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Homogenization of Dirichlet parabolic problems for coefficients and open sets simultaneously variable and applications to optimal design

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Abstract

In a previous paper, we studied the homogenization of a sequence of parabolic linear Dirichlet problems, when the coefficients and the domains vary arbitrarily. Here, we improve the convergence result given in this paper by showing the strong convergence in L^2 every time. This is applied to obtain an existence result for control problems in optimal design written in a relaxed form. The control variables are the material and the shape.

Keywords: Asymptotic behavior; Homogenization; Control; Dirichlet problems; Parabolic equations; Perforated domains

1. Introduction

We are interested in the asymptotic behavior of a sequence of parabolic Dirichlet problems when the coefficients and the open sets where they are posed simultaneously vary. Specifically, for $T > 0$, $\Omega \subset \mathbb{R}^N$, open, $A_n : \Omega \times (0, T) \to \mathbb{R}^{N\times N}$, elliptic and bounded, $\Omega_n \subset \Omega$ open, and $f \in L^2(0, T; H^{-1}(\Omega))$, let us consider the homogenization problem

$$\partial_t y_n - div\, A_n(x, t)\nabla y_n = f \quad \text{in } \Omega \times (0, T),$$

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$$y = 0 \quad \text{on } (\Omega \times \{0\}) \cup (\partial\Omega \times (0, T)). \tag{1.1}$$

We do not introduce any hypotheses about Ω_n (only the fact that they are all contained in Ω). For A_n, we only assume it to be uniformly elliptic, and bounded. As it is usual in the homogenization of Dirichlet problems in varying domains (see, e.g., [5,6,9–18,26,27]), it is proved in [9] that the limit problem of (1.1) does not have the same structure. In the place of an equation such as

$$\partial_t y - div\, A(x, t)\nabla y = f \quad \text{in } \Omega \times (0, T),$$

we find a bounded and elliptic matrix A, a nonnegative measure μ and a positive and bounded μ-measurable function F, such that the limit equation is

$$\partial_t y - div\, A(x, t)\nabla y + F(x, t) y\mu = f \quad \text{in } \Omega \times (0, T). \tag{1.2}$$

The measure μ vanishes on the sets of capacity zero, and then the functions in $H_0^1(\Omega)$ have a representative which is well defined for it. However, it is not in general in $H^{-1}(\Omega)$, and not even a Radon measure. So, Eq. (1.2) does not hold in general in the sense of the distributions. Thus, we will prefer to write it in a variational form better than as a partial differential equation.

The above result is closely related to the fact that a control problem like

$$\min_{\widetilde{\Omega} \subset \Omega \text{ open}} \int_\Omega |y - y_d|^2 \,\mathrm{d}x \qquad \begin{cases} \partial_t y - \Delta y = f & \text{in } \widetilde{\Omega} \times (0, T), \\ y = 0 & \text{on } (\Omega \times \{0\}) \cup (\partial\Omega \times (0, T)), \end{cases}$$

with y_d in $L^2(\Omega)$, and f in $L^2(0, T; H^{-1}(\Omega))$, does not have a solution in general.

At the place of (1.1), we will prefer to consider the problem

$$\begin{aligned} & y_n \in L^2(0, T; H_0^1(\Omega) \cap L^2_{\mu_n}(\Omega)), \quad y_n(x, 0) = 0 \text{ a.e. in } \Omega, \\ & \langle \partial_t y_n, v\rangle + \int_\Omega A_n(x, t)\nabla y_n \nabla v \,\mathrm{d}x + \int_\Omega F_n(x, t) y_n v \,\mathrm{d}\mu_n = \langle f, v\rangle \quad \text{in } \mathscr{D}'(0, T), \\ & \forall v \in L^2(0, T; H_0^1(\Omega) \cap L^2_{\mu_n}(\Omega)), \end{aligned} \tag{1.3}$$

