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Extended Half-Power Exponential Distribution with Applications to COVID-19 Data

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Abstract: In this paper, the Extended Half-Power Exponential (EHPE) distribution is built on the basis of the Power Exponential model. The properties of the EHPE model are discussed: the cumulative distribution function, the hazard function, moments, and the skewness and kurtosis coefficients. Estimation is carried out by applying maximum likelihood (ML) methods. A Monte Carlo simulation study is carried out to assess the performance of ML estimates. To illustrate the usefulness and applicability of EHPE distribution, two real applications to COVID-19 data in Chile are discussed.

Keywords: symmetric distributions; nonnegative distributions; kurtosis; maximum likelihood; COVID-19 data

MSC: 62E15; 62E20

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1. Introduction

Nonnegative data sets are quite usual in different scientific fields. To carry out inference for these kinds of data, models of continuous distributions with a nonnegative support are required. In this sense, we can cite the following models: Half-Normal (HN), Right Half Bimodal Normal (RHBN) (Alavi [1]), the Generalized Gamma (Meeker and Escobar [2]), Truncated Positive Normal (Gómez et al. [3]), Slash Truncated Positive Normal (Gómez et al. [4]), Truncated Skewed Bimodal Normal (Sharifipanah et al. [5]), Extended generalized half-normal (Duarte Sánchez et al. [6]), Extended generalized half-normal with progressive type-I interval censoring (Ahmadi and Yousefzadeh [7]), Gamma-Exponential (Kudryavtsev and Shestakov [8]). In a more general setting, families of distributions with nonnegative support can be found, in which the model is built on the basis of given probability density functions symmetric around zero, f_0. In this sense, we highlight the seminar paper by Elal-Olivero et al. [9]. Specifically, let Y be a continuous random variable (rv) with support in $(-\infty, \infty)$ and probability density function (pdf) f_0 symmetric about zero, satisfying that $E[Y^2] = k < \infty$. Then, Elal-Olivero et al. [9] proposed to obtain a family of nonnegative continuous distributions, X, whose pdf is given by:

$$f_X(x;\alpha,\delta) = \frac{2}{\delta^3}\left(\frac{\alpha\delta^2 + x^2}{\alpha + k}\right) f_0\left(\frac{x}{\delta}\right), \quad x \geq 0, \tag{1}$$

where $\alpha > 0$ is a shape parameter, and $\delta > 0$ is a scale parameter.

If $f_0 = \phi$, where ϕ denotes the pdf of the $N(0,1)$ distribution, then the Extended Half-Normal (EHN) distribution, proposed in [9], is obtained, whose pdf is:

$$f_X(x;\alpha,\delta) = \frac{2}{\delta}\left(\frac{\alpha + (x/\delta)^2}{\alpha+1}\right)\phi\left(\frac{x}{\delta}\right), \quad x \geq 0. \tag{2}$$

The EHN model was studied in depth in Elal-Olivero et al. [9]. The aim of this paper is to introduce a new model of distributions based on (1) and taking as f_0, the pdf of the Power Exponential (PE) distribution, PE(β), which was introduced in Subbotin [10]. Recall that the pdf of a PE(β) distribution is:

$$f_0(y) \ = \ c\ \exp\left\{-\frac{1}{2}|y|^{2/(\beta+1)}\right\}, \quad y \in (-\infty,\infty), \quad \beta > -1\,, \tag{3}$$

where the normalising constant $c = c(\beta)$ is:

$$c = \frac{1}{\Gamma\left(1+\frac{\beta+1}{2}\right)2^{1+\frac{\beta+1}{2}}}\,. \tag{4}$$

Moreover, for the $PE(\beta)$ model, $k = k(\beta) = E[Y^2]$ is:

$$k = \frac{2^{1+\beta}\Gamma\left(\frac{3(1+\beta)}{2}\right)}{\Gamma\left(\frac{1+\beta}{2}\right)}\,. \tag{5}$$

The PE(β) model has received a great deal of interest in papers dealing with robust inference, such as Box [11] and Box et al. [12]. Its statistical properties can be seen, for instance, in Nadarajah [13]. Details about its use as a prior robust in Bayesian Statistics are given in Choy and Walker [14]. We highlight that, in the PE(β) model, the kurtosis coefficient depends on β. The PE(β) model includes the normal and the Laplace distribution, along with other symmetric distributions around zero with lighter or heavier tails than the normal distribution. So, our proposal will be to obtain a new model, called the Extended Half-Power Exponential (EHPE), by using (1) and (3), which will be more flexible for its kurtosis than the EHN distribution, and moreover, it contains the EHN as a particular case. As for the outline of this paper, in Section 2, the EHPE distribution is defined, and its properties are studied: pdf and cumulative distribution function (cdf), reliability and hazard functions, moments, and skewness and kurtosis coefficients. In Section 3, the estimation of the parameters is discussed by using the maximum likelihood (ML) method. In Section 4, a Montecarlo simulation study is carried out, which shows the good asymptotic behaviour of ML estimates. In Section 5, two applications to COVID-19 data are given. It will be shown there that this model can be used to describe nonnegative asymmetric data with light or heavy tails. Section 6 is devoted to the final conclusions about our study.

2. EHPE Distribution

In this section, the EHPE distribution is introduced. Its pdf and cumulative distribution function (cdf) are given, along with some properties of interest in reliability and survival analysis, such as the reliability and hazard function. The section is completed with the study of moments, which allow us to study the skewness and kurtosis.

2.1. Probability Density Function

Proposition 1. *Let $X \sim EHPE(\alpha,\delta,\beta)$. Then, the pdf of X is given by:*

$$f_X(x;\alpha,\delta,\beta) = \frac{2c}{\delta^3}\left(\frac{\alpha\delta^2 + x^2}{\alpha+k}\right)\exp\left\{-\frac{1}{2}\left(\frac{x}{\delta}\right)^{2/(1+\beta)}\right\}, \ x \geq 0, \tag{6}$$

where $\alpha > 0$, $\delta > 0$ and $\beta > -1$ are parameters of this model. On the other hand, c and k are features of the $PE(\beta)$ model, which were introduced in (4) and (5), respectively.

Proof. By applying (1) and taking into account the expression for the pdf of the Power Exponential distribution, PE(β), given in (3), the result proposed in (6) is obtained. □

Remark 1 (Interpretation of parameters in (6))**.** *$\alpha > 0$ is a shape parameter, $\delta > 0$ is a scale parameter, and it will be seen in Corollary 2 that $\beta > -1$ is a parameter mainly related to the skewness and kurtosis of EHPE model.*

By construction, the following models are particular cases for the EHPE distribution:

- EHPE $(\alpha, \delta, \beta = 0) \equiv$ EHN (α, δ);
- EHPE $(\alpha \to \infty, \delta, \beta = 0) \equiv$ HN (δ);
- EHPE $(\alpha = 0, \delta = 1, \beta = 0) \equiv$ RHBN (2).

Figure 1 summarizes the relationships among the EHPE and the particular cases previously cited.

