J. Math. Pures Appl.,
76, 1997, *p.* 431-476

HOMOGENIZATION OF GENERAL QUASI-LINEAR DIRICHLET PROBLEMS WITH QUADRATIC GROWTH IN PERFORATED DOMAINS

By Juan CASADO-DÍAZ

ABSTRACT. – In this paper, we study the homogenization of a Dirichlet problem in perforated domains for an operator which is the perturbation of the Laplace operator by a general nonlinear term with quadratic growth in the gradient. We show that a new term, which does not depend on the gradient, but which is nonlinear, appears in the limit problem. We also give a corrector result.

0. Introduction

The goal of the present paper is to study the homogenization problem ([1])

$$(0.1)\qquad \begin{cases} -\Delta u^\varepsilon + H(x, u^\varepsilon, \nabla u^\varepsilon) = f \ \text{ in } \mathcal{D}'(\Omega^\varepsilon), \\ u^\varepsilon \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon), \end{cases}$$

where Ω^ε is a sequence of open sets which are contained in a fixed bounded open set $\Omega \subset \mathbb{R}^N$, f is a function in $L^\infty(\Omega)$ and $H(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ is a Carathéodory function which has a quadratic growth in the variable ξ and is of classe C^2 in the variable (s, ξ).

The existence of a solution for the problem (0.1) has been established by L. Boccardo, F. Murat and J.P. Puel in [B M P] and its uniqueness by G. Barles and F. Murat in [B M]. From these works, we also deduce that u^ε is bounded in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore, extracting a subsequence, u^ε converges weakly in $H_0^1(\Omega)$ and weakly-$*$ in $L^\infty(\Omega)$ to a function u. The questions which we address here is to find the problem satisfied by the function u and a corrector result.

It is well known (see [C M], [DM M1], [DM M2], [DM G]) than even for the linear problem

$$(0.2)\qquad \begin{cases} -\Delta u^\varepsilon = f \ \text{ in } \mathcal{D}'(\Omega^\varepsilon), \\ u^\varepsilon \in H_0^1(\Omega^\varepsilon), \end{cases}$$

([1]) Here and in what follows, we consider the functions u^ε as defined on the whole of Ω by setting $u^\varepsilon = 0$ on $\Omega \setminus \Omega^\varepsilon$ (see Appendix: Notation).

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES. – 0021-7824/1997/05/$ 7.00

the equation satisfied by the function u is not in general

$$-\Delta u = f \quad \text{in } \mathcal{D}'(\Omega),$$

but is of the type

$$-\Delta u + \mu u = f \quad \text{in } \mathcal{D}'(\Omega),$$

where a positive measure μ vanishing on the sets of zero capacity appears.

In the case of equation (0.1), the nonlinear term $H(x, u^\varepsilon, \nabla u^\varepsilon)$ leads us to a more complex equation, in which the new term which appears is no more linear in u, but is of the form $T(x, u)\mu$ for some nonlinear function $T(x, s)$ and for the same measure μ which appears in the linear case. In [C3], we have studied the particular case where $H(x, u^\varepsilon, \nabla u^\varepsilon) = \lambda u^\varepsilon - \gamma|\nabla u^\varepsilon|^2$, $\lambda > 0$, and have proved that in that case

$$T(x, s) = \frac{e^{\gamma s} - 1}{\gamma e^{\gamma s}}.$$

For what concerns the sequence Ω^ε, we will assume in the present paper that

$$\text{(0.3)} \qquad \begin{cases} \exists\, z^\varepsilon \in H^1(\Omega), \\ z^\varepsilon = 0 \quad \text{in } \Omega \setminus \Omega^\varepsilon, \\ z^\varepsilon \rightharpoonup 1 \quad \text{in } H^1(\Omega) \text{ weakly.} \end{cases}$$

This implies in particular that the holes $\Omega \setminus \Omega^\varepsilon$ are sufficiently small. As proved in [C2], this hypothesis is very close to the hypotheses assumed in [C M] (see also [K M]) to study the homogenization problem (0.2). A typical example is the case $\Omega^\varepsilon = \Omega \setminus T^\varepsilon$, where T^ε is the union of balls of radius $\varepsilon^{\frac{N}{N-2}}$ the centers of which are periodically distributed at the edges of a cubic network of size ε.

Hypothesis (0.3) implies the existence of a subsequence w^ε which vanishes in $\Omega \setminus \Omega^\varepsilon$ and which converges weakly to 1 in $H^1(\Omega)$ (see the precise properties of w^ε in Section 1, Theorem 1.3) such that the following corrector result holds: If $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, the solution u^ε of (0.2) satisfies

$$\text{(0.4)} \qquad u^\varepsilon - w^\varepsilon u \to 0 \quad \text{in } H_0^1(\Omega) \quad \text{strongly}$$

or equivalently

$$\nabla u^\varepsilon - \nabla u - u \nabla w^\varepsilon \to 0 \quad \text{in } L^2(\Omega)^N \quad \text{strongly.}$$

It is however proved in [C3], by the study of the example $H(x, u^\varepsilon, \nabla u^\varepsilon) = \lambda u^\varepsilon - \gamma\,|\nabla u^\varepsilon|^2$, that (0.4) does not hold in general for the quasi-linear problem (0.1). We will nevertheless use $w^\varepsilon u$ as a test function in the proofs below to estimate ∇u^ε when u^ε is the solution of the quasi-linear problem (0.1): In some sense, we will compare ∇u^ε and $\nabla u + u \nabla w^\varepsilon$. We follow the general method designed by L. Tartar (see [T]) to study a homogenization problem which consists to test the equation by special test functions. We introduce here an original variant of this method, which consists to make a comparison between ∇u^ε and $\nabla u + u\nabla w^\varepsilon$ when u^ε is the solution of (0.1), u its weak limit and w^ε the corrector for the linear problem. The important fact is that $w^\varepsilon u$ is no more a corrector for the nonlinear equation (0.1) (*i.e.* (0.4) does not hold here) but this comparison will nevertheless to reconstruct the limit equation. Our proof will also make use of nonlinear test functions (as done in [B M P]) and of a change of unknown function (as done in [B M]) to pass from the quasi-linear equation (0.1) to another equivalent quasi-linear equation which satisfies a good "structure condition".

The homogenization of the quasi-linear problem (0.1) could as well be carried out without assuming any hypothesis on the sequence Ω^ε. In this case it is sufficient to replace the sequence $w^\varepsilon u$ by the corrector given in [DM G] (see also [DM Mu1], [DM Mu2]). One could as well consider the case where $-\Delta u^\varepsilon$ is replaced by a monotone or even pseudo-monotone operator $-div\, a(x, u^\varepsilon, \nabla u^\varepsilon)$ acting on $W_0^{1,p}(\Omega)$ (in this case the function H has to have a growth less than $|\xi|^p$): One has to use in this latest case the corrector results of [DM Mu2]. In view of the technical difficulties which appear in the present paper, and which are mostly due to the use of the technique of change of unknown function which traces back to [B M], we have prefered to limit ourselves to the case where we assume that (0.3) holds. We hope that the reader will be happy of our choice.

The method we use in the present paper (*i.e.* the comparison of ∇u^ε with $\nabla u + u\nabla w^\varepsilon$) is also sucessful in the study of the homogenization of Dirichlet problems for nonlinear monotone and pseudo-monotne operators of Leray-Lions type. The mehod is presented in [C4] in the simple case where monotone operators defined on $W_0^{1,p}(\Omega)$ are considered and where an hypothesis similar to (0.3) is made on the sequence Ω^ε. The general case of monotone systems without any hypotheses on the sets Ω^ε is treated in [C G]. Note finally that even if the basis of the technique used in [C4] and [C G] is the same as in the present paper, the situation is simpler there since no change of unknown function is necessary when no "quadratic" perturbation occurs.

The main results obtained in the present paper can be summarized as follows: Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and let Ω^ε be a sequence of open sets contained in Ω such that (0.3) holds true. We consider a Carathéodory function $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ such that for almost every $x \in \Omega$ the function $H(x, s, \xi)$ is of class C^2 in s and ξ and has at most a quadratic growth in ξ. We also assume that for a strictly positive constant λ and for almost every $x \in \Omega$ we have

$$\frac{\partial H(x,s,\xi)}{\partial s} > \lambda, \quad \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N$$

and that the first and second derivatives in (s, ξ) of H satisfy reasonable growth conditions (actually the same as the derivatives of $|\xi|^2$, see (1.2) and (4.3) for the precise hypotheses made on H).

Then we have the following homogenization theorem for the quasi-linear problem (0.1) (which easily results from Theorem 5.1, Theorem 6.3 and Remark 6.3):

THEOREM 0.1. – *There exists a subsequence of ε (still denoted by ε), a positive bounded Borel measure μ which vanishes on the sets of zero capacity (μ is the same measure which* *appears in the homogenization of the linear problem (0.2)) and a Carathéodory function* $T : \Omega \times \mathbb{R} \mapsto \mathbb{R}$, *such that for any function* $f \in L^\infty(\Omega)$, *the unique solution* u^ε *of (0.1) converges strongly in* $W_0^{1,p}(\Omega)$, $1 \leq p < 2$, *weakly in* $H_0^1(\Omega)$, *and weakly-∗ in* $L^\infty(\Omega)$ *to a function* u *which is the unique solution of the problem:*

$$(0.5) \qquad \begin{cases} -\Delta u + T(x,u)\mu + H(x,u,\nabla u) = f \quad in\ \mathcal{D}'(\Omega), \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases}$$

For μ*-almost every* $x \in \Omega$, *the function* $T(x,.)$ *is increasing, satisfies* $T(x,0) = 0$, *and is locally Hölder continuous,* i.e. *satisfies*

$$|T(x,s_1) - T(x,s_2)| \leq C(s)|s_1 - s_2|^{\frac{1}{\lambda(s)}},$$

where $s = \max\{|s_1|,|s_2|\}$ *and where* $C : [0,+\infty) \mapsto [0,+\infty)$ *and* $\lambda : [0,+\infty) \mapsto [1,+\infty)$ *are increasing.*

We also have the following corrector result (this result is stated in Theorem 7.1):

THEOREM 2. – *Let* ε *be the subsequence extracted in Theorem 0.1. Define for* $s \in \mathbb{R}$ *and* $n \in \mathbb{N}$ *the function* s_n^ε *as the solution of*

$$(0.6) \qquad \begin{cases} -\Delta s_n^\varepsilon + n s_n^\varepsilon + H(x,s_n^\varepsilon,\nabla s_n^\varepsilon) = ns \quad in\ \mathcal{D}'(\Omega^\varepsilon), \\ s_n^\varepsilon \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon) \end{cases}$$

and set $P_n^\varepsilon(s) = \nabla s_n^\varepsilon$. *Consider on the other hand a step function* $y(x) = \sum_{i=1}^m s_i \chi_{Q_i}(x)$, *where the* Q_i *are closed subsets of* Ω *which satisfy* $\mu(Q_i \cap Q_j) = 0$ *for* $i \neq j$ *and where the* s_i *are real numbers. Let* Q *and* t *be defined by:*

$$Q = \bigcup_{i=1}^m Q_i, \quad t = \max\{\sup\{\| u_\varepsilon \|_{L^\infty(\Omega)}\}, \| y \|_{L^\infty(\Omega)}\}.$$

Then

$$(0.7) \qquad \begin{cases} \limsup_{n\to\infty} \limsup_{\varepsilon\to 0} \sum_{i=1}^m \int_{Q_i} |\nabla u^\varepsilon - \nabla u - P_n^\varepsilon(x,y(x))|^2\,dx \\ \leq \bar{C}(t)(\mu(Q))^{1-\frac{1}{\bar{\lambda}(t)}} \left(\int_Q |u-y|\,d\mu \right)^{\frac{1}{\bar{\lambda}(t)}}, \end{cases}$$

where $\bar{C} : [0,+\infty) \mapsto [0,+\infty)$ *and* $\bar{\lambda} : [0,+\infty) \mapsto [1,+\infty)$ *are increasing functions which do not depend neither on the sets* Q_i, *nor on the function* y *and nor on the right hand side* f *of (0.1).*

The above result provides an approximation of ∇u^ε in $L^2(\Omega)^N$. Indeed when $y(x) = \sum_{i=1}^m s_i \chi_{Q_i}(x)$ is a step function which is defined on closed sets Q_i with $\mu(Q_i \cap Q_j) = 0$ and when y is close to u in $L^1(\Omega, d\mu)$ (it is possible to construct such test functions, see Remark 7.4), then $\nabla u + P_n^\varepsilon(x,y(x))$ is close to ∇u^ε in $L^2(\Omega)^N$. Formally the idea is to replace ∇u^ε by ∇s_n^ε, where s is the value of $u(x)$ at the (frozen) point x. This replacement is nevertheless impossible, since the function $\nabla s_n^\varepsilon = P_n^\varepsilon(x,s)$ is not a continuous function with respect to s and this leads us to the use of an approximation by a step function y.

The idea for the introduction of s_n^ε is the following: For $s \in \mathbb{R}$ given (which will be $u(x_0)$ for a given x_0), we would like to find some $f_s \in L^\infty(\Omega)$ such that the solution s^ε of

$$\begin{cases} -\Delta s^\varepsilon + H(x, s^\varepsilon, \nabla s^\varepsilon) = f_s & \text{in } \mathcal{D}'(\Omega^\varepsilon) \\ s^\varepsilon \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon) \end{cases} \tag{0.8}$$

has the property that s^ε tends to s. This is impossible for several reasons: The first one is that s does not belong to $H_0^1(\Omega)$, since $s \neq 0$ on $\partial\Omega$. This could be solved by replacing s by $s\varphi(x)$ with $\varphi \in \mathcal{D}(\Omega)$, but a new difficulty appears: Passing to the limit in (0.8) would give, according to Theorem 0.1,

$$-\Delta(s\varphi) + T(x, s\varphi)d\mu + H(x, s\varphi, s\nabla\varphi) = f_s \quad \text{in } \mathcal{D}'(\Omega)$$

and in general f_s does not belong to $L^\infty(\Omega)$. For this last reason, we introduce a new parameter n and the penalization $n(s_n^\varepsilon - s)$ in (0.6); passing to the limit in (0.6) for n fixed implies that s_n^ε tends to s^n in $H_0^1(\Omega)$ weak, with:

$$\begin{cases} -\Delta s_n + T(x, s_n)\mu + H(x, s_n, \nabla s_n) = ns & \text{in } \mathcal{D}'(\Omega), \\ s_n \in H_0^1(\Omega) \cap L^\infty(\Omega) \end{cases}$$

and it can be proved that s_n tends to s when n tends to infinity.

Contents

1. Some preliminary results about quasi-linear problems with quadratic growth and homogenization in perforated domains

1.1. Quasilinear problems with quadratic growth

We first recall some results about the existence and uniqueness of the solution of the problem

$$\begin{cases} -\Delta u + H(x,u,\nabla u) = f \quad \text{in } \mathcal{D}'(\Theta), \\ u \in H_0^1(\Theta) \cap L^\infty(\Theta), \end{cases} \tag{1.1}$$

where Θ is an open set contained in Ω. Following the ideas of [B M P] and [B M], we will obtain for this problem some estimates which will be useful in the homogenization of (0.1).

Let us assume that the Carathéodory function $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ satisfies the following hypotheses:

i) For almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^N$, the function $H(x,.,\xi)$ is continuously differentiable and there exists a constant $\lambda > 0$ such that

$$\frac{\partial H}{\partial s}(x,s,\xi) \geq \lambda, \quad \text{a.e. } x \in \Omega, \ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.2}$$

ii) There exist two increasing functions v_0 and $v : [0,+\infty) \mapsto [0,+\infty)$, such that

$$|H(x,s,\xi)| \leq v_0(|s|) + v(|s|)|\xi|^2, \quad \text{a.e. } x \text{ in } \Omega, \ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.3}$$

REMARK 1.1. – It is enough to assume that v_0 and v are just bounded on the bounded sets of $[0,+\infty)$. We then obtain increasing functions by defining $\hat{v}_0(s) = \sup_{0\leq t\leq s} v_0(t)$ and $\hat{v}(s) = \sup_{0\leq t\leq s} v(t)$.

L. Boccardo, F. Murat and J.P. Puel proved in [B M P] (see also [C1]) the following existence result for (1.1).

THEOREM. – *Assume that Θ is an open set, $\Theta \subset \Omega$, and that H satisfies (1.2) and (1.3) and that $f \in L^\infty(\Theta)$. Then there exists a solution u of (1.1) such that $\| u \|_{H_0^1(\Theta)}$ and $\| u \|_{L^\infty(\Theta)}$ are bounded by constants which depend only on λ, v_0, v, $\| f \|_{L^\infty(\Omega)}$ and the measure of Θ.*

In fact, we have

$$\| u \|_{L^\infty(\Theta)} \leq \frac{v_0(0) + \| f \|_{L^\infty(\Theta)}}{\lambda}. \tag{1.4}$$

The estimate for $\| u \|_{H_0^1(\Theta)}$ is more complicated and will not be given explicitly. (It is easily deduced from the following Lemma by taking $\varphi = 1$ and $r = 0$ in (1.7).

LEMMA 1. – *(see [B M P]) Assume that Θ is an open set, $\Theta \subset \Omega$, and that H satisfies (1.2) and (1.3) and that $f \in L^\infty(\Theta)$. Consider a constant $M > 0$ and a function $u \in H^1(\Theta) \cap L^\infty(\Theta)$ such that $\| u \|_{L^\infty(\Theta)} \leq M$, and define $f \in H^{-1}(\Theta) + L^1(\Theta)$ by:*

$$-\Delta u + H(x,u,\nabla u) = f \quad \textit{in } \mathcal{D}'(\Theta). \tag{1.5}$$

Let $h \in C^1(\mathbb{R})$ *be the function defined by*

$$h(s) = 2se^{(v(M)s)^2}$$

which depends only on M. *This function satisfies*

$$\begin{cases} h(0) = 0, \\ h'(s) - 2v(M)|h(s)| \geq 1, \ \forall s \in \mathbb{R}. \end{cases} \tag{1.6}$$

Then, for any r *and* φ *such that*

$$r \in H^1(\Theta) \cap L^\infty(\Theta), \quad \varphi \in H^1(\Theta) \cap L^\infty(\Theta), \quad \varphi \geq 0, \quad (u-r)\varphi \in H_0^1(\Theta),$$

we have [1]

$$\begin{cases} \displaystyle\int_\Theta |\nabla(u-r)|^2\varphi \leq \langle f, h(u-r)\varphi\rangle_\Theta - \int_\Theta \varphi \nabla r \nabla h(u-r) \\ \displaystyle - \int_\Theta h(u-r)\nabla u \nabla \varphi + v_0(M)\int_\Theta |h(u-r)|\varphi + 2v(M)\int_\Theta |\nabla r|^2 |h(u-r)|\varphi. \end{cases} \tag{1.7}$$

Proof. – Taking $h(u-r)\varphi \in H_0^1(\Theta) \cap L^\infty(\Theta)$ as test function in (1.5), we obtain

$$\int_\Theta \nabla u \nabla h(u-r)\varphi + \int_\Theta h(u-r)\nabla u \nabla \varphi + \int_\Theta H(x,u,\nabla u)h(u-r)\varphi$$
$$= \langle f, h(u-r)\varphi\rangle_\Theta.$$

Since $\nabla h(u-r) = h'(u-r)\nabla(u-r)$ and (1.3) we have:

$$\begin{cases} \displaystyle\int_\Theta h'(u-r)|\nabla(u-r)|^2\varphi \leq \langle f, h(u-r)\varphi\rangle_\Theta \\ \displaystyle - \int_\Theta \varphi\nabla r \nabla h(u-r) - \int_\Theta h(u-r)\nabla u \nabla \varphi \\ \displaystyle + v_0(M)\int_\Theta |h(u-r)|\varphi + v(M)\int_\Theta |\nabla u|^2|h(u-r)|\varphi. \end{cases} \tag{1.8}$$

Using the inequality $|\nabla u|^2 \leq 2|\nabla(u-r)|^2 + 2|\nabla r|^2$ in (1.8) and carrying the term $2v(M)\int_\Theta |\nabla(u-r)|^2|h(u-r)|\varphi$ to the right-hand side of (1.8) and using (1.6), we deduce (1.7). ■

We will use stronger hypotheses about H to obtain a uniqueness result for Problem (1.1). Specifically let us assume that for almost every $x \in \Omega$ the function $H(x,.,.)$ is continuously differentiable and that there exists an increasing function $\alpha : [0,+\infty) \mapsto [0,+\infty)$ such that

$$\begin{cases} H(.,0,0) \in L^\infty(\Omega), \\ \left|\dfrac{\partial H}{\partial s}(x,s,\xi)\right| \leq \alpha(|s|)(1+|\xi|^2), \quad \text{a.e. } x \in \Omega, \ \forall (s,\xi) \in \mathbb{R}\times\mathbb{R}^N, \\ \left|\dfrac{\partial H}{\partial \xi}(x,s,\xi)\right| \leq \alpha(|s|)(1+|\xi|), \quad \text{a.e. } x \in \Omega, \ \forall (s,\xi) \in \mathbb{R}\times\mathbb{R}^N. \end{cases} \tag{1.9}$$

---

[1] Here and in what follows, $\langle f, v\rangle_\Theta$ denotes the duality pairing between $H^{-1}(\Theta) + L^1(\Theta)$ and $H_0^1(\Theta) \cap L^\infty(\Theta)$ (see Appendix: Notation).

REMARK 1.2. – Assumption (1.9) implies the existence of increasing functions $v_0, v : [0, +\infty) \mapsto [0, +\infty)$ such that the function H satisfies inequality (1.3).

