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Annales de l'Institut Henri Poincaré – Analyse non linéaire 38 (2021) 877–909

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Semilinear problems with right-hand sides singular at $u = 0$ which change sign

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Received 23 November 2019; received in revised form 10 August 2020; accepted 7 September 2020
Available online 15 September 2020

Abstract

The present paper is devoted to the study of the existence of a solution u for a quasilinear second order differential equation with homogeneous Dirichlet conditions, where the right-hand side tends to infinity at $u = 0$. The problem has been considered by several authors since the 70's. Mainly, nonnegative right-hand sides were considered and thus only nonnegative solutions were possible. Here we consider the case where the right-hand side can change sign but is non negative (finite or infinite) at $u = 0$, while no restriction on its growth at $u = 0$ is assumed on its positive part. We show that there exists a nonnegative solution in a sense introduced in the paper; moreover, this solution is stable with respect to the right-hand side and is unique if the right-hand side is nonincreasing in u. We also show that if the right-hand side goes to infinity at zero faster than $1/|u|$, then only nonnegative solutions are possible. We finally prove by means of the study of a one-dimensional example that nonnegative solutions and even many solutions which change sign can exist if the growth of the right-hand side is $1/|u|^\gamma$ with $0 < \gamma < 1$.

MSC: 35J25

Keywords: Singular equations; Monotone operators; Existence; Uniqueness; Positive and nonpositive solutions

1. Introduction

The existence of a nonnegative solution for a singular semilinear second order partial differential problem with homogeneous Dirichlet conditions such as

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = F(x,u) \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

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https://doi.org/10.1016/j.anihpc.2020.09.001

is a classical problem which has been first considered by several authors. Here Ω is a bounded open set of $\mathbb{R}^N$, $A \in L^\infty(\Omega)^{N\times N}$ is uniformly elliptic and the right-hand side $F : \Omega \times (0, \infty) \to \mathbb{R}$ satisfies

$$\lim_{s\to 0} F(x,s) = +\infty, \quad \text{a.e. } x \in \Omega. \tag{1.2}$$

In [8], it is shown the existence and uniqueness of a nonnegative solution assuming that the limit in (1.1) is uniform in Ω and $F = F(x,s)$ decreasing in s. Indeed, in [8] the equation is not written in a divergence form and the solution is a classical solution, i.e. it is in $C^2(\Omega) \cap C^0(\bar{\Omega})$, which is strictly positive in Ω. The authors also study the behavior of the solution at the boundary and give some partial results about the existence of solution when F is not necessarily decreasing in s. In [7] it is considered the case where A is the identity and $F = F(x,s) = 1/s^\gamma + (\lambda s)^q$ with $\gamma, \lambda, q > 0$. It is shown that for every $\gamma > 0$ and every $q > 1$, there exists $\tilde{\lambda} \in (0,\infty)$ such that a solution exists if $\lambda < \tilde{\lambda}$ and there is not solution if $\lambda \geq \tilde{\lambda}$. The existence and uniqueness of classical solution for every $\lambda \geq 0$ and $p < 1$ was before proved in [19].

Looking for solutions in a Sobolev space, the problem has been considered in [3], where it is studied the case $F(x,s) = f(x)/s^\gamma$, with $\gamma > 0$, $f \geq 0$ not identically zero, $f \in L^r(\Omega)$, for some $r \geq 1$. It is proved the existence of a solution in $H^1_{loc}(\Omega) \cap W^{1,1}_0(\Omega)$, which is strictly positive in Ω. The authors also study the integrability of u and ∇u depending on γ and r. In that paper the function u is a solution in the distributional sense, that is, taking test functions in $\mathcal{D}(\Omega)$. It is proved in [1] that more general test functions can be considered and that there is just one solution which can be obtained as the limit of the solutions corresponding to replacing F by $F_n(x,s) = f_n(x)/(s + 1/n)$ where $f_n \geq 0$ increases to f. Moreover it is studied the homogenization result corresponding to replacing A in (1.1) by a sequence of matrix functions A_n.