where A_n and f are as in (1.1), μ_n is a sequence of nonnegative Borel measures which vanish on the sets of capacity zero, and F_n are in $L^\infty_{\mu_n}(\Omega)$, uniformly positive, and bounded. Following Dal Maso and Mosco [16], we remark that if Ω_n is a sequence of open sets contained in Ω, then, defining μ_n as

$$\mu_n(B) = \begin{cases} +\infty & \text{if } Cap(B \cap (\Omega \backslash \Omega_n), \Omega) > 0, \\ 0 & \text{if } Cap(B \cap (\Omega \backslash \Omega_n), \Omega) = 0, \end{cases} \qquad \forall B \subset \Omega \text{ Borel},$$

and, e.g., $F_n = \chi_{\Omega_n}$, problem (1.1) is equivalent to (1.3), and so (1.3) generalizes (1.1).

The homogenization problem (1.1) has been studied in [9] (see also [5,17], for elliptic problems, and [6] for nonlinear parabolic problems where the coefficients do not depend on the time), where the existence of a limit problem is proved (for a subsequence), which has the same structure as (1.3). The convergence of y_n is proved to hold strong in $L^2(\Omega \times (0, T))$ and weak in $L^2(0, T; H_0^1(\Omega_n))$. In the present paper, let us also show that for every $t \in [0, T]$, $y_n(., t)$ converges strongly in $L^2(\Omega)$. As an application of these results, we prove the existence of solutions for control problems in the coefficients and the domains. These problems must be written in a relaxed form. In other cases, it is well known that a solution does not exist in general (see, e.g., [3,7,22]). We refer to [1,3,4,7,8,21,22,24] for the study of control problems in optimal design.

2. Notations

We denote by $\Omega \subset \mathbb{R}^N$ a bounded open set, by Q_R, $R>0$, the cylinder $Q_R = \Omega \times (0, R)$, and by Q_R^S, $0<R<S$, the cylinder $Q_R = \Omega \times (R, S)$.

For a measure $\hat{\mu}$ in Q_R, we denote by $L^p_{\hat{\mu}}(Q_R)$, $1 \leqslant p \leqslant +\infty$, the usual Lebesgue spaces relatives to $\hat{\mu}$. If $\hat{\mu}$ is the Lebesgue measure, we write $L^p(Q_R)$. Analogously, for a measure μ in Ω, we use the notations $L^p_\mu \Omega$, $L^p(\Omega)$.

For a normed space X, $x \in X$, $x' \in X'$ (the dual space of X), we denote by $\langle x', x\rangle_{x',x}$ the duality product between x' and x. When the spaces are understood, we just write $\langle x', x\rangle$.

For every $B \subset \Omega$, $Cap(B, \Omega)$ denotes the capacity of B (in Ω), which is defined as the infimum of

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x$$

over the set of $u \in H^1_0(\Omega)$ such that $u \geqslant 1$ a.e. in a neighborhood of B.

A function $u : \Omega \to \mathbb{R}$ is said to be quasi-continuous if for every $\varepsilon > 0$ there exists $N \subset \Omega$, with $C(N, \Omega) < \varepsilon$, such that the restriction of u to $\Omega \backslash N$ is continuous. It is well known that every function $u \in H^1_0(\Omega)$ has a quasi-continuous representative (see [19,20,30]). We always identify u with its quasi-continuous representative.

A set $\Theta \subset \Omega$ is said to be quasi-open, if for every $\varepsilon > 0$ there exists N with $C(N, \Omega) < \varepsilon$ such that $\Theta \cup N$ is open.

We denote by $\mathscr{M}^2_0(\Omega)$ the class of all nonnegative Borel measures which vanish on the sets of capacity zero and satisfy

$$\mu(B) = inf\{\mu(\Theta) : \Theta \text{ quasi-open}, \ B \subseteq \Theta \subseteq \Omega\}, \quad \forall B \subset \Omega \text{ Borel}.$$

For a measure $\mu \in \mathscr{M}^2_0(\Omega)$, we denote by $\hat{\mu}$ the measure in Q_T defined by $\hat{\mu} = \mu \otimes \mathrm{d}t$.