The following lemma results from a computation which is due to G. Barles and F. Murat (see proof of Theorem II.1 in [B M]).

LEMMA 1.2. – *Assume that Θ is an open set, $\Theta \subset \Omega$, and that H satisfies (1.2) and (1.9). Let $u \in H^1(\Theta) \cap L^\infty(\Theta)$ and $f \in H^{-1}(\Theta) + L^1(\Theta)$ which satisfy*

$$\begin{cases} -\Delta u + H(x, u, \nabla u) = f \ \text{ in } \mathcal{D}'(\Theta), \\ u \in H^1(\Theta) \cap L^\infty(\Theta), \ \| u \|_{L^\infty(\Omega)} \leq M, \end{cases} \tag{1.10}$$

for some $M > 0$. Define for $A > 0$ and $K > 0$ the functions ψ and $\vartheta = \psi^{-1}$ by:

$$\begin{cases} \psi : (-\infty, +\infty) \mapsto \left(-\infty, \dfrac{1}{A} logK \right) \\ \psi(\hat{s}) = -\dfrac{1}{A} log\left(e^{-KA\hat{s}} + \dfrac{1}{K} \right), \ \forall \hat{s} \in \mathbb{R}, \end{cases} \tag{1.11}$$

$$\begin{cases} \vartheta : \left(-\infty, \dfrac{1}{A} logK \right) \mapsto (-\infty, +\infty) \\ \vartheta(s) = \psi^{-1}(s) = -\dfrac{1}{KA} log\left(e^{-As} - \dfrac{1}{K} \right), \ \forall s \text{ such that } s < \dfrac{1}{A} log(K). \end{cases} \tag{1.12}$$

Then there exists two constants $A, K > 0$, which are increasing with respect to M such that the function $\hat{u} = \vartheta(u)$ satisfies

$$\begin{cases} -\Delta \hat{u} + B(x, \hat{u}, \nabla \hat{u}) = \hat{f} \ \text{ in } \mathcal{D}'(\Theta), \\ \hat{u} \in H^1(\Theta) \cap L^\infty(\Theta), \end{cases} \tag{1.13}$$

where $\hat{f}$ is defined by

$$\hat{f} = \frac{f}{\psi'(\hat{u})}$$

and where the function $B : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ is a Carathéodory function which satisfies a property similar to (1.9): For almost every $x \in \Omega$ the function $B(x, ., .)$ is continuously differentiable and there exists an increasing function $\beta : [0, +\infty) \mapsto [0, +\infty)$ such that

$$\begin{cases} B(., 0, 0) \in L^\infty(\Omega), \\ \left| \dfrac{\partial B}{\partial \hat{s}}(x, \hat{s}, \hat{\xi}) \right| \leq \beta(|\hat{s}|)(1 + |\hat{\xi}|^2), \ \text{a.e. } x \in \Omega, \ \forall (\hat{s}, \hat{\xi}) \in \mathbb{R} \times \mathbb{R}^N, \\ \left| \dfrac{\partial B}{\partial \hat{\xi}}(x, \hat{s}, \hat{\xi}) \right| \leq \beta(|\hat{s}|)(1 + |\hat{\xi}|), \ \text{a.e. } x \in \Omega, \ \forall (\hat{s}, \hat{\xi}) \in \mathbb{R} \times \mathbb{R}^N. \end{cases} \tag{1.14}$$

Finally, there exists also a constant $n > 0$, which depends on A and K, and thus on M, and which is increasing with respect to M such that

$$\frac{\partial B}{\partial \hat{s}}(x, \hat{s}, \hat{\xi}) - \frac{1}{2n} \left| \frac{\partial B}{\partial \hat{\xi}}(x, \hat{s}, \hat{\xi}) \right|^2 \geq 0 \text{ a.e. } x \in \Omega, \ \forall (\hat{s}, \hat{\xi}) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.15}$$

Proof. – The results of Lemma 1.2 are proved in [B M] (proof of Theorem II.1). Let us recall the main points of this proof.

When $K > e^{AM}$, the domain of definition of ϑ covers $[-M, M]$. When u is a solution of (1.1) the change of unknown function $\hat{u} = \vartheta(u)$ implies that $\hat{u}$ is a solution of (1.13), where

$$B(x, \hat{s}, \hat{\xi}) = -\frac{\psi''(\hat{s})}{\psi'(\hat{s})}|\hat{\xi}|^2 + \frac{1}{\psi'(\hat{s})}H(x, \psi(\hat{s}), \psi'(\hat{s})\xi), \quad \text{a.e. } x \in \Omega, \ \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N.$$

It is then easy to prove that (1.14) holds true.

In order to prove (1.15), it is sufficient to follow the proof of [B M]. One first fixed A sufficiently large, $A = A_0(M)$ (with $A_0(M) = K_2 + 1$ in the notation of [B M]). Then for K large enough (more precisely, $K \geq K_0(M, \lambda)$ where λ is the constant which appears in (1.2)) one obtains, when n is large enough ($n \geq n_0(M, \lambda)$)

$$\frac{\partial B}{\partial \hat{s}}(x, \hat{s}, \hat{\xi}) - \frac{1}{2n}\left|\frac{\partial B}{\partial \hat{\xi}}(x, \hat{s}, \hat{\xi})\right|^2 \geq 0, \tag{1.16}$$

which is the desired result.

REMARK 1.3. – As in Remark 1.2, the inequalities in (1.14) imply the existence of two increasing functions $\hat{v}_0, \hat{v} : [0, +\infty) \mapsto [0, +\infty)$ such that the function B satisfies an inequality similar to (1.3).

The following result provides similar estimates to those of Lemma 1.1, which will be used later for the homogenization of problem (0.1). The ideas used in the proof follow from [B M].

LEMMA 1.3. – *Assume that Θ is an open set, $\Theta \subset \Omega$ and let $B : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto R$ be a Carathéodory function which satisfies properties (1.14) and (1.15). Consider $\hat{u}, \hat{v} \in H^1(\Omega) \cap L^\infty(\Omega)$, with $\| \hat{u} \|_{L^\infty(\Theta)} \leq M$, $\| \hat{v} \|_{L^\infty(\Theta)} \leq M$, and $\hat{f}, \hat{g} \in H^{-1}(\Theta) + L^1(\Theta)$ which satisfy*

$$\begin{cases} -\Delta \hat{u} + B(x, \hat{u}, \nabla \hat{u}) = \hat{f} \ \text{ in } \mathcal{D}'(\Theta), \\ -\Delta \hat{v} + B(x, \hat{v}, \nabla \hat{v}) = \hat{g} \ \text{ in } \mathcal{D}'(\Theta). \end{cases} \tag{1.17}$$

Set $S(s) = |s|^{n-1}s$. Then, for any function $\varphi \in H^1(\Theta) \cap L^\infty(\Theta)$, $\varphi \geq 0$, we have the following estimates:

i) *Let $\hat{\omega} = \hat{u} - \hat{v}$. Assume that $\hat{\omega}\varphi \in H_0^1(\Theta)$, we have:*

(1.18)

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_\Theta S'(\hat{\omega})|\nabla \hat{\omega}|^2\varphi + \int_\Theta \left[\big(B(x, \hat{u}, \nabla \hat{u}) - B(x, \hat{v}, \nabla \hat{v})\big)S(\hat{\omega}) + \frac{1}{2}S'(\hat{\omega})|\nabla \hat{\omega}|^2\right]\varphi \\ = \langle \hat{f} - \hat{g}, S(\hat{\omega})\varphi\rangle_\Theta - \displaystyle\int_\Theta S(\hat{\omega})\,\nabla(\hat{\omega})\,\nabla \varphi. \end{cases}$$

Moreover

$$\left[\big(B(x, \hat{u}, \nabla \hat{u}) - B(x, \hat{v}, \nabla \hat{v})\big)S(\hat{\omega}) + \frac{1}{2}S'(\hat{\omega})|\nabla \hat{\omega}|^2\right] \geq 0 \ \text{ a.e. in } \Theta.$$

ii) *Let* $\hat{r} \in H^1(\Theta) \cap L^\infty(\Theta)$ *and* $\hat{\omega} = \hat{u} - \hat{v} - \hat{r}$. *Assume that* $\hat{\omega}\varphi \in H_0^1(\Theta)$, *we have:*

$$(1.19) \quad \begin{cases} \dfrac{1}{2}\displaystyle\int_\Theta S'(\hat{\omega})|\nabla \hat{\omega}|^2\varphi \le \langle \hat{f} - \hat{g}, S(\hat{\omega})\varphi\rangle_\Theta \\ -\displaystyle\int_\Theta \nabla \hat{r}\, \nabla S(\hat{\omega})\,\varphi - \int_\Theta S(\hat{\omega})\,\nabla(\hat{u} - \hat{v})\,\nabla\varphi \\ + C_M \displaystyle\int_\Theta \big[(1 + |\nabla \hat{u}|^2 + |\nabla \hat{v}|^2)|\hat{r}| + (1 + |\nabla \hat{u}| + |\nabla \hat{v}|)|\nabla \hat{r}|\big]|\hat{\omega}|^n\varphi. \end{cases}$$

iii) *Let* $\hat{\omega} = \hat{u} - \hat{v}$. *Assume that* $\hat{\omega}^+ \in H_0^1(\Theta)$, *we have:*

$$(1.20) \quad \frac{1}{2}\int_\Theta S'(\hat{\omega}^+)|\nabla \hat{\omega}^+|^2 \le \langle \hat{f} - \hat{g}, S(\hat{\omega}^+)\rangle_\Theta.$$

Proof. – Let $\hat{r} \in H^1(\Theta) \cap L^\infty(\Theta)$ and set $\hat{\omega} = \hat{u} - \hat{v} - \hat{r}$. Let $\varphi \in H^1(\Theta) \cap L^\infty(\Theta)$, $\varphi \ge 0$ be such that $\hat{\omega}\varphi \in H_0^1(\Theta)$. Using $S(\hat{\omega})\varphi$ (which belongs to $H_0^1(\Theta) \cap L^\infty(\Theta)$) as test function in the difference between the two equations of (1.17), we obtain

$$\int_\Theta \varphi\nabla(\hat{u} - \hat{v})\,\nabla S(\hat{\omega}) + \int_\Theta S(\hat{\omega})\,\nabla(\hat{u} - \hat{v})\,\nabla\varphi$$
$$+ \int_\Theta [B(x, \hat{u}, \nabla \hat{u}) - B(x, \hat{v}, \nabla \hat{v})]S(\hat{\omega})\varphi = \langle \hat{f} - \hat{g}, S(\hat{\omega})\varphi\rangle_\Theta.$$

The equality $\nabla S(\hat{\omega}) = S'(\hat{\omega})\,\nabla\hat{\omega}$, yields

$$(1.21) \quad \begin{cases} \displaystyle\int_\Theta S'(\hat{\omega})|\nabla \hat{\omega}|^2\varphi + \int_\Theta \big[B(x, \hat{u}, \nabla \hat{u}) - B(x, \hat{v}, \nabla \hat{v})\big]S(\hat{\omega})\varphi \\ = \langle \hat{f} - \hat{g}, S(\hat{\omega})\varphi\rangle_\Theta - \displaystyle\int_\Theta \varphi\nabla \hat{r}\,\nabla S(\hat{\omega}) - \int_\Theta S(\hat{\omega})\,\nabla(\hat{u} - \hat{v})\,\nabla\varphi. \end{cases}$$

Define the measurable functions $b_{\hat{s}}, b_{\hat{\xi}}$ by

$$(1.22) \quad \begin{cases} b_{\hat{s}} : [0, 1] \times \Omega \mapsto \mathbb{R} \\ b_{\hat{s}}(t, x) = \dfrac{\partial B}{\partial \hat{s}}(x, t\hat{u}(x) + (1 - t)\hat{v}(x), t\nabla \hat{u}(x) + (1 - t)\nabla \hat{v}(x)), \\ b_{\hat{\xi}} : [0, 1] \times \Omega \mapsto \mathbb{R}^N \\ b_{\hat{\xi}}(t, x) = \dfrac{\partial B}{\partial \hat{\xi}}(x, t\hat{u}(x) + (1 - t)\hat{v}(x), t\nabla \hat{u}(x) + (1 - t)\nabla \hat{v}(x)). \end{cases}$$

Since B is continously differentiable, $b_{\hat{s}}$ and $b_{\hat{\xi}}$ are measurable on $[0, 1] \times \Omega$. By (1.14); we have (2)

$$(1.23) \quad \begin{cases} |b_{\hat{s}}(t, x)| \le C_M(1 + |\nabla\hat{u}|^2 + |\nabla\hat{v}|^2), \\ |b_{\hat{\xi}}(t, x)| \le C_M(1 + |\nabla\hat{u}| + |\nabla\hat{v}|). \end{cases}$$

(2) Here and in what follows, C_M denotes a generic constant which can change from a line to another and which is increasing with respect to M (*see* Appendix: Notation).

We have then using Taylor's formula:

$$(1.24)\quad \begin{cases} [B(x,\hat u,\nabla \hat u) - B(x,\hat v,\nabla \hat v)]S(\hat\omega) \\ = \displaystyle\int_0^1 \frac{d}{dt}[B(x, t\hat u + (1-t)\hat v, t\nabla \hat u + (1-t)\nabla \hat v)]dt\ S(\hat\omega) \\ = \displaystyle\int_0^1 \Big[b_{\hat s}(t,x)(\hat u - \hat v) + b_{\hat\xi}(t,x)\nabla(\hat u - \hat v)\Big]dt\ S(\hat\omega) \\ = \displaystyle\int_0^1 \Big[b_{\hat s}(t,x)\hat\omega S(\hat\omega) + b_{\hat\xi}(t,x)S(\hat\omega)\nabla\hat\omega\Big]dt \\ + \displaystyle\int_0^1 \Big[b_{\hat s}(t,x)\hat r + b_{\hat\xi}(t,x)\nabla \hat r\Big]dt\ S(\hat\omega). \end{cases}$$

Using Young's inequality and (1.23) yields

$$|b_{\hat\xi}(t,x)S(\hat\omega)\nabla\hat\omega| \le \frac{1}{2}\frac{S(\hat\omega)^2}{S'(\hat\omega)}|b_{\hat\xi}(t,x)|^2 + \frac{1}{2}S'(\hat\omega)|\nabla\hat\omega|^2,$$

$$\int_0^1 \Big[b_{\hat s}(t,x)\hat r + b_{\hat\xi}(t,x)\nabla \hat r\Big]dt\ S(\hat\omega)$$
$$\ge -C_M\big[(1+|\nabla\hat u|^2+|\nabla\hat v|^2)|\hat r| + (1+|\nabla\hat u|+|\nabla\hat v|)|\nabla\hat r|\big]\,|S(\hat\omega)|.$$

Therefore

$$(1.25)\quad \begin{cases} [B(x,\hat u,\nabla \hat u) - B(x,\hat v,\nabla \hat v)]S(\hat\omega) \\ \ge -\dfrac{1}{2}S'(\hat\omega)|\nabla\hat\omega|^2 + \displaystyle\int_0^1 \Big[b_{\hat s}(t,x)\hat\omega S(\hat\omega) - \frac{1}{2}\frac{S(\hat\omega)^2}{S'(\hat\omega)}|b_{\hat\xi}(t,x)|^2\Big]dt \\ - C_M\big[(1+|\nabla\hat u|^2+|\nabla\hat v|^2)|\hat r| + (1+|\nabla\hat u|+|\nabla\hat v|)|\nabla\hat r|\big]\,|S(\hat\omega)|. \end{cases}$$

Using $S(s) = |s|^{n-1}s$ and (1.15), we have:

$$b_{\hat s}(t,x)\hat\omega S(\hat\omega) - \frac{1}{2}\frac{S(\hat\omega)^2}{S'(\hat\omega)}|b_{\hat\xi}(t,x)|^2$$
$$= |\hat\omega|^{n+1}\left(b_{\hat s}(t,x) - \frac{1}{2n}|b_{\hat\xi}(t,x)|^2\right) \ge 0,$$

and therefore we finally deduce from (1.25) that

$$(1.26)\quad \begin{cases} [B(x,\hat u,\nabla \hat u) - B(x,\hat v,\nabla \hat v)]S(\hat\omega) \ge -\dfrac{1}{2}S'(\hat\omega)|\nabla\hat\omega|^2 \\ - C_M\big[(1+|\nabla\hat u|^2+|\nabla\hat v|^2)|\hat r| + (1+|\nabla\hat u|+|\nabla\hat v|)|\nabla\hat r|\big]\,|\hat\omega|^n. \end{cases}$$

When $\hat r = 0$, (1.18) is nothing but (1.21). Moreover (1.26) proves the positivity of the second term. Similarly, (1.21) and (1.26) prove (1.19). The proof of (1.20) is similar to the proof of (1.19) with $\hat r = 0$ and $\varphi = 1$ taking as test function $S(\hat\omega)^+$ instead of $S(\hat\omega)\varphi$ in the difference of the equations of (1.17). ■

The uniqueness result obtained by G. Barles and F. Murat in [B M] for equation (1.1) is now easily derived:

THEOREM 1.2. – *Assume that H satisfies (1.2) and (1.9) and let $\Theta \subset \Omega$ be an open set. Consider u and v in $H^1(\Theta) \cap L^\infty(\Theta)$ such that:*

$$- \Delta u + H(x, u, \nabla u) \leq 0 \ \ in\ \mathcal{D}'(\Theta),$$
$$- \Delta v + H(x, v, \nabla v) \geq 0 \ \ in\ \mathcal{D}'(\Theta).$$

Then inequality $u \leq v$ in $\partial\Theta$, (i.e. $(u - v)^+ \in H_0^1(\Theta)$) implies $u \leq v$ almost everywhere in Θ.

In particular the problem

$$\begin{cases} - \Delta u + H(x, u, \nabla u) = f \ \ in\ \mathcal{D}'(\Theta) \\ u \in H_0^1(\Theta) \cap L^\infty(\Theta) \end{cases}$$

has a unique solution when H satisfies (1.2) and (1.9) and $f \in L^\infty(\Theta)$.

Proof. – Using Lemma 1.2, the functions $\hat{u} = \vartheta(u)$, $\hat{v} = \vartheta(v)$ satisfy, since $\psi' > 0$

$$\begin{cases} - \Delta \hat{u} + B(x, \hat{u}, \nabla \hat{u}) \leq 0 \ \ \text{in } \mathcal{D}'(\Theta), \\ - \Delta \hat{v} + B(x, \hat{v}, \nabla \hat{v}) \geq 0 \ \ \text{in } \mathcal{D}'(\Theta). \end{cases}$$

The result follows applying (1.20) to these equations. ■

1.2. Homogenization in perforated domains

Let us now recall some results related with the homogenization of the Poisson's equation with Dirichlet boundary conditions in perforated domains. In the whole of the present paper, assume that the sequence Ω^ε of open sets with $\Omega^\varepsilon \subset \Omega$, satisfies the following condition:

$$\text{(1.27)} \qquad \begin{cases} \exists\, z^\varepsilon, z \in H^1(\Omega),\ z \geq \rho > 0 \ (\rho \text{ constant}),\ \text{such that} \\ z^\varepsilon = 0 \text{ in } \Omega \setminus \Omega^\varepsilon,\ z^\varepsilon \rightharpoonup z \text{ in } H^1(\Omega). \end{cases}$$

The following theorem has been proved in [C2].

THEOREM 1.3. – *Assume that (1.27) holds true. Then for a subsequence of ε, that we still denote by ε, there exist a sequence of functions w^ε and a distribution μ which satisfy* ([3])

(*P1*) $w^\varepsilon \in H^1(\Omega)$

(*P2*) $w^\varepsilon = 0$ *in* $\Omega \setminus \Omega^\varepsilon$

(*P3*) $0 \leq w^\varepsilon \leq 1$

(*P4*) $w^\varepsilon \rightharpoonup 1$ *weakly in* $H^1(\Omega)$ *and strongly in* $W^{1,p}(\Omega)$, $1 \leq p < 2$

(*P5*) $\mu \in \mathcal{M}_b^0(\Omega)$

([3]) Here and in what follows, $\mathcal{M}_b^0(\Omega)$ denotes the set of bounded positive Borel measures which vanish on the sets of zero capacity (see Apendix: Notation)

$$(P6)\quad \begin{cases} \forall\, u, \varphi \in H^1(\Omega) \cap L^\infty(\Omega), \\ \forall\, v^\varepsilon \in H_0^1(\Omega^\varepsilon) \text{ and } \forall\, v \in H_0^1(\Omega) \text{ such that } v^\varepsilon \rightharpoonup v \text{ in } H_0^1(\Omega), \\ \text{we have} \\ v \in L^2(\Omega, d\mu) \text{ and } \displaystyle\int_\Omega \varphi \nabla (w^\varepsilon u) \nabla v^\varepsilon \to \int_\Omega \varphi \nabla u \nabla v + \int_\Omega uv\varphi\, d\mu. \end{cases}$$

$$(P7)\quad \begin{cases} \forall\, \varphi \in H^1(\Omega) \cap L^\infty(\Omega), \\ \forall\, v^\varepsilon \in H^1(\Omega),\ v^\varepsilon = 0 \text{ in } \Omega \setminus \Omega^\varepsilon, \text{ such that } v^\varepsilon \rightharpoonup 0 \text{ in } H^1(\Omega), \\ \text{we have} \\ \displaystyle\int_\Omega \varphi \nabla w^\varepsilon \nabla v^\varepsilon \to 0. \end{cases}$$

The following results follow easily from these properties of w^ε and μ.