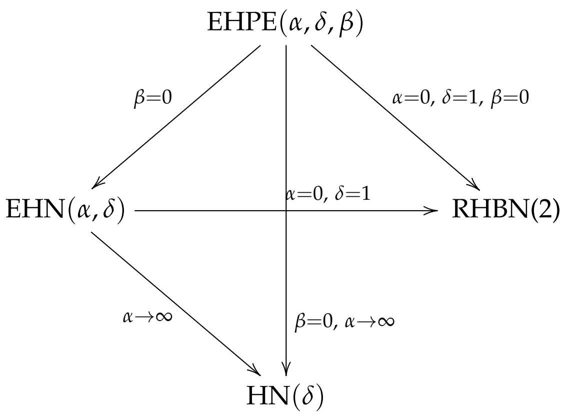


Figure 1. Particular cases for the EHPE distribution.

In the next proposition, we highlight that (6) can be expressed as a mixture of two half densities. This result is useful to interpret the parameters in Figure 2.

Proposition 2. *Let $X \sim EHPE(\alpha, \delta, \beta)$.*

1. *The pdf of X can be written as a mixture of two half densities:*

$$f_X(x; \alpha, \delta, \beta) \;=\; \frac{\alpha}{\alpha + k} f_1(x; \delta, \beta) \;+\; \frac{k}{\alpha + k} f_2(x; \delta, \beta), \qquad x \geq 0, \tag{7}$$

where

$$f_1(x; \delta, \beta) \;=\; 2\frac{1}{\delta}\, f_0\left(\frac{x}{\delta}\right), \qquad x \geq 0, \tag{8}$$

$$f_2(x; \delta, \beta) \;=\; 2\frac{x^2}{k\delta^2}\, f_0\left(\frac{x}{\delta}\right), \qquad x \geq 0. \tag{9}$$

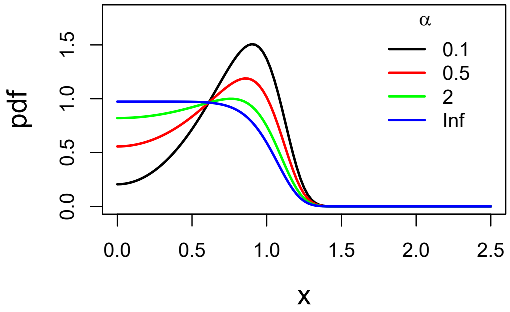
The pdf's given in (8) and (9) are the half pdf's built from $f_0(x)$ and $(\frac{x^2}{k}) f_0(x)$, respectively.

2. *If $\alpha \to \infty$, then*

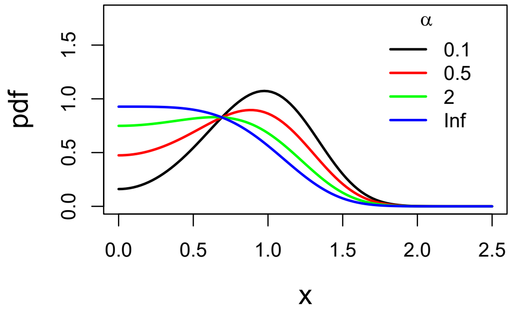
$$\lim_{\alpha \to \infty} f_X(x; \alpha, \delta, \beta) \;=\; f_1(x; \delta, \beta)\,.$$

3. *If $\alpha \to 0^+$, then*

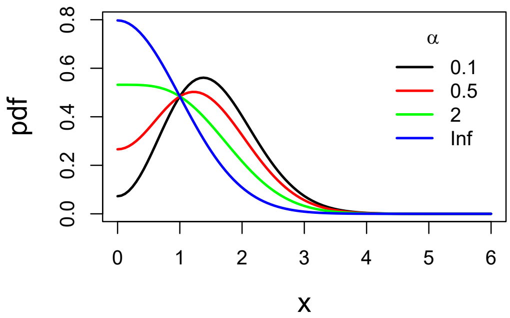
$$\lim_{\alpha \to 0^+} f_X(x; \alpha, \delta, \beta) \;=\; f_2(x; \delta, \beta)\,.$$



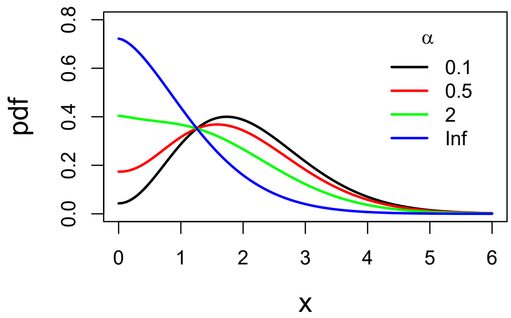
(**a**) pdf of EHPE $(\alpha, \delta = 1, \beta = -0.75)$



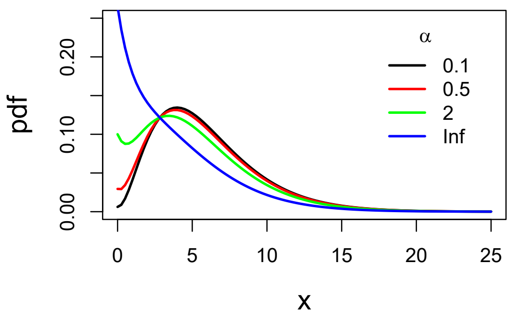
(**b**) pdf of EHPE $(\alpha, \delta = 1, \beta = -0.5)$



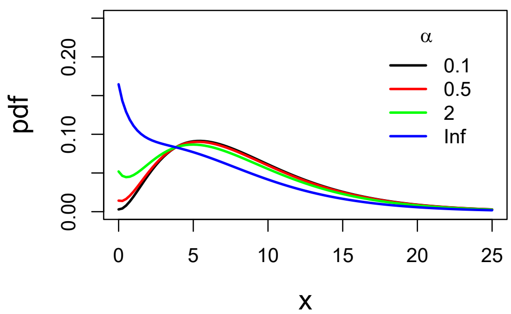
(**c**) pdf of EHPE $(\alpha, \delta = 1, \beta = 0)$



(**d**) pdf of EHPE $(\alpha, \delta = 1, \beta = 0.25)$



(**e**) pdf of EHPE $(\alpha, \delta = 1, \beta = 1)$



(**f**) pdf of EHPE $(\alpha, \delta = 1, \beta = 1.25)$

Figure 2. EHPE $(\alpha, 1, \beta)$ pdf's for different values of α and β. In all panels, $\alpha = 0.1$ (black), $\alpha = 0.5$ (red), $\alpha = 2$ (green), and $\alpha = \infty$ (blue). As for β value: (**a**) $\beta = -0.75$, (**b**) $\beta = -0.5$, (**c**) $\beta = 0$, (**d**) $\beta = 0.25$, (**e**) $\beta = 1$, and (**f**) $\beta = 1.25$.

Proof.

1. The expression given in (7) is immediate from (6). Note that $f_2(\cdot)$ given in (9) is a half-density, since $\frac{x^2}{k\delta^2} f_0\left(\frac{x}{\delta}\right)$ is a pdf symmetrical to about zero.
   Moreover, 2. and 3. are immediate from (7).

