n(Du_n):D\big[w_{n,m}T_m(\bar u_{n,m})-\bar u_n\big]\varphi\,\mathrm{d}x+\int_\Omega F_n(u_n)w_{n,m}T_m(\bar u_{n,m})\varphi\,\mathrm{d}\mu_n+O_{m,n}\\
&\quad=\int_\Omega a_n(Du_n):DT_m(\bar u_{n,m})(w_{n,m}-1)\varphi\,\mathrm{d}x+\int_\Omega a_n(Du_n):D\big[T_m(\bar u_{n,m})-\bar u_{n,m}\big]\varphi\,\mathrm{d}x\\
&\qquad+\int_\Omega a_n(Du_n):D(\bar u_{n,m}-\bar u_n)\varphi\,\mathrm{d}x+\int_\Omega a_n(Du_n):\bigg(T_m(\bar u_{n,m})\otimes\frac{\nabla(w_n-w)}{w\vee\frac{1}{m}}\bigg)\varphi\,\mathrm{d}x\\
&\qquad+\int_{\{w>1/m\}} a_n(Du_n):\big(T_m(\bar u_{n,m})\otimes\nabla w\big)\frac{(w-w_n)}{w^2}\varphi\,\mathrm{d}x+m\int_{\{w\leqslant 1/m\}} a_n(Du_n):\big(T_m(\bar u_{n,m})\otimes\nabla w\big)\varphi\,\mathrm{d}x\\
&\qquad+\int_\Omega F_n(u_n)w_{n,m}T_m(\bar u_{n,m})\varphi\mathrm{d}\mu_n+O_{m,n}.
\end{aligned}\tag{3.24}
$$

Let us estimate the first term of the right-hand side of (3.24). It can be descomposed as

$$
\int_{\{0\leqslant w\leqslant 1/m\}} a_n(Du_n):DT_m(\bar u_{n,m})(w_{n,m}-1)\varphi\,\mathrm{d}x+\int_{\{w>1/m\}} a_n(Du_n):DT_m(\bar u_{n,m})(w_{n,m}-1)\varphi\,\mathrm{d}x,\tag{3.25}
$$

where from (3.5) applied to $\bar u_{n,m}$, the first term tends to zero when n tends to infinity, for every $m\in\mathbb{N}$. The equintegrability of $|D\bar u_{n,m}|^p\varphi$ and the convergence in measure of $(w_{n,m}-1)\varphi\chi_{w>1/m}$ to zero also implies that the second term in (3.25) is equal to O_n (use Egorov's theorem), for each $m\in\mathbb{N}$.

From the equiintegrability of $|D\bar u_{n,m}|^p\varphi$ we easily show

$$
\int_\Omega \big|D\big(\bar u_{n,m}-T_m(\bar u_{n,m})\big)\big|^p\varphi\,\mathrm{d}x=O_{m,n},
$$

and then, the second term in the right-hand side of (3.24) is equal to $O_{m,n}$.

Using $\bar u_n-\bar u_{n,m}$ as a test function in the difference of (3.4) and (3.21), we deduce

$$
\int_\Omega \big|D(\bar u_n-\bar u_{n,m})\big|^p\,\mathrm{d}x=O_{m,n}.
$$

Thus, the third term in (3.24) is equals to $O_{m,n}$. The fifth and sixth terms of (3.24) converge clearly to zero when n tends to zero for every $m\in\mathbb{N}$ (use that $\psi_m=0$ C_p-q.e. in $w\leqslant 1/m$).

Now, by (iii′) we have

$$
\begin{aligned}
&\big|\big[a_n(Du_n)-a_n\big(D(u_n-\bar u_n)\big)\big]D(u_n-\bar u_n)\big|\varphi\\
&\quad\leqslant\gamma'\big(r'(x)+|Du_n|+\big|D(u_n-\bar u_n)\big|\big)^{\frac{p(p-1-\sigma)}{p-\sigma}}|D\bar u_n|^{\frac{\sigma}{p-\sigma}}\big|D(u_n-\bar u_n)\big|\varphi,
\end{aligned}\tag{3.26}
$$

where from Lemmas 2.6 and 3.4, the right-hand side tends to zero in $L^1(\Omega,\mathbb{R}^M)$ and then

$$
\big[a_n(Du_n)-a_n\big(D(u_n-\bar u_n)\big)\big]:D(u_n-\bar u_n)\varphi\to 0\quad\text{in } L^1\big(\Omega,\mathbb{R}^M\big).
$$

Analogously, we can prove

$$
\big(a_n(Du_n)-a_n\big(D(u_n-\bar u_n)\big)\big)\frac{\nabla(w_n-w)}{w\vee\frac{1}{m}}\varphi\to 0\quad\text{in } L^1\big(\Omega,\mathbb{R}^M\big).
$$

So, from (3.24) and the properties of a_n and F_n, we get

$$
\int_\Omega \big|D(u_n-\bar u_n)\big|^p\varphi\,\mathrm{d}x+\int_\Omega |u_n|^p\varphi\,\mathrm{d}\mu_n
$$

$$\begin{aligned}
&\leqslant C\int_\Omega \big[h(x)+\big|D(u_n-\bar u_n)\big|^{p-1}\big]:\frac{|\nabla(w_n-w)|}{w\vee 1/m}\big|T_m(\bar u_{n,m})\big|\varphi\,\mathrm{d}x\\
&\quad + C\int_\Omega |u_n|^{p-1}\frac{w_n}{w\vee 1/m}\big|T_m(\bar u_{n,m})\big|\varphi\,\mathrm{d}\mu_n + O_{m,n}\\
&\leqslant C\bigg(\int_\Omega \big|D(u_n-u)\big|^p\varphi\,\mathrm{d}x+\int_\Omega |u_n|^p\varphi\,\mathrm{d}\mu_n\bigg)^{\frac{p-1}{p}}\\
&\quad\times\bigg(\int_\Omega \big|\nabla(w_n-w)\big|^p\frac{|T_m(\bar u_{n,m})|^p}{(w\vee 1/m)^p}\varphi\,\mathrm{d}x+\int_\Omega \frac{|w_n|^p}{(w\vee 1/m)^p}\big|T_m(\bar u_{n,m})\big|^p\varphi\,\mathrm{d}x\bigg)^{\frac1p}+O_{m,n},
\end{aligned}$$

where we have used that $\nabla(w_n-w)$ converges to zero in measure and then, that $|\nabla(w_n-w)|$ converges weakly to zero in $L^p(\Omega,\mathbb{R}^M)$. Young's inequality and Proposition 2.3 imply

$$\begin{aligned}
&\int_\Omega \big|D(u_n-\bar u_n)\big|^p\varphi\,\mathrm{d}x+\int_\Omega |u_n|^p\varphi\,\mathrm{d}\mu_n\\
&\quad\leqslant C\bigg(\int_\Omega \big|\nabla(w_n-w)\big|^p\frac{|T_m(\bar u_{n,m})|^p}{(w\vee 1/m)^p}\varphi\,\mathrm{d}x+\int_\Omega \frac{|w_n|^p}{(w\vee 1/m)^p}\big|T_m(\bar u_{n,m})\big|^p\varphi\,\mathrm{d}x\bigg)+O_{m,n}\\
&\quad= C\int_\Omega T_m(w\psi_m)^p\varphi\,\mathrm{d}\mu+O_{m,n}=C\int_\Omega |u|^p\varphi\,\mathrm{d}\mu+O_{m,n}.
\end{aligned}$$

This finishes the proof of (3.20). □

Lemma (3.7) gives a first representation of the problem satisfied by u.