COROLLARY 1.1. – *For every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have*

$$\int_\Omega |\nabla w^\varepsilon|^2 \varphi \to \int_\Omega \varphi\, d\mu. \tag{1.28}$$

Proof. – Use $u = 1$, $\varphi = 1$ and $v^\varepsilon = w^\varepsilon \varphi$ in (P6). ■

COROLLARY 1.2. – *Consider a sequence ψ^ε such that*

$$\psi^\varepsilon \in L^\infty(\Omega), \quad \| \psi^\varepsilon \|_{L^\infty(\Omega)} \leq C. \tag{1.29}$$

Then for any $u \in H^1(\Omega) \cap L^\infty(\Omega)$ we have [4]

$$\int_\Omega |\nabla (w^\varepsilon u - u)|^2 \psi^\varepsilon = \int_\Omega |\nabla w^\varepsilon|^2 u^2 \psi^\varepsilon + O_\varepsilon. \tag{1.30}$$

If ψ^ε also belongs to $H^1(\Omega)$ and converges almost everywhere to zero and if $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$, we have:

$$\int_\Omega |\nabla w^\varepsilon|^2 \psi^\varepsilon \varphi \leq \left(\int_\Omega |\nabla w^\varepsilon|^2 \varphi\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla \psi^\varepsilon|^2 \varphi\right)^{\frac{1}{2}} + O_\varepsilon. \tag{1.31}$$

Proof. – To obtain (1.30) use the fact that $\nabla (w^\varepsilon u - u) = u \nabla w^\varepsilon + (w^\varepsilon - 1) \nabla u$ and then (P4) and (1.29).

To obtain (1.31) use $v^\varepsilon = w^\varepsilon \psi^\varepsilon$ in (P7) and then (P3) and Cauchy-Schwarz's inequality. ■

REMARK 1.4. – The properties of the sequence of functions w^ε and the distribution μ are very close to the hypotheses imposed in [C M] (see also [K M]) for the study of the homogenization problem:

$$\begin{cases} -\Delta u^\varepsilon = f \text{ in } \mathcal{D}'(\Omega^\varepsilon) \\ u^\varepsilon \in H_0^1(\Omega^\varepsilon). \end{cases} \tag{1.32}$$

---

[4] Here and in what follows O_ε denotes a sequence of real numbers which converges to zero and which can change from a line to another (see Appendix: Notation)

For this problem, D. Cioranescu and F. Murat ([C M]) have shown that under their hypotheses the sequence u^ε converges weakly in $H_0^1(\Omega)$ to the unique solution u of the problem

$$\begin{cases} -\Delta u + u\mu = f & \text{in } \mathcal{D}'(\Omega) \\ u \in H_0^1(\Omega). \end{cases} \tag{1.33}$$

It is also known (see [C M] and [C2]) that when $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ one has for the linear problem (1.32) the following corrector result:

$$u^\varepsilon - w^\varepsilon u \to 0 \quad \text{in } H_0^1(\Omega). \tag{1.34}$$

It is proved however in [C3] by means of an example, that the corrector result (1.34) is no more true in general when u^ε is the solution of the quasi-linear problem (0.1). In spite of this, the main idea of the present paper will be to make a comparison between the gradient of u^ε, solution of (0.1), and the gradient of $w^\varepsilon u$; this will provides us with some estimates for the gradient of u^ε which are similar of the properties of the gradient of w^ε.

In constrast with the work of D. Cioranescu and F. Murat the linear problem (1.32) is solved without any hypothesis about the sequence Ω^ε in [DM M1], [DM M2], [DM G]. For what concerns the nonlinear problem (0.1), it is possible to eliminate hypothesis (1.27) about Ω^ε at the expense of replacing in what follows the sequence $w^\varepsilon u$ by the corrector given in [DM G] (see also [DM Mu1], [DM Mu2]) for the linear problem (1.32). This will however make the exposition of the quasi-linear problem much more tedious. We have therefore prefered to remain in the more restictive case in which hypothesis (1.27) is assumed. To see how it is possible to extend the results given in the present paper to the case where no hypothesis is made on the domains Ω^ε, the reader is referred to [C G] where this task is carried out for monotone systems in which the corresponding estimates are easier to obtain than for the problem (0.1).

From now on we will assume that (1.27) holds true or more exactly that the properties (P1) to (P7) of Theorem 1.3 hold.

2. Estimates on the gradients of the solutions and first results on the homogenization problem

In this Section we obtain some estimates on ∇u^ε when u^ε is the solution of (0.1). As a consequence, we obtain a first representation for the limit problem of (0.1).

We will actually consider a problem which is more general than (0.1): more pecisely, we consider the case where the right-hand sides are a sequence of distributions f^ε which satisfies

$$\begin{cases} f^\varepsilon \in H^{-1}(\Omega^\varepsilon) + L^1(\Omega^\varepsilon),\ f \in H^{-1}(\Omega) + L^1(\Omega) \text{ such that} \\ \text{for any sequence } v^\varepsilon \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon) \text{ such that} \\ v^\varepsilon \rightharpoonup v \text{ in } H_0^1(\Omega) \text{ weak and in } L^\infty(\Omega) \text{ weak-}*, \\ \text{we have } \langle f^\varepsilon, v^\varepsilon \rangle_{\Omega^\varepsilon} \to \langle f, v \rangle_\Omega \end{cases} \tag{2.1}$$

and such that the following equation holds

$$-\Delta u^\varepsilon + H(x, u^\varepsilon, \nabla u^\varepsilon) = f^\varepsilon \quad \text{in } \mathcal{D}'(\Omega^\varepsilon) \tag{2.2}$$

Let us thus usually to consider f^ε, f, u^ε and u such that:

$$\begin{cases} f^\varepsilon \text{ and } f \text{ satisfy (2.1), } u^\varepsilon \text{ and } f^\varepsilon \text{ satisfy (2.2),} \\ u^\varepsilon \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon), \; u \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \| u^\varepsilon \|_{L^\infty(\Omega^\varepsilon)} \leq M, \\ u^\varepsilon \rightharpoonup u \text{ in } H_0^1(\Omega) \text{ weak.} \end{cases} \tag{2.3}$$

REMARK 2.1. – Consider the solution u^ε of problem (0.1) for f given in $L^\infty(\Omega)$. Applying Theorem 1.1 implies that u^ε is bounded in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Thus extracting a subsequence such that u^ε converges to some u, and setting $f^\varepsilon = f$, we deduce that f^ε, f, u^ε and u satisfy (2.3).

This provides an example which proves that the set of the f^ε, f, u^ε and u which satisfy (2.3) is not empty (once a subsequence has been extracted). We will prove most of the results of the present paper in the framework (2.3), which has the advantage of to avoid the extraction of a subsequence, since u^ε is already assumed to converge to some u.

2.1. Strong $W_0^{1,p}(\Omega)$ ($p < 2$) convergence

Our first result states the pointwise convergence of the gradient of the sequence u^ε, in the spirit of Boccardo-Murat [Bo M].

THEOREM 2.1. – *Assume that H satisfies (1.2) and (1.9). Consider f^ε, f, u^ε and u which satisfy (2.3). Then the sequence u^ε converges strongly to u in $W^{1,p}(\Omega)$, for any p with $1 \leq p < 2$.*

Proof. – The sequence $z^\varepsilon = u^\varepsilon - w^\varepsilon u$ converges to zero in measure, so by the Egorov's theorem, there exists a sequence $z^{\varepsilon'}$ which converges to zero almost uniformly, *i.e.*, for every $\delta > 0$ there exists a set A_δ with $|\Omega \setminus A_\delta| < \delta$ such that $z^{\varepsilon'}$ converges uniformly to zero in A_δ.

Given $\rho > 0$, we take $T_\rho(z^{\varepsilon'}) \in H_0^1(\Omega^{\varepsilon'}) \cap L^\infty(\Omega^{\varepsilon'})$ as test function in (2.2), where $T_\rho : \mathbb{R} \mapsto \mathbb{R}$ is the truncation defined by

$$T_\rho(s) = \begin{cases} \rho & \text{if } s \geq \rho \\ s & \text{if } -\rho \leq s \leq \rho \\ -\rho & \text{if } s \leq -\rho. \end{cases}$$

We obtain

$$\int_\Omega \nabla u^{\varepsilon'} \nabla T_\rho(z^{\varepsilon'}) + \int_\Omega H(x, u^{\varepsilon'}, \nabla u^{\varepsilon'}) T_\rho(z^{\varepsilon'}) = \langle f^{\varepsilon'}, T_\rho(z^{\varepsilon'}) \rangle_{\Omega^\varepsilon}.$$

Note that the integrals are written on the whole of Ω.

Using (2.1) and the fact that $H(x, u^{\varepsilon'}, \nabla u^{\varepsilon'})$ is bounded in $L^1(\Omega)$ independently of ε', we conclude to the existence of a constant $C > 0$ such that

$$\int_\Omega |\nabla z^{\varepsilon'}|^2 \chi_{\{|z^{\varepsilon'}|<\rho\}} \leq O_\varepsilon + C\rho - \int_\Omega \nabla (w^{\varepsilon'} u) \nabla T_\rho(z^{\varepsilon'}).$$

By (P6) the last term converges to zero. Since for ε' small enough we have $A_\delta \subset \{x \in \Omega : |z^{\varepsilon'}(x)| < \rho\}$, we deduce that

$$\limsup_{\varepsilon' \to 0} \int_{A_\delta} |\nabla z^{\varepsilon'}|^2 \leq C\rho.$$

Since ρ is arbitrary, this implies that

$$\lim_{\varepsilon' \to 0} \int_{A_\delta} |\nabla z^{\varepsilon'}|^2 = 0. \tag{2.4}$$

Writing

$$\int_\Omega |\nabla z^{\varepsilon'}|^p = \int_{\Omega \setminus A_\delta} |\nabla z^{\varepsilon'}|^p + \int_{A_\delta} |\nabla z^{\varepsilon'}|^p$$
$$\leq \| \nabla z^{\varepsilon'} \|^p_{L^2(\Omega)} \delta^{1-\frac{p}{2}} + \| \nabla z^{\varepsilon'} \|^p_{L^2(A_\delta)} |\Omega|^{1-\frac{p}{2}},$$

we obtain that for any p such that $1 \leq p < 2$, $z^{\varepsilon'}$ converges strongly to zero in $W_0^{1,p}(\Omega)$. Since $w^{\varepsilon'} u$ converges strongly to u in $W_0^{1,p}(\Omega)$ by (P4), we have that $u^{\varepsilon'}$ converges strongly to u in $W^{1,p}(\Omega)$. Finally, since the above reasoning holds not only for u^ε but for any subsequence of u^ε, Theorem 2.1 is proved. ■

2.2. First estimates on ∇u^ε

The following lemma extends in some sense the resuts of Corollary 1.2 to the case where $w^\varepsilon u$ is replaced by u^ε, the solution of (0.1).

LEMMA 2.1. – *Assume that H satisfies (1.2) and (1.9). Consider f^ε, f, u^ε and u which satisfy (2.3), a function $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \geq 0$ and a sequence of functions ψ^ε such that*

$$\begin{cases} \varphi \in H^1(\Omega) \cap L^\infty(\Omega), \quad \varphi \geq 0, \\ \psi^\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega), \quad \psi^\varepsilon \geq 0, \\ \| \psi^\varepsilon \|_{H^1(\Omega)} + \| \psi^\varepsilon \|_{L^\infty(\Omega)} \leq C. \end{cases}$$

Then there exists a sequence of functions ρ^ε which satisfies

$$\begin{cases} \rho^\varepsilon \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon), \ \rho^\varepsilon \geq 0 \\ \rho^\varepsilon \rightharpoonup 0 \ \text{ in } H_0^1(\Omega) \text{ weak}, \ \| \rho^\varepsilon \|_{L^\infty(\Omega)} \leq C_M \end{cases} \tag{2.5}$$

such that

$$\begin{cases} \displaystyle\int_\Omega |\nabla (u^\varepsilon - u)|^2 \psi^\varepsilon \varphi \leq C_M \int_\Omega |\nabla w^\varepsilon|^2 \psi^\varepsilon \varphi \\ \displaystyle\qquad\qquad + \left(\int_\Omega |\nabla w^\varepsilon|^2 \varphi\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla \psi^\varepsilon|^2 \rho^\varepsilon \varphi\right)^{\frac{1}{2}} + O_\varepsilon. \end{cases} \tag{2.6}$$

Proof. – Applying estimate (1.7) to equation (2.2) with $\Theta = \Omega^\varepsilon$, $u = u^\varepsilon$, $r = w^\varepsilon u$, $\varphi = \psi^\varepsilon \varphi$ and setting

$$z^\varepsilon = u^\varepsilon - w^\varepsilon u,$$

we obtain

(2.7)
$$\begin{cases} \displaystyle\int_{\Omega^\varepsilon} |\nabla z^\varepsilon|^2 \psi^\varepsilon \varphi \le \langle f^\varepsilon, h(z^\varepsilon)\psi^\varepsilon\varphi\rangle_{\Omega^\varepsilon} - \int_{\Omega^\varepsilon} \psi^\varepsilon \varphi \nabla (w^\varepsilon u) \nabla h(z^\varepsilon) \\ \displaystyle - \int_{\Omega^\varepsilon} h(z^\varepsilon) \nabla u^\varepsilon \nabla (\psi^\varepsilon \varphi) + C_M \int_{\Omega^\varepsilon} |h(z^\varepsilon)| \psi^\varepsilon \varphi + C_M \int_{\Omega^\varepsilon} |\nabla (w^\varepsilon u)|^2 |h(z^\varepsilon)| \psi^\varepsilon \varphi. \end{cases}$$

Let us now estimate the various terms of the right-hand side of (2.7):

By (2.1) we have $\langle f^\varepsilon, h(z^\varepsilon)\psi^\varepsilon\varphi\rangle_{\Omega^\varepsilon} = O_\varepsilon$ since $h(z^\varepsilon) \rightharpoonup 0$ in $H^1(\Omega)$ weak.

Applying (P6) to $v^\varepsilon = u h(z^\varepsilon)\psi^\varepsilon\varphi$ which belongs to $H_0^1(\Omega^\varepsilon)$ and tends weakly to zero in $H_0^1(\Omega)$, and using $\| u \|_{L^\infty(\Omega)} \le M$, we obtain for the second term

$$\begin{aligned} &- \int_{\Omega^\varepsilon} \psi^\varepsilon \varphi \nabla (w^\varepsilon u) \nabla h(z^\varepsilon) = - \int_\Omega \psi^\varepsilon \varphi (w^\varepsilon \nabla u + u \nabla w^\varepsilon) \nabla h(z^\varepsilon) \\ &= O_\varepsilon - \int_\Omega \psi^\varepsilon \varphi u \nabla w^\varepsilon \nabla h(z^\varepsilon) \\ &= O_\varepsilon - \int_\Omega \nabla w^\varepsilon \nabla (\psi^\varepsilon \varphi u h(z^\varepsilon)) + \int_\Omega \psi^\varepsilon \varphi h(z^\varepsilon) \nabla w^\varepsilon \nabla u \\ &+ \int_\Omega \varphi u h(z^\varepsilon) \nabla w^\varepsilon \nabla \psi^\varepsilon + \int_\Omega \psi^\varepsilon u h(z^\varepsilon) \nabla w^\varepsilon \nabla \varphi \\ &= \int_\Omega \varphi u h(z^\varepsilon) \nabla w^\varepsilon \nabla \psi^\varepsilon + O_\varepsilon \\ &\le C_M \Big(\int_\Omega |\nabla w^\varepsilon|^2 \varphi \Big)^{\frac12} \Big(\int_\Omega |\nabla \psi^\varepsilon|^2 |h(z^\varepsilon)|^2 \varphi \Big)^{\frac12} + O_\varepsilon. \end{aligned}$$

For what concerns the third term of the right-hand side of (2.7) we write

$$\begin{aligned} - \int_{\Omega^\varepsilon} h(z^\varepsilon) \nabla u^\varepsilon \nabla (\psi^\varepsilon \varphi) &= - \int_\Omega h(z^\varepsilon) \nabla (u^\varepsilon - u) \nabla (\psi^\varepsilon \varphi) + O_\varepsilon \\ &= - \int_\Omega h(z^\varepsilon) \varphi \nabla (u^\varepsilon - u) \nabla \psi^\varepsilon + O_\varepsilon \\ &\le \Big(\int_\Omega |\nabla (u^\varepsilon - u)|^2 \varphi \Big)^{\frac12} \Big(\int_\Omega |\nabla \psi^\varepsilon|^2 |h(z^\varepsilon)|^2 \varphi \Big)^{\frac12} + O_\varepsilon. \end{aligned}$$

The fourth term is

$$C_M \int_{\Omega^\varepsilon} |h(z^\varepsilon)| \psi^\varepsilon \varphi = O_\varepsilon.$$

For the fifth term, we have:

$$\begin{aligned} &C_M \int_{\Omega^\varepsilon} |\nabla (w^\varepsilon u)|^2 |h(z^\varepsilon)| \psi^\varepsilon \varphi \le C_M \int_\Omega \big[|\nabla w^\varepsilon|^2 u^2 + |w^\varepsilon|^2 |\nabla u|^2 \big] |h(z^\varepsilon)| \psi^\varepsilon \varphi \\ &= C_M \int_\Omega |\nabla w^\varepsilon|^2 u^2 |h(z^\varepsilon)| \psi^\varepsilon \varphi + O_\varepsilon \le C_M \int_\Omega |\nabla w^\varepsilon|^2 \psi^\varepsilon \varphi + O_\varepsilon. \end{aligned}$$

Taking into account the estimates obtained for the right-hand side of (2.7) we have proved that:

(2.8)
$$\begin{cases} \displaystyle\int_\Omega |\nabla (u^\varepsilon - w^\varepsilon u)|^2 \psi^\varepsilon \varphi \leq C_M \int_\Omega |\nabla w^\varepsilon|^2 \psi^\varepsilon \varphi + O_\varepsilon \\ \displaystyle + C_M \left(\left(\int_\Omega |\nabla w^\varepsilon|^2 \varphi \right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla (u^\varepsilon - u)|^2 \varphi \right)^{\frac{1}{2}} \right) \left(\int_\Omega |\nabla \psi^\varepsilon|^2 |h(z^\varepsilon)|^2 \varphi \right)^{\frac{1}{2}}. \end{cases}$$

Using (1.30) with $\psi^\varepsilon = \psi^\varepsilon \varphi$ and the fact that $\| u \|_{L^\infty(\Omega)} \leq M$, we have

(2.9)
$$\begin{cases} \displaystyle\int_\Omega |\nabla (u^\varepsilon - u)|^2 \psi^\varepsilon \varphi \leq 2 \int_\Omega |\nabla (u^\varepsilon - w^\varepsilon u)|^2 \psi^\varepsilon \varphi + 2 \int_\Omega |\nabla (w^\varepsilon u - u)|^2 \psi^\varepsilon \varphi \\ \displaystyle \leq C_M \left(\left(\int_\Omega |\nabla w^\varepsilon|^2 \varphi \right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla (u^\varepsilon - u)|^2 \varphi \right)^{\frac{1}{2}} \right) \left(\int_\Omega |\nabla \psi^\varepsilon|^2 |h(z^\varepsilon)|^2 \varphi \right)^{\frac{1}{2}} \\ \displaystyle + C_M \int_\Omega |\nabla w^\varepsilon|^2 \psi^\varepsilon \varphi + O_\varepsilon. \end{cases}$$

Taking in (2.9) $\psi^\varepsilon = 1$ (which is licit), we obtain

$$\int_\Omega |\nabla (u^\varepsilon - u)|^2 \varphi \leq C_M \int_\Omega |\nabla w^\varepsilon|^2 \varphi + O_\varepsilon,$$

which substituted in (2.9), implies (2.6) with $\rho^\varepsilon = C_M |h(z^\varepsilon)|^2$. ■

COROLLARY 2.1. – *Assume that H satisfies (1.2) and (1.9). Consider f^ε, f, u^ε and u which satisfy (2.3), as well as $\psi^\varepsilon, \psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, with $\psi^\varepsilon \geq 0$, ψ^ε bounded in $L^\infty(\Omega)$ and ψ^ε converging strongly in $H_0^1(\Omega)$ to ψ. Then*

$$\limsup_{\varepsilon \to 0} \int_\Omega |\nabla (u^\varepsilon - u)|^2 \psi^\varepsilon \leq C_M \int_\Omega \psi \, d\mu. \tag{2.10}$$

Proof. – Take $\varphi = 1$ in (2.6) and observe that $\int_\Omega |\nabla w^\varepsilon|^2 \psi^\varepsilon \to \int_\Omega \psi \, d\mu$ by (P6) with $v^\varepsilon = w^\varepsilon \psi^\varepsilon$ and $u = \varphi = 1$. ■

COROLLARY 2.2. – *Assume that H satisfies (1.2) and (1.9). Consider f^ε, f, u^ε and u which satisfy (2.3) and let ψ^ε and φ be such that*

$$\begin{cases} \varphi \in H^1(\Omega) \cap L^\infty(\Omega), \ \varphi \geq 0, \\ \psi^\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega), \ \psi^\varepsilon \geq 0, \\ \psi^\varepsilon \rightharpoonup 0 \ \textit{in } H_0^1(\Omega) \cap L^\infty(\Omega) \ \textit{weak-}* . \end{cases}$$

Then we have

$$\int_\Omega |\nabla (u^\varepsilon - u)|^2 \psi^\varepsilon \varphi \leq C_M \left(\int_\Omega |\nabla w^\varepsilon|^2 \varphi \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla \psi^\varepsilon|^2 \varphi \right)^{\frac{1}{2}} + O_\varepsilon, \tag{2.11}$$

$$\int_\Omega |\nabla u^\varepsilon|^2 \psi^\varepsilon \varphi \leq C_M \left(\int_\Omega |\nabla w^\varepsilon|^2 \varphi \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla \psi^\varepsilon|^2 \varphi \right)^{\frac{1}{2}} + O_\varepsilon. \tag{2.12}$$

Proof. – Inequality (2.11) follows from (2.6), (1.31) and $\| \rho_\varepsilon \|_{L^\infty(\Omega)} \leq C_M$, while inequality (2.12) is deduced from (2.11) and from $|\nabla u^\varepsilon|^2 \leq 2|\nabla (u^\varepsilon - u)|^2 + 2|\nabla u|^2$. ■

2.3. Structure of the limit of (0.1)

The estimates above obtained allow us to give a first result about the structure of the problem obtained by passing to the limit in $(0, 1)$.