□

Remark 2. *In Figure 2, plots for the pdf in the EHPE (α, δ, β) model for different values of $\alpha > 0$ and $\beta > -1$ are given ($\delta = 1$). It can be seen in [14] that for $-1 < \beta < 0$, the tails of the PE(β) distribution are more platykurtic than the normal ones. To asses the effect of this fact on the EHPE model, the values $\beta = -0.75$ and $\beta = -0.5$ are considered in Figure 2, panels (a) and (b), respectively. As for α, its values vary from $\alpha = 0.1$ (black), 0.5 (red), 2 (green), and ∞ (blue). In this way, (a) and (b) panels show the effect of considering $-1 < \beta < 0$, an increasing value of β, and for fixed β, the effect of increasing the value of α.*

Panel (c) is devoted to $\beta = 0$. Recall that, in this case, the PE(β) distribution reduces to the $N(0,1)$ and the EHPE to the EHN [9].

On the other hand, for $\beta > 0$, the tails in PE(β) distribution are more leptokurtic than the normal ones, see [14]. The positive values of β are considered in panels (d), (e), and (f). The effect of *increasing the β value (and for a fixed β, to increase the value of α), can be appreciated there. It is also worth highlighting that the case β = 1 corresponds to the EHPE distributions built from the Laplace or double exponential model, PE(1).*

In all panels, it can be appreciated, that increasing the value of α, a higher coordinate in the origin is obtained for the pdf.

Remark 3. *(a) Note that the results given in Proposition 2, along with plots in Figure 2, show that by applying (1), it is possible to obtain a plethora of pdf's whose shapes varying from f_2 to f_1 depending on the value of $\alpha \geq 0$.*

(b) Also note that if $\alpha = 0$, then f_X reduces to f_2.

2.2. Some Properties

Next, the cdf of $X \sim$ EHPE (α, δ, β) is obtained. Recall that this function is defined as $F_X(x) = P[X \leq x]$.

Proposition 3. *Let $X \sim EHPE(\alpha, \delta, \beta)$. Then, the cdf of X is given by:*

$$F_X(x; \alpha, \delta, \beta) = \frac{\alpha}{\alpha + k} F_W\left(\frac{x}{\delta}\right) + \frac{2^{\beta+1}}{(\alpha + k)\Gamma\left(\frac{\beta+1}{2}\right)} \gamma\left(\frac{3(\beta+1)}{2}, \frac{1}{2}\left(\frac{x}{\delta}\right)^{2/(\beta+1)}\right), \; x \geq 0, \quad (10)$$

where $F_W(\cdot)$ is the cdf of a nonnegative rv W with pdf $f_W(x) = 2f_0(x)$, $x \geq 0$, and $\gamma(a, x)$ is the lower incomplete gamma function, $\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt$.

Proof.

$$F_X(x) = \int_0^x \frac{2c\alpha}{\delta(\alpha + k)} \exp\left\{-\frac{1}{2}\left(\frac{y}{\delta}\right)^{2/(1+\beta)}\right\} dy + \int_0^x \frac{2c}{\delta(\alpha + k)} \left(\frac{y}{\delta}\right)^2 \exp\left\{-\frac{1}{2}\left(\frac{y}{\delta}\right)^{2/(1+\beta)}\right\} dy$$

Making the change of variable, $\delta t = y$, we have that:

$$F_X(x) = \frac{\alpha}{\alpha + k} \int_0^{x/\delta} 2c \exp\left\{-\frac{1}{2} t^{2/(1+\beta)}\right\} dt + \frac{2c}{\alpha + k} \int_0^{x/\delta} t^2 \exp\left\{-\frac{1}{2} t^{2/(1+\beta)}\right\} dt.$$

Note that the first integral is the cdf of the nonnegative rv W with pdf $f_W(x) = 2f_0(x)$, for $x \geq 0$. As for the second integral, changing the variable, $u = \frac{1}{2} t^{2/(1+\beta)}$, the lower incomplete gamma function is obtained, $\gamma(a, x) = \int_0^x u^{a-1} e^{-u} du$, with $a > 0$. □

Proposition 4. *Let $X \sim EHPE(\alpha, \delta, \beta)$. Then:*

(i) *The reliability function, $R(t)$, of X is given by:*

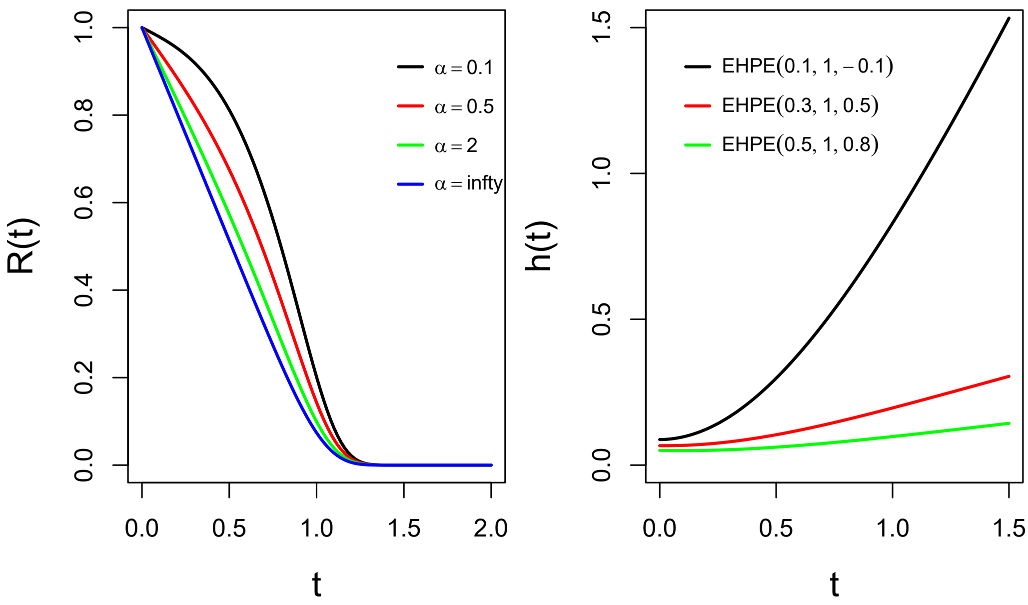
$$R(t) = 1 - \frac{\alpha}{\alpha + k} F_W\left(\frac{t}{\delta}\right) - \frac{2^{\beta+1}}{(\alpha + k)\Gamma\left(\frac{\beta+1}{2}\right)} \gamma\left(\frac{3(\beta+1)}{2}, \frac{1}{2}\left(\frac{t}{\delta}\right)^{2/(\beta+1)}\right), \quad t > 0\,.$$

(ii) *The hazard function, $h(t)$, of X is given by:*

$$h(t) = \frac{(\alpha\delta^2 + t^2)\exp\left\{-\frac{1}{2}\left(\frac{t}{\delta}\right)^{2/(1+\beta)}\right\}}{\delta^3 2^{\frac{(\beta-1)}{2}}(1+\beta)\left\{\Gamma\left(\frac{\beta+1}{2}\right)\left[(\alpha + k) - \alpha F_W\left(\frac{t}{\delta}\right)\right] - 2^{\beta+1}\gamma\left(\frac{3(\beta+1)}{2}, \frac{1}{2}\left(\frac{t}{\delta}\right)^{2/(\beta+1)}\right)\right\}}, \quad (11)$$

$$t > 0\,.$$

Remark 4. *Plots for the reliability and hazard function are given in Figure 3 for several values of α and β, (δ = 1). Although the hazard functions in Figure 3 are increasing functions of t, we point* *out that other shapes are also possible. For instance, for $\alpha = \delta = 1$, $\beta = 1$ or $\beta = 2$, $h(t)$ is first decreasing and later increasing.*



(**a**) (**b**)

Figure 3. Reliability and hazard function for different values of α and β: (**a**) $R(t)$ of EHPE ($\alpha, \delta = 1$, $\beta = -0.75$); (**b**) $h(t)$ of EHPE ($\alpha, \delta = 1, \beta$).