Definition 2.1. For $T>0$, and two constants $\gamma > \alpha > 0$, we denote by $M^\gamma_\alpha(Q_T)$ (see [23]) the set of all the matrices A in $L^\infty(Q_T)^{N\times N}$, such that

(i) $A(x,t)\xi\xi \geqslant \alpha|\xi|^2$, $\forall \xi \in \mathbb{R}^N$, a.e. $(x,t) \in Q_T$.
(ii) $A^{-1}(x,t)\xi\xi \geqslant \gamma^{-1}|\xi|^2$, $\forall \xi \in \mathbb{R}^N$, a.e. $(x,t) \in Q_T$.

We also denote by $\mathscr{F}^\gamma_\alpha(Q_T)$ the set of pairs (F, μ) such that $\mu \in \mathscr{M}^2_0(\Omega)$, F belongs to $L^\infty_{\hat{\mu}}(Q_T)$, and

$$\gamma \geqslant F(x,t) \geqslant \alpha, \quad \hat{\mu}\text{-a.e. in } Q_T. \tag{2.4}$$

Remark 2.2. We recall (see [23]) that (ii) implies

(iii) $|A(x,t)| \leqslant \gamma$, a.e. $(x,t) \in Q_T$.

Reciprocally, if A satisfies (i) and (iii), then

$$A^{-1}(x,t)\xi\xi \geqslant \frac{\alpha}{\gamma^2}|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \ \text{a.e. } (x,t) \in Q_T.$$

3. Homogenization results

We recall in this section the following compactness result, which gives the homogenization of (1.3) (see also [5,18], for the case of elliptic equations, and [6] for the case of nonlinear parabolic problems with coefficients independent of the time variable.

Theorem 3.1. *For $T > 0$, $\gamma > \alpha > 0$, and two sequences $A_n \in \mathscr{M}_\alpha^\gamma(Q_T)$ and $(F_n, \mu_n) \in \mathscr{F}_\alpha^\gamma(Q_T)$, there exist a subsequence of n, still denoted by n, $A \in \mathscr{M}_\alpha^\gamma(Q_T)$ and $(F, \mu) \in \mathscr{F}_\alpha^\gamma(Q_T)$, such that for every distribution $f \in L^2(0, T; H^{-1}(\Omega))$, the solution y_n of* (1.3) *converges weakly in $L^2(0, T; H_0^1(\Omega))$ and strongly in $L^2(Q_T)$ to the unique solution y of*

$$\begin{aligned}
&y \in L^2(0, T; H_0^1(\Omega) \cap L_\mu^2(\Omega)), \quad y(x, 0) = 0 \text{ a.e. in } \Omega,\\
&\langle \partial_t y, v\rangle + \int_\Omega A(x, t)\nabla y \nabla v \,\mathrm{d}x + \int_\Omega F(x, t) y v \,\mathrm{d}\mu = \langle f, v\rangle \quad \text{in } \mathscr{D}'(0, T),\\
&\forall v \in L^2(0, T; H_0^1(\Omega) \cap L_\mu^2(\Omega)).
\end{aligned} \tag{3.5}$$

The matrix A coincides with the H-limit of A_n (*see, e.g.,* [23,25,28]), *and then, it does not depend on* (F_n, μ_n). *The measure μ can be chosen* (*note that only the product $F\mu$ is uniquely defined*) *as the unique element of $\mathscr{M}_0^2(\Omega)$* (*see* [15]), *such that the unique solution w_n of*

$$\begin{aligned}
&w_n \in H_0^1(\Omega) \cap L_{\mu_n}^2(\Omega),\\
&\int_\Omega \nabla w_n \nabla v \,\mathrm{d}x + \int_\Omega w_n v \,\mathrm{d}\mu_n = \int_\Omega w_n v \,\mathrm{d}x,\\
&\forall v \in H_0^1(\Omega) \cap L_{\mu_n}^2(\Omega)
\end{aligned}$$

converges weakly in $H_0^1(\Omega)$ to the unique solution w of

$$\begin{aligned}
&w \in H_0^1(\Omega) \cap L_\mu^2(\Omega),\\
&\int_\Omega \nabla w \nabla v \,\mathrm{d}x + \int_\Omega w v \,\mathrm{d}\mu = \int_\Omega w v \,\mathrm{d}x,\\
&\forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega),
\end{aligned} \tag{3.6}$$

and then, it can be chosen independently of A_n and F_n.