THEOREM 2.2. – *Assume that H satisfies (1.2) and (1.9). Consider f^ε, f, u^ε and u which satisfy (2.3). Then, there exists a function $E \in L^\infty(\Omega, d\mu)$ with $\| E \|_{L^\infty(\Omega,d\mu)} \leq C_M$, such that u is a solution of the problem:*

$$(2.13) \qquad \begin{cases} -\Delta u + E\mu + H(x, u, \nabla u) = f \text{ in } \mathcal{D}'(\Omega), \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases}$$

or equivalently (5):

$$(2.14) \qquad \begin{cases} u \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \int_\Omega \nabla u \nabla z + \int_\Omega Ez\, d\mu + \int_\Omega H(x, u, \nabla u)z = \langle f, z\rangle_\Omega, \\ \forall z \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases}$$

The function E is defined by:

(2.15)

$$\int_\Omega E\varphi\, d\mu = \lim_{\varepsilon\to 0} \int_\Omega H(x, u^\varepsilon, \nabla u^\varepsilon)w^\varepsilon\varphi - \int_\Omega H(x, u, \nabla u)\varphi + \int_\Omega u\varphi\, d\mu, \quad \forall \varphi \in \mathcal{D}(\Omega)$$

i.e., *since $H(x, u^\varepsilon, \nabla u^\varepsilon)w^\varepsilon$ is bounded in $L^1(\Omega)$,*

$$E\mu - u\mu + H(x, u, \nabla u) = \lim_{\varepsilon\to 0} H(x, u^\varepsilon, \nabla u^\varepsilon)w^\varepsilon \text{ in } \mathcal{M}_b(\Omega) \text{ weak-}* .$$

Proof. – For $\varphi \in \mathcal{D}(\Omega)$ we use (as for the linear case, see [C M]) $w^\varepsilon\varphi \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$ as test function in (2.2). We obtain (note that the integrals can be written on the whole of Ω)

$$\int_\Omega \nabla u^\varepsilon \nabla (w^\varepsilon\varphi) + \int_\Omega H(x, u^\varepsilon, \nabla u^\varepsilon)w^\varepsilon\varphi = \langle f^\varepsilon, w^\varepsilon\varphi\rangle_{\Omega^\varepsilon}.$$

Using (P6) and (2.1) we deduce that, as $\varepsilon \to 0$

$$(2.16) \qquad \int_\Omega H(x, u^\varepsilon, \nabla u^\varepsilon)w^\varepsilon\varphi \to \langle f, \varphi\rangle_\Omega - \int_\Omega \nabla u \nabla \varphi - \int_\Omega u\varphi\, d\mu, \ \forall \varphi \in \mathcal{D}(\Omega).$$

Since $H(x, u^\varepsilon, \nabla u^\varepsilon)w^\varepsilon$ is bounded in $L^1(\Omega)$, we deduce from (2.16) that:

$$H(x, u^\varepsilon, \nabla u^\varepsilon)w^\varepsilon \rightharpoonup \nu = f + \Delta u - u\mu \text{ in } \mathcal{M}_b(\Omega) \text{ weak-}* .$$

(5) Recall that $H_0^1(\Omega) \cap L^\infty(\Omega) \subset L^\infty(\Omega, d\mu)$ (*see* Appendix: Notation)

On the other hand, using (1.9) and $0 \leq w^\varepsilon \leq 1$, we have for every function $\varphi \in \mathcal{D}(\Omega)$,

$$\left|\int_\Omega [H(x,u^\varepsilon,\nabla u^\varepsilon)w^\varepsilon\varphi - H(x,u,\nabla u)w^\varepsilon\varphi]\right|$$
$$\leq C_M\left(\int_\Omega (1+|\nabla u^\varepsilon|^2+|\nabla u|^2)|u^\varepsilon-u||\varphi| + \int_\Omega (1+|\nabla u^\varepsilon|+|\nabla u|)|\nabla(u^\varepsilon-u)||\varphi|\right).$$

Using that $u^\varepsilon - u$ converges almost everywhere to zero, that $\| u^\varepsilon - u \|_{L^\infty(\Omega)} \leq 2M$, that

$$(2.17) \qquad \begin{cases} |\nabla u^\varepsilon|^2 \leq 2|\nabla(u^\varepsilon-u)|^2 + 2|\nabla u|^2, \\ |\nabla u^\varepsilon| \leq |\nabla(u^\varepsilon-u)| + |\nabla u| \end{cases}$$

and that

$$(2.18) \qquad |\nabla(u^\varepsilon-u)| \rightharpoonup 0 \ \text{ in } L^2(\Omega),$$

which is deduced from Theorem 2.1, we get:

$$\left|\int_\Omega [H(x,u^\varepsilon,\nabla u^\varepsilon)w^\varepsilon\varphi - H(x,u,\nabla u)w^\varepsilon\varphi]\right| \leq C_M\int_\Omega |\nabla(u^\varepsilon-u)|^2|\varphi| + O_\varepsilon.$$

Passing to the limit in this expression and using (2.10) with $\psi^\varepsilon = |\varphi|$, we deduce

$$\left|\int_\Omega \varphi\, d\nu - \int_\Omega H(x,u,\nabla u)\varphi\right| \leq C_M\int_\Omega |\varphi|\, d\mu, \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

By the Radon-Nikodym's theorem, we deduce that there exists a function $E' \in L^\infty(\Omega, d\mu)$ with $\| E' \|_{L^\infty(\Omega,d\mu)} \leq C_M$ such that

$$\int_\Omega E'\varphi\, d\mu = \int_\Omega \varphi\, d\nu - \int_\Omega H(x,u,\nabla u)\varphi$$
$$= \langle f,\varphi\rangle_\Omega - \int_\Omega \nabla u\nabla\varphi - \int_\Omega u\varphi\, d\mu - \int_\Omega H(x,u,\nabla u)\varphi, \quad \forall\, \varphi \in \mathcal{D}(\Omega)$$

which implies that E defined by $E = E' + u$ satisfies

$$E\mu = \nu - H(x,u,\nabla u) + u\mu = f + \Delta u - H(x,u,\nabla u).$$

This proves (2.13) and (2.15). The equivalence between (2.13) and (2.14) follows from a result of J. Deny ([D], see also [Z]) which implies that:

$$(2.19) \qquad \begin{cases} \text{when } \mu \in \mathcal{M}_b^0(\Omega),\ E \in L^\infty(\Omega,d\mu),\ z \in H_0^1(\Omega)\cap L^\infty(\Omega); \\ \text{then } \langle E\mu, z\rangle_\Omega = \displaystyle\int_\Omega Ez\, d\mu. \quad \blacksquare \end{cases}$$

3. Comparison of the gradients of two sequences of solutions

This section is devoted to the proof of the following Lemma, which shows that when u^ε and v^ε are the solutions of two problems (0.1) with right-hand sides f and g, which weakly converge to u and v, then $\| u^\varepsilon - v^\varepsilon - u + v \|_{H_0^1(\Omega)}$ can be estimated by $\| u - v \|_{L^1(\Omega, d\mu)}$.

LEMMA 3.1. – *Assume that H satisfies (1.2) and (1.9). Consider f^ε, f, u^ε and u, and g^ε, g, v^ε and v which respectively satisfy (2.3). Define:*

$$\tau^\varepsilon = u^\varepsilon - v^\varepsilon - u + v.$$

Then, for any function $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \geq 0$, we have (6)

$$\int_\Omega |\nabla \tau^\varepsilon|^2 \varphi \leq C_M \Big(\int_\Omega |\nabla w^\varepsilon|^2 \varphi \Big)^{1-\frac{1}{\lambda_M}} \Big(\int_\Omega |\nabla w^\varepsilon|^2 |u-v| \varphi \, d\mu \Big)^{\frac{1}{\lambda_M}} + O_\varepsilon. \tag{3.1}$$

Proof. – It will be performed in eight steps.

STEP 1. – In view of Lemma 1.2, there exist two functions ψ and $\vartheta = \psi^{-1}$ given by (1.11) and (1.12) such that denoting

$$\begin{cases} \hat{u}^\varepsilon = \vartheta(u^\varepsilon), & \hat{v}^\varepsilon = \vartheta(v^\varepsilon), \\ \hat{u} = \vartheta(u), & \hat{v} = \vartheta(v), \\ \hat{f}^\varepsilon = \dfrac{f^\varepsilon}{\psi'(u^\varepsilon)}, & \hat{g}^\varepsilon = \dfrac{g^\varepsilon}{\psi'(v^\varepsilon)}, \end{cases} \tag{3.2}$$

we have

$$\begin{cases} -\Delta \hat{u}^\varepsilon + B(x, \hat{u}^\varepsilon, \nabla \hat{u}^\varepsilon) = \hat{f}^\varepsilon & \text{in } \mathcal{D}'(\Omega^\varepsilon), \\ -\Delta \hat{v}^\varepsilon + B(x, \hat{v}^\varepsilon, \nabla \hat{v}^\varepsilon) = \hat{g}^\varepsilon & \text{in } \mathcal{D}'(\Omega^\varepsilon), \end{cases} \tag{3.3}$$

where the function B satisfies (1.14) and (1.15).

STEP 2. We have (7)

$$|\hat{u} - \hat{v}| \leq C_M |u - v|, \tag{3.4}$$

$$\begin{cases} |\nabla \hat{u}^\varepsilon| \leq C_M |\nabla u^\varepsilon| \\ |\nabla \hat{v}^\varepsilon| \leq C_M |\nabla v^\varepsilon|, \end{cases} \tag{3.5}$$

$$\begin{cases} |\nabla (\hat{u}^\varepsilon - \hat{u})| \leq C_M |\nabla (u^\varepsilon - u)| + O_\varepsilon^{L^2} \\ |\nabla (\hat{v}^\varepsilon - \hat{v})| \leq C_M |\nabla (v^\varepsilon - v)| + O_\varepsilon^{L^2}, \end{cases} \tag{3.6}$$

$$|\nabla \tau^\varepsilon|^2 \leq C_M (|\nabla \hat{\tau}^\varepsilon|^2 + |\nabla (v^\varepsilon - v)|^2 |\hat{u}^\varepsilon - \hat{v}^\varepsilon|^2) + O_\varepsilon^{L^1}, \tag{3.7}$$

(6) Here and in what follows λ_M denotes a generic constant with $\lambda_M \geq 0$, which can change from a line to another and which is increasing with respect to M (see Appendix: Notation)

(7) Here and in what follows O_ε^X denotes a sequence which converges to zero in X and which can change from a line to another (see Appendix: Notation)

where analogously to τ^ε, $\hat{\tau}^\varepsilon$ denotes

$$\hat{\tau}^\varepsilon = \hat{u}^\varepsilon - \hat{v}^\varepsilon - \hat{u} + \hat{v}.$$

Proof. – Inequality (3.4) is clear since ϑ is locally Lipschitz-continuous.

The equality $\nabla \hat{u}^\varepsilon = \vartheta'(u^\varepsilon) \nabla u^\varepsilon$ implies $|\nabla \hat{u}^\varepsilon| \leq C_M |\nabla u^\varepsilon|$ and analogously, we have $|\nabla \hat{v}^\varepsilon| \leq C_M |\nabla v^\varepsilon|$.

The first inequality of (3.6) (the second one is similar) is deduced from

$$\begin{aligned}\nabla (\hat{u}^\varepsilon - \hat{u}) &= \vartheta'(u^\varepsilon) \nabla u^\varepsilon - \vartheta'(u) \nabla u \\ &= \vartheta'(u^\varepsilon) \nabla (u^\varepsilon - u) + (\vartheta'(u^\varepsilon) - \vartheta'(u)) \nabla u = \vartheta'(u^\varepsilon) \nabla (u^\varepsilon - u) + O_\varepsilon^{L^2}.\end{aligned}$$

In order to prove (3.7), we write

$$\begin{aligned}\nabla \tau^\varepsilon &= \psi'(\hat{u}^\varepsilon) \nabla \hat{u}^\varepsilon - \psi'(\hat{u}) \nabla \hat{u} - \psi'(\hat{v}^\varepsilon) \nabla \hat{v}^\varepsilon + \psi'(\hat{v}) \nabla \hat{v} \\ &= \psi'(\hat{u}^\varepsilon) \nabla (\hat{u}^\varepsilon - \hat{u}) + (\psi'(\hat{u}^\varepsilon) - \psi'(\hat{u})) \nabla \hat{u} \\ &- \psi'(\hat{v}^\varepsilon) \nabla (\hat{v}^\varepsilon - \hat{v}) - (\psi'(\hat{v}^\varepsilon) - \psi'(\hat{v})) \nabla \hat{v} \\ &= \psi'(\hat{u}^\varepsilon) \nabla \hat{\tau}^\varepsilon + (\psi'(\hat{u}^\varepsilon) - \psi'(\hat{v}^\varepsilon)) \nabla (\hat{v}^\varepsilon - \hat{v}) + O_\varepsilon^{L^2},\end{aligned}$$

which implies (3.7).

STEP 3. – Define

$$\hat{\eta}^\varepsilon = \hat{u}^\varepsilon - \hat{v}^\varepsilon - w^\varepsilon(\hat{u} - \hat{v}).$$

Then, for any function $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \geq 0$, we have

$$\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2 \varphi \leq C_M \int_\Omega |\nabla w^\varepsilon|^2 \varphi + O_\varepsilon. \tag{3.8}$$

Proof. – Write

$$\begin{aligned}|\nabla \hat{\eta}^\varepsilon|^2 \leq 4|\nabla (\hat{u}^\varepsilon - \hat{u})|^2 + 4|\nabla (\hat{v}^\varepsilon - \hat{v})|^2 \\ + 4|\nabla (w^\varepsilon \hat{u} - \hat{u})|^2 + 4|\nabla (w^\varepsilon \hat{v} - \hat{v})|^2\end{aligned}$$

and then apply (1.30) with $\psi^\varepsilon = \varphi$, (3.6) and (2.6) with $\psi^\varepsilon = 1$.

STEP 4. – For any function $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \geq 0$, we have

$$\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2 |\hat{\eta}^\varepsilon|^{n-1} \varphi \leq C_M \int_\Omega |\nabla w^\varepsilon|^2 |u - v| \varphi + O_\varepsilon, \tag{3.9}$$

where the constant n is defined in Lemma 1.2 and is increasing with respect to M.

Proof. – Applying (1.19), with $\Theta = \Omega^\varepsilon$, $\hat{u} = \hat{u}^\varepsilon$, $\hat{v} = \hat{v}^\varepsilon$, $\hat{r} = w^\varepsilon(\hat{u} - \hat{v})$ and $\hat{\omega} = \hat{\eta}^\varepsilon$ to the difference between the two equations of (3.3), (and writing the integrals on the whole of Ω) we have

$$\begin{aligned}\frac{1}{2}\int_\Omega S'(\hat{\eta}^\varepsilon)|\nabla \hat{\eta}^\varepsilon|^2\varphi &\le \langle f^\varepsilon, \frac{S(\hat{\eta}^\varepsilon)}{\psi'(u^\varepsilon)}\varphi\rangle_{\Omega^\varepsilon} - \langle g^\varepsilon, \frac{S(\hat{\eta}^\varepsilon)}{\psi'(v^\varepsilon)}\varphi\rangle_{\Omega^\varepsilon}\\ &- \int_\Omega \nabla(w^\varepsilon(\hat{u} - \hat{v}))\nabla S(\hat{\eta}^\varepsilon)\,\varphi - \int_\Omega S(\hat{\eta}^\varepsilon)\,\nabla(\hat{u}^\varepsilon - \hat{v}^\varepsilon)\nabla\varphi\\ &+ C_M \int_{\Omega^\varepsilon}\Big[(1 + |\nabla \hat{u}^\varepsilon|^2 + |\nabla \hat{v}^\varepsilon|^2)|w^\varepsilon(\hat{u} - \hat{v})|\\ &\qquad + (1 + |\nabla \hat{u}^\varepsilon| + |\nabla \hat{v}^\varepsilon|)|\nabla(w^\varepsilon(\hat{u} - \hat{v}))|\Big]|\hat{\eta}^\varepsilon|^n\varphi.\end{aligned}$$

Using in this inequality the property (2.1) of f^ε and g^ε, and taking into account (P6), (P3), (3.4), (3.5), that $\hat{\eta}^\varepsilon$ converges almost everywhere to 0 and

$$\|\hat{\eta}^\varepsilon\|_{L^\infty(\Omega)} \le C_M, \tag{3.10}$$

we have:

$$\left\{\begin{aligned}&\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2|\hat{\eta}^\varepsilon|^{n-1}\varphi \le C_M \int_\Omega (|\nabla u^\varepsilon|^2 + |\nabla v^\varepsilon|^2)|u - v||\hat{\eta}^\varepsilon|^n\varphi\\ &+ \int_\Omega (|\nabla u^\varepsilon| + |\nabla v^\varepsilon|)|\nabla(w^\varepsilon(\hat{u} - \hat{v}))||\hat{\eta}^\varepsilon|^n\varphi + O_\varepsilon.\end{aligned}\right. \tag{3.11}$$

Inequality (2.12) with $\psi^\varepsilon = |\hat{\eta}^\varepsilon|^n$, $\varphi = |u - v|\varphi$, (3.8) and (3.10) implies

$$\left\{\begin{aligned}&\int_\Omega (|\nabla u^\varepsilon|^2 + |\nabla v^\varepsilon|^2)|u - v||\hat{\eta}^\varepsilon|^n\varphi\\ &\le C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2|u - v|\varphi\Big)^{\frac{1}{2}}\Big(\int_\Omega |\hat{\eta}^\varepsilon|^{2(n-1)}|\nabla \hat{\eta}^\varepsilon|^2|u - v|\varphi\Big)^{\frac{1}{2}} + O_\varepsilon\\ &\le C_M \int_\Omega |\nabla w^\varepsilon|^2|u - v|\varphi + O_\varepsilon.\end{aligned}\right. \tag{3.12}$$

Writing

$$|\nabla u^\varepsilon| \le |\nabla(u^\varepsilon - u)| + |\nabla u|,\ |\nabla v^\varepsilon| \le |\nabla(v^\varepsilon - v)| + |\nabla v|$$

and

$$|\nabla(w^\varepsilon(\hat{u} - \hat{v}))| \le |w^\varepsilon||\nabla(\hat{u} - \hat{v})| + |\hat{u} - \hat{v}||\nabla w^\varepsilon|,$$

and taking into account (3.10) and then using (3.4), Cauchy-Schwarz's inequality and finally (2.6) with $\psi^\varepsilon = 1$ and $\varphi = |u - v|\varphi$, we obtain:

$$(3.13)\quad \begin{cases} \displaystyle\int_\Omega (|\nabla u^\varepsilon| + |\nabla v^\varepsilon|)|\nabla (w^\varepsilon(\hat{u} - \hat{v}))||\hat{\eta}^\varepsilon|^n \varphi \\ \displaystyle\le C_M \int_\Omega (|\nabla (u^\varepsilon - u)| + |\nabla (v^\varepsilon - v)|)|\nabla w^\varepsilon||\hat{u} - \hat{v}|\varphi + O_\varepsilon \\ \displaystyle\le C_M \left[\left(\int_\Omega |\nabla (u^\varepsilon - u)|^2 |u - v|\varphi\right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla (v^\varepsilon - v)|^2 |u - v|\varphi\right)^{\frac{1}{2}}\right] \\ \displaystyle\cdot \left(\int_\Omega |\nabla w^\varepsilon|^2 |u - v|\varphi\right)^{\frac{1}{2}} + O_\varepsilon \\ \displaystyle\le C_M \int_\Omega |\nabla w^\varepsilon|^2 |u - v|\varphi + O_\varepsilon. \end{cases}$$

Inequalities (3.11), (3.12) and (3.13) now give (3.9).

STEP 5. – For any function $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \ge 0$ and for any fixed $k \ge 1$, we have

$$(3.14)\quad \begin{cases} \displaystyle\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2 |\hat{\eta}^\varepsilon|^{k-1}\varphi \le C_M(k)\left[\int_\Omega |\nabla w^\varepsilon|^2 |u - v|\varphi \right. \\ \displaystyle\qquad \left. + \left(\int_\Omega |\nabla w^\varepsilon|^2 \varphi\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2 |\hat{\eta}^\varepsilon|^{2k}\varphi\right)^{\frac{1}{2}}\right] + O_\varepsilon, \end{cases}$$

where we have written $C_M(k)$ to remark that the constant depend on k.