In next proposition, we prove that the cdf in a EHPE model can be expressed as a mixture of two gamma cdf's.

Proposition 5. *Let $X \sim EHPE(\alpha, \delta, \beta)$. Then, the cdf of X can be written as:*

$$F_X(x; \alpha, \delta, \beta) = \frac{\alpha}{\alpha + k} F_1\left(\frac{1}{2}\left(\frac{x}{\delta}\right)^{2/(1+\beta)}\right) + \frac{k}{\alpha + k} F_2\left(\frac{1}{2}\left(\frac{x}{\delta}\right)^{2/(1+\beta)}\right), \tag{12}$$

where $F_1(\cdot)$ is the cdf of a Gamma $G\left(\frac{1+\beta}{2}, 1\right)$ distribution, and $F_2(\cdot)$ is the cdf of a Gamma $G\left(\frac{3(1+\beta)}{2}, 1\right)$ distribution.

Proof. Recall (10). Changing variable $u = \frac{1}{2} t^{2/(1+\beta)}$ in $F_W\left(\frac{x}{\delta}\right)$, after some algebra, we have that:

$$F_W\left(\frac{x}{\delta}\right) = \int_0^{x/\delta} 2c \exp\left\{-\frac{1}{2} t^{2/(1+\beta)}\right\} dt = \frac{1}{\Gamma\left(\frac{1+\beta}{2}\right)} \gamma\left(\frac{1+\beta}{2}, \frac{1}{2}\left(\frac{x}{\delta}\right)^{2/(1+\beta)}\right) . \tag{13}$$

It can be seen in Abramowitz and Stegun [15] that the cdf of a Gamma distribution, $Ga(a, 1)$, denoted as $P(a, x)$ can be obtained as:

$$P(a, x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt = \frac{1}{\Gamma(a)} \gamma(a, x), \quad x > 0, \; a > 0\,. \tag{14}$$

Therefore, by applying (14), (13) can be obtained as:

$$F_W\left(\frac{x}{\delta}\right) = F_1\left(\frac{1}{2}\left(\frac{x}{\delta}\right)^{2/(1+\beta)}\right),$$

where $F_1(\cdot)$ is the cdf of a Gamma $G\left(\frac{1+\beta}{2}, 1\right)$ distribution. As for the second summand in (10), taking into account that $\gamma(a,x) = \Gamma(a)\ P(a,x)$, and the expression of $k = k(\beta)$, given in (5), (12) is obtained. □

Remark 5.

1. *The result given in Proposition 5 is similar to the one given in [9] for the EHN distribution.*
2. *The fact that the EHPE cdf can be expressed as a mixture of gamma cdf's may motivate the use of this model as a competitor of Rayleigh type models such as those introduced in [16,17].*

2.3. Moments

The moments of the EHPE distribution are given in the next proposition.

Proposition 6. *Let $X \sim EHPE(\alpha, \delta, \beta)$. Then, the nth moment of X, where n is a positive integer, is given by:*

$$\mu_n = E[X^n] \quad = \quad \frac{\delta^n}{(\alpha+k)\Gamma\left(\frac{\beta+1}{2}\right)}[\alpha \mathrm{M}_n(\beta) + \mathrm{M}_{n+2}(\beta)], \tag{15}$$

with $\mathrm{M}_n = \mathrm{M}_n(\beta) = 2^{\frac{n(\beta+1)}{2}}\Gamma\left(\frac{(1+n)(\beta+1)}{2}\right)$.

Proof. Let W with $f_W(x) = 2f_0(x)$ for $x \geq 0$. Then, the nth moment of W is:

$$E[W^n] \quad = \quad \frac{\mathrm{M}_n(\beta)}{\Gamma\left(\frac{\beta+1}{2}\right)}. \tag{16}$$

By noting that

$$E[X^n] = \frac{\alpha\delta^n}{\alpha+k}E[W^n] + \frac{\delta^n}{\alpha+k}E[W^{n+2}],$$

and using (16), (15) is obtained. □

Corollary 1. *If $X \sim EHPE(\alpha, \delta, \beta)$, then:*

1. $\mu_1 = E[X] = \frac{\delta}{(\alpha+k)\Gamma\left(\frac{\beta+1}{2}\right)}[\alpha\mathrm{M}_1(\beta) + \mathrm{M}_3(\beta)]$;
2. $\mu_2 = E\left[X^2\right] = \frac{\delta^2}{(\alpha+k)\Gamma\left(\frac{\beta+1}{2}\right)}[\alpha\mathrm{M}_2(\beta) + \mathrm{M}_4(\beta)]$;
3. $\mu_3 = E[X^3] = \frac{\delta^3}{(\alpha+k)\Gamma\left(\frac{\beta+1}{2}\right)}[\alpha\mathrm{M}_3(\beta) + \mathrm{M}_5(\beta)]$;
4. $\mu_4 = E[X^4] = \frac{\delta^4}{(\alpha+k)\Gamma\left(\frac{\beta+1}{2}\right)}[\alpha\mathrm{M}_4(\beta) + \mathrm{M}_6(\beta)]$;
5. *The variance of X, $V[X] = E[X^2] - E^2[X]$, is:*

$$V[X] = \frac{\delta^2}{(\alpha+k)\Gamma\left(\frac{\beta+1}{2}\right)}\left[\alpha\mathrm{M}_2(\beta) + \mathrm{M}_4(\beta) - \frac{(\alpha\mathrm{M}_1(\beta) + \mathrm{M}_3(\beta))^2}{(\alpha+k)\Gamma\left(\frac{\beta+1}{2}\right)}\right].$$