Let us improve the above result by showing the following:

Proposition 3.2. *In Theorem* (3.1), *we also have*

$$y_n(., t) \to y(., t) \quad \text{in } L^2(\Omega), \ \ \forall t \in [0, T]. \tag{3.7}$$

Proof. Let t be in $[0, T]$; there is nothing to prove $t = 0$. So, we can assume $t \in (0, T]$. Moreover, it is not restrictive to assume that y_n and y are defined in Q_S for some $S > T$, and that Theorem 3.1 holds with T replaced by S. For this, it will be enough to extend A_n, and F_n to Q_S.

For $\varepsilon > 0$, we consider $h \in (0, \min\{t/2, (S-t)/2\})$ such that

$$\|\nabla y\|_{L^2(Q_{t-2h}^{t+2h})} + \|y\|_{L^2_{\hat\mu}(Q_{t-2h}^{t+2h})} + \frac{\alpha}{\gamma^2}\|f\|_{L^2(t-h,t+h;H^{-1}(\Omega))} < \varepsilon. \tag{3.8}$$

Since the solutions y_n of (1.3) are in $C^0([0,S];L^2(\Omega))$, for every $n \in N$, there exists $h_n \in (0,h)$ such that

$$\left\| y_n(.,t) - \frac{1}{2h_n}\int_{t-h_n}^{t+h_n} y_n(.,s)\right\|_{L^2(\Omega)} < \varepsilon. \tag{3.9}$$

Using (1.3), for every $n \in N$, and a.e. $(r,s) \in (t-h, t+h)^2$, we have

$$\begin{aligned}
&\left\langle \frac{\partial y_n}{\partial r}(x,r), y_n(.,r) - y_n(.,s)\right\rangle_{(H_0^1(\Omega)\cap L^2_{\mu_n}(\Omega))', H_0^1(\Omega)\cap L^2_{\mu_n}(\Omega)} \\
&\quad + \int_\Omega A_n(x,r)\nabla y_n(x,r)\nabla(y_n(x,r) - y_n(x,s))\,\mathrm{d}x \\
&\quad + \int_\Omega F_n(x,r)y_n(x,r)(y_n(x,r) - y_n(x,s))\,\mathrm{d}\mu_n \\
&\quad = \langle f, y_n(.,r) - y_n(.,s)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.
\end{aligned}$$

Integrating in $r \in (q,s)$, for $q \in (t-h, s)$, or in $r \in (s,q)$, for $q \in (s, t+h)$, we get

$$\begin{aligned}
\int_\Omega |y_n(x,q) - y_n(x,s)|^2\,\mathrm{d}x \leqslant{}& \gamma\|\nabla y_n\|_{L^2(Q_{t-h}^{t+h})}\|\nabla(y_n - y_n(.,s))\|_{L^2(Q_{t-h}^{t+h})} \\
&+ \gamma\|y_n\|_{L^2_{\mu_n}(Q_{t-h}^{t+h})}\|y_n - y_n(.,s)\|_{L^2_{\mu_n}(Q_{t-h}^{t+h})} \\
&+ \|f\|_{L^2(t-h,t+h;H^{-1}(\Omega))}\|\nabla(y_n - y_n(.,s))\|_{L^2(Q_{t-h}^{t+h})},
\end{aligned}$$

for a.e. $(q,s) \in (t-h,t+h)^2$. Integrating in $(q,s) \in (t-h_n, t+h_n) \times (t-h, t+h)$, and dividing by $4h_n h$ we obtain