Proof. – Let $S_k(s) = |s|^{k-1}s$. Using $S_k(\hat{\eta}^\varepsilon)\varphi$ as test function in the difference between the two equations of (3.3), we have

$$\int_{\Omega^\varepsilon} \nabla (\hat{u}^\varepsilon - \hat{v}^\varepsilon) \nabla S_k(\hat{\eta}^\varepsilon)\varphi + \int_{\Omega^\varepsilon} S_k(\hat{\eta}^\varepsilon) \nabla (\hat{u}^\varepsilon - \hat{v}^\varepsilon) \nabla \varphi$$
$$+ \int_{\Omega^\varepsilon} (B(x, \hat{u}^\varepsilon, \nabla \hat{u}^\varepsilon) - B(x, \hat{v}^\varepsilon, \nabla \hat{v}^\varepsilon)) S_k(\hat{\eta}^\varepsilon)\varphi = \langle f^\varepsilon, \frac{S_k(\hat{\eta}^\varepsilon)}{\psi'(\hat{u}^\varepsilon)}\varphi\rangle_{\Omega^\varepsilon} - \langle g^\varepsilon, \frac{S_k(\hat{\eta}^\varepsilon)}{\psi'(\hat{v}^\varepsilon)}\varphi\rangle_{\Omega^\varepsilon},$$

which using (1.14) and (2.1), implies

$$\int_\Omega S_k'(\hat{\eta}^\varepsilon)|\nabla \hat{\eta}^\varepsilon|^2 \varphi + \int_\Omega \varphi \nabla (w^\varepsilon(\hat{u} - \hat{v})) \nabla S_k(\hat{\eta}^\varepsilon)$$
$$\le C_M \int_\Omega \Big[(1 + |\nabla \hat{u}^\varepsilon|^2 + |\nabla \hat{v}^\varepsilon|^2)|\hat{u}^\varepsilon - \hat{v}^\varepsilon|$$
$$+ (1 + |\nabla \hat{u}^\varepsilon| + |\nabla \hat{v}^\varepsilon|)|\nabla (\hat{u}^\varepsilon - \hat{v}^\varepsilon)|\Big] |S_k(\hat{\eta}^\varepsilon)|\varphi + O_\varepsilon.$$

Using in this inequality, (P6) and (3.5), we have:

$$(3.15)\quad \begin{cases} \displaystyle k\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2 |\hat{\eta}^\varepsilon|^{k-1}\varphi \\ \displaystyle\le C_M \int_\Omega \Big[(|\nabla u^\varepsilon|^2 + |\nabla v^\varepsilon|^2)|\hat{u}^\varepsilon - \hat{v}^\varepsilon| \\ \displaystyle\qquad + (|\nabla u^\varepsilon| + |\nabla v^\varepsilon|)|\nabla (\hat{u}^\varepsilon - \hat{v}^\varepsilon)|\Big] |\hat{\eta}^\varepsilon|^k \varphi + O_\varepsilon. \end{cases}$$

To estimate the first term of the right-hand side of (3.15), we use the triangle inequality, (P3) and (2.12) with $\psi^\varepsilon = |\hat{\eta}^\varepsilon|^{k+1}$. Therefore we have

$$\begin{aligned}
&\int_\Omega (|\nabla u^\varepsilon|^2 + |\nabla v^\varepsilon|^2)|\hat{u}^\varepsilon - \hat{v}^\varepsilon||\hat{\eta}^\varepsilon|^k\varphi \\
&\le \int_\Omega (|\nabla u^\varepsilon|^2 + |\nabla v^\varepsilon|^2)|\hat{\eta}^\varepsilon||\hat{\eta}^\varepsilon|^k\varphi + \int_\Omega (|\nabla u^\varepsilon|^2 + |\nabla v^\varepsilon|^2)|u-v||\hat{\eta}^\varepsilon|^k\varphi \\
&\le C_M(k)\Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{\frac12}\Big(\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2|\hat{\eta}^\varepsilon|^{2k}\varphi\Big)^{\frac12} \\
&+ C_M(k)\Big(\int_\Omega |\nabla w^\varepsilon|^2|u-v|\varphi\Big)^{\frac12}\Big(\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2|\hat{\eta}^\varepsilon|^{2(k-1)}|u-v|\varphi\Big)^{\frac12} + O_\varepsilon.
\end{aligned}$$

Inequalities (3.10) and (3.8) then give

$$\begin{aligned}
&\int_\Omega (|\nabla u^\varepsilon|^2 + |\nabla v^\varepsilon|^2)|\hat{u}^\varepsilon - \hat{v}^\varepsilon||\hat{\eta}^\varepsilon|^k\varphi \\
&\le C_M(k)\left[\Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{\frac12}\Big(\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2|\hat{\eta}^\varepsilon|^{2k}\varphi\Big)^{\frac12} + \int_\Omega |\nabla w^\varepsilon|^2|u-v|\varphi\right] + O_\varepsilon.
\end{aligned}$$

For what concerns the second term of the right-hand side of (3.15) we use the triangle inequality

$$\begin{aligned}
&\int_\Omega (|\nabla u^\varepsilon| + |\nabla v^\varepsilon|)|\nabla(\hat{u}^\varepsilon - \hat{v}^\varepsilon)||\hat{\eta}^\varepsilon|^k\varphi \\
&\le \int_\Omega \big(|\nabla u^\varepsilon| + |\nabla v^\varepsilon|\big)\big(|\nabla \hat{\eta}^\varepsilon| + |\nabla(w^\varepsilon(\hat{u} - \hat{v}))|\big)|\hat{\eta}^\varepsilon|^k\varphi
\end{aligned}$$

and we estimate the two terms of the right-hand side. For the first term, we use the triangle inequality, Cauchy-Schwarz's inequality and (2.6) with $\psi^\varepsilon = 1$. This gives

$$\begin{aligned}
&\int_\Omega (|\nabla u^\varepsilon| + |\nabla v^\varepsilon|)|\nabla \hat{\eta}^\varepsilon||\hat{\eta}^\varepsilon|^k\varphi = \int_\Omega (|\nabla(u^\varepsilon - u)| + |\nabla(v^\varepsilon - v)|)|\nabla \hat{\eta}^\varepsilon||\hat{\eta}^\varepsilon|^k\varphi + O_\varepsilon \\
&\le \left[\Big(\int_\Omega |\nabla(u^\varepsilon - u)|^2\varphi\Big)^{\frac12} + \Big(\int_\Omega |\nabla(v^\varepsilon - v)|^2\varphi\Big)^{\frac12}\right]\Big(\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2|\hat{\eta}^\varepsilon|^{2k}\varphi\Big)^{\frac12} + O_\varepsilon \\
&\le C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{\frac12}\Big(\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2|\hat{\eta}^\varepsilon|^{2k}\varphi\Big)^{\frac12} + O_\varepsilon.
\end{aligned}$$

For the second term we make again the computation that we did in (3.13) with now n replaced by k. We obtain

$$\int_\Omega (|\nabla u^\varepsilon| + |\nabla v^\varepsilon|)|\nabla(w^\varepsilon(\hat{u} - \hat{v}))||\hat{\eta}^\varepsilon|^k\varphi \le C_M(k)\int_\Omega |\nabla w^\varepsilon|^2|u-v|\varphi + O_\varepsilon.$$

The estimates we obtained for each term of the right-hand side of (3.15) now give (3.14).

STEP 6. – For any function $\varphi \in H^1(\Omega)\cap L^\infty(\Omega)$, $\varphi \ge 0$, we have:

$$\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2\varphi \le C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{1-\frac{1}{\lambda_M}}\Big(\int_\Omega |\nabla w^\varepsilon|^2|u-v|\varphi\Big)^{\frac{1}{\lambda_M}} + O_\varepsilon. \tag{3.16}$$

Proof. – We claim that for any $j \geq 1$ one has

$$(3.17)\quad \begin{cases} \displaystyle\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2\varphi \leq C_M(j)\Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{1-\frac{1}{2^j-1}}\Big[\Big(\int_\Omega |\nabla w^\varepsilon|^2|u-v|\varphi\Big)^{\frac{1}{2^j-1}} \\ \displaystyle + \Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{\frac{1}{2^j}}\Big(\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2|\hat{\eta}^\varepsilon|^{2(2^j-1)}\varphi\Big)^{\frac{1}{2^j}}\Big] + O_\varepsilon, \end{cases}$$

an estimate that we now prove by induction. Indeed when $j = 1$, (3.17) is nothing but (3.14) with $k = 1$. Assume that (3.17) holds true for some j, *i.e.* that

$$X_0 \leq C_M(j)A^{1-\frac{1}{2^j-1}}[B^{\frac{1}{2^j-1}} + A^{\frac{1}{2^j}}X_j^{\frac{1}{2^j}}] + O_\varepsilon,$$

where we denote

$$A = \int_\Omega |\nabla w^\varepsilon|^2\varphi,\quad B = \int_\Omega |\nabla w^\varepsilon|^2|u-v|\varphi\quad X_j = \int_\Omega |\nabla \hat{\eta}^\varepsilon|^2|\hat{\eta}^\varepsilon|^{2(2^j-1)}\varphi.$$

Then using (3.14) and (use that $\| u - v \|_{L^\infty(\Omega)} \leq 2M$) that $B^{\frac{1}{2^j-1}} \leq C_M(j)A^{\frac{1}{2^j}}B^{\frac{1}{2^j}}$ it is easy to prove that (3.17) holds for $j + 1$.

Taking the first integer j such that $2(2^j - 1)$ is bigger than $n - 1$ (which only depends on n and then on M) and using (3.10), we have

$$(3.18)\qquad \int_\Omega |\nabla \hat{\eta}^\varepsilon|^2|\hat{\eta}^\varepsilon|^{2(2^j-1)}\varphi \leq C_M\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2|\hat{\eta}^\varepsilon|^{n-1}\varphi.$$

Inequalities (3.17), (3.18) and (3.9) now give (3.16).

STEP 7. – For any function $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \geq 0$, we have

$$(3.19)\qquad \int_\Omega |\nabla \hat{\tau}^\varepsilon|^2\varphi \leq C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{1-\frac{1}{\lambda_M}}\Big(\int_\Omega |\nabla w^\varepsilon|^2|u-v|\varphi\Big)^{\frac{1}{\lambda_M}} + O_\varepsilon,$$

Proof. – The result is easily obtained by writing

$$(3.20)\qquad \int_\Omega |\nabla \hat{\tau}^\varepsilon|^2\varphi \leq 2\int_\Omega |\nabla \hat{\eta}^\varepsilon|^2\varphi + 2\int_\Omega |\nabla\left((w^\varepsilon - 1)(\hat{u} - \hat{v})\right)|^2\varphi.$$

and then, using (3.16), $\int_\Omega |w^\varepsilon - 1|^2|\nabla(\hat{u} - \hat{v})|^2\varphi = O_\varepsilon$, (3.4) and the inequality

$$\int_\Omega |\nabla w^\varepsilon|^2|u-v|^2\varphi \leq C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{1-\frac{1}{\lambda_M}}\Big(\int_\Omega |\nabla w^\varepsilon|^2|u-v|\varphi\Big)^{\frac{1}{\lambda_M}},$$

which follows from $\| u - v \|_{L^\infty(\Omega)} \leq 2M$.

STEP 8. – Proof of (3.1).

Using (3.7), the inequality $|\hat{u}^\varepsilon - \hat{v}^\varepsilon|^2 \le C_M|\hat{u}^\varepsilon - \hat{v}^\varepsilon|$ almost everywhere in Ω and the triangle inequality, we have

(3.21)
$$\begin{cases} \displaystyle\int_\Omega |\nabla \tau^\varepsilon|^2\varphi \le C_M\Big(\int_\Omega |\nabla \hat{\tau}^\varepsilon|^2\varphi + \int_\Omega |\nabla (v^\varepsilon - v)|^2|\hat{u}^\varepsilon - \hat{v}^\varepsilon|^2\varphi\Big) + O_\varepsilon \\ \displaystyle\le C_M\Big(\int_\Omega |\nabla \hat{\tau}^\varepsilon|^2\varphi + \int_\Omega |\nabla (v^\varepsilon - v)|^2|\hat{\tau}^\varepsilon|\varphi + \int_\Omega |\nabla (v^\varepsilon - v)|^2|\hat{u} - \hat{v}|\varphi\Big) + O_\varepsilon. \end{cases}$$

Inequality (2.11) with $u^\varepsilon = v^\varepsilon$, $u = v$, $\psi^\varepsilon = |\hat{\tau}^\varepsilon|$ gives

$$\int_\Omega |\nabla (v^\varepsilon - v)|^2|\hat{\tau}^\varepsilon|\varphi \le C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla \hat{\tau}^\varepsilon|^2\varphi\Big)^{\frac{1}{2}} + O_\varepsilon. \tag{3.22}$$

Inequality (3.4) and then (2.6) with $\psi^\varepsilon = 1$, $\varphi = |u - v|\varphi$ implies

$$\int_\Omega |\nabla (u^\varepsilon - u)|^2|\hat{u} - \hat{v}|\varphi \le C_M\int_\Omega |\nabla w^\varepsilon|^2|u - v|\varphi + O_\varepsilon. \tag{3.23}$$

From (3.21), (3.19), (3.22), and (3.23), we deduce

$$\begin{cases} \displaystyle\int_\Omega |\nabla \tau^\varepsilon|^2\varphi \le C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{1-\frac{1}{\lambda_M}}\Big(\int_\Omega |\nabla w^\varepsilon|^2|u - v|\varphi\Big)^{\frac{1}{\lambda_M}} \\ \displaystyle\qquad + C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{1-\frac{1}{2\lambda_M}}\Big(\int_\Omega |\nabla w^\varepsilon|^2|u - v|\varphi\Big)^{\frac{1}{2\lambda_M}} \\ \displaystyle\qquad + C_M\int_\Omega |\nabla w^\varepsilon|^2|u - v|\varphi + O_\varepsilon. \end{cases} \tag{3.24}$$

Since $\| u - v \|_{L^\infty(\Omega)} \le 2M$ we have

$$\Big(\int_\Omega |\nabla w^\varepsilon|^2|u - v|\varphi\Big)^{\frac{1}{\lambda_M}} \le C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{\frac{1}{2\lambda_M}}\Big(\int_\Omega |\nabla w^\varepsilon|^2|u - v|\varphi\Big)^{\frac{1}{2\lambda_M}}$$
$$\int_\Omega |\nabla w^\varepsilon|^2|u - v|\varphi \le C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2\varphi\Big)^{1-\frac{1}{2\lambda_M}}\Big(\int_\Omega |\nabla w^\varepsilon|^2|u - v|\varphi\Big)^{\frac{1}{2\lambda_M}},$$

and inequality (3.24) implies (3.1). ■

4. Dependence of the function E with respect to u

Consider f^ε, f, u^ε and u, and g^ε, g, v^ε and v which respectively satisfy (2.3). By Theorem 2.2, there exist two functions E and F in $L^\infty(\Omega, d\mu)$ such that u and v satisfy

$$-\Delta u + E\mu + H(x, u, \nabla u) = f \quad \text{in } \mathcal{D}'(\Omega), \tag{4.1}$$
$$-\Delta v + F\mu + H(x, v, \nabla v) = g \quad \text{in } \mathcal{D}'(\Omega). \tag{4.2}$$

The goal of this Section is to prove estimate (4.4), which in particular implies that

$$E(x) = F(x) \;\; \mu\text{-a.e. on } \{x \in \Omega : u(x) = v(x)\}$$

and therefore that there exists a function $T : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ such that E is of the form $E = T(x, u)$. In order to obtain these results, we need an hypothesis which is stronger than (1.9). Further to (1.2), we will assume that:

i) For almost every $x \in \Omega$, $H(x, ., .)$ is continuously differentiable, and there exists an increasing function $\gamma : [0, +\infty) \mapsto [0, +\infty)$ such that for any $(s_1, \xi_1), (s_2, \xi_2) \in \mathbb{R} \times \mathbb{R}^N$ we have for $s = \max\{|s_1|, |s_2|\}$

$$(4.3) \quad \begin{cases} H(., 0, 0), \dfrac{\partial H}{\partial s}(., 0, 0), \dfrac{\partial H}{\partial \xi}(., 0, 0) \in L^\infty(\Omega), \\ \left|\dfrac{\partial H}{\partial s}(x, s_1, \xi_1) - \dfrac{\partial H}{\partial s}(x, s_2, \xi_2)\right| \\ \leq \gamma(s)\big[(1 + |\xi_1|^2 + |\xi_2|^2)|s_1 - s_2| + (1 + |\xi_1| + |\xi_2|)|\xi_1 - \xi_2|\big], \\ \left|\dfrac{\partial H}{\partial \xi}(x, s_1, \xi_1) - \dfrac{\partial H}{\partial \xi}(x, s_2, \xi_2)\right| \leq \gamma(s)\big[(1 + |\xi_1| + |\xi_2|)|s_1 - s_2| + |\xi_1 - \xi_2|\big]. \end{cases}$$

REMARK 4.1. – In other terms, H is assumed to be sufficiently smooth (two times differentiable in (s, ξ)) and such that $\frac{\partial^2 H}{\partial s^2}$ has a quadratic growth in ξ, $\frac{\partial^2 H}{\partial s \partial \xi}$ has a linear growth in ξ while $\frac{\partial^2 H}{\partial \xi^2}$ is bounded when s varies in a bounded set. A model example which satisfies all the required hipotheses is $H(x, s, \xi) = A(x, s)\xi\xi + \lambda s$, where A is a matrix which is sufficiently smooth in s and is such that $\frac{\partial A}{\partial s}(x, s) \geq 0$ in the sense of matrices.

REMARK 4.2. – Hypothesis (4.3) implies (1.9) and hence (1.3).

The goal of this Section is to prove the following Lemma:

LEMMA 4.1 – *Assume that H satisfies (1.2) and (4.3). Consider f^ε, f, u^ε and u, and g^ε, g, v^ε and v which respectively satify (2.3) and let E and F in $L^\infty(\Omega, d\mu)$ be the functions defined in Theorem 2.2, which thus satisfy (4.1) and (4.2). Then, we have*

$$(4.4) \qquad |E - F| \leq C_M |u - v|^{\frac{1}{\lambda_M}} \;\; \mu\text{-a.e. in } \Omega.$$

Proof.

STEP 1. – Let us first prove that for any function $\varphi \in \mathcal{D}(\Omega)$, we have

$$(4.5) \quad \begin{cases} \displaystyle\int_\Omega |H(x, u^\varepsilon, \nabla u^\varepsilon) - H(x, v^\varepsilon, \nabla v^\varepsilon) - H(x, v, \nabla v) + H(x, v, \nabla v)||\varphi| \\ \displaystyle\leq C_M \Big(\int_\Omega |\varphi|\, d\mu\Big)^{1 - \frac{1}{\lambda_M}} \Big(\int_\Omega |u - v||\varphi|\, d\mu\Big)^{\frac{1}{\lambda_M}} + O_\varepsilon. \end{cases}$$

For almost every $x \in \Omega$, we define the functions $h_s^\varepsilon, h_s : [0,1] \times \Omega \mapsto \mathbb{R}$ and $h_\xi^\varepsilon, h_\xi : [0,1] \times \Omega \mapsto \mathbb{R}$ by:

$$\begin{cases} h_s^\varepsilon(t,x) = \dfrac{\partial H}{\partial s}(x, tu^\varepsilon + (1-t)v^\varepsilon, t\nabla u^\varepsilon + (1-t)\nabla v^\varepsilon), \\ h_s(t,s) = \dfrac{\partial H}{\partial s}(x, tu + (1-t)v, t\nabla u + (1-t)\nabla v), \\ h_\xi^\varepsilon(t,x) = \dfrac{\partial H}{\partial \xi}(x, tu^\varepsilon + (1-t)v^\varepsilon, t\nabla u^\varepsilon + (1-t)\nabla v^\varepsilon), \\ h_\xi(t,x) = \dfrac{\partial H}{\partial \xi}(x, tu + (1-t)v, t\nabla u + (1-t)\nabla v). \end{cases}$$

By (1.9) and (4.3), there exists a constant C_M such that for almost every $x \in \Omega$, we have:

$$|h_s^\varepsilon(t,x)| \le C_M\big(1 + |\nabla u^\varepsilon|^2 + |\nabla v^\varepsilon|^2\big),$$

$$|h_\xi^\varepsilon(t,x)| \le C_M(1 + |\nabla u^\varepsilon| + |\nabla v^\varepsilon|),$$

$$\begin{aligned} |h_s^\varepsilon(t,x) - h_s(t,x)| \le{}& C_M\big[(1 + |\nabla u^\varepsilon|^2 + |\nabla v^\varepsilon|^2 + |\nabla u|^2 + |\nabla v|^2)(|u^\varepsilon - u| + |v^\varepsilon - v|)\big] \\ &+ C_M\big[(1 + |\nabla u^\varepsilon| + |\nabla v^\varepsilon| + |\nabla u| + |\nabla v|) \\ &\cdot (|\nabla(u^\varepsilon - u)| + |\nabla(v^\varepsilon - v)|)\big], \end{aligned}$$

$$\begin{aligned} |h_\xi^\varepsilon(t,x) - h_\xi(t,x)| \le{}& C_M\big[(1 + |\nabla u^\varepsilon| + |\nabla v^\varepsilon| + |\nabla u| + |\nabla v|)\big](|u^\varepsilon - u| + |v^\varepsilon - v|) \\ &+ C_M(|\nabla(u^\varepsilon - u)| + |\nabla(v^\varepsilon - v)|). \end{aligned}$$