Corollary 2. *Let $X \sim EHPE(\alpha, \delta, \beta)$. Then, the skewness coefficient, $(\sqrt{\beta_1})$, and the kurtosis coefficient, (β_2), are given by:*

$$\sqrt{\beta_1} \quad = \quad \frac{(\alpha+k)^2\Gamma^2\left(\frac{\beta+1}{2}\right)[\alpha\mathrm{M}_3 + \mathrm{M}_5] - 3(\alpha+k)\Gamma\left(\frac{\beta+1}{2}\right)[\alpha\mathrm{M}_1 + \mathrm{M}_3][\alpha\mathrm{M}_2 + \mathrm{M}_4] + 2[\alpha\mathrm{M}_1 + \mathrm{M}_3]}{(\alpha+k)^2\Gamma^2\left(\frac{\beta+1}{2}\right)\left[\alpha\mathrm{M}_2 + \mathrm{M}_4 - [\alpha\mathrm{M}_1 + \mathrm{M}_3]^2\right]^{3/2}}$$

$$\beta_2 \quad = \quad \frac{(\alpha+k)^3\Gamma^3\left(\frac{\beta+1}{2}\right)[\alpha\mathrm{M}_4 + \mathrm{M}_6] - 4(\alpha+k)^2\Gamma^2\left(\frac{\beta+1}{2}\right)[\alpha\mathrm{M}_1 + \mathrm{M}_3][\alpha\mathrm{M}_3 + \mathrm{M}_5] + 3[\alpha\mathrm{M}_1 + \mathrm{M}_3]^2\mathrm{A}}{(\alpha+k)^2\Gamma^2\left(\frac{\beta+1}{2}\right)\left[\alpha\mathrm{M}_2 + \mathrm{M}_4 - [\alpha\mathrm{M}_1 + \mathrm{M}_3]^2\right]^2}$$

where $A = 2(\alpha + k)\Gamma\left(\frac{\beta+1}{2}\right)[\alpha M_2 + M_4] - [\alpha M_1 + M_3]^4$.

Remark 6. *The expressions for the skewness and kurtosis coefficients given in Corollary 2 are obtained by using:*

$$\sqrt{\beta_1} = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{3/2}} \quad y \quad \beta_2 = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}.$$

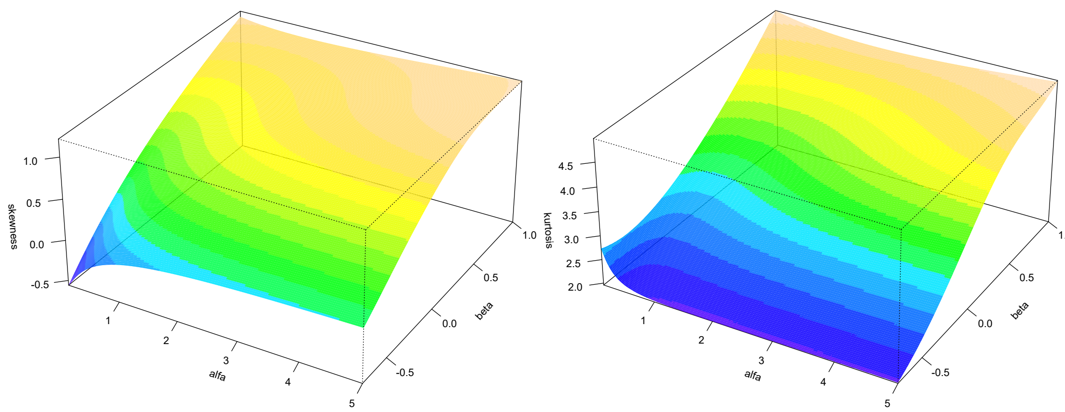
Remark 7. *Proposition 6 shows that the moments of EHPE distribution basically depend on moments of the PE(β) model. Plots for the skewness and kurtosis coefficients of EHPE distribution are given in Figure 4 for different values of α and β parameters. In this figure, the effect of the β parameter can be seen. A greater value of β produces a higher value of the skewness and kurtosis coefficients. This fact can also be appreciated in Tables 1 and 2.*

Table 1. Skewness coefficient in EHPE $(\alpha, 1, \beta)$ model for different values of α and β.

	β					
α	**−0.75**	**−0.50**	**0**	**0.25**	**1**	**1.25**
0.1	−0.5595	−0.1769	0.3932	0.6145	1.1413	1.2892
0.5	−0.2531	0.0181	0.4284	0.5932	1.1043	1.2659
1	−0.0976	0.1650	0.5545	0.6783	1.0840	1.2464
2	0.0156	0.2841	0.7167	0.8377	1.0892	1.2295
4	0.0849	0.3598	0.8553	1.0204	1.1654	1.2440
6	0.1102	0.3871	0.9111	1.1110	1.2586	1.2867
∞	0.1637	0.4428	0.9952	1.2529	2.0000	2.2523

Table 2. Kurtosis coefficient in EHPE $(\alpha, 1, \beta)$ model for different values of α and β.

	β					
α	**−0.75**	**−0.50**	**0**	**0.25**	**1**	**1.25**
0.1	2.7332	2.6261	3.0323	3.4161	4.9681	5.5913
0.5	2.0975	2.2571	2.8550	3.2508	4.8661	5.5223
1	1.9847	2.2254	2.9018	3.2661	4.7849	5.4553
2	1.9633	2.2767	3.1238	3.4932	4.7275	5.3719
4	1.9722	2.3437	3.4287	3.9369	4.8335	5.3400
6	1.9820	2.3735	3.5847	4.2270	5.0630	5.4131
∞	2.0043	2.4446	3.8688	4.8437	9.0000	10.8975



(**a**) Skewness coefficient (**b**) Kurtosis coefficient

Figure 4. (**a**) Skewness coefficient for EHPE $(\alpha, \delta = 1, \beta)$ model and (**b**) Kurtosis coefficient for EHPE $(\alpha, \delta = 1, \beta)$ model.

3. Inference

In this section, classical inferential results in the EHPE distribution are given. ML estimators are discussed in depth.

3.1. ML Estimation

Let $X_1 \ldots, X_n$ be a sample from EHPE (α, δ, β). The log-likelihood function for $\boldsymbol{\theta} = (\alpha, \delta, \beta)$ is given by:

$$\ell(\boldsymbol{\theta}) = n\log(2c) - 3n\log(\delta) - n\log(\alpha + k) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{2/(1+\beta)} + \sum_{i=1}^{n}\log(\alpha\delta^2 + x_i^2). \quad (17)$$

Taking the first derivatives of (17) with respect to the components of $\boldsymbol{\theta}$, after some algebra, the likelihood equations are:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\delta^2}{\alpha\delta + x_i^2} = \frac{1}{\alpha + k}, \quad (18)$$

$$\frac{2\alpha}{n}\sum_{i=1}^{n}\left(\alpha + \frac{x_i^2}{\delta^2}\right)^{-1} + \frac{1}{n\,(1+\beta)}\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{\frac{2}{(1+\beta)}} = 3, \quad \text{and} \quad (19)$$

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{2/(1+\beta)}\log\left(\frac{x_i}{\delta}\right) = (1+\beta)^2\left(\frac{k_0}{\alpha + k} - \frac{c_0}{c}\right), \quad (20)$$

where c_0 and k_0 denote the derivatives with respect to β of $c = c(\beta)$ and $k = k(\beta)$, which were introduced in (4) and (5). As for the likelihood Equations (18)–(20), they can be solved by using numerical methods, such as the Newton–Raphson procedure, to obtain ML estimators, $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha}, \widehat{\delta}, \widehat{\beta})$. Other maximisation techniques could also be applied, which directly maximizes the log-likelihood function; for instance, the method proposed in McDonald [18].