Theorem 3.7. *Assume* (3.1), (3.2) *and* (3.3)*. Then there exists a μ-measurable function $H\in L^{p'}_\mu(\Omega,\mathbb{R}^M)$, such that u is solution of the variational problem*

$$\begin{cases} u\in W^{1,p}_0\big(\Omega,\mathbb{R}^M\big)\cap L^p_\mu\big(\Omega,\mathbb{R}^M\big),\\ \displaystyle\int_\Omega a(Du):Dz\,\mathrm{d}x+\int_\Omega Hz\,\mathrm{d}\mu=\langle f,z\rangle,\\ \forall z\in W^{1,p}_0\big(\Omega,\mathbb{R}^M\big)\cap L^p_\mu\big(\Omega,\mathbb{R}^M\big).\end{cases}\tag{3.27}$$

The function H satisfies

$$|H|\leqslant C|u|^{p-1}\quad \mu\text{-a.e. in }\Omega\quad\text{and}\tag{3.28}$$

$$\int_\Omega Hw\psi\,\mathrm{d}\mu=\lim_{n\to\infty}\bigg(\int_\Omega a_n\big(D(u_n-\bar u_n)\big):\big(\psi\otimes\nabla(w_n-w)\big)\,\mathrm{d}x+\int_\Omega F_n(u_n)w_n\psi\,\mathrm{d}\mu_n\bigg),\tag{3.29}$$

for every $\psi\in W^{1,p}_c(\Omega,\mathbb{R}^M)\cap L^\infty(\Omega,\mathbb{R}^M)$.

Proof. For $\psi\in W^{1,p}_c(\Omega,\mathbb{R}^M)\cap L^\infty(\Omega,\mathbb{R}^M)$, we take $w_n\psi$ as a test function in (3.3). This gives

$$\begin{aligned}
&\int_\Omega a_n(Du_n):D\psi\,w_n\,\mathrm{d}x+\int_\Omega a_n(Du_n):(\psi\otimes\nabla w)\,\mathrm{d}x+\int_\Omega a_n(Du_n):\big(\psi\otimes\nabla(w_n-w)\big)\,\mathrm{d}x\\
&\quad+\int_\Omega F_n(u_n)w_n\psi\,\mathrm{d}\mu=\langle f_n,w_n\psi\rangle.
\end{aligned}\tag{3.30}$$

The strong convergence in $W^{-1,p'}(\Omega,\mathbb{R}^M)$ of f_n, implies

$$\langle f_n,w_n\psi\rangle=\langle f,w\psi\rangle+O_n,\tag{3.31}$$

and Corollary 3.5 easily gives

$$\int_\Omega a_n(Du_n):D\psi\, w_n\,\mathrm{d}x+\int_\Omega a_n(Du_n):(\psi\otimes\nabla w)\,\mathrm{d}x=\int_\Omega a(Du)D(w\psi)\,\mathrm{d}x+O_n.$$

On the other hand, reasoning as in (3.26) (in the proof of Lemma 3.6), we get

$$\big[a_n(Du_n)-a_n\big(D(u_n-\bar u_n)\big)\big]:\big[\psi\otimes\nabla(w_n-w)\big]\to 0\quad\text{in } L^1\big(\Omega,\mathbb{R}^M\big).$$

Therefore, by (3.30), we get

$$\int_\Omega a(Du)D(w\psi)\,\mathrm{d}x+\int_\Omega a_n\big(D(u_n-\bar u_n)\big):\big[\psi\otimes\nabla(w_n-w)\big]\,\mathrm{d}x+\int_\Omega F_n(u_n)w_n\psi\,\mathrm{d}\mu_n=\langle f,w\psi\rangle+O_n. \tag{3.32}$$

In order to characterize the second term in (3.32) we use

$$\int_\Omega \big|a_n\big(\nabla(u_n-\bar u_n)\big)\big|\big|\nabla(w_n-w)\big|\,\mathrm{d}x+\int_\Omega\big|F_n(u_n)\big||w_n|\,\mathrm{d}\mu_n\leqslant C.$$

So, there exists a vector Radon measure ν such that for $\psi\in C_c(\Omega,\mathbb{R}^M)$, we have

$$\int_\Omega\psi\,\mathrm{d}\nu=\lim_{n\to\infty}\bigg(\int_\Omega a_n\big(D(u_n-\bar u_n)\big):\big[\psi\otimes\nabla(w_n-w)\big]\,\mathrm{d}x+\int_\Omega F_n(u_n)w_n\psi\,\mathrm{d}\mu_n\bigg).$$

By (iv′), (C′), Hölder inequality, $|\nabla(w_n-w)|$ converging weakly to zero in $L^p(\Omega)$, (2.8) and (3.20), we get

$$\begin{aligned}\bigg|\int_\Omega\psi\,\mathrm{d}\nu\bigg| &\leqslant \int_\Omega\big[h+\beta\big|D(u_n-\bar u_n)\big|^{p-1}\big]\big|\nabla(w_n-w)\big||\psi|\,\mathrm{d}x+C\int_\Omega|u_n|^{p-1}w_n|\psi|\,\mathrm{d}\mu_n+O_n\\ &\leqslant C\bigg(\int_\Omega\big|D(u_n-\bar u_n)\big|^p|\psi|\,\mathrm{d}x+\int_\Omega|u_n|^p|\psi|\,\mathrm{d}\mu_n\bigg)^{\frac{p-1}{p}}\bigg(\int_\Omega\big|\nabla(w_n-w)\big|^p|\psi|\,\mathrm{d}x+\int_\Omega w_n^p|\psi|\,\mathrm{d}\mu_n\bigg)^{\frac1p}+O_n\\ &\leqslant C\bigg(\int_\Omega|u|^p|\psi|\,\mathrm{d}\mu\bigg)^{\frac{p-1}{p}}\bigg(\int_\Omega w^p|\psi|\,\mathrm{d}\mu\bigg)^{\frac1p}.\end{aligned}$$