$$\begin{aligned}
&\frac{1}{4h_n h}\int_{t-h_n}^{t+h_n}\int_{t-h}^{t+h}\int_\Omega |y_n(x,q) - y_n(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}q \\
&\quad\leqslant (\gamma\|\nabla y_n\|_{L^2(Q_{t-h}^{t+h})} + \|f\|_{L^2(t-h,t+h;H^{-1}(\Omega))}) \\
&\qquad \times \left(\frac{1}{2h}\int_{t-h}^{t+h}\int_{t-h}^{t+h}\int_\Omega |\nabla(y_n(x,s) - y_n(x,r))|^2\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}r\right)^{1/2} \\
&\qquad + \gamma\|y_n\|_{L^2_{\mu_n}(Q_{t-h}^{t+h})}\left(\frac{1}{2h}\int_{t-h}^{t+h}\int_{t-h}^{t+h}\int_\Omega |y_n(x,s) - y_n(x,r)|^2\,\mathrm{d}\mu_n\,\mathrm{d}s\,\mathrm{d}r\right)^{1/2} \\
&\quad\leqslant \sqrt{2}(\gamma\|\nabla y_n\|_{L^2(Q_{t-h}^{t+h})} + \gamma\|y_n\|_{L^2_{\mu_n}(Q_{t-h}^{t+h})} + \|f\|_{L^2(t-h,t+h;H^{-1}(\Omega))}) \\
&\qquad \times (\|\nabla y_n\|_{L^2(Q_{t-h}^{t+h})} + \|y_n\|_{L^2_{\mu_n}(Q_{t-h}^{t+h})}).
\end{aligned} \tag{3.10}$$

Now, for $\varphi \in \mathscr{D}(t-2h, t+2h)$, $\varphi \geqslant 0$, $\varphi = 1$ in $(t-h, t+h)$, we take the application $(x,t) \to y_n(x,t)\varphi(t)$ as test function in (1.3), and the application $(x,t) \to y(x,t)\varphi(t)$ as test function in (3.5). Using then that y_n converges to y strongly in $L^2(Q_S)$ and weakly in $L^2(0, S; H^{-1}(\Omega))$, we have

$$\begin{aligned}\int_{Q_S} A_n \nabla y_n \nabla y_n \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_S} F_n y_n^2 \varphi \,\mathrm{d}\mu_n\,\mathrm{d}t &= \frac{1}{2}\int_{\Omega} y_n^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t}\,\mathrm{d}x\,\mathrm{d}t + \langle f, y_n\varphi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &\to \frac{1}{2}\int_{\Omega} y^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t}\,\mathrm{d}x\,\mathrm{d}t + \langle f, y\varphi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &= \int_{Q_S} A\nabla y \nabla y \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_S} F y^2 \varphi \,\mathrm{d}\mu\,\mathrm{d}t.\end{aligned}$$

So, using the properties of A_n, F_n, A, and F, we get the following estimate to the right-hand side of (3.10):

$$\limsup_{n\to\infty}(\|\nabla y_n\|_{L^2(Q_{t-h}^{t+h})} + \|y_n\|_{L^2_{\mu_n}(Q_{t-h}^{t+h})}) \leqslant \frac{\gamma}{\alpha}(\|\nabla y\|_{L^2(Q_{t-h}^{t+h})} + \|y\|_{L^2_{\mu}(Q_{t-h}^{t+h})}). \tag{3.11}$$

Let us now consider the inequality

$$\begin{aligned}\|y_n(.,t) - y(.,t)\|_{L^2(\Omega)} \leqslant & \left\| y_n(,t) - \frac{1}{2h_n}\int_{t-h_n}^{t+h_n} y_n(.,q)\,\mathrm{d}q \right\|_{L^2(\Omega)} \\ &+ \left\| \frac{1}{4h_n h}\int_{t-h_n}^{t+h_n}\int_{t-h}^{t+h} (y_n(.,q) - y_n(.,s))\,\mathrm{d}s\,\mathrm{d}q \right\|_{L^2(\Omega)} \\ &+ \left\| \frac{1}{2h}\int_{t-h}^{t+h} (y_n(.,s) - y(.,s))\,\mathrm{d}s \right\|_{L^2(\Omega)} \\ &+ \left\| \frac{1}{2h}\int_{t-h}^{t+h} y(.,s)\,\mathrm{d}s - y(.,t) \right\|_{L^2(\Omega)}.\end{aligned}$$