3.2. Observed Fisher Information Matrix

The asymptotic variance of ML estimators, $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha}, \widehat{\delta}, \widehat{\beta})$, can be estimated from the Fisher information matrix, given by $\mathcal{I}(\boldsymbol{\theta}) = -E\left[\partial^2\ell(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top\right]$ with $\ell(\boldsymbol{\theta})$ given in (17). Recall that, under regularity conditions:

$$\mathcal{I}(\boldsymbol{\theta})^{1/2}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) \xrightarrow{\mathcal{D}} N_3(\mathbf{0}_3, \boldsymbol{I}_3), \quad \text{as } n \to +\infty, \quad (21)$$

where $\mathcal{D}$ stands convergence in distribution and $N_3(\mathbf{0}_3, \boldsymbol{I}_3)$ denotes the standard trivariate normal distribution. Moreover, $\mathcal{I}(\boldsymbol{\theta})$ can be obtained from the matrix $-\partial^2\ell(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top$, whose elements are given by $I_{\alpha\alpha} = -\partial^2\ell(\boldsymbol{\theta})/\partial\alpha^2$, $I_{\alpha\delta} = -\partial^2\ell(\boldsymbol{\theta})/\partial\alpha\partial\delta$, and so on. Explicitly, we have:

$$I_{\alpha\alpha} = \sum_{i=1}^{n}\frac{\delta^4}{(\alpha\delta^2 + x_i^2)^2} - \frac{n}{(\alpha + k)^2},$$

$$I_{\alpha\delta} = -\sum_{i=1}^{n}\frac{2\delta}{\alpha\delta^2 + x_i^2} + \sum_{i=1}^{n}\frac{2\alpha\delta^3}{(\alpha\delta^2 + x_i^2)^2},$$

$$I_{\alpha\beta} = \frac{n\,k_0}{(\alpha + k)^2},$$

$$I_{\delta\delta} = -\frac{3n}{\delta^2} - 2\alpha\sum_{i=1}^{n}\frac{x_i^2 - \alpha\delta^2}{(\alpha\delta^2 + x_i^2)^2} + \frac{(3+\beta)}{\delta^2(1+\beta)^2}\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{2/(1+\beta)},$$

$$I_{\delta\beta} = \frac{1}{\delta\,(1+\beta)^2}\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{2/(1+\beta)} + \frac{2}{\delta(1+\beta)^3}\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{2/(1+\beta)}\log\left(\frac{x_i}{\delta}\right),$$

$$I_{\beta\beta} = \frac{2}{(1+\beta)^3}\sum_{i=1}^{n}\log\left(\frac{x_i}{\delta}\right)\left(\frac{x_i}{\delta}\right)^{2/(1+\beta)}\left[1 + \frac{1}{1+\beta}\log\left(\frac{x_i}{\delta}\right)\right] + \frac{n}{\alpha + k}\left[k_{00} - \frac{k_0^2}{(\alpha + k)}\right] - \frac{n}{c}\left[c_{00} - \frac{c_0^2}{c}\right],$$

where c_{00} and k_{00} denote the derivatives of c_0 and k_0 with respect to β, respectively. In practice, it is not possible to obtain the expected value of previous expressions. So, the covariance matrix of ML estimators, $\mathcal{I}(\boldsymbol{\theta})^{-1}$, is estimated by $I(\widehat{\boldsymbol{\theta}})^{-1}$, where $I(\widehat{\boldsymbol{\theta}})$ denotes the observed information matrix, which is obtained by evaluating the previous derivatives at the ML estimators $\widehat{\boldsymbol{\theta}}$, i.e.:

$$I(\widehat{\boldsymbol{\theta}}) = -\partial^2 \ell(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}. \tag{22}$$

The asymptotic variances of $\widehat{\alpha}$, $\widehat{\delta}$, and $\widehat{\beta}$ are estimated by the diagonal elements of $I(\widehat{\boldsymbol{\theta}})^{-1}$, and their standard errors by the square root of asymptotic variances. So, by using (21) and (22), approximate confidence intervals for the parameters can be given. Asymptotic confidence intervals follow from these results. For instance, an approximate confidence interval for α with confidence level $(1-\gamma)$, $0<\gamma<1$, would be:

$$\left(\widehat{\alpha} - z_{1-\gamma/2} s.e.(\widehat{\alpha}),\ \ \widehat{\alpha} + z_{1-\gamma/2} s.e.(\widehat{\alpha})\right)$$

where $z_{1-\gamma/2}$ denotes the quantile of order $1-\gamma/2$ in the $N(0,1)$ distribution, and $s.e.(\widehat{\alpha})$ is the standard error of $\widehat{\alpha}$.

4. Simulation Study

In this section, a Monte Carlo simulation study is carried out to illustrate the behaviour of ML estimators. For the sample size n, the values $n=100$, 200 and 500 are considered; without loss of generality $\delta=1$; for $\alpha=$0.1 and 0.2; and for $\beta=-0.3$, 0 and 0.1. For every combination of α, δ and β, 1000 samples of size n are generated from the EHPE (α,δ,β) model. To generate random numbers of $X \sim$ EHPE $(\alpha,\delta=1,\beta)$ an acceptance–rejection technique is proposed, which proceeds as follows:

Step 1. Simulate Y from pdf g;
Step 2. Simulate $U \sim Uniform(0,1)$;
Step 3. If $U \leq \frac{f(Y)}{c\,g(Y)}$ define $X=Y$. Otherwise, repeat Step 1.

For $-1<\beta<0$, g is taken as the pdf of an EHN $(\alpha,\delta=1)$ distribution introduced in [9], which corresponds to the EHPE model with $\beta=0$. On the other hand, for $0<\beta<1$, g is the pdf of an EHPE $(\alpha,1,\beta=1)$ model. In both settings, c is taken as the maximum value of $\frac{f(y)}{g(y)}$. For every sample, the ML estimates are obtained by applying Newton–Raphson algorithm. As initial values to start the MLE recursion algorithms, the estimate of parameter $\widehat{\alpha}_0$ in the EHN distribution, $\widehat{\delta}_0=1$ and $\widehat{\beta}_0=0$ are taken. In Table 3, the empirical bias, the mean of the standard errors (SEs) and the root of the empirical mean squared error (RMSE) are given for the estimators of the parameters. Note that, although the ML estimators are biased, the bias decreases when the sample size increases. The SEs and RMSEs also decrease with the sample size. These facts suggest that the ML estimators are consistent. Moreover, approximate confidence intervals to 95% level were obtained, by using the asymptotic distribution of ML estimators. Their empirical coverage probabilities (CP) have also been included in Table 3. In this table, it can be appreciated that, in general, the performance is good.

Table 3. Empirical bias, SE, RMSE and 95% CP for the ML estimates of α, δ and β in the EHPE distribution with different combinations of parameters (case true $\delta = 1$).