Using the derivation theorem for measures (see [17,26]), it is easy to deduce that there exists a μ-measurable vector function $G=(G_1,\dots,G_M)$ such that

$$\int_\Omega\psi\,\mathrm{d}\nu=\int_\Omega G\psi\,\mathrm{d}\mu,\quad\forall\psi\in C_c\big(\Omega,\mathbb{R}^M\big)\quad\text{and}\quad |G|\leqslant C|u|^{p-1}w\quad \mu\text{-a.e. in }\Omega.$$

Defining then $H=G/w\in L^{p'}_\mu(\Omega,\mathbb{R}^M)$, we deduce that H satisfies (3.28) and

$$\lim_{n\to\infty}\bigg(\int_\Omega a_n\big(D(u_n-\bar u_n)\big):\big[\psi\otimes\nabla(w_n-w)\big]\,\mathrm{d}x+\int_\Omega F_n(u_n)w_n\psi\,\mathrm{d}\mu_n\bigg)=\int_\Omega\psi\,\mathrm{d}\nu=\int_\Omega Hw\psi\,\mathrm{d}\mu,$$

$\forall\psi\in C_c(\Omega,\mathbb{R}^M)$. So, from (3.32) we get

$$\int_\Omega a(Du)D(w\psi)\,\mathrm{d}x+\int_\Omega Hw\psi\,\mathrm{d}\mu=\langle f,w\psi\rangle, \tag{3.33}$$

for every $\psi\in W^{1,p}_0(\Omega,\mathbb{R}^M)\cap C_c(\Omega,\mathbb{R}^M)$. By Theorem 2.2(a), we conclude that u satisfies (3.27). Returning to (3.32), we deduce that (3.29) holds for ψ in $W^{1,p}_c(\Omega,\mathbb{R}^M)\cap L^\infty(\Omega,\mathbb{R}^M)$. □

4. Dependence of H with respect to u

As in the previous section, we consider u_n, u, f_n and f which satisfy (3.1)–(3.3). We also consider v_n, g_n, v and g such that

$$\begin{cases} g_n, g \in W^{-1,p'}(\Omega, \mathbb{R}^M), \\ g_n \to g \text{ in } W^{-1,p'}(\Omega, \mathbb{R}^M), \end{cases} \tag{4.1}$$

$$\begin{cases} v_n \in W_0^{1,p}(\Omega_n, \mathbb{R}^M) \cap L^p_{\mu_n}(\Omega, \mathbb{R}^M), \\ \displaystyle\int_\Omega a_n(Dv_n)Dz\,\mathrm{d}x + \int_\Omega F_n(v_n)z\,\mathrm{d}\mu_n = \langle g_n, z\rangle, \\ \quad \forall z \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_{\mu_n}(\Omega, \mathbb{R}^M), \end{cases} \tag{4.2}$$

$$\begin{cases} v \in W^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_\mu(\Omega, \mathbb{R}^M), \\ v_n \rightharpoonup v \text{ in } W_0^{1,p}(\Omega, \mathbb{R}^M). \end{cases} \tag{4.3}$$

As for u_n, we define $\bar{v}_n \in W_0^{1,p}(\Omega, \mathbb{R}^M)$ by

$$\begin{cases} -\operatorname{div} a_n(D\bar{v}_n) = -\operatorname{div} a(Dv) \quad \text{in } W^{-1,p'}(\Omega, \mathbb{R}^M), \\ \bar{v}_n \in W_0^{1,p}(\Omega, \mathbb{R}^M). \end{cases} \tag{4.4}$$

By Theorem 3.7, there exits two μ-measurable functions $H, H' \in L^{p'}_\mu(\Omega, \mathbb{R}^M)$, such that u and v respectively satisfy

$$\int_\Omega a(Du) : Dz + \int_\Omega Hz\,\mathrm{d}\mu = \langle f, z\rangle, \quad \forall z \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_\mu(\Omega, \mathbb{R}^M) \quad \text{and} \tag{4.5}$$

$$\int_\Omega a(Dv) : Dz + \int_\Omega H'z\,\mathrm{d}\mu = \langle g, z\rangle, \quad \forall z \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_\mu(\Omega, \mathbb{R}^M). \tag{4.6}$$

Our aim in the present section is to prove Lemma 4.2, where we estimate the difference of H and H'. First, we show:

Lemma 4.1. *For every $\varphi \in \mathcal{D}(\Omega)$, we have*

$$\lim_{n\to\infty}\left(\int_\Omega \hat{a}_n\big(D(u_n - \bar{u}_n)\big)\varphi\,\mathrm{d}x + \int_\Omega \widehat{F}_n(u_n)\varphi\,\mathrm{d}\mu_n\right) = \int_\Omega Hu\varphi\,\mathrm{d}\mu, \tag{4.7}$$

$$\lim_{n\to\infty}\left(\int_\Omega \widetilde{a}_n\big(D(u_n - \bar{u}_n), D(v_n - \bar{v}_n)\big)\varphi\,\mathrm{d}x + \int_\Omega \widetilde{F}_n(u_n, v_n)\varphi\,\mathrm{d}\mu_n\right) = \int_\Omega (H - H')(u - v)\varphi\,\mathrm{d}\mu. \tag{4.8}$$

Proof. For $\varphi \in \mathcal{D}(\Omega)$, we take $(u_n - v_n)\varphi$ as a test function in the difference of (3.3) and (4.2). This gives

$$\int_\Omega \widetilde{a}_n(Du_n, Dv_n)\varphi\,\mathrm{d}x + \int_\Omega \big[a_n(Du_n) - a_n(Dv_n)\big] : \big[(u_n - v_n) \otimes \nabla\varphi\big]\,\mathrm{d}x + \int_\Omega \widetilde{F}_n(u_n, v_n)\varphi\,\mathrm{d}\mu_n \\ = \big\langle f_n - g_n, (u_n - v_n)\varphi\big\rangle. \tag{4.9}$$

In the second term of (4.9), we use that $a_n(Du_n)$ and $a_n(Dv_n)$ respectively converge to $a(Du)$ and $a(Dv)$ weakly in $L^{p'}(\Omega, \mathbb{R}^M)$. Using also the Rellich–Kondrachov's compactness theorem, we get

$$\int_\Omega \big[a_n(Du_n) - a_n(Dv_n)\big] : \big[(u_n - v_n) \otimes \nabla\varphi\big]\,\mathrm{d}x = \int_\Omega \big[a(Du) - a(Dv)\big] : \big[(u - v) \otimes \nabla\varphi\big]\,\mathrm{d}x + O_n. \tag{4.10}$$