By the previous estimates, we have

$$\begin{aligned} &|H(x,u^\varepsilon,\nabla u^\varepsilon) - H(x,v^\varepsilon,\nabla v^\varepsilon) - H(x,u,\nabla u) + H(x,v,\nabla v)| \\ &= \left| \int_0^1 \big[h_s^\varepsilon(t,x)(u^\varepsilon - v^\varepsilon) + h_\xi^\varepsilon(t,x)\nabla(u^\varepsilon - v^\varepsilon) - h_s(t,x)(u - v) - h_\xi(t,x)\nabla(u - v)\big]\,dt \right| \\ &\le \int_0^1 |h_s^\varepsilon(t,x)||\tau^\varepsilon|\,dt + \int_0^1 |h_s^\varepsilon(t,x) - h_s(t,x)||u - v|\,dt \\ &+ \int_0^1 |h_\xi^\varepsilon(t,x)||\nabla\tau^\varepsilon|\,dt + \int_0^1 |h_\xi^\varepsilon(t,x) - h_\xi(t,x)||\nabla(u - v)|\,dt, \end{aligned}$$

where as in Lemma 3.1, τ^ε denotes

$$\tau^\varepsilon = u^\varepsilon - v^\varepsilon - u + v.$$

Using (2.17) and (2.18) (applied to u^ε and v^ε) and the fact that $|\tau^\varepsilon|$ and $|\nabla\tau^\varepsilon|$ tend to zero in $L^2(\Omega)$ weak, we have for any function $\varphi \in \mathcal{D}(\Omega)$

$$\text{(4.6)} \quad \begin{cases} \displaystyle\int_\Omega |H(x,u^\varepsilon,\nabla u^\varepsilon) - H(x,v^\varepsilon,\nabla v^\varepsilon) - H(x,u,\nabla u) + H(x,v,\nabla v)||\varphi| \\ \displaystyle\le C_M \int_\Omega (|\nabla u^\varepsilon|^2 + |\nabla v^\varepsilon|^2)|\tau^\varepsilon||\varphi| \\ \displaystyle+ C_M \int_\Omega \big(|\nabla(u^\varepsilon - u)|^2 + |\nabla(v^\varepsilon - v)|^2\big)\big(1 + |u^\varepsilon - u| + |v^\varepsilon - v|\big)|u - v||\varphi| \\ \displaystyle+ C_M \int_\Omega (|\nabla(u^\varepsilon - u)| + |\nabla(v^\varepsilon - v)|)|\nabla\tau^\varepsilon||\varphi| + O_\varepsilon. \end{cases}$$

Let us estimate each integral of the right-hand side of (4.6). For the first integral, (2.12) with $\psi^\varepsilon = |\tau^\varepsilon|$, $\varphi = |\varphi|$ gives:

$$\int_\Omega (|\nabla u^\varepsilon|^2 + |\nabla v^\varepsilon|^2)|\tau^\varepsilon||\varphi| \le C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2|\varphi|\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla \tau^\varepsilon|^2|\varphi|\Big)^{\frac{1}{2}} + O_\varepsilon.$$

For the second integral, we use the fact that $1 + |u^\varepsilon - u| + |v^\varepsilon - v| \le C_M$, then (2.6) with $\psi^\varepsilon = 1$, $\varphi = |u - v||\varphi|$ to obtain

$$\begin{aligned}&\int_\Omega (|\nabla (u^\varepsilon - u)|^2 + |\nabla (v^\varepsilon - v)|^2)(1 + |u^\varepsilon - u| + |v^\varepsilon - v|)|u - v||\varphi|\\ &\le C_M \int_\Omega \big(|\nabla (u^\varepsilon - u)|^2 + |\nabla (v^\varepsilon - v)|^2\big)|u - v||\varphi|\\ &\le C_M \int_\Omega |\nabla w^\varepsilon|^2|u - v||\varphi| + O_\varepsilon.\end{aligned}$$

For the third integral, we have using Cauchy-Schwarz's inequality and then (2.6) with $\psi^\varepsilon = 1$, $\varphi = |\varphi|$

$$\begin{aligned}&\int_\Omega (|\nabla (u^\varepsilon - u)| + |\nabla (v^\varepsilon - v)|)|\nabla \tau^\varepsilon||\varphi|\\ &\le \left(\Big(\int_\Omega |\nabla (u^\varepsilon - u)|^2\varphi\Big)^{\frac{1}{2}} + \Big(\int_\Omega |\nabla (v^\varepsilon - v)|^2\varphi\Big)^{\frac{1}{2}}\right)\Big(\int_\Omega |\nabla \tau^\varepsilon|^2\varphi\Big)^{\frac{1}{2}}\\ &\le C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2|\varphi|\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla \tau^\varepsilon|^2|\varphi|\Big)^{\frac{1}{2}} + O_\varepsilon.\end{aligned}$$

These estimates of the right-hand side of (4.6) give

$$\begin{aligned}&\int_\Omega |\, H(x, u^\varepsilon, \nabla u^\varepsilon) - H(x, v^\varepsilon, \nabla v^\varepsilon) - H(x, u, \nabla u) + H(x, v, \nabla v)\,|\,|\varphi|\\ &\le C_M\Big[\int_\Omega |\nabla w^\varepsilon|^2\,|\,u - v\,|\,|\varphi| + \Big(\int_\Omega |\nabla w^\varepsilon|^2|\varphi|\Big)^{\frac{1}{2}}\Big(\int_\Omega |\,\nabla \tau^\varepsilon\,|^2\,|\varphi|\Big)^{\frac{1}{2}}\Big] + O_\varepsilon.\end{aligned}$$

We now use the fact that for any $\lambda_M \ge 1$, then

(4.7)
$$\left\{\begin{aligned}&\int_\Omega |\nabla w^\varepsilon|^2|u - v||\varphi| \le C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2|\varphi|\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla w^\varepsilon|^2|u - v||\varphi|\Big)^{\frac{1}{2}}\\ &\le C_M\Big(\int_\Omega |\nabla w^\varepsilon|^2|\varphi|\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla w^\varepsilon|^2|\varphi|\Big)^{\frac{1}{2}(1-\frac{1}{\lambda_M})}\Big(\int_\Omega |\nabla w^\varepsilon|^2|u - v||\varphi|\Big)^{\frac{1}{2}\frac{1}{\lambda_M}}.\end{aligned}\right.$$

then estimate (3.1) and finally the facts (which are respectively deduced from (1.28) with $\varphi = |\varphi|$ and $\varphi = |u - v|\varphi$) that

$$\begin{aligned}\int_\Omega |\nabla w^\varepsilon|^2|\varphi| &= \int_\Omega |\varphi|\, d\mu + O_\varepsilon\\ \int_\Omega |\nabla w^\varepsilon|^2|u - v||\varphi| &= \int_\Omega |u - v||\varphi|\, d\mu + O_\varepsilon,\end{aligned}$$

complete the proof of (4.5).

STEP 2. – The functions E and F are defined by (2.15). Thus, we have for any function $\varphi \in \mathcal{D}(\Omega)$

$$\begin{cases} \int_\Omega E\,\varphi\, d\mu = \lim_{\varepsilon\to 0}\int_\Omega (H(x,u^\varepsilon,\nabla u^\varepsilon) - H(x,u,\nabla u))w^\varepsilon\varphi + \int_\Omega u\varphi\, d\mu, \\ \int_\Omega F\varphi\, d\mu = \lim_{\varepsilon\to 0}\int_\Omega (H(x,v^\varepsilon,\nabla v^\varepsilon) - H(x,v,\nabla v))w^\varepsilon\varphi + \int_\Omega v\varphi\, d\mu, \end{cases}$$

which implies

$$\left|\int_\Omega (E-F)\varphi\, d\mu\right| \le \int_\Omega |u-v||\varphi| d\mu$$
$$+ \limsup_{\varepsilon\to 0}\int_\Omega |H(x,u^\varepsilon,\nabla u^\varepsilon) - H(x,v^\varepsilon,\nabla v^\varepsilon) - H(x,u,\nabla u) + H(x,v,\nabla v)||\varphi| d\mu,$$

therefore using

$$\int_\Omega |u-v||\varphi|\, d\mu \le C_M\left(\int_\Omega |\varphi|\, d\mu\right)^{1-\frac{1}{\lambda_M}}\left(\int_\Omega |u-v||\varphi|\, d\mu\right)^{\frac{1}{\lambda_M}}$$

and (4.5) we have

$$\left|\int_\Omega (E-F)\varphi\, d\mu\right| \le C_M\left(\int_\Omega |\varphi|\, d\mu\right)^{1-\frac{1}{\lambda_M}}\left(\int_\Omega |u-v||\varphi|\, d\mu\right)^{\frac{1}{\lambda_M}}. \tag{4.8}$$

Since for every open set $A \subset \Omega$, we have (see [Fo])

$$\int_A |E-F|\, d\mu = sup\left\{\left|\int_\Omega (E-F)\varphi\, d\mu\right| : \varphi \in \mathcal{D}(A),\ 0 \le \varphi \le 1\right\},$$

we deduce from (4.8) that for any open set $A \subset \Omega$, we have

$$\int_A |E-F|\, d\mu \le C_M \mu(A)^{1-\frac{1}{\lambda_M}}\left(\int_A |u-v|\, d\mu\right)^{\frac{1}{\lambda_M}}.$$

Then, for any open ball $B(x,r) \subset \Omega$, with $\mu(B(x,r)) > 0$, we have

$$\frac{\int_{B(x,r)} |E-F|\, d\mu}{\mu(B(x,r))} \le C_M\left(\frac{\int_{B(x,r)} |u-v|\, d\mu}{\mu(B(x,r))}\right)^{\frac{1}{\lambda_M}},$$

which letting r tend to zero and using the measure derivation Theorem, proves 4.4. ■

5. Construction of the function T

We have seen in the previous Section that the function E is of the form $E(x) = T(x, u(x))$ for some function T. However the function T is only defined for the pairs of the form $(x, u(x))$ where u is such that there exists f^ε, f and u^ε, where f^ε, f, u^ε and u satisfy (2.3). We begin this Section by showing that for every $s \in \mathbb{R}$ there exists a sequence of such functions u (which we denote by s_n) which converges to s in $H^1_{loc}(\Omega) \cap L^\infty(\Omega, d\mu)$ where $H^1_{loc}(\Omega)$ is endowed with its strong topology and $L^\infty(\Omega, d\mu)$ with its weak-$*$ topology.

LEMMA 5.1. – *Assume that H satisfies (1.2) and (4.3). Consider $s \in \mathbb{R}$. For any $n \in \mathbb{N}$, define s_n^ε as the solution of the problem*

$$(5.1) \qquad \begin{cases} -\Delta s_n^\varepsilon + n s_n^\varepsilon + H(x, s_n^\varepsilon, \nabla s_n^\varepsilon) = ns \text{ in } \mathcal{D}'(\Omega^\varepsilon), \\ s_n^\varepsilon \in H^1_0(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon). \end{cases}$$

Then, there exists a subsequence of ε (which in order to simplify the notation we still denote by ε), two sequences of functions s_n and S_n and a function S such that

(5.2) $\quad s_n \in H^1_0(\Omega) \cap L^\infty(\Omega),\ S_n \in L^\infty(\Omega, d\mu),\ S \in L^\infty(\Omega, d\mu),$

(5.3) $\quad -\Delta s_n + S_n\mu + ns_n + H(x, s_n, \nabla s_n) = ns$ in $\mathcal{D}'(\Omega)$,

(5.4) $\quad$ for any $n \in \mathbb{N}$ fixed, $s_n^\varepsilon \rightharpoonup s_n$ in $H^1_0(\Omega)$ weak as $\varepsilon \to 0$,

(5.5) $\quad \| s_n^\varepsilon \|_{L^\infty(\Omega)} \leq C_{|s|}$ and thus $\| s_n \|_{L^\infty(\Omega)} \leq C_{|s|}$,

(5.6) $\quad s_n \to s$ in $H^1_{loc}(\Omega)$ and $L^p(\Omega, d\mu)$ $(1 \leq p < +\infty)$ strong,

(5.7) $\quad S_n \rightharpoonup S$ in $L^\infty(\Omega, d\mu)$ weak-$*$ and in $L^p(\Omega, d\mu)$ $(1 \leq p < +\infty)$ strong,

(5.8) $\quad \| S_n \|_{L^\infty(\Omega, d\mu)} \leq C_{|s|}$ and thus $\| S \|_{L^\infty(\Omega, d\mu)} \leq C_{|s|}$.

Proof.

STEP 1. – Let $n \in \mathbb{N}$ be fixed. By Theorems 1.1 and 1.2, there exists a unique solution s_n^ε of problem (5.1) such that $\| s_n^\varepsilon \|_{H^1_0(\Omega) \cap L^\infty(\Omega)}$ is bounded by a constant which is independent on ε but could depend on n. In fact the $L^\infty(\Omega)$ norm of s_n^ε is bounded independently of ε and n since in view of (1.4), we have:

$$(5.9) \qquad \| s_n^\varepsilon \|_{L^\infty(\Omega)} \leq \frac{\omega_0(0) + n|s|}{\lambda + n} \leq \frac{\omega_0(0)}{\lambda} + |s| = C_{|s|}.$$

By the diagonal process, we can thus assume that there exists a subsequence of ε and a sequence s_n such that s_n belongs to $H^1_0(\Omega) \cap L^\infty(\Omega)$ and that (5.4) and (5.5) hold true.

By Theorem 2.2, there exists for each $n \in \mathbb{N}$ a function $S_n \in L^\infty(\Omega, d\mu)$, with

$$(5.10) \qquad \| S_n \|_{L^\infty(\Omega, d\mu)} \leq C_{|s|},$$

such that s_n satisfies (5.3).

STEP 2. – For $\varphi \in \mathcal{D}(\Omega)$, inequality (1.7) with $\Theta = \Omega$, $u = s_n$, $r = s$, $\varphi = \varphi^2$ and $f = n(s - s_n) - S_n\mu$ (observe that $S_n\mu \in H^{-1}(\Omega) + L^1(\Omega)$ by (5.3)) yields

$$\int_\Omega |\nabla s_n|^2\varphi^2 \leq n\int_\Omega (s - s_n)h(s_n - s)\varphi^2 - \int_\Omega S_n h(s_n - s)\varphi^2\, d\mu$$
$$- 2\int_\Omega h(s_n - s)\varphi\nabla s_n \nabla \varphi + C_{|s|}\int_\Omega |h(s_n - s)|\varphi^2,$$

where we used (2.19) with $E = S_n$, $z = h(s - s_n)\varphi^2$.

Since $h' > 1$ and $h(0) = 0$ (see (1.6)) we have $(s_n - s)h(s_n - s) \geq |s_n - s|^2$ almost everywhere in Ω, which implies:

$$(5.11)\qquad \begin{cases} \displaystyle\int_\Omega |\nabla s_n|^2\varphi^2 + n\int_\Omega |s_n - s|^2\varphi^2 \leq -\int_\Omega S_n h(s_n - s)\varphi^2\, d\mu \\ \displaystyle - 2\int_\Omega h(s_n - s)\varphi\nabla s_n \nabla \varphi + C_{|s|}\int_\Omega |h(s_n - s)|\varphi^2. \end{cases}$$

By (5.5) and (5.10), the two terms

$$\left| -\int_\Omega S_n h(s_n - s)\varphi^2\, d\mu \right|, \quad \left| \int_\Omega |h(s_n - s)|\varphi^2 \right|$$

are bounded independently on n, while for the remaining term, we have

$$\left| \int_\Omega h(s_n - s)\varphi\nabla s_n \nabla \varphi \right| \leq \| h(s_n - s) \|_{L^\infty(\Omega)} \| \varphi\nabla s_n \|_{L^2(\Omega)^N} \| \nabla \varphi \|_{L^2(\Omega)^N} .$$

Thus for each $\varphi \in \mathcal{D}(\Omega)$ there exists two positive constants $a(\varphi)$, $b(\varphi)$ such that:

$$\| \varphi\nabla s_n \|^2_{L^2(\Omega)^N} + n \| (s_n - s)\varphi \|^2_{L^2(\Omega)} \leq a(\varphi) + b(\varphi) \| \varphi\nabla s_n \|_{L^2(\Omega)^N} .$$

This implies that s_n is bounded in $H^1_{loc}(\Omega)$ and that $\sqrt{n}(s_n - s)$ is bounded in $L^2_{loc}(\Omega)$. Thus $s_n - s$ converges to zero strongly in $L^2_{loc}(\Omega)$, and weakly in $H^1_{loc}(\Omega)$. By Theorem A6 in [C1], the weak convergence in $H^1_{loc}(\Omega)$ implies the μ-almost everywhere convergence. It is now easy to see that the right-hand side of (5.11) converges to zero. This implies that (5.6) holds true.

STEP 3. – By Lemma 4.1, the sequence S_n satisfies

$$|S_n - S_m| \leq C_{|s|}|s_n - s_m| \quad \mu\text{-a.e. in } \Omega$$

and then, since s_n is a Cauchy sequence in $L^1(\Omega, d\mu)$ we deduce that S_n converges strongly to a function S in $L^1(\Omega, d\mu)$. Since $\| S_n \|_{L^\infty(\Omega,d\mu)} \leq C_{|s|}$, this proves (5.7) which completes the proof of Lemma 5.1. ■

REMARK 5.1. – In order to define the function T, the idea is now to set:

$$T(x, s) = S(x) \quad \mu\text{-a.e. } x \in \Omega, \ \forall s \in \mathbb{R},$$

where for $s \in \mathbb{R}$, $S = S(x)$ is the function defined in Lemma 5.1. The problem in this definition is the fact that the subsequence of ε given by Theorem 5.1 depends on s. In order to avoid this problem we thus define $T(x, s)$ only when s is a rational number, and then extend the definition to any real number s by a limit argument. This will be carried out in Theorem 5.1. Moreover, we will prove in Section 6 that the subsequence ε given in Lemma 5.1 may be choosen independently on s and that the functions S_n and S which appear in Lemma 5.1 satisfy

$$S_n = T(x, s_n), \; S = T(x, s) \;\; \mu\text{-a.e. } x \in \Omega.$$

But in order to prove this result, we need a uniqueness result for the limit problem which cannot be proved at this stage.

THEOREM 5.1. – *Assume that H satisfies (1.2) and (4.3). For any $q \in Q$ and for any $n \in \mathbb{N}$, define q_n^ε by*

$$(5.12) \qquad \begin{cases} -\Delta q_n^\varepsilon + n q_n^\varepsilon + H(x, q_n^\varepsilon, \nabla q_n^\varepsilon) = nq \text{ in } \mathcal{D}'(\Omega^\varepsilon) \\ q_n^\varepsilon \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon). \end{cases}$$

Then there exist a subsequence of ε which does not depend neither on q nor on n (and which to simplify the notation we still denote by ε), two sequences of functions q_n and Q_n, and a function Q such that (5.2),..., (5.8) hold true with $s = q$, $s_n = q_n$, $S_n = Q_n$, $S = Q$.

Define the function $T : \Omega \times Q \mapsto \mathbb{R}$ by

$$(5.13) \qquad T(x, q) = Q(x) \;\; \mu\text{-a.e. } x \text{ in } \Omega, \; \forall q \in Q.$$

If $s \in \mathbb{R}$ and if q^k is a sequence in Q which converges to s, then the sequence $T(x, q^k)$ converges in $L^p(\Omega, d\mu)$ strong $(1 \le p < +\infty)$ and in $L^\infty(\Omega, d\mu)$ weak-$*$. Define now $T : \Omega \to \mathbb{R}$ by

$$(5.14) \; T(x, s) = \lim_{k\to\infty} T(x, q^k) \text{ in } L^p(\Omega, d\mu) \text{ strong } (1 \le p < +\infty) \text{ and in } L^\infty(\Omega, d\mu) \text{ weak-}*$$

The function $T : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ defined in this way, satisfies:

$$(5.15) \qquad T(., s) \in L^\infty(\Omega, d\mu), \;\; \forall s \in \mathbb{R}, \;\; \text{with } \| T(., s) \|_{L^\infty(\Omega, d\mu)} \le C(|s|)$$

$$(5.16) \qquad |T(x, s_1) - T(x, s_2)| \le C(s)|s_1 - s_2|^{\frac{1}{\lambda(s)}}, \;\; s = \max\{|s_1|, |s_2|\},$$

where $C : [0, +\infty) \mapsto [0, +\infty)$ and $\lambda : [1, +\infty) \mapsto [0, +\infty)$ are increasing functions.

Proof.

STEP 1. – Since Q is a countable set, by Lemma 5.1 and the diagonal process we can extract a subsequence ε such that for any $q \in$ Q the results of Lemma 5.1 hold true. This proves the first part of Theorem 5.1.