True Value			$n = 100$				$n = 200$				$n = 500$			
α	β	par.	bias	SE	RMSE	CP	bias	SE	RMSE	CP	bias	SE	RMSE	CP
0.1	−0.3	α	0.01	0.08	0.08	0.93	0.01	0.05	0.05	0.94	0.00	0.03	0.03	0.93
		δ	0.03	0.19	0.19	0.92	0.02	0.13	0.14	0.93	0.00	0.08	0.09	0.94
		β	−0.02	0.16	0.17	0.91	−0.01	0.12	0.12	0.94	−0.00	0.07	0.07	0.94
	0	α	0.01	0.11	0.10	0.98	0.00	0.06	0.06	0.94	0.00	0.04	0.03	0.94
		δ	0.07	0.29	0.31	0.92	0.03	0.20	0.21	0.94	0.01	0.12	0.12	0.94
		β	−0.03	0.23	0.23	0.91	−0.02	0.15	0.16	0.92	−0.00	0.10	0.10	0.94
	0.1	α	0.01	0.10	0.09	0.96	−0.00	0.06	0.06	0.93	0.00	0.04	0.04	0.93
		δ	0.06	0.29	0.30	0.92	0.03	0.19	0.20	0.94	0.01	0.12	0.13	0.92
		β	−0.12	0.22	0.26	0.84	−0.11	0.15	0.19	0.83	−0.10	0.10	0.14	0.77
0.2	−0.3	α	0.03	0.16	0.19	0.90	0.01	0.08	0.09	0.94	0.00	0.05	0.05	0.96
		δ	0.03	0.22	0.24	0.89	0.02	0.15	0.17	0.92	0.01	0.10	0.10	0.94
		β	−0.02	0.19	0.20	0.90	−0.01	0.13	0.14	0.91	−0.00	0.08	0.08	0.94
	0	α	0.01	0.14	0.14	0.92	0.00	0.08	0.09	0.93	0.00	0.05	0.05	0.94
		δ	0.07	0.32	0.34	0.92	0.03	0.23	0.22	0.94	0.02	0.14	0.15	0.94
		β	−0.03	0.25	0.25	0.91	−0.01	0.17	0.17	0.94	−0.01	0.11	0.11	0.94
	0.1	α	0.02	0.16	0.19	0.94	0.02	0.16	0.28	0.93	0.01	0.05	0.05	0.95
		δ	0.07	0.32	0.36	0.90	0.04	0.23	0.24	0.94	0.01	0.14	0.15	0.94
		β	−0.12	0.25	0.29	0.84	−0.12	0.17	0.21	0.85	−0.10	0.11	0.15	0.82

5. Applications

In this section, two real applications are given. The aim is to compare the EHPE model with other competing models. Specifically, the EHPE model is compared to the EHN distribution introduced in (2) and the Slash Truncation Positive Normal (STPN) distribution introduced in Gómez et al. [4], whose pdf is:

$$f_Y(y; \alpha, \delta, \beta) = \frac{\beta}{\delta \Phi(\alpha)} \int_0^1 w^\beta \phi\left(\frac{yw}{\delta} - \alpha\right) dw, \tag{23}$$

where $y > 0$, $\delta > 0$, $\beta > 0$ and $\alpha \in \mathbb{R}$.

5.1. Application 1

A real dataset related to COVID-19 in Chile is considered. Specifically, the data represent the incidence rate per 10,000 inhabitants affected by the virus without symptoms during the second quarter of 2020. These data were recorded from April 29 (see https://coronavirus.mat.uc.cl, accessed on 23 January 2022). For interested readers, the dataset is given in Appendix A.1. In Table 4, the following descriptive summaries are provided: sample size, sample mean, sample variance, sample skewness and sample kurtosis coefficient. We highlight that we obtained a low value for the sample kurtosis coefficient, $b_2 = 2.615$, which suggests that a distribution with flexible values for this coefficient, such as the EHPE can be used to model this dataset.

The estimates of parameters and their standard errors, in parentheses, are provided in Table 5 for the models under consideration, along with criteria based on the maximized likelihood to compare these models. The standard errors have been obtained from the square root of the diagonal elements in the inverse of the observed information matrix. As for the measurements to compare models, the Akaike Information Criterion (AIC) [19] and Bayesian Information Criterion (BIC) [20] are considered. Both criteria penalize the maximized likelihood function by the excess of parameters in the model. From values in Table 5, it can be concluded that the EHPE model provides the best fit to this dataset.

Table 4. Descriptive statistics for data in Application 1.

	n	$\overline{X}$	S^2	$\sqrt{b_1}$	b_2
Incidence rate	63	0.208	0.006	−0.360	2.615

Table 5. Estimated parameters and their standard errors (in parentheses) for the EHPE.

Estimated	EHPE	EHN	STPN
α	0.069 (0.066)	0.001 (0.063)	0.076 (0.008)
δ	0.250 (0.028)	0.128 (0.007)	2.677 (0.294)
β	−0.627 (0.116)	-	40.04 (71.082)
AIC	−142.257	−132.029	−139.509
BIC	−135.828	−125.600	−133.080

Figure 5 compares the fit provided by the models under consideration, whereas in Figure 6, the empirical cdf is plotted along with the cdf estimated for the EHPE model. We highlight the good agreement existing between both cdf's.

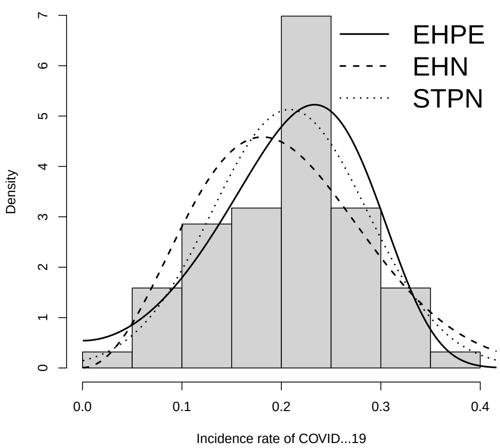


Figure 5. Fit of the distributions for the incidence rate of COVID-19 data set.

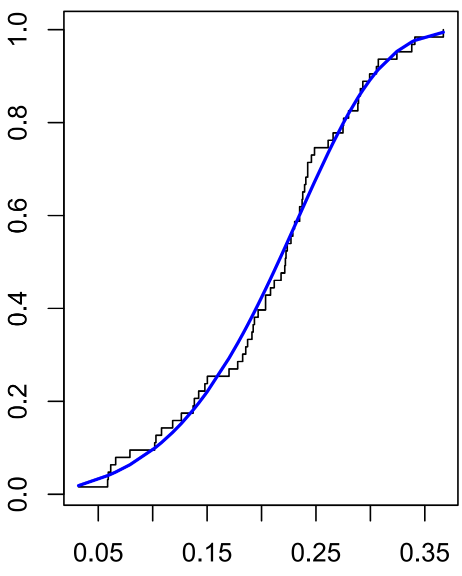


Figure 6. Empirical (black) and fitted EHPE (blue) cdf for the incidence rate of COVID-19 in dataset 1.