For the fourth term of (4.9), we have

$$\big\langle f_n - g_n, (u_n - v_n)\varphi\big\rangle = \big\langle f - g, (u - v)\varphi\big\rangle + O_n.$$

The first term of (4.9) is the most difficult to estimate. We use

$$\begin{aligned}&\int_\Omega \widetilde{a}_n(Du_n, Dv_n)\varphi\,\mathrm{d}x\\&\quad=\int_\Omega \bigl[a_n(Du_n)-a_n(Dv_n)\bigr]:D(u_n-v_n-\bar{u}_n+\bar{v}_n)\varphi\,\mathrm{d}x+\int_\Omega \bigl[a_n(Du_n)-a_n(D\bar{u}_n)\bigr]:D(\bar{u}_n-\bar{v}_n)\varphi\,\mathrm{d}x\\&\qquad-\int_\Omega \bigl[a_n(Dv_n)-a_n(D\bar{v}_n)\bigr]:D(\bar{u}_n-\bar{v}_n)\varphi\,\mathrm{d}x+\int_\Omega \bigl[a_n(D\bar{u}_n)-a_n(D\bar{v}_n)\bigr]:D(\bar{u}_n-\bar{v}_n)\varphi\,\mathrm{d}x.\end{aligned}\tag{4.11}$$

In the first term of the second member of (4.11) we use that

$$\bigl(a_n(Du_n)-a_n\bigl(D(u_n-\bar{u}_n)\bigr)\bigr):D(u_n-v_n-\bar{u}_n+\bar{v}_n)\varphi$$

is equiintegrable and by Lemma 3.4 pointwise converges in measure to zero. So, it converges strongly to zero in $L^1(\Omega,\mathbb{R}^M)$. Reasoning analogously with

$$\bigl(a_n(Dv_n)-a_n\bigl(D(v_n-\bar{v}_n)\bigr)\bigr):D(u_n-v_n-\bar{u}_n+\bar{v}_n)\varphi,$$

we get

$$\int_\Omega \bigl[a_n(Du_n)-a_n(Dv_n)\bigr]:D(u_n-v_n-\bar{u}_n+\bar{v}_n)\varphi\,\mathrm{d}x=\int_\Omega \widetilde{a}_n\bigl(D(u_n-\bar{u}_n),D(v_n-\bar{v}_n)\bigr)\varphi\,\mathrm{d}x+O_n.\tag{4.12}$$

For the second term of the right side of (4.11), we use that $a_n(Du_n)-a_n(D\bar{u}_n)$, converges weakly to 0 in $L^{p'}(\Omega,\mathbb{R}^M)$ and strongly in $L^r(\Omega,\mathbb{R}^M)$ for $1\leqslant r<p'$ (use (iii$'$) and Lemma 3.4). Since the power p of $D(\bar{u}_n-\bar{v}_n)\varphi$ is equiintegrable, the Egorov's theorem implies

$$\int_\Omega \bigl[a_n(Du_n)-a_n(D\bar{u}_n)\bigr]:D(\bar{u}_n-\bar{v}_n)\varphi\,\mathrm{d}x=O_n,\tag{4.13}$$

and analogously

$$\int_\Omega \bigl[a_n(Dv_n)-a_n(D\bar{v}_n)\bigr]:D(\bar{u}_n-\bar{v}_n)\varphi\,\mathrm{d}x=O_n.$$

For the fourth term of the right-hand side of (4.11), taking $(\bar{u}_n-\bar{v}_n)$ as a test function in the difference of (3.4) and (4.4), we easily get

$$\int_\Omega \widetilde{a}_n(D\bar{u}_n,D\bar{v}_n)\varphi\,\mathrm{d}x=\int_\Omega \widetilde{a}(Du,Dv)\varphi\,\mathrm{d}x+O_n.\tag{4.14}$$

So, from (4.11) we get

$$\int_\Omega \widetilde{a}_n(Du_n,Dv_n)\varphi\,\mathrm{d}x=\int_\Omega \widetilde{a}_n\bigl(D(u_n-\bar{u}_n),D(v_n-\bar{v}_n)\bigr)\varphi\,\mathrm{d}x+\int_\Omega \widetilde{a}(Du,Dv)\varphi\,\mathrm{d}x+O_n.\tag{4.15}$$

So, (4.9), (4.10) and (4.15) give

$$\begin{aligned}&\int_\Omega \bigl[a(Du)-a(Dv)\bigr]:D\bigl((u-v)\varphi\bigr)\,\mathrm{d}x+\int_\Omega \widetilde{a}_n\bigl(D(u_n-\bar{u}_n),D(v_n-\bar{v}_n)\bigr)\varphi\,\mathrm{d}x+\int_\Omega \widetilde{F}_n(u_n,v_n)\varphi\,\mathrm{d}\mu_n\\&\quad=\langle f-g,(u-v)\varphi\rangle+O_n.\end{aligned}\tag{4.16}$$

On the other hand, using $(u-v)\varphi$ as a test function in the difference of (4.5) and (4.6) we have

$$\int_\Omega \bigl(a(Du)-a(Dv)\bigr):D\bigl((u-v)\varphi\bigr)\,\mathrm{d}x+\int_\Omega (H-H')(u-v)\varphi\,\mathrm{d}\mu=\langle f-g,(u-v)\varphi\rangle.\tag{4.17}$$

From (4.16) and (4.17) we deduce (4.8). In order to obtain (4.7), it is enough to take in (4.8) $v_n=\bar{v}_n=v=0$. □

Lemma 4.2. *The functions H and H' satisfy the following inequalities*

$$(H-H')(u-v) \geqslant \alpha|u-v|^p, \quad \mu\text{-a.e. in } \Omega \quad \text{and} \tag{4.18}$$

$$|H-H'| \leqslant \gamma(Hu+H'v)^{\frac{p-1-\sigma}{p}}\big[(H-H')(u-v)\big]^{\frac{\sigma}{p}}, \quad \mu\text{-a.e. in } \Omega. \tag{4.19}$$

Proof. In order to obtain (4.18), we use (4.8) and the properties (ii) and (B) of a_n and $\widetilde{F}_n$. Then, for $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geqslant 0$ in Ω, we get

$$\alpha\int_\Omega \big|D(u_n-v_n-\bar{u}_n+\bar{v}_n)\big|^p\varphi\,\mathrm{d}x + \alpha\int_\Omega |u_n-v_n|^p\varphi\,\mathrm{d}\mu_n \leqslant \int_\Omega (H-H')(u-v)\varphi\,\mathrm{d}\mu. \tag{4.20}$$