From the strong convergence in $L^2(Q_S)$ of y_n to y, the Cauchy–Schwartz inequality, (3.8)–(3.11), we can pass to the limit in this inequality to get

$$\limsup_{n\to\infty} \|y_n(.,t) - y(.,t)\|_{L^2(\Omega)} \leqslant \varepsilon + \frac{\sqrt{2}\gamma^3}{\alpha^2}\varepsilon^2 + \left\| \frac{1}{2h}\int_{t-h}^{t+h} y(.,s)\,\mathrm{d}s - y(.,t) \right\|_{L^2(\Omega)}.$$

In this inequality h can be chosen as small as we want, since u belongs to $C^0([0,S]; L^2(\Omega))$. We can then pass to the limit when h tends to zero to obtain

$$\limsup_{n\to\infty} \|y_n(.,t) - y(.,t)\|_{L^2(\Omega)} \leqslant \varepsilon + \frac{\sqrt{2}\gamma^3}{\alpha^2}\varepsilon^2, \quad \forall \varepsilon > 0,$$

and then (3.7). □

4. Existence of solution for optimal design problems

In this section, we investigate the existence of solution for the following control problem:

$$\min_{\widetilde{\Omega}\in\mathcal{O},\, A\in\mathcal{A}} J(y) \quad \begin{cases} \partial_t y - div\, A(x,t)\nabla y = f & \text{in } \widetilde{\Omega}\times(0,T),\\ y=0 & \text{on } (\widetilde{\Omega}\times\{0\})\cup(\partial\widetilde{\Omega}\times(0,T)), \end{cases} \tag{4.12}$$

where f belongs to $L^2(0,T;H_0^1(\Omega))$, J is a functional in $L^2(0,T;H_0^1(\Omega))\cap C^0([0,T];L^2(\Omega))$, $\mathcal{O}$ is composed by open subsets of Ω, and $\mathcal{A}$ is a subset of $\mathcal{M}_\alpha^\gamma(Q_T)$. This type of problems arise in the optimization of materials (represented by the matrix A) and shapes (represented by the open set $\widetilde{\Omega}$). It is well known that a problem like (4.12) has not a solution in general (see, e.g., [3,7,22]), and then, it is necessary to take a relaxation. In fact, because from Theorem 3.1, it is problem (3.5) which is stable by homogenization, it is better to replace (4.12) by

$$\min_{(A,(F,\mu))\in\mathcal{E}} J(y) \quad \begin{cases} y\in L^2(0,T;H_0^1(\Omega)\cap L^2_\mu(\Omega)), \quad y(x,0)=0, \text{ a.e. in } \Omega,\\ \langle\partial_t y, v\rangle + \int_\Omega A(x,t)\nabla y\nabla v\,\mathrm{d}x\\ \quad + \int_\Omega F(x,t) y v\,\mathrm{d}\mu = \langle f, v\rangle \quad \text{in } \mathcal{D}'(0,T),\\ \forall\, v\in L^2(0,T;H_0^1(\Omega)\cap L^2_\mu(\Omega)), \end{cases} \tag{4.13}$$

with $\mathcal{E}$ a subset of $\mathcal{M}_\alpha^\gamma(Q_T)\times\mathcal{F}_\alpha^\gamma(Q_T)$. Using the direct method of the calculus of variations, Theorem 3.1 and Proposition 3.2 can be immediately proved.