5.2. Application 2

In this second application, the data considered are also related to COVID-19 in Chile. Specifically, it is the incidence rate per every 10,000 inhabitants affected by COVID-19, with and without symptoms, in the first two months of the pandemic, these data were recorded starting on 2 March 2020 (see https://coronavirus.mat.uc.cl, last accessed on 23 January 2022). The dataset is explicitly given in Appendix A.2. For this dataset, again, the performance of the EHPE model is analyzed in comparison with the EHN and STPN models. Descriptive summaries are given in Table 6, where a high sample kurtosis coefficient is observed, which suggests the use of a heavy-tailed distribution such as EHPE.

Table 6. Descriptive statistics for data in Application 2.

	n	$\overline{X}$	S^2	$\sqrt{b_1}$	b_2
Incidence rate of COVID-19	62	0.169	0.024	1.445	6.611

Table 7 shows the estimated parameters and their standard errors for the three models under consideration. Based on AIC [19] and BIC [20] criteria, the EHPE model provides a better fit to this data.

Table 7. Estimated parameters and their standard errors (in parentheses).

	EHPE	EHN	STPN
α	106.709 (60.098)	4289.033 (968.649)	0.375 (0.443)
δ	0.005 (0.001)	0.228 (0.020)	−1.747 (2.723)
β	2.258 (0.173)	-	9.505 (32.835)
AIC	−106.739	−89.094	−90.955
BIC	−100.357	−84.840	−84.573

In Figure 7, the fit provided by the models under consideration is compared. The empirical cdf and the cdf estimated for the proposed EHPE model are plotted in Figure 8. Again, the results are satisfactory.

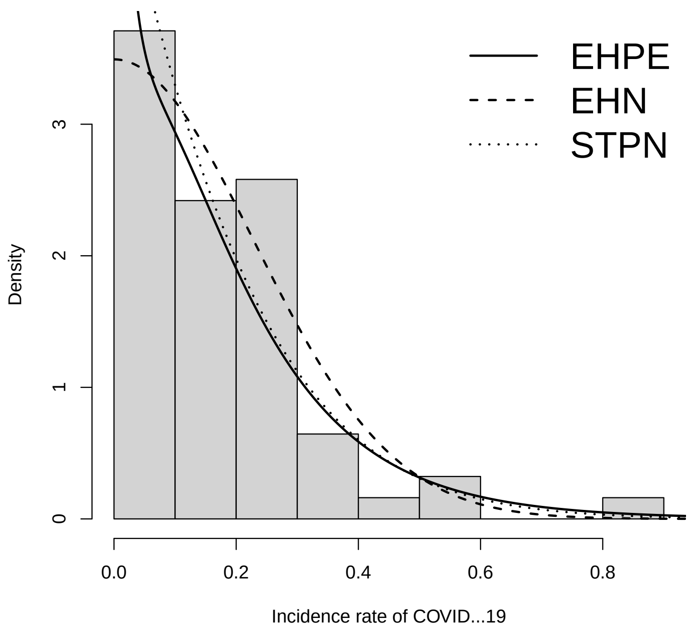


Figure 7. Fit of the distributions for the incidence rate of COVID-19 in the second data set.

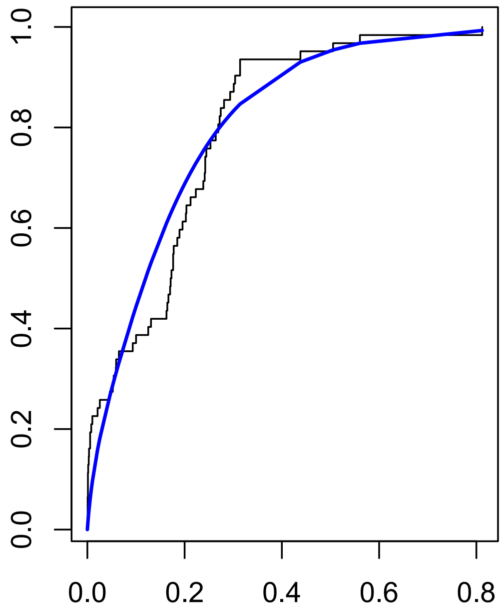


Figure 8. Empirical (black) and fitted EHPE (blue) cdf for the incidence rate of COVID-19 in dataset 2.

6. Conclusions

In this paper, the EHPE distribution has been introduced. This is a new continuous model with positive support, which can be considered more general than the EHN distribution. We highlight its flexibility in the kurtosis coefficient, which allows us to model datasets with lower and greater kurtosis than three. Due to this fact, it can be useful for modelling nonnegative data presenting these values. Its properties are studied in depth, with emphasis on those of interest in reliability. The estimation of parameters has been studied from a theoretical and computational point of view. Finally, the model is applied to two real datasets related to COVID-19. These applications are of interest due to the challenging nature of these data. We point out that this application should be considered from a descriptive point of view; that is, the aim is to describe these datasets. All computations have been carried out by using R software [21]. As future research in this model, we propose to spread out the range of applications and the study of their theoretical properties.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

In this appendix the data sets used in the applications are provided.

Appendix A.1. Dataset Used in Application 1

```
0.14225558 0.06145441 0.05860930 0.10299304 0.06600659 0.05917832
0.03186525 0.07909410 0.10811424 0.11835664 0.13827242 0.13713438
0.12632296 0.10185500 0.28906334 0.20370999 0.22191871 0.17810399
0.18720834 0.14794580 0.23955840 0.22248773 0.24240351 0.23500622
0.17013767 0.20826217 0.26573342 0.19688172 0.21167630 0.29930574
0.19346759 0.27540680 0.22362577 0.22134968 0.18265617 0.23045404
0.27995898 0.23728231 0.20370999 0.36701940 0.26118125 0.18550128
0.29077041 0.28849432 0.30556499 0.30727205 0.32434272 0.27483778
0.29304650 0.24866275 0.21793555 0.34084437 0.23785133 0.33799926
0.22703991 0.19119150 0.15022189 0.24069644 0.23500622 0.24240351
0.22874697 0.24581764 0.19232954
```

Appendix A.2. Dataset Used in Application 2

```
0.0005690223 0.0000000000 0.0011380446 0.0005690223 0.0005690223
0.0011380446 0.0017070670 0.0028451116 0.0011380446 0.0034141339
0.0056902232 0.0056902232 0.0102424018 0.0079663125 0.0460908081
0.0256060045 0.0210538259 0.0591783215 0.0523500537 0.0586092992
0.0540571206 0.0648685447 0.1001479287 0.1251849109 0.0933196609
0.1729827860 0.1701376744 0.1308751341 0.1763969199 0.1644474512
0.1667235404 0.2122453262 0.1894844333 0.2412654647 0.1763969199
0.1957436789 0.1712757190 0.2446795986 0.2424035093 0.3010128085
0.2424035093 0.1627403842 0.1775349646 0.2230567504 0.2025719468
0.3038579201 0.2532149334 0.2719926701 0.2037099914 0.2384203531
0.1849322548 0.2640263576 0.2936155183 0.2810970272 0.3141003219
0.2691475585 0.2742687594 0.3141003219 0.4381471882 0.5052918222
0.5604869875 0.8119948540
```

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