Let us estimate the left-hand side of (4.20). For that, we take a sequence $\psi_m \in \mathcal{D}(\Omega)$, such that $w\psi_m \to u-v$ in $W_0^{1,p}(\Omega,\mathbb{R}^M) \cap L^p_\mu(\Omega,\mathbb{R}^M)$ (use Theorem 2.2(a)). By convexity, we have

$$\begin{aligned}
&\int_\Omega \big|D(u_n-v_n-\bar{u}_n+\bar{v}_n)\big|^p\varphi\,\mathrm{d}x + \int_\Omega |u_n-v_n|^p\varphi\,\mathrm{d}\mu_n \\
&\quad\geqslant \int_\Omega \big|D(w_n\psi_m-u+v)\big|^p\varphi\,\mathrm{d}x + p\int_\Omega \big|D(w_n\psi_m-u+v)\big|^{p-2}D(w_n\psi_m-u+v) : Dz_{n,m}\varphi\,\mathrm{d}x \\
&\qquad + \int_\Omega |w_n\psi_m|^p\varphi\,\mathrm{d}\mu_n + p\int_\Omega |w_n\psi_m|^{p-2}w_n\psi_m(u_n-v_n-w_n\psi_m)\varphi\,\mathrm{d}\mu_n,
\end{aligned} \tag{4.21}$$

where $z_{n,m} = u_n - v_n - \bar{u}_n + \bar{v}_n - w_n\psi_m + u - v$.

By (2.4), we deduce

$$\int_\Omega \big|D(w_n\psi_m-u+v)\big|^p\varphi\,\mathrm{d}x + \int_\Omega |w_n\psi_m|^p\varphi\,\mathrm{d}\mu_n \geqslant \int_\Omega |u-v|^p\varphi\,\mathrm{d}\mu + O_{m,n}. \tag{4.22}$$

On the other hand, for every $m \in \mathbb{N}$, we have that

$$\big|D(w_n\psi_m-u+v)\big|^{p-2}D(w_n\psi_m-u+v)\varphi - \big|D(w_n\psi_m)\big|^{p-2}D(w_n\psi_m)\varphi$$

pointwise converges a.e. and has a p'th power equiintegrable. So, it converges strongly in $L^{p'}(\Omega,\mathbb{R}^M)$. Thus,

$$\begin{aligned}
&\int_\Omega \big|D(w_n\psi_m-u+v)\big|^{p-2}D(w_n\psi_m-u+v) : Dz_{n,m}\varphi\,\mathrm{d}x + \int_\Omega |w_n\psi_m|^{p-2}w_n\psi_m(u_n-v_n-w_n\psi_m)\varphi\,\mathrm{d}\mu_n \\
&\quad= \int_\Omega \big|D(w_n\psi_m)\big|^{p-2}D(w_n\psi_m) : D(u_n-v_n-w_n\psi_m)\varphi\,\mathrm{d}x + \int_\Omega |w_n\psi_m|^{p-2}w_n\psi_m(u_n-v_n-w_n\psi_m)\varphi\,\mathrm{d}\mu_n \\
&\qquad - \int_\Omega \big|D(w_n\psi_m)\big|^{p-2}D(w_n\psi_m) : D(\bar{u}_n-\bar{v}_n-u+v)\varphi\,\mathrm{d}x + O_{m,n}.
\end{aligned} \tag{4.23}$$

By (2.9), we have

$$\begin{aligned}
&\int_\Omega \big|D(w_n\psi_m)\big|^{p-2}D(w_n\psi_m) : D(u_n-v_n-w_n\psi_m)\varphi\,\mathrm{d}x + \int_\Omega |w_n\psi_m|^{p-2}w_n\psi_m(u_n-v_n-w_n\psi_m)\varphi\,\mathrm{d}\mu_n \\
&\quad= O_{m,n}.
\end{aligned} \tag{4.24}$$

Using that for every $m \in \mathbb{N}$, $|D(w_n\psi_m)|^{p-2}D(w_n\psi_m)$ converges weakly in $L^{p'}(\Omega,\mathbb{R}^M)$ and strongly in $L^r(\Omega,\mathbb{R}^M)$ for $1 \leqslant r < p'$ and that $D(\bar{u}_n-\bar{v}_n-u+v)\varphi$ converges weakly in $L^p(\Omega,\mathbb{R}^M)$ and its power p is equiintegrable, the Egorov's theorem gives

$$\int_\Omega \big|D(w_n\psi_m)\big|^{p-2}D(w_n\psi_m) : D(\bar{u}_n-\bar{v}_n-u+v)\varphi\,\mathrm{d}x = O_n, \quad \forall m \in \mathbb{N}. \tag{4.25}$$

Thus, (4.20), (4.21), (4.22), (4.23), (4.24) and (4.25), imply

$$\alpha\int_\Omega |u-v|^p\varphi\,\mathrm{d}\mu \leqslant \int_\Omega (H-H')(u-v)\varphi\,\mathrm{d}\mu, \quad \forall\varphi \in \mathcal{D}(\Omega),\ \varphi \geqslant 0 \text{ in } \Omega.$$

An easy application of the derivation measures theorem then gives (4.18).

Let us now prove (4.19). By (3.29) and Hölder's inequality, for every $\psi \in W_c^{1,p}(\Omega, \mathbb{R}^M) \cap L^\infty(\Omega, \mathbb{R}^M)$, we have

$$\begin{aligned}
&\left|\int_\Omega (H - H')w\psi \,\mathrm{d}\mu\right| \\
&\quad\leqslant \left|\int_\Omega \big(a_n\big(D(u_n - \bar{u}_n)\big) - a_n\big(D(v_n - \bar{v}_n)\big)\big) : \big[\psi \otimes \nabla(w_n - w)\big]\,\mathrm{d}x\right| + \left|\int_\Omega \big(F_n(u_n) - F_n(v_n)\big)w_n\psi \,\mathrm{d}\mu_n\right| + O_n \\
&\quad\leqslant I_1^{\frac{p-1}{p}} \cdot I_2^{\frac{1}{p}} + O_n,
\end{aligned} \tag{4.26}$$

with

$$I_1 = \int_\Omega \big|a_n\big(D(u_n - \bar{u}_n)\big) - a_n\big(D(v_n - \bar{v}_n)\big)\big|^{\frac{p}{p-1}}|\psi|\,\mathrm{d}x + \int_\Omega \big|F_n(u_n) - F_n(v_n)\big|^{\frac{p}{p-1}}|\psi|\,\mathrm{d}\mu_n \quad \text{and}$$

$$I_2 = \int_\Omega \big|\nabla(w_n - w)\big|^p|\psi|\,\mathrm{d}x + \int_\Omega |w_n|^p|\psi|\,\mathrm{d}\mu_n.$$