Theorem 4.1. *For $T>0$, $\gamma>\alpha>0$, let $\mathcal{E}$ be a subset of $\mathcal{M}_\alpha^\gamma(Q_T)\times\mathcal{F}_\alpha^\gamma(Q_T)$ stable by homogenization, i.e., such that the limit of a sequence of problems like* (1.3), *with $(A_n,(F_n,\mu_n))\in\mathcal{E}$, is of the form* (3.5), *with $(A,(F,\mu))\in\mathcal{E}$, and let $J: L^2(0,T;H_0^1(\Omega))\cap C^0([0,T];L^2(\Omega))\to\mathbb{R}$ be a functional which is semicontinuous in the following sense*:

For every sequence $y_n\in L^2(0,T;H_0^1(\Omega))\cap C^0([0,T];L^2(\Omega))$, which is bounded in $L^2(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$, and converges to $y\in L^2(0,T;H_0^1(\Omega))\cap C^0([0,T];L^2(\Omega))$, weakly in $L^2(0,T;H_0^1(\Omega))$, strongly in $L^2(Q_T)$, and also $y_n(.,t)$ converges strongly in $L^2(\Omega)$ to $y(.,t)$, for every $t\in[0,T]$, we have

$$\liminf_{n\to\infty} J(y_n)\geqslant J(y).$$

Then, for every $f\in L^2(0,T;H^{-1}(\Omega))$, problem (4.13) *has at least a solution.*

As examples of functionals J in the conditions of Theorem 4.1, we have

$$y\to\int_\Omega |y(x,T)-y_d|^2\,\mathrm{d}x, \quad y_d\in L^2(\Omega),$$

$$y\to\int_{Q_t} |y(x,T)-y_d|^2\,\mathrm{d}x\,\mathrm{d}t, \quad y_d\in L^2(Q_T),$$

$$y\to\int_{Q_t} |\nabla(y(x,T)-y_d)|^2\,\mathrm{d}x\,\mathrm{d}t, \quad y_d\in L^2(0,T;H_0^1(\Omega)),$$

with respect to subsets $\mathcal{E}$ in the conditions of Theorem 4.1. Thanks to Theorem 3.1, we can take $\mathcal{E}=\mathcal{M}_\alpha^\gamma(Q_T)\times\mathcal{F}_\alpha^\gamma(Q_T)$, but it is too large. In practice we only dispose of a few of materials and shapes.

Moreover, the question remains whether problem (4.13) is a relaxation of problem (4.12) or not. In this sense, the following definition is useful:

Definition 4.2. Given a subset $\mathscr{E}$ of $\mathscr{M}_\alpha^\gamma(Q_T) \times \mathscr{F}_\alpha^\gamma(Q_T)$, we define the closure by homogenization of $\mathscr{E}$, and we denote it by $C_H(\mathscr{E})$, as the set of pairs $(A, (F, \mu)) \in \mathscr{M}_\alpha^\gamma(Q_T) \times \mathscr{F}_\alpha^\gamma(Q_T)$, such that there exists $(A_n, (F_n, \mu_n)) \in \mathscr{E}$, which satisfies that for every $f \in L^2(0, T; H^{-1}(\Omega))$, the unique solution of (1.3) converges weakly in $L^2(0, T; H_0^1(\Omega))$ to the unique solution of (3.5).

From Theorem 3.1, it is clear that the closure by homogenization of a set $\mathscr{E}$ is stable by homogenization, and then, it is in the conditions of Theorem 4.1. We easily prove the following:

Proposition 4.3. *For* $T > 0$, $\gamma > \alpha > 0$, *let* $\mathscr{E}$ *be a subset of* $\mathscr{M}_\alpha^\gamma(Q_T) \times \mathscr{F}_\alpha^\gamma(Q_T)$, *and let* $J : L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega)) \to \mathbb{R}$ *be a functional which satisfies the following continuity property*:

For every sequence $y_n \in L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$, *which is bounded in* $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, *and converges to* $y \in L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$, *weakly in* $L^2(0, T; H_0^1(\Omega))$, *strongly in* $L^2(Q_T)$, *and also* $y_n(., t)$ *converges strongly in* $L^2(\Omega)$ *to* $y(., t)$, *for every* $t \in [0, T]$, *we have*

$$\lim_{n\to\infty} J(y_n) = J(y).$$

Then, for every distribution $f \in L^2(0, T; H^{-1}(\Omega))$, *we get a relaxation of problem* 4.13, *just by replacing* $\mathscr{E}$ *by* $C_H(\mathscr{E})$.