STEP 2. – Define the function $T : \Omega \times \mathrm{Q} \mapsto \mathbb{R}$ by (5.13). By (5.8), the function T satisfies

$$(5.17) \qquad |T(x, q)| \le C_{|q|} \;\; \mu\text{-a.e. } x \text{ in } \Omega, \; \forall q \in \mathrm{Q}.$$

Consider q and q' in Q. By the definitions of Q_n, Q'_n and Lemma 4.1, we have

$$|Q_n - Q'_n| \le C_M |q - q'|^{\frac{1}{\lambda_M}} \;\; \mu\text{-a.e. in } \Omega,$$

where $M = \max\{|q|, |q'|\}$ and then, passing to the limit in n, we obtain:

$$(5.18) \qquad |T(x, q) - T(x, q')| \le C_M |q - q'|^{\frac{1}{\lambda_M}} \;\; \mu\text{-a.e. } x \text{ in } \Omega,$$

where $M = \max\{|q|, |q'|\}$. This uniform continuity of the mapping $q \in \mathrm{Q} \mapsto T(., q) \in L^1(\Omega, d\mu)$ and (5.17) allows one to define $T(., s)$ for any $s \in \mathbb{R}$ by (5.14). By (5.17) and (5.18), the function T satisfies (5.15) and (5.16). ■

6. The homogenization result and a property of the function T

6.1. A first homogenization result

In this subsection we prove that the function $T : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ defined in Theorem 5.1 is such that the function E defined by (2.15) may be expressed in the form $E = T(x, u)$. Indeed, we will prove:

THEOREM 6.1. – *Assume that H satisfies (1.2) and (4.3). Then for the subsequence ε and the function T defined in Theorem 5.1 we have the following homogenization result:*

If f^ε, f, u^ε and u satisfy (2.3), the function u satisfies

$$(6.1)\qquad \begin{cases} -\Delta u + T(x,u)\mu + H(x,u,\nabla u) = f \text{ in } \mathcal{D}'(\Omega), \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases}$$

or equivalently

$$(6.2)\qquad \begin{cases} u \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega \nabla u \nabla z + \int_\Omega T(x,u) z\, d\mu + \int_\Omega H(x,u,\nabla u) z = \langle f, z \rangle_\Omega, \\ \forall z \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases}$$

Proof. – Consider f^ε, f, u^ε and u which satisfy (2.3) and let E be defined by (2.15). By applying (4.4) to the problems (2.2) and (5.12) with $f^\varepsilon = f^\varepsilon$ and $g^\varepsilon = nq - nq_n^\varepsilon$ we have for any $q \in \mathrm{Q}$ and any $n \in \mathbb{N}$

$$|E - Q_n| \le C_M |u - q_n|^{\frac{1}{\lambda_M}} \quad \mu\text{-a.e. in } \Omega, \quad M = \max\{\sup\{\| u^\varepsilon \|_{L^\infty(\Omega)}\}, |q|\}$$

and then by passing to the limit in n, we obtain that for any $q \in \mathrm{Q}$

$$|E - T(x,q)| \le C_M |u - q|^{\frac{1}{\lambda_M}} \quad \mu\text{-a.e. } x \in \Omega, \quad M = \max\{\sup\{\| u^\varepsilon \|_{L^\infty(\Omega)}\}, |q|\}.$$

If now s belongs to $\mathbb{R}$, taking a sequence of rational numbers q^i which converges to s and using the continuity (5.16) of T, we get

$$(6.3)\quad |E - T(x,s)| \le C_M |u - s|^{\frac{1}{\lambda_M}}, \ \mu\text{-a.e. } x \in \Omega, \quad M = \max\{\sup\{\| u^\varepsilon \|_{L^\infty(\Omega)}\}, |s|\}.$$

Considering the points x where $u(x)$ and $E(x)$ are defined by their representatives and then taking $s = u(x)$, inequality (6.3) implies that

$$E(x) = T(x, u(x)) \quad \mu\text{-a.e. } x \in \Omega. \quad \blacksquare$$

REMARK 6.1. – Using Theorem 1.1, we can prove that there exists a subsequence (still denoted by ε) of the subsequence extracted in Theorem 5.1, such that the corresponding subsequence of u^ε, solution of (0.1), converges weakly in $H_0^1(\Omega)$ to a function u which is a solution of (6.1). Once uniqueness will be proved for problem (6.1), we will deduce that for the subsequence ε extracted in Theorem 5.1, the whole subsequence u^ε, solution of (0.1), converges to u, without extracting another subsequence.

6.2. The uniqueness of the limit problem

We will prove in this Subsection the uniqueness of the solution of the limit problem (6.1).

LEMMA 6.1. – *Assume that H satisfies (1.2) and (4.3) and consider the function T defined in Theorem 5.1. For any $M > 0$, there exist two constants A and K which only depend on M and are increasing in M such that for the functions ϑ and ψ defined by (1.11) and (1.12) we have*

$$\left(\frac{T(x,s)}{\psi'(\vartheta(s))}-\frac{T(x,t)}{\psi'(\vartheta(t))}\right)(\vartheta(s)-\vartheta(t))\geq 0 \quad \mu\text{-a.e. in } \Omega, \tag{6.4}$$

for any s and t in $\mathbb{R}$ such that $\max\{|s|,|t|\}\leq M$.

Since $\psi=\vartheta^{-1}$ and ϑ are increasing functions, the result of Lemma 6.1 states that the function $s\to\frac{T(x,s)}{\psi'(\vartheta(s))}$ (or equivalently $\frac{T(x,\psi(s))}{\psi'(s)}$) is increasing.

Proof.

STEP 1. – We will first prove the following result: Consider f^ε, f, u^ε, u and g^ε, g, v^ε, v which satisfy (2.3). Then, there exist two constants A and K (which are increasing with respect to M) such that for the functions ψ and $\vartheta=\psi^{-1}$ defined by (1.11) and (1.12), the functions u and v satisfy

$$\left(\frac{T(x,u)}{\psi'(\vartheta(u))}-\frac{T(x,v)}{\psi'(\vartheta(v))}\right)(\vartheta(u)-\vartheta(v))\geq 0 \quad \mu\text{-a.e. } x\in\Omega. \tag{6.5}$$

Proof. – By Lemma 1.2 applied to equation (2.2), with $\Theta=\Omega^\varepsilon$, $u=u^\varepsilon$ and $f=f^\varepsilon$ (respectively $u=v^\varepsilon$ and $f=g^\varepsilon$) there exist two constants A and K which are increasing with respect to M, such that the functions $\hat{u}^\varepsilon=\vartheta(u^\varepsilon)$ and $\hat{v}^\varepsilon=\vartheta(v^\varepsilon)$ satisfy

$$\begin{cases} -\Delta\hat{u}^\varepsilon+B(x,\hat{u}^\varepsilon,\nabla\hat{u}^\varepsilon)=\dfrac{f^\varepsilon}{\psi'(\hat{u}^\varepsilon)} & \text{in } \mathcal{D}'(\Omega^\varepsilon),\\ -\Delta\hat{v}^\varepsilon+B(x,\hat{v}^\varepsilon,\nabla\hat{v}^\varepsilon)=\dfrac{g^\varepsilon}{\psi'(\hat{v}^\varepsilon)} & \text{in } \mathcal{D}'(\Omega^\varepsilon),\end{cases} \tag{6.6}$$

where the function B satisfies properties (1.14) and (1.15). Estimate (1.18) applied to these equations with $\Theta=\Omega^\varepsilon$, $\hat{u}=\hat{u}^\varepsilon$, $\hat{v}=\hat{v}^\varepsilon$, $\hat{f}=\hat{f}^\varepsilon$ and $\hat{g}=\hat{g}^\varepsilon$, implies that for $\hat{\omega}^\varepsilon=\hat{u}^\varepsilon-\hat{v}^\varepsilon$ and for any function $\varphi\in\mathcal{D}(\Omega)$, $\varphi\geq 0$, we have

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega S'(\hat{\omega}^\varepsilon)|\nabla\hat{\omega}^\varepsilon|^2\varphi\\ +\displaystyle\int_\Omega\left[\big(B(x,\hat{u}^\varepsilon,\nabla\hat{u}^\varepsilon)-B(x,\hat{v}^\varepsilon,\nabla\hat{v}^\varepsilon)\big)S(\hat{\omega}^\varepsilon)+\frac{1}{2}S'(\hat{\omega}^\varepsilon)|\nabla\hat{\omega}^\varepsilon|^2\right]\varphi\\ =\langle f^\varepsilon,\dfrac{S(\hat{\omega}^\varepsilon)}{\psi'(\hat{u}^\varepsilon)}\varphi\rangle_{\Omega^\varepsilon}-\langle g^\varepsilon,\dfrac{S(\hat{\omega}^\varepsilon)}{\psi'(\hat{v}^\varepsilon)}\varphi\rangle_{\Omega^\varepsilon}-\displaystyle\int_\Omega S(\hat{\omega}^\varepsilon)\nabla\hat{\omega}^\varepsilon\nabla\varphi,\end{cases} \tag{6.7}$$

where the integrand of the second term is nonnegative. By Theorem 2.1, this integrand converges almost everywhere to

$$\left[\big(B(x,\hat{u},\nabla\hat{u})-B(x,\hat{v},\nabla\hat{v})\big)S(\hat{\omega})+\frac{1}{2}S'(\hat{\omega})|\nabla\hat{\omega}|^2\right]\varphi,$$

with $\hat{u} = \vartheta(u)$, $\hat{v} = \vartheta(v)$ and $\hat{\omega} = \hat{u} - \hat{v}$. Therefore Fatou's lemma permit us to pass to the limit in (6.7) and to obtain

$$(6.8)\quad \begin{cases} \int_\Omega S'(\hat{\omega})|\nabla \hat{\omega}|^2\varphi + \int_\Omega [B(x,\hat{u},\nabla \hat{u}) - B(x,\hat{v},\nabla \hat{v})]S(\hat{\omega})\varphi \\ \leq \langle f, \frac{S(\hat{\omega})}{\psi'(\hat{u})}\varphi\rangle_\Omega - \langle g, \frac{S(\hat{\omega})}{\psi'(\hat{v})}\varphi\rangle_\Omega - \int_\Omega S(\hat{\omega})\nabla \hat{\omega}\nabla\varphi, \quad \forall\varphi \in \mathcal{D}(\Omega),\ \varphi \geq 0. \end{cases}$$

On the other hand, by Theorem 6.1 the functions u and v satisfy the equations:

$$\begin{cases} -\Delta u + T(x,u)\mu + H(x,u,\nabla u) = f \text{ in } \mathcal{D}'(\Omega), \\ -\Delta v + T(x,v)\mu + H(x,v,\nabla v) = g \text{ in } \mathcal{D}'(\Omega), \end{cases}$$

where $f - T(x,v)$ and $g - T(x,v)\mu$ belong to $H^{-1}(\Omega) + L^1(\Omega)$. Applying Lemma 1.2 to these two equations with the same functions ϑ and ψ as above (since $\| u \|_{L^\infty(\Omega)} \leq M$ and $\| v \|_{L^\infty(\Omega)} \leq M$) and $\Theta = \Omega$ implies that $\hat{u}$ and $\hat{v}$ satisfy:

$$(6.9)\quad \begin{cases} -\Delta\hat{u} + \frac{T(x,u)}{\psi'(\hat{u})}\mu + B(x,\hat{u},\nabla \hat{u}) = \frac{f}{\psi'(\hat{u})} \text{ in } \mathcal{D}'(\Omega^\varepsilon), \\ -\Delta\hat{v} + \frac{T(x,v)}{\psi'(\hat{v})}\mu + B(x,\hat{v},\nabla \hat{v}) = \frac{g}{\psi'(\hat{v})} \text{ in } \mathcal{D}'(\Omega^\varepsilon). \end{cases}$$

Taking $S(\hat{\omega})\varphi$ with $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$ as test function in the difference of the two equations of (6.9) and applying (2.19) with $E = \frac{T(x,u)}{\psi'(\hat{u})}\mu - \frac{T(x,v)}{\psi'(\hat{v})}\mu$ and $z = S(\hat{\omega})\varphi$, we get

$$\int_\Omega S'(\hat{\omega})|\nabla \hat{\omega}|^2\varphi + \int_\Omega S(\hat{\omega})\nabla \hat{\omega}\nabla\varphi + \int_\Omega \left(\frac{T(x,u)}{\psi'(\hat{u})} - \frac{T(x,v)}{\psi'(\hat{v})}\right)S(\hat{\omega})\varphi\, d\mu$$
$$+ \int_\Omega [B(x,\hat{u},\nabla \hat{u}) - B(x,\hat{v},\nabla \hat{v})]S(\hat{\omega})\varphi = \langle f, \frac{S(\hat{\omega})}{\psi'(\hat{u})}\varphi\rangle_\Omega - \langle g, \frac{S(\hat{\omega})}{\psi'(\hat{v})}\varphi\rangle_\Omega.$$

Comparison with (6.8) implies that

$$\int_\Omega \left(\frac{T(x,u)}{\psi'(\hat{u})} - \frac{T(x,v)}{\psi'(\hat{v})}\right)S(\hat{\omega})\varphi\, d\mu \geq 0, \quad \forall\varphi \in \mathcal{D}(\Omega).$$

Since the sign of $S(\hat{\omega})$ coincides with the sign of $\hat{\omega}$, we have proved that (6.6) holds true.

STEP 2. – Let q and q' be rational numbers with $\max\{|q|,|q'|\} \leq M$. From (6.5) applied to the sequences $u^\varepsilon = q_n^\varepsilon$ and $v^\varepsilon = (q')_n^\varepsilon$ defined by (5.12), we deduce that there exists two constants A and K which are increasing with respect to M such that for the functions ψ and ϑ defined by (1.11) and (1.12) we have

$$\left(\frac{T(x,q_n)}{\psi'(\vartheta(q_n))} - \frac{T(x,q'_n)}{\psi'(\vartheta(q'_n))}\right)(\vartheta(q_n) - \vartheta(q'_n)) \geq 0 \quad \mu\text{-a.e. in } \Omega,\ \forall n \in \mathbb{N},$$

where the functions q_n and q'_n are defined by Theorem 5.1. Taking in this expression the limit in n we deduce that

$$\left(\frac{T(x,q)}{\psi'(\vartheta(q))} - \frac{T(x,q')}{\psi'(\vartheta(q'))}\right)(\vartheta(q) - \vartheta(q')) \geq 0 \quad \mu\text{-a.e. in } \Omega.$$

The continuity (5.16) of T then implies (6.4). ■

We are now in position to prove a maximum principle.

THEOREM 6.2. – *Let $\bar{H} : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ be a Carathéodory function (note that $\bar{H}$ can be different of H) which satisfies hypotheses similar to (1.2) and (1.9),* i.e.:

i) *For almost every $x \in \Omega$ the function $\bar{H}(x,.,.)$ is continuously derivable and there exists a constant $\bar{\lambda} > 0$, such that for almost every $x \in \Omega$ we have*

$$\frac{\partial \bar{H}}{\partial s}(x,s,\xi) \geq \bar{\lambda} > 0, \ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N. \tag{6.10}$$

ii) *There exists an increasing function $\bar{\alpha} : [0,+\infty) \mapsto [0,+\infty)$ such that*

$$\begin{cases} \bar{H}(.,0,0) \in L^\infty(\Omega), \\ \left| \dfrac{\partial \bar{H}}{\partial s}(x,s,\xi) \right| \leq \bar{\alpha}(|s|)(1+|\xi|^2), \ \ a.e. \ x \in \Omega, \ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N \\ \left| \dfrac{\partial \bar{H}}{\partial \xi}(x,s,\xi) \right| \leq \bar{\alpha}(|s|)(1+|\xi|), \ \ a.e. \ x \in \Omega, \ \ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N. \end{cases} \tag{6.11}$$

Assume that H satisfies (1.2) and (4.3) and let T be the function defined in Theorem 5.1. Consider u and v in $H^1(\Omega) \cap L^\infty(\Omega)$ such that there exist f and g in $H^{-1}(\Omega) + L^1(\Omega)$ which satisfy

$$\begin{cases} -\Delta u + T(x,u)\mu + \bar{H}(x,u,\nabla u) = f \leq 0 \ \ in \ \mathcal{D}'(\Omega), \\ -\Delta v + T(x,v)\mu + \bar{H}(x,v,\nabla v) = g \geq 0 \ \ in \ \mathcal{D}'(\Omega). \end{cases} \tag{6.12}$$

*Then inequality $u \leq v$ in $\partial\Omega$ (*i.e.* $(u-v)^+ \in H_0^1(\Omega)$) implies that $u \leq v$ almost everywhere in Ω.*

In particular for $f \in H^{-1}(\Omega) + L^1(\Omega)$, the problem

$$\begin{cases} -\Delta u + T(x,u)\mu + \bar{H}(x,u,\nabla u) = f \ \ in \ \mathcal{D}'(\Omega) \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{6.13}$$

has at most one solution.

Proof. – By Lemma 1.2 with $\Theta = \Omega$ and $H = \bar{H}$, there exist two constants A and K such that for the functions ψ and $\vartheta = \psi^{-1}$ defined by (1.11) and (1.12), the functions $\hat{u} = \vartheta(u)$ and $\hat{v} = \vartheta(v)$ respectively satisfy (recall that by (6.12) $T(x,u)\mu$ and $T(x,v)\mu$ belong to $H^{-1}(\Omega) + L^1(\Omega)$ and observe that $\psi' > 0$)

$$\begin{cases} -\Delta \hat{u} + \dfrac{T(x,u)}{\psi'(\hat{u})}\mu + \bar{B}(x,\hat{u},\nabla \hat{u}) \leq 0 \ \text{ in } \mathcal{D}'(\Omega), \\ -\Delta \hat{v} + \dfrac{T(x,v)}{\psi'(\hat{v})}\mu + \bar{B}(x,\hat{v},\nabla \hat{v}) \geq 0 \ \text{ in } \mathcal{D}'(\Omega), \end{cases} \tag{6.14}$$

where the function $\bar{B}$ satisfies properties analogous to (1.14) and (1.15). Moreover, by (6.4) we have

$$\left(\frac{T(x,u)}{\psi'(\hat{u})}-\frac{T(x,v)}{\psi'(\hat{v})}\right)(\hat{u}-\hat{v})\geq 0 \quad \mu\text{-a.e. in } \Omega, \tag{6.15}$$

where $\hat{u} = \vartheta(u)$ and $\hat{v} = \vartheta(v)$.

Define $\hat{\omega} = \hat{u} - \hat{v}$ and apply estimate (1.20) to the two equations of (6.14). By (6.15) we have

$$\frac{1}{2}\int_\Omega S'(\hat{\omega}^+)|\nabla\hat{\omega}^+|^2 \leq -\int_\Omega\left(\frac{T(x,u)}{\psi'(\hat{u})}-\frac{T(x,v)}{\psi'(\hat{v})}\right)S(\hat{\omega}^+)d\mu\leq 0 \quad \mu\text{-a.e. in } \Omega$$

and thus $\hat{\omega}^+ = 0$ almost everywhere in Ω, *i.e.* $\hat{u} \leq \hat{v}$ and thus $u \leq v$ almost everywhere in Ω. ■

6.3. The homogenization result

As a consequence of Theorem 6.2, we can now prove the following homogenization Theorem.

THEOREM. – *Assume that H satisfies (1.2) and (4.3) and consider the subsequence ε and the function T defined in Theorem 5.1.*

Let $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \mapsto \mathbb{R}$ be a Carathéodory function such that:

i) *There exist two increasing functions $\Upsilon_0, \Upsilon_1 : [0,+\infty) \mapsto [0,+\infty)$ and there exists $\alpha \in [0,2)$ such that*

$$|f(x,s,\xi)| \leq \Upsilon_0(|s|) + \Upsilon_1(|s|)|\xi|^\alpha \quad \text{a.e. } x\in\Omega,\ \forall (s,\xi)\in\mathbb{R}\times\mathbb{R}^N. \tag{6.16}$$

ii) *For every $x \in \Omega$ the function $f(x,.,.)$ is continuously derivable and there exists an increasing function $\zeta : [0,+\infty) \mapsto [0,+\infty)$ such that*

$$\begin{cases} f(.,0,0)\in L^\infty(\Omega) \\ \left|\dfrac{\partial f}{\partial s}(x,s,\xi)\right| \leq \zeta(|s|)(1+|\xi|^2), & a.e.\ x\in\Omega,\ \forall (s,\xi)\in\mathbb{R}\times\mathbb{R}^N \\ \left|\dfrac{\partial f}{\partial \xi}(x,s,\xi)\right| \leq \zeta(|s|)(1+|\xi|), & a.e.\ x\in\Omega,\ \forall (s,\xi)\in\mathbb{R}\times\mathbb{R}^N, \end{cases} \tag{6.17}$$

iii) *There exists $\sigma > 0$ such that for almost every $x \in \Omega$*

$$\lambda - \frac{\partial f}{\partial s}(x,s,\xi) > \sigma, \quad a.e.\ in\ \Omega,\ \forall (s,\xi)\in\mathbb{R}\times\mathbb{R}^N, \tag{6.18}$$

where λ is defined by (1.2).

Then the unique solution u^ε of the problem:

$$\begin{cases} -\Delta u^\varepsilon + H(x,u^\varepsilon,\nabla u^\varepsilon) = f(x,u^\varepsilon,\nabla u^\varepsilon), & in\ \mathcal{D}'(\Omega^\varepsilon), \\ u^\varepsilon \in H_0^1(\Omega^\varepsilon)\cap L^\infty(\Omega^\varepsilon), \end{cases} \tag{6.19}$$

converges weakly in $H_0^1(\Omega)$*, strongly in* $W_0^{1,p}(\Omega)$ $(1 \le p < 2)$ *and weakly-∗ in* $L^\infty(\Omega)$ *to the unique solution* u *of the problem*

$$(6.20)\qquad \begin{cases} -\Delta u + T(x,u)\mu + H(x,u,\nabla u) = f(x,u,\nabla u) \ \text{ in } \mathcal{D}'(\Omega) \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases}$$

REMARK 6.1. – Taking $f(x,s,\xi) = f(x) \in L^\infty(\Omega)$ and taking into account Remark 6.3 below, we inmediately deduce Theorem 0.1 from Theorem 6.3.