By (1.2), (3.12) and (C), we can write

$$\begin{aligned}
I_1 &\leqslant \gamma^{\frac{p}{p-1}} \int_\Omega \big(\hat{a}_n\big(D(u_n - \bar{u}_n)\big) + \hat{a}_n\big(D(v_n - \bar{v}_n)\big)\big)^{\frac{p-1-\sigma}{p-1}} \widetilde{a}_n\big(D(u_n - \bar{u}_n), D(v_n - \bar{v}_n)\big)^{\frac{\sigma}{p-1}}|\psi|\,\mathrm{d}x \\
&\quad + \gamma^{\frac{p}{p-1}} \int_\Omega \big(\widehat{F}_n(u_n) + \widehat{F}_n(v_n)\big)^{\frac{p-1-\sigma}{p-1}} \widetilde{F}_n(u_n, v_n)^{\frac{\sigma}{p-1}}|\psi|\,\mathrm{d}\mu_n + O_n \\
&\leqslant \gamma^{\frac{p}{p-1}} I_3^{\frac{p-1-\sigma}{p-1}} I_4^{\frac{\sigma}{p-1}} + O_n,
\end{aligned}$$

with

$$I_3 = \int_\Omega \big(\hat{a}_n\big(D(u_n - \bar{u}_n)\big) + \hat{a}_n\big(D(v_n - \bar{v}_n)\big)\big)|\psi|\,\mathrm{d}x + \int_\Omega \big(\widehat{F}_n(u_n) + \widehat{F}_n(v_n)\big)|\psi|\,\mathrm{d}\mu_n \quad \text{and}$$

$$I_4 = \int_\Omega \widetilde{a}_n\big(D(u_n - \bar{u}_n), D(v_n - \bar{v}_n)\big)|\psi|\,\mathrm{d}x + \int_\Omega \widetilde{F}_n(u_n, v_n)|\psi|\,\mathrm{d}\mu_n.$$

From (4.7) and (4.8), we get

$$I_1 \leqslant \gamma^{\frac{p}{p-1}} \left(\int_\Omega (Hu + H'v)|\psi|\,\mathrm{d}\mu\right)^{\frac{p-1-\sigma}{p-1}} \cdot \left(\int_\Omega (H - H')(u - v)|\psi|\,\mathrm{d}\mu\right)^{\frac{\sigma}{p-1}} + O_n.$$

On the other hand, by (2.8) we have

$$I_2 = \int_\Omega w^p\psi\,\mathrm{d}\mu + O_n.$$

Using in (4.26) the estimates obtained for I_1 and I_2 and applying the derivation measures theorem we easily deduce (4.19). □

5. The homogenization and corrector results

In this section, we will obtain a representation theorem for the function H which appears in Theorem 3.7. Indeed, from Lemma 4.2, the pointwise values of $H(x)$ depend only on the pointwise values of $u(x)$, i.e., there exists F such that $H(x) = F(x, u(x))$ μ-a.e. in Ω, but F is only defined on the pairs (x_0, s_0) such that $s_0 = u(x_0)$, where u is the limit of a sequence u_n which satisfies (3.3) for some f_n which converges strongly in $W^{-1,p'}(\Omega, \mathbb{R}^M)$ to a distribution f. The following lemma shows that the set of such (x_0, s_0) is dense in $\{w > 0\} \times \mathbb{R}^M$. The result is analogue with Theorem 6.9 in [5] and has a similar proof. So, we do not prove it.

Lemma 5.1. *For every $q \in \mathbb{Q}^M$ and every $m, n \in \mathbb{N}$, we denote by q_n^m the solution of the problem*

$$\begin{cases} q_n^m \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_{\mu_n}(\Omega, \mathbb{R}^M), \\ \displaystyle\int_\Omega a_n(Dq_n^m) Dv \, dx + \int_\Omega F_n(q_n^m) v \, d\mu_n = m \int_\Omega \big[|w_n q|^{p-2} w_n q - |q_n^m|^{p-2} q_n^m\big] v \, dx, \\ \forall v \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_{\mu_n}(\Omega, \mathbb{R}^M). \end{cases} \tag{5.1}$$

Then, there exists a subsequence of n, still denoted by n, such that for every $m \in \mathbb{N}$, the sequence q_n^m converges to a function q^m weakly in $W_0^{1,p}(\Omega, \mathbb{R}^M)$. This sequence q^m converges to wq strongly in $W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_\mu(\Omega, \mathbb{R}^M)$ and there exists a μ-measurable function Q^m, such that q^m satisfies

$$\begin{cases} q^m \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_{\mu}(\Omega, \mathbb{R}^M), \\ \displaystyle\int_\Omega a(Dq^m) v \, dx + \int_\Omega Q^m v \mu = m \int_\Omega \big[|wq|^{p-2} wq - |q^m|^{p-2} q^m\big] v \, dx, \\ \forall v \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_{\mu}(\Omega, \mathbb{R}^M). \end{cases} \tag{5.2}$$

The sequence Q^m converges strongly in $L^{p'}_\mu(\Omega, \mathbb{R}^M)$ to a function Q.

Definition 5.2. We consider the subsequence of n given by Lemma 5.1. Then, we define $\mathcal{F} : \Omega \times \mathbb{Q}^M \to \mathbb{R}^M$ by

$$\mathcal{F}(x, q) = Q(x), \quad \forall q \in \mathbb{Q}^M, \ \mu\text{-a.e. } x \in \Omega.$$

By (3.28), (4.18) and (4.19) it is easy to show that for every $q_1, q_2 \in \mathbb{Q}^M$ and μ-a.e. $x \in \Omega$, we have

$$\mathcal{F}(x, 0) = 0, \tag{5.3}$$

$$\big(\mathcal{F}(x, q_2) - \mathcal{F}(x, q_1)\big)(q_2 - q_1) \geqslant \alpha |q_2 - q_1|^p w(x)^p, \tag{5.4}$$

$$\big|\mathcal{F}(x, q_2) - \mathcal{F}(x, q_1)\big| \leqslant C\big(\mathcal{F}(x, q_1) q_1 + \mathcal{F}(x, q_2) q_2\big)^{\frac{p-1-\sigma}{p}} \big|\big(\mathcal{F}(x, q_2) - \mathcal{F}(x, q_1)\big)(q_2 - q_1)\big|^{\frac{\sigma}{p}}. \tag{5.5}$$