Remark 4.4. The functional

$$y \to \int_{Q_t} |\nabla(y(x, T) - y_d)|^2 \, dx \, dt, \qquad y_d \in L^2(0, T; H_0^1(\Omega))$$

satisfies the assumptions of Theorem 4.1 but not those of Proposition 4.3.

Remark 4.5. Since in Theorem 3.1 A is the homogenized matrix of the sequence A_n, it is clear that for $\mathscr{E} \subset \mathscr{M}_\alpha^\gamma(Q_T) \times \mathscr{F}_\alpha^\gamma(Q_T)$, the projection of $C_H(\mathscr{E})$ on $\mathscr{M}_\alpha^\gamma(Q_T)$ coincides with the closure by H-convergence (H-closure) (see, e.g., [23,25,28]) of the projection of $\mathscr{E}$ on $\mathscr{M}_\alpha^\gamma(Q_T)$.

From Proposition 4.3, in order to obtain a relaxation of (4.12), we need to obtain the closure by homogenization of the set of pairs $\widetilde{\Omega} \in \mathscr{O}$, $A \in \mathscr{A}$, where $\mathscr{O}$ is composed of open subsets of Ω, and $\mathscr{A}$ is contained in $\mathscr{M}_\alpha^\gamma(Q_T)$. Here, we identify an open set $\widetilde{\Omega} \subset \Omega$, with the pair $(F, \mu) \in \mathscr{F}_\alpha^\gamma(Q_T)$, given by

$$\mu(B) = \begin{cases} +\infty & \text{if } Cap(B \cap (\Omega \backslash \widetilde{\Omega}), \Omega) > 0, \\ 0 & \text{if } Cap(B \cap (\Omega \backslash \widetilde{\Omega}), \Omega) = 0, \end{cases} \qquad \forall B \subset \Omega \text{ Borel},$$

and $F = \frac{\alpha+\gamma}{2} \chi_{\widetilde{\Omega}}$.

When $\mathscr{E}$ is of the form

$$\mathscr{E} = \mathscr{A} \times \{\widetilde{\Omega} : \widetilde{\Omega} \subset \Omega \text{ open}\},$$

with $\mathscr{A}$ a subset of $\mathscr{M}_\alpha^\gamma(Q_T)$ composed of constant matrices with respect to the time variable, we can use the results which appear in [2] to prove

$$C_H(\mathscr{E}) = \bar{\mathscr{A}} \times \{(F, \mu) \in \mathscr{F}_\alpha^\gamma(Q_T) : F(x, t) \text{ constant with respect to } t\},$$

with $\bar{\mathscr{A}}$ the H-closure of $\mathscr{A}$. So, in this case the relaxation of problem (4.12) is reduced to the calculus of the H-closure of $\mathscr{A}$ (which is only known to a very few choices of sets $\mathscr{A}$, see, e.g., [1,21,29]). Indeed, because for $\mu \in \mathscr{M}_0^2(\Omega)$, and $F \in L_\mu^\infty(\Omega)$ constant with respect to the time variable, the product $F\mu$ also gives a measure in $\mu \in \mathscr{M}_0^2$, for the above choice of $\mathscr{E}$, a relaxation of (4.13) is given by

$$\min_{(A,\mu)\in\bar{\mathscr{A}}\times\mathscr{M}_0^2(\Omega)} J(y) \quad \begin{cases} y \in L^2(0,T;H_0^1(\Omega)\cap L_\mu^2(\Omega)), \quad y(x,0)=0 \text{ a.e. in } \Omega, \\ \langle \partial_t y, v\rangle + \int_\Omega A(x)\nabla y \nabla v\,\mathrm{d}x + \int_\Omega yv\,\mathrm{d}\mu = \langle f, v\rangle \quad \text{in } \mathscr{D}'(0,T), \\ \forall\, v \in L^2(0,T;H_0^1(\Omega)\cap L_\mu^2(\Omega)). \end{cases}$$

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