REMARK 6.2. – As announced in Remark 5.1, Theorem 6.3 implies that the subsequence ε which appears in the statement of Lemma 5.1 may be chosen as the subsequence ε given in Theorem 5.1, and thus independently of u. Moreover Theorem 6.3 implies that the functions S_n and S defined in Lemma 5.1 satisfy

$$S_n = T(x,s_n), \quad S = T(x,s), \quad \mu\text{-a.e. } x \in \Omega.$$

Proof of Theorem 6.3. – Theorems 1.1 and 1.2 applied to $\Theta = \Omega^\varepsilon$ and $H = H - f$, imply that there exists a unique solution u^ε of (6.19) and that u^ε is bounded in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore there exists a subsequence ε' such that $u^{\varepsilon'}$ converges weakly in $H_0^1(\Omega)$ and weakly-∗ in $L^\infty(\Omega)$ to a function u. By Theorem 2.1, applied to $u^\varepsilon = u^{\varepsilon'}$, $H = H - f$ and $f^\varepsilon = f^{\varepsilon'} = 0$, we also have that $u^{\varepsilon'}$ converges strongly in $W^{1,p}(\Omega)$ $(1 \le p < 2)$ to u. Inequality (6.16) and Lebesgue's dominated convergence theorem imply thus that $f(x, u^{\varepsilon'}, \nabla u^{\varepsilon'})$ converges strongly in $L^1(\Omega)$ (and thus in the sense of (2.1)), to $f(x,u,\nabla u)$. By Theorem 6.1 we then have that u is a solution of (6.20). Theorem 6.2 applied to $\bar{H} = H - f$ implies the uniqueness of u and therefore the convergence for the whole sequence. ■

REMARK 6.3. – As a consequence of the results of the present Section, we also could prove the following monotonicity property of the function $T(x,.)$.

$$(6.21)\qquad \begin{cases} T(x,0) = 0, \ \ \mu\text{-a.e. } x \in \Omega, \\ (T(x,s_1) - T(x,s_2))(s_1 - s_2) \ge 0, \ \ \mu\text{-a.e. } x \in \Omega, \ \forall s_1, s_2 \in \mathbb{R}. \end{cases}$$

Actually this monotonicity property is not very important (except for esthetic reasons). What is important for uniqueness is property (6.4), *i.e.* that $\frac{T(x,\psi(s))}{\psi'(s)}$ is increasing with respect to s, and thus we do not give the proof of (6.21).

7. Corrector

In this Section, we use Lemma 3.1 and Lemma 5.1 to give an approximation of ∇u^ε in the strong topology of $L^2(\Omega)^N$.

DEFINITION 7.1. – *Define* $P_n^\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}^N$ *by* $P_n^\varepsilon(x,s) = \nabla s_n^\varepsilon(x)$*, where for any* $s \in \mathbb{R}$ *and* $n \in \mathbb{N}$*,* s_n^ε *is defined as the unique solution of problem* (5.1).

THEOREM 7.1. – *Assume that* H *satisfies (1.2) and (4.3) and let* ε *be the subsequence defined in Theorem 5.1. Consider* f^ε*,* f*,* u^ε *and* u *which satisfy (2.3). Then, for any step* *function* $y(x) = \sum_{i=1}^{m} s_i \chi_{Q_i}(x)$ *with* $s_i \in \mathbb{R}$ *and* Q_i *closed subsets of* $\mathbb{R}^N$ *with* $Q_i \subset \Omega$, *which satisfy* $\mu(Q_i \cap Q_j) = 0$ *for* $i \neq j$, *we have*

$$(7.1) \qquad \begin{cases} \limsup\limits_{n\to\infty} \limsup\limits_{\varepsilon\to 0} \displaystyle\int_Q |\nabla u^\varepsilon - \nabla u - P_n^\varepsilon(x,y)|^2 \\ \leq C_M \mu(Q)^{1-\frac{1}{\lambda_M}} \left(\displaystyle\int_Q |u-y| d\mu \right)^{\frac{1}{\lambda_M}}, \end{cases}$$

where

$$Q = \bigcup_{i=1}^{m} Q_i \quad \textit{and} \quad M = \max\{\sup\{\| u^\varepsilon \|_{L^\infty(\Omega)}\}, \| y \|_{L^\infty(\Omega)}\}$$

and where the constant C_M *does not depend on* Q.

REMARK 7.1. – The meaning of Theorem 7.1 is the following: If we could take $y = u$ in (7.1) we would obtain

$$\limsup_{n\to\infty} \limsup_{\varepsilon\to 0} \int_Q |\nabla u^\varepsilon - \nabla u - P_n^\varepsilon(x,u)|^2 = 0$$

which says that $\nabla u + P_n^\varepsilon(x,u)$ is a good approximation of ∇u^ε in $L^2(Q)^N$ strongly. However this choice is not possible, since P_n^ε is not a Carathéodory function in general and since therefore $P_n^\varepsilon(x,u(x))$ has not reason to be measurable. This is why we approximate u by the step function y.

REMARK 7.2. – In the statement of Theorem 7.1, the value of the function y on the set $Q_i \cap Q_j$, $i \neq j$, does play any role since $\mu(Q_i \cap Q_j) = 0$ by hypothesis. Indeed estimate (7.2) applied to $Q = Q_i \cap Q_j$ and $s = s_1$ and s_2, together with the triangle inequality, shows that

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \| P_n^\varepsilon(x,s_1) - P_n^\varepsilon(x,s_2) \|^2_{L^2(Q_i\cap Q_j)} = 0.$$

REMARK 7.3. – Consider a closed set Q such that $\mu(Q) = 0$. Applying estimate (7.1) to $y = 0$, we obtain

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \int_Q |\nabla u^\varepsilon - \nabla u - P_n^\varepsilon(x,0)|^2 = 0.$$

On the other hand by taking $u^\varepsilon = u = 0$ and $y = 0$ in (7.1), which is possible since $u^\varepsilon = 0$ satisfies

$$\begin{cases} -\Delta u^\varepsilon + H(x, u^\varepsilon, \nabla u^\varepsilon) = H(x,0,0) \\ u^\varepsilon \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon), \end{cases}$$

we have

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \int_Q |P_n^\varepsilon(x,0)|^2 = 0.$$

These inequalities show that for u^ε and u as in the statement of Theorem 7.1, one has

$$\nabla u^\varepsilon - \nabla u \to 0 \text{ strongly in } L^2(Q)^N,$$

for any closed set Q contained in Ω such that $\mu(Q) = 0$.

REMARK 7.4. – It is easy to approach a function u in $L^1(\Omega, d\mu)$ by a step function $y(x) = \sum_{i=1}^m s_i \chi_{Q_i}(x)$ with $s_i \in \mathbb{R}$ and Q_i closed subsets of $\mathbb{R}^N$ with $Q_i \subset \Omega$, such that $\mu(Q_i \cap Q_j) = 0$ for $i \neq j$. for example we can reason in the following way:

Given $\delta > 0$, we choose $M > 0$ such that

$$\int_{\{|u| \geq M\}} |u|\, d\mu < \delta.$$

Since $\mu(\Omega) < +\infty$, the set of $s \in \mathbb{R}$ such that $\mu(\{x \in \Omega : u(x) = s\}) > 0$ is at most a countable set and thus there exist $s_1, \ldots, s_{m+1}$ in $\mathbb{R}$ such that:

$$\begin{cases} -M = s_1 < s_2 < \ldots. < s_m < s_{m+1} = M, \\ s_{i+1} - s_i < \delta, \quad \forall i \text{ with } 1 \leq i \leq m, \\ \mu(\{x \in \Omega : u(x) = s_i\}) = 0, \quad \forall i \text{ with } 2 \leq i \leq m. \end{cases}$$

Defining $y(x) = \sum_{i=1}^m s_i \chi_{Q_i}(x)$ where for any i with $1 \leq i \leq m$, Q_i is a closed set of $\mathbb{R}^N$ contained in $\{x \in \Omega : s_i \leq u(x) \leq s_{i+1}\}$ such that

$$\mu(\{x \in \Omega : s_i \leq u(x) \leq s_{i+1}\} \setminus Q_i) < \frac{\delta}{mM},$$

we have

$$\int_\Omega |u - y|\, d\mu$$
$$\leq \int_{\{|u| \geq M\}} |u|\, d\mu + \sum_{i=1}^m \int_{\{s_i \leq u \leq s_{i+1}\} \setminus Q_i} |u|\, d\mu + \sum_{i=1}^m \int_{Q_i} |u - s_i|\, d\mu \leq \delta(2 + \mu(\Omega)).$$

Proof of Theorem 7.1. – Let $s \in \mathbb{R}$ be and Q be a closed set of $\mathbb{R}^N$ with $Q \subset \Omega$. By Lemma 5.1 and Remark 6.2, the sequence s_n^ε defined by (5.1) converges weakly in $H_0^1(\Omega)$ to a function s_n and the sequence s_n converges to s strongly in $H^1_{loc}(\Omega)$ and μ-almost everywhere. Moreover $\| s_n^\varepsilon \|_{L^\infty(\Omega)} \leq C_{|s|}$.

Lemma 3.1 implies that for any function $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq \chi_Q$ we have:

$$\limsup_{\varepsilon \to 0} \int_Q |\nabla u^\varepsilon - \nabla u - P_n^\varepsilon(x, s) + \nabla s_n|^2 \leq C_M \left(\int_\Omega \varphi\, d\mu \right)^{1 - \frac{1}{\lambda_M}} \left(\int_\Omega |u - s_n| \varphi\, d\mu \right)^{\frac{1}{\lambda_M}}$$

where $M = \max\{\sup\{\| u^\varepsilon \|_{L^\infty(\Omega)}\}, |s|\}$. Since φ is arbitrary, we have:

$$\limsup_{\varepsilon \to 0} \int_Q |\nabla u^\varepsilon - \nabla u - P_n^\varepsilon(x, s) + \nabla s_n|^2 \leq C_M (\mu(Q))^{1 - \frac{1}{\lambda_M}} \left(\int_Q |u - s_n|\, d\mu \right)^{\frac{1}{\lambda_M}},$$

where $M = \max\{\sup\{\| u^\varepsilon \|_{L^\infty(\Omega)}\}, |s|\}$. Taking in this expression the limit in n and using that ∇s_n converges strongly to zero in $L^2(Q)^N$ we get
(7.2)

$$\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\int_Q |\nabla u^\varepsilon - \nabla u - P_n^\varepsilon(x,s)|^2 \le C_M(\mu(Q))^{1-\frac{1}{\lambda_M}}\left(\int_Q |u-s|\,d\mu\right)^{\frac{1}{\lambda_M}},$$

where $M = \max\{\sup\{\| u^\varepsilon \|_{L^\infty(\Omega)}\}, |s|\}$.

Let now $y = \sum_{i=1}^m s_i \chi_{Q_i}$, $Q = \bigcup_{i=1}^m Q_i$ and $M = \max\{\sup\{\| u^\varepsilon \|_{L^\infty(\Omega)}\}, \| y \|_{L^\infty(\Omega)}\}$ be as in the statement of Theorem 7.1. Adding the inequalities (7.2) for $Q = Q_i$ and $s = s_i$, and then using Hölder's inequality and the fact that $\mu(Q_i \cap Q_j) = 0$ for $i \neq j$ we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\int_Q |\nabla u^\varepsilon - \nabla u - P_n^\varepsilon(x,y)|^2 \\
&\le \limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\sum_{i=1}^m \int_{Q_i} |\nabla u^\varepsilon - \nabla u - P_n^\varepsilon(x,s_i)|^2 \\
&\le C_M \sum_{i=1}^m (\mu(Q_i))^{1-\frac{1}{\lambda_M}}\left(\int_{Q_i} |u-s_i|\,d\mu\right)^{\frac{1}{\lambda_M}} \\
&\le C_M \left(\sum_{i=1}^m \mu(Q_i)\right)^{1-\frac{1}{\lambda_M}}\left(\sum_{i=1}^m\int_{Q_i} |u-s_i|\,d\mu\right)^{\frac{1}{\lambda_M}} \\
&= C_M(\mu(Q))^{1-\frac{1}{\lambda_M}}\left(\int_Q |u-y|\,d\mu\right)^{\frac{1}{\lambda_M}}.
\end{aligned}$$

which proves (7.1). ■

Appendix: Notation

A.1. Standard notation

We denote by ε a parameter which takes its values in a sequence of strictly positive real numbers which converges to zero; the subsequences are also denoted by ε.

Ω denotes a bounded open set of $\mathbb{R}^N$ and Ω^ε a sequence of open sets of $\mathbb{R}^N$ which are contained in Ω. In the whole of the paper we assume that (1.27) (or more exactly (P1),...,(P7) see Theorem 1.3) hold true.

$\mathcal{D}(\Omega)$ denotes the space of smooth functions with compact support in Ω. Its dual space is the space of distributions which is denoted by $\mathcal{D}'(\Omega)$.

$\mathcal{M}_b(\Omega)$ denotes the space of bounded Borel measures in Ω.

Given a measure μ in Ω, we define $L^p(\Omega, d\mu)$, $1 \le p < +\infty$, as the space of those functions v which are μ-measurable and such that $\int_\Omega |v|^p\,d\mu < +\infty$. The space $L^\infty(\Omega, d\mu)$ is defined as the space of functions μ-essentially bounded. When the measure under consideration is the Lebesgue measure, we simplify the notation by writing $L^p(\Omega)$ and $L^\infty(\Omega)$, respectively.

$W^{1,p}(\Omega)$ denotes the space of those functions $u \in L^p(\Omega)$ whose first derivatives in the sense of distributions belongs to $L^p(\Omega)$. The space $W^{1,2}(\Omega)$ is denoted by $H^1(\Omega)$.

$L^p_{loc}(\Omega, d\mu)$ (respectively $W^{1,p}_{loc}(\Omega)$) denotes the space of functions which belong to $L^p(K, d\mu)$ (respectively $W^{1,p}(\Omega)$) for any compact set $K \subset \Omega$.

$W^{1,p}_0(\Omega)$ denotes the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$.

The characteristic function of the set $A \subset \mathbb{R}^N$ is denoted by χ_A.

The Lebesgue measure of the set $A \subset \mathbb{R}^N$ is denoted by $|A|$.

The capacity of a subset A of Ω is defined as in the following way:

If A is a compact set, the capacity of A is defined by

$$cap(A) = inf\Big\{ \int_\Omega |\nabla \varphi|^2 : \ \varphi \in \mathcal{D}(\Omega), \ \varphi \geq \chi_A \Big\}.$$

If A is an open set, the capacity of A is defined by

$$cap(A) = sup\Big\{ cap(K) : \ K \subset A, \ K \text{ compact} \Big\}.$$

If A is an arbitrary set, the capacity of A is defined by

$$cap(A) = inf\Big\{ cap(G) : \ A \subset G \subset \Omega, \ G \text{ open} \Big\}.$$

$\mathcal{M}^0_b(\Omega)$ denotes the set of bounded positive Borel measures which vanish on the sets of zero capacity.

It is well known (see [F Z], [Z], [E G]) that a function $u \in H^1(\Omega)$ has a representative which is well defined except on a set of zero capacity. We always identify u with this representative. If $\mu \in \mathcal{M}^0_b(\Omega)$, a consequence of this result and of the fact that μ is bounded is that

$$H^1_0(\Omega) \cap L^\infty(\Omega) \subset L^\infty(\Omega, d\mu) \subset L^q(\Omega, d\mu) \quad \text{for any } q, \ 1 \leq q < +\infty.$$

A.2. Specific notation

The functions $u^\varepsilon \in W^{1,p}_0(\Omega^\varepsilon)$ will be extended to the whole of Ω by setting

$$u^\varepsilon = \begin{cases} u^\varepsilon & \text{in } \Omega, \\ 0 & \text{in } \Omega \setminus \Omega^\varepsilon \end{cases}$$

and thus they will be considered as elements of $W^{1,p}_0(\Omega)$.

We denote by O_ε a sequence of real numbers which converges to zero when ε tends to zero and which can change from a line to another. Similarly, for a Banach space X (which will be $L^1(\Omega)$ or $L^2(\Omega)$), we denote by $O^X_\varepsilon \in X$ a sequence which strongly converges to zero in X and which can change from a line to another.

For a real parameter M, we denote by C_M and λ_M generic constants which can change from a line to another and which are increasing with respect to M; The constants λ_M will allways be assumed to satisfy $\lambda_M \geq 1$. These constants will neither depend on ε nor on the right-hand side of the homogenization problem (0.1), but can depend on the function H and on Ω.

For an open set $\Theta \subset \Omega$ we denote by $\langle f, v\rangle_\Theta$ the duality pairing between $f \in H^{-1}(\Theta) + L^1(\Theta)$ and $v \in H^1_0(\Theta) \cap L^\infty(\Theta)$.

Aknowledgment

I thank F. Murat for interesting discussions and many suggestions concerning a previous version of the present paper.

This work has been partially supported by the Project EURHomogenization, Contrat SC1-CT91-0732 of the Program SCIENCE of the Commission of the European Communities, and by the Project PB92-0696 of the DGICYT of Spain.

REFERENCES

[A] H. ATTOUCH, Variational convergence for functions and operators. *Pitman,* London, 1984.,

[B M] G. BARLES and F. MURAT, Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions. *Arch. Rat. Mech. Anal., 133,* 1, 1995, pp. 70-101.

[Bo M] L. BOCCARDO, F. MURAT, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Analysis, Theory, Methods and Applications, 19,* 6, 1992, pp. 581-597.

[B M P] L. BOCCARDO, F. MURAT and J.P. PUEL, Existence de solutions faibles pour des équations elliptiques quasi-linéaires à croissance quadratique. In Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, Vol. IV, ed. by H. Brézis, J.L. Lions. *Research Notes in Math.* 84 1983, Pitman, London, pp. 19-73.

[C1] J. CASADO-DÍAZ, Sobre la homogeneización de problemas no coercivos y problemas en dominios con agujeros. Ph.D. Thesis, *University of Seville,* 1993.

[C2] J. CASADO-DÍAZ, Existence of a sequence satisfyng Cioranescu-Murat conditions in homogenization of Dirichlet problems in perforated domains. *Rendiconti di Mathematica VII,* 16, 1996, pp. 387-413.

[C3] J. CASADO-DÍAZ. Homogenization of a quasi-linear problem with quadratic growth in perforated domains: An example. *To appear in Ann. I. H. Poincare, Analyse non Linéaire.*

[C4] J. CASADO-DÍAZ. Homogenization of Dirichlet problems for monotone operators in varying domains. *To appear in P. Roy. Soc. Edinburgh.*

[C G] J. CASADO-DÍAZ and A. GARRONI, Asymptotic behaviour of Dirichlet solutions of nonlinear elliptic systems in varying domains. *To appear.*

[C M] D. CIORANESCU and F. MURAT, Un terme étrange venu d'ailleurs. In Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, Vol. II and III, ed. by H. Brézis, J.L. Lions. *Research Notes in Math. 60 and 70 (1982), Pitman, London,* pp. 98-138 and pp. 154-178.

[DM G] G. DAL MASO and A. GARRONI, New results on the asymptotic behaviour of Dirichlet problems in perforated domains. *Math. Mod. Meth. Appl. Sci.* 3, 1994, pp. 373-407.

[DM Mu1] G. DAL MASO and F. MURAT, Dirichlet problems in perforated domains for homogeneous monotone operators on H_0^1. In Calculus of Variations, Homogenization and Continuum Mechanics (Proceedings, Cirm-Lumigny, Marseille, June 21-25, 1993), ed. by G. Bouchitté, G. Buttazzo, P. Suquet. Series on Advances in Mathematics for Applied Sciences 18, *World Scientific, Singapore,* 1994, pp. 177-202.

[DM Mu2] G. DAL MASO and F. MURAT, Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators. *To appear.*

[D] J. DENY, Les potentiels d'énergie finie. *Acta Math.* 82 1950, pp. 107-183.

[E G] L.C. EVANS and R.F. GARIEPY, Measure theory and fine properties of functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, 1992.

[F Z] H. FEDERER and W. ZIEMER, The Lebesgue set of a function whose distribution derivatives are p-th power summable. *Indiana U. Math. J.* 22, 1972, pp. 139-158.

[F T] S. FINZI VITA and N.A. TCHOU, Corrector results for relaxed Dirichlet problems. *Asymptotic Analysis* 5, 1992, pp. 269-281.

[F M T] S. FINZI VITA, F. MURAT and N.A. TCHOU, Quasilinear relaxed Dirichlet problems. *Siam J. Math. Anal.*, *To appear.*

[K M] H. KACIMI, F. MURAT, Estimation de l'erreur dans des problèmes de Dirichlet où apparaît un terme étrange. In Partial Differential Equations and the Calculus of Variations, Vol. II, Essays in honor of E. De Giorgi, ed. by F. Colombini, L. Modica, A. Marino, S. Spagnolo. *Progress in Nonlinear Differential Equations and their Applications*, 2, Birkhäuser, Boston, 1989, pp. 661-696.

[L P] N. LABANI and C. PICARD, Homogenization of a nonlinear Dirichlet problem in a periodically perforated domain. In Recent advances in nonlinear elliptic and parabolic problems, ed. by P. Bénilan, M. Chipot, L.C. Evans, M. Pierre. *Pitman Research Notes in Math.*, 208, *Longman, Harlow*, 1989, pp. 294-305.

[S1] I.V. SKRYPNIK, Asymptotic behaviour of solutions of nonlinear elliptic problems in perforated domains. *Math. Sbornik* 184, 10, 1993, pp. 67-90.

[S2] I.V. SKRYPNIK, Homogenization of nonlinear Dirichlet problems in perforated domains of general structure. *To appear.*

[T] L. TARTAR, Homogenization and compensated compactness, Cours Peccot, Collège de France, March 1977. Partially written in F. MURAT, H-convergence, Séminaire d'Analyse Fonctionnelle et Numérique de l'Université d'Alger 1977/1978, mimeographed notes. English translation: F. Murat, L. Tartar, H-convergence. In Mathematical Modelling of Composites Materials, ed. by R. V. Kohn. *Progress in Nonlinear Differential Equations and their Applications,* Birkhäuser, Boston. *To appear.*

[Z] W. P. ZIEMER, Weakly differentiable functions. Springer-Verlag, New York, 1989.

(Manuscript received February 1996)

J. CASADO-DÍAZ
Departamento de Ecuaciones
Diferenciales y Análisis Numérico,
Universidad de Sevilla, Facultad de Matemáticas,
C. Tarfía s/n, 41012 Sevilla, Spain.