From (5.5), we can extend by continuity $\mathcal{F}$ to $\Omega \times \mathbb{R}^M$. We then define $F : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ by

$$F(x, s) = \begin{cases} \mathcal{F}(x, s/w(x)) & \text{if } w(x) > 0, \\ 0 & \text{if } w(x) = 0. \end{cases}$$

Analogously to a_n and F_n, we respectively note by $\widehat{F} : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ and $\widetilde{F} : \Omega \times \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M$ the functions:

$$\widehat{F}(x, s) = F(x, s) s, \qquad \widetilde{F}(x, s_1, s_2) = \big(F(x, s_1) - F(x, s_2)\big)(s_1 - s_2),$$

$\forall s, s_1, s_2 \in \mathbb{R}^M$, μ-a.e. $x \in \Omega$. For every $s_1, s_2 \in \mathbb{R}^M$ and μ-a.e. $x \in \Omega$, the function F (as usual, we do not specify the dependence on x) satisfies

$$F(0) = 0, \tag{5.6}$$

$$\big|F(s_2) - F(s_1)\big| \leqslant C\big(\widehat{F}(s_1) + \widehat{F}(s_2)\big)^{\frac{p-1-\sigma}{p}} \big|\widetilde{F}(s_2, s_1)\big|^{\frac{\sigma}{p}}, \tag{5.7}$$

$$\widetilde{F}(s_2, s_1) \geqslant \alpha |s_2 - s_1|^p, \quad \forall s_1, s_2 \in \mathbb{R}^M, \ \mu\text{-a.e. } x \in \Omega. \tag{5.8}$$

Theorem 3.7 and estimate (4.19) give the following homogenization result for problem (3.3). The proof is similar to the corresponding one of Theorem 2.1 in [2].

Theorem 5.3. *We consider the subsequence of n given by Lemma* 5.1 *and the function F given by Definition* 5.2. *Then, for every sequence $f_n \in W^{-1,p'}(\Omega, \mathbb{R}^M)$ which converges to f in $W^{-1,p'}(\Omega, \mathbb{R}^M)$, the solution u_n of*

$$\begin{cases} u_n \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_{\mu_n}(\Omega, \mathbb{R}^M), \\ \displaystyle\int_\Omega a_n(Du_n) : Dv \, dx + \int_\Omega F_n(u_n) v \, d\mu_n = \langle f_n, v \rangle, \\ \forall v \in W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L^p_{\mu}(\Omega, \mathbb{R}^M), \end{cases} \tag{5.9}$$

converges weakly in $W_0^{1,p}(\Omega,\mathbb{R}^M)$ *to the unique solution* u *of*

$$\begin{cases} u\in W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_\mu(\Omega,\mathbb{R}^M),\\ \displaystyle\int_\Omega a(Du):Dv\,\mathrm{d}x+\int_\Omega F(u)v\,\mathrm{d}\mu=\langle f,v\rangle,\\ \forall v\in W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_\mu(\Omega,\mathbb{R}^M).\end{cases}\tag{5.10}$$

To finish the paper, we give in the present section a corrector result (i.e., an approach in the strong topology of $L^p(\Omega,M_{M\times N})$) of the gradient of the solutions u_n of (5.9). We will use the following estimate.

Lemma 5.4. *We consider the subsequence of* n *given by Lemma* 5.1. *Assume* $u_n,v_n\in W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_{\mu_n}(\Omega,\mathbb{R}^M)$, $f_n,g_n\in W^{-1,p'}(\Omega,\mathbb{R}^M)$, $u,v\in W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_\mu(\Omega,\mathbb{R}^M)$, $f,g\in W^{-1,p'}(\Omega,\mathbb{R}^M)$ *such that* (3.1)–(3.4), (4.1)–(4.4) *hold. Then*

$$\limsup_{n\to\infty}\left(\int_\Omega\left|D(u_n-v_n-\bar u_n+\bar v_n)\right|^p\varphi\,\mathrm{d}x+\int_\Omega|u_n-v_n|^p\varphi\,\mathrm{d}\mu_n\right)\leqslant C\int_\Omega\big(|u|+|v|\big)^{\frac{p(p-1-\sigma)}{p-\sigma}}|u-v|^{\frac{p}{p-\sigma}}\varphi\,\mathrm{d}\mu,\quad\forall\varphi\in\mathcal{D}(\Omega),\ \varphi\geqslant0.\tag{5.11}$$

Proof. The result follows from (4.8), the properties (ii) and (B) of a_n and F_n, Theorem 5.3 and (C′) (applied to the function F which appear in Theorem 5.3). □

Definition 5.5. We consider the subsequence of n given by Lemma 5.1. For any $m,n\in\mathbb{N}$ and $s\in\mathbb{R}^M$, we define $R_n^m:\Omega\times\mathbb{R}^M\to M_{M\times N}$ by

$$R_n^m(x,s)=Ds_n^m-D(\bar s_n)\ \text{a.e. in}\ \Omega,\tag{5.12}$$

where s_n^m is the unique solution of

$$\begin{cases} s_n^m\in W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_{\mu_n}(\Omega,\mathbb{R}^M),\\ \displaystyle\int_\Omega a_n(Ds_n^m):Dv\,\mathrm{d}x+\int_\Omega F_n(s_n^m)v\,\mathrm{d}\mu_n=m\int_\Omega\left[|w_ns|^{p-2}w_ns-\left|s_n^m\right|^{p-2}s_n^m\right]v\,\mathrm{d}x,\\ \forall v\in W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_{\mu_n}(\Omega,\mathbb{R}^M),\end{cases}\tag{5.13}$$

and $\bar s_n$ is the unique solution of

$$\begin{cases} -\operatorname{div}a_n(D\bar s_n)=-\operatorname{div}a(sw)\quad\text{in }W^{-1,p'}(\Omega,\mathbb{R}^M),\\ \bar s_n\in W_0^{1,p}(\Omega,\mathbb{R}^M).\end{cases}$$

By Theorem 5.3, the sequence s_n^m converges to the unique solution s^m of

$$\begin{cases} s^m\in W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_\mu(\Omega,\mathbb{R}^M),\\ \displaystyle\int_\Omega a(Ds^m):Dv\,\mathrm{d}x+\int_\Omega F(x,s^m)v\,\mathrm{d}\mu=m\int_\Omega\left[|ws|^{p-2}ws-|s^m|^{p-2}s^m\right]v\,\mathrm{d}x,\\ \forall v\in W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_\mu(\Omega,\mathbb{R}^M).\end{cases}$$

Reasoning as in Lemma 5.1, we deduce

$$s_n^m\to sw\ \text{in}\ W_0^{1,p}(\Omega,\mathbb{R}^M)\cap L^p_\mu(\Omega,\mathbb{R}^M).$$

Remark 5.6. The function $R_n^m(x,s)$ is measurable in x for s fixed but in general is not continuous in s for x fixed. Hence, R_n^m is not a Carathéodory function.