In [13], it is considered the case where F satisfies

$$0 \leq F(x,s) \leq h(x)\left(\frac{1}{s^\gamma} + 1\right), \quad \text{a.e. } x \in \Omega, \ \forall s \geq 0, \tag{1.3}$$

with $0 < \gamma \leq 1$ and h in a certain space $L^r(\Omega)$, $r > 1$. In this case, the authors provide a new definition of nonnegative solution which does not need the use of the strong maximum principle. It is shown that this solution is stable when we replace the function F by a sequence F_n converging pointwise to F and satisfying (1.3) with h and γ independent on F. In particular, this shows that these are the solutions we find by approximating the singular function F by non-singular functions. It is also proved the uniqueness of solution if F is decreasing in s. The results are used to carry out the homogenization when the open set Ω is replaced by a sequence of open sets Ω_ε satisfying similar conditions to those which appear in [6]. The extension of the existence, stability and uniqueness results to the "strong singular"case where F satisfies the weaker assumption

$$0 \leq F(x,s) \leq \frac{h(x)}{\Gamma(s)}, \quad \text{a.e. } x \in \Omega, \ \forall s \geq 0, \tag{1.4}$$

with h as above and Γ a Lipschitz function, strictly increasing, and such that $\Gamma(0) = 0$ is carried out in [14] (see also [12] and [16]). The corresponding extension of the homogenization result for varying domains is considered in [15].

Excepting [8] where $F(x,s)$ decreasing in s implies that $F(x,s)$ can take negative values for s bigger enough, the rest of the papers we have mentioned above consider a function $F : \Omega \times [0,\infty) \to [0,\infty)$ in (1.1). Thus, by the maximum principle, if there exists a solution for (1.1), it must be nonnegative. However, taking F which can take nonnegative values introduces several questions such as: Is there still a positive solution? Can we find nonnegative solutions or even solutions changing sign? To give an answer to these questions is the first motivation of the present paper. Indeed, with respect to the existence of nonpositive solutions we must refer for example to [9], [10], where it is considered the existence of nonnegative solutions for

$$\begin{cases} -\Delta u = \lambda G(x,u) - \dfrac{1}{|u|^\beta} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

with $\lambda > 0$, G nonsingular and strictly positive (some other conditions are needed) and $\beta \in (0,1)$. Then, it is proved that a nonnegative solution exists if and only if λ is bigger than a certain $\tilde{\lambda} > 0$. Replacing u by $-u$, this provides an example of a singular function

$$F(x,s)=\frac{1}{|s|^{\beta}}-\lambda G(x,s)$$

satisfying (1.2) for which problem (1.1) has a negative solution.

In [11], it is also considered the case $F(x,s)=g(x,s)+f(x)$, with $|g(x,s)|\le\lambda|s|^{p-1}$, $0<p<1$, $f\in L^m(\Omega)$, $m>N/2$. Assuming A symmetric, the solution is defined as a minimum point for the functional

$$\frac{1}{2}\int_\Omega A\nabla u\cdot\nabla u\,dx-\int_\Omega G(x,u)\,dx,\quad G(x,s)=\int_0^s F(x,t)\,dt,$$

and can change sign. The definition of solution given by the authors uses test functions which vanish at $u=0$ and thus the equation is satisfied in $\Omega\setminus\{u=0\}$. It is proved the uniqueness for nonnegative solutions and $g(x,.)$ decreasing.

In the present paper, more generally than (1.1) we deal with the problem

$$\begin{cases}-\operatorname{div}a(x,u,\nabla u)=F(x,u) \text{ in } \Omega\\ u=0 \text{ on } \partial\Omega,\end{cases}\tag{1.6}$$

with $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ a Carathéodory function satisfying usual assumptions in such way that the operator

$$v\in W_0^{1,p}(\Omega)\mapsto-\operatorname{div}a(x,v,\nabla v)\in W^{-1,p'}(\Omega),\tag{1.7}$$

is pseudo-monotone in the sense of Leray-Lions ([17], [18]), for some $p>1$ and it is such that $a(x,0,0)=0$ (see (2.6), (2.7), (2.8) and (2.9) below for the exact assumptions on a).

In the first part of the paper we are interested in the existence of nonnegative solutions for (1.6) although F can take nonnegative values. We assume $F=F(x,s)$ a Carathéodory function from $\Omega\times[0,\infty)$ into $\mathbb{R}\cup\{\infty\}$. It is nonnegative at $s=0$ (being able to take an infinite value) and has a growth at $s\to+\infty$ at most of order $p-1$. Namely, there exists $\nu>0$ not too large such that for every $\delta>0$, there exists $k_\delta\in L^{p'}(\Omega)$, $k_\delta\ge0$, such that

$$F(x,s)\le k_\delta(x)+\nu s^{p-1},\quad \text{a.e. } x\in\Omega,\ \forall s\ge\delta.$$

Observe that this assumption is guaranteed if F satisfies (1.4). Since we do not assume F to be nonnegative, we also introduce a below estimate assuring that $F^-(x,v)$ is in $W^{-1,p'}(\Omega)$ for every $v\in W_0^{1,p}(\Omega)$ (see (2.15)). With these general hypotheses we show that a nonnegative solution still exists. In fact, similarly to [13] and [14] we introduce a suitable definition of solution (it is in particular a solution in the distribution sense) for which we show existence, stability with respect to the right-hand side and uniqueness for $F(x,s)$ non increasing in s and a independent of s. The estimates we use to get the result are inspired by [13] and [14] although we use more general test functions. In particular, we observe that due to the nonlinearity of the operator given by (1.7), some duality arguments used in the choice of the test functions in [13] and [14] cannot be used in the present setting.

In the second part of the paper we consider the question relative to the existence of nonpositive solutions or even solutions changing sign. Thus, the function F is now assumed to be defined in $\Omega\times\mathbb{R}$. In Section 4 we show that if there exists $\delta,\tau>0$ such that

$$F(x,s)\ge\frac{\tau}{|s|},\qquad\forall s\in(-\delta,0),\tag{1.8}$$

then every solution of (1.6) is necessarily nonnegative. That is, although, now F can take nonnegative values, the fact that it is very large near $s=0$ is enough to preclude the existence of solutions taking negative values in a portion of Ω. Taking in particular F of the form

$$F(x,s)=\frac{f(x)}{|s|^{\gamma}}-g(x),\tag{1.9}$$

with $f(x)\ge\rho$ a.e. in Ω for some $\rho>0$, and $\gamma\ge1$, we get that all the solutions for (1.6) are nonnegatives. Now, the question is What happens if $\gamma<1$? As we said above, references [9], and [10] show that it is possible to get nonpositive solutions. Here we give in Section 5 a different example in the simple case $\Omega=(0,l)\subset\mathbb{R}$, $a(x,s,\xi)=\xi$ and F given by (1.9) with f and g positive constants and $\gamma\in(0,1)$. By a suitable change of variables, the problem is transformed into

$$\begin{cases} -w'' = \dfrac{1}{|w|^{\gamma}} - 1 \ \text{ in } (0, l) \\ w(0) = w(l) = 0. \end{cases} \tag{1.10}$$

We show that although there is a unique strictly positive solution, there also exist nonpositive solutions and even solutions changing sign, but only if l is large enough. In fact the number of solutions increases with l and tends to infinity when l tends to infinity.

The paper is organized as follows:

- In Section 2 we show the existence of nonnegative solutions for (1.6) in the singular case and the stability and uniqueness of these solutions.
- In Section 3 we prove the results corresponding to Section 2.
- In Section 4 we show that (1.8) implies that only nonnegative solutions are possible. The proof of these results is given at the end of the Section.
- In Subsection 5.1 we state the results corresponding to the one-dimensional example. The corresponding proofs are given in Subsection 5.2.

2. Existence, stability and uniqueness of nonnegative solutions for the singular problem

The present section is devoted to the existence, in a sense which we describe below, of nonnegative solutions for problem

$$\begin{cases} -\operatorname{div} a(x, u, \nabla u) = F(x, u) \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where $F = F(x, s)$ can take the value $+\infty$ at $s = 0$. We also give a stability result with respect to F for this kind of solutions. To finish, we show a uniqueness result when F is decreasing in s and $a = a(x, s, \xi)$ does not depend on s and is strictly monotone in ξ. The