The following result gives an approach in $L^p(\Omega, M_{M\times N})$ of the gradient of the solution u_n of problem (3.3).

Theorem 5.7. *Let n be the subsequence of n given by Lemma* 5.1. *Then, there exits a constant $C>0$ which satisfies the following property*:

Consider $f_n \in W^{-1,p'}(\Omega, \mathbb{R}^M)$ which converges strongly to f in $W^{-1,p'}(\Omega, \mathbb{R}^M)$ and define u_n, u, $\bar{u}_n$ respectively by (3.3), (5.10) *and* (3.4). *Then, for every simple function $\psi = \sum_{i=1}^{l} s_i \chi_{K_i}$ with $s_i \in \mathbb{R}^M$, $K_i \subset \Omega$ compact and $w=0$ μ-a.e. on $K_i \cap K_j$, $i \neq j$, we have*

$$\limsup_{m\to\infty} \lim_{n\to\infty} \int_{\bigcup_{i=1}^{l} K_i} \left|D(u_n - \bar{u}_n) - R_n^m(x,\psi)\right|^p \mathrm{d}x \leqslant \int_{\bigcup_{i=1}^{l} K_i} \left(|u| + |w\psi|\right)^{\frac{p(p-1-\sigma)}{p-\sigma}} |u - w\psi|^{\frac{p}{p-\sigma}} \mathrm{d}\mu. \tag{5.14}$$

Proof. Let $s \in \mathbb{R}$ given. Using the definitions (5.12) and (5.13) of R_n^m and s^m, Lemma 5.4 implies that for any function $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geqslant 0$, and any $m \in \mathbb{N}$, we have

$$\limsup_{n\to\infty} \int_{\Omega} \left|D\left(\bar{u}_n - s_n^m\right) - D\left(u - \bar{s}_n^m\right)\right|^p \varphi \,\mathrm{d}x \leqslant C \int_{\Omega} \left(|u| + |s^m|\right)^{\frac{p(p-1-\sigma)}{p-\sigma}} |u - s^m|^{\frac{p}{p-\sigma}} \varphi \,\mathrm{d}\mu, \tag{5.15}$$

where $\bar{s}_n^m$ is the solution of

$$\begin{cases} -\operatorname{div} a_n\left(D\bar{s}_n^m\right) = -\operatorname{div} a_n\left(Ds^m\right), \\ \bar{s}_n^m \in W_0^{1,p}(\Omega). \end{cases}$$

Using that s^m converges strongly to ws in $W_0^{1,p}(\Omega, \mathbb{R}^M) \cap L_\mu^p(\Omega, \mathbb{R}^M)$, we easily deduce from (5.15)

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \int_K \left|D(u_n - \bar{u}_n) - R_n^m(x,s)\right|^p \mathrm{d}x \leqslant C \int_K \left(|u| + |sw|\right)^{\frac{p(p-1-\sigma)}{p-\sigma}} |u - sw|^{\frac{p}{p-\sigma}} \mathrm{d}\mu \tag{5.16}$$

for every compact set $K \subset \Omega$.

If now $\psi = \sum_{i=1}^{l} s_i \chi_{K_i}$ is the function which appears in the statement of Theorem 5.7, then, writing (5.16) for s_i, K_i and adding in i, we deduce (5.14). □

Remark 5.8. The meaning of Theorem 5.7 is that

$$Du_n \sim D\bar{u}_n + R_n^m\left(x, \frac{u}{w}\right)$$

in $L^p(\Omega, \mathbb{R}^M)$, however $R_n^m(x, u/w)$ is not well defined (see Remark 5.6). So, we need to write (5.14).

Analogously as it has been proved in [5], there are some properties, about the sequence a_n, as homogeneity (Proposition 5.9) or linearity (Proposition 5.10) which are inherited by the functions F and a in the limit problem. More exactly, we have:

Proposition 5.9. *Let a_n and (F_n, μ_n) in the conditions in Section* 2. *Let us also assume the following homogeneity conditions*:

$$\begin{aligned} a_n(x, \lambda\xi) &= |\lambda|^{p-2}\lambda a_n(x,\xi), \quad \forall \xi \in \mathcal{M}_{M\times N},\ \forall \lambda \in \mathbb{R},\ \text{a.e. } x \in \Omega, \\ F_n(x, \lambda s) &= |\lambda|^{p-2}\lambda F_n(x,s), \quad \forall s \in \mathbb{R}^M,\ \forall t \in \mathbb{R},\ \text{a.e. } x \in \Omega. \end{aligned}$$

Under these hypotheses, in Theorem 5.3 *the functions a and F satisfy the same homogeneity conditions.*

In the linear case, we analogously have:

Proposition 5.10. *Let us consider now that the functions $a_n(x,\xi)$ are in the form $a_n(x)\xi$, where $a_n(x)$ are measurable functions in Ω, valuated in the space of the linear functions in $\mathcal{M}_{M\times N}$ and satisfy*: *there exists two constants α, $\gamma > 0$ such that*

$$a_n(x)(\xi_1 - \xi_2) : (\xi_1 - \xi_2) \geqslant \max\left\{\alpha|\xi_1 - \xi_2|^2, \frac{1}{\gamma}\left|a_n(x)(\xi_1 - \xi_2)\right|^2\right\}, \quad \forall n \in \mathbb{N},\ \forall \xi_1, \xi_2 \in \mathcal{M}_{M\times N},\ \text{a.e. } x \in \Omega.$$

For a given sequence $\mu_n \in \mathcal{M}_0^p(\Omega)$, we also assume that the functions F_n are linear in the second argument, i.e., of the form $F_n(x)s$, were F_n are μ_n-measurable functions in Ω valuated in the space of the linear functions in $\mathbb{R}^M$ and satisfy

$$F_n(x)(s_1 - s_2)(s_1 - s_2) \geqslant \max\left\{\alpha|s_1 - s_2|^2, \frac{1}{\gamma}\left|F_n(x)(s_1 - s_2)\right|^2\right\}, \quad \forall n \in \mathbb{N},\ \forall s_1, s_2 \in \mathbb{R}^M,\ \mu_n\text{-a.e. } x \in \Omega.$$

Under these hypotheses, it can be proved (*see* [5]) *that the functions F and a in the limit problem* (5.10), *satisfy the same conditions of linearity with the same constants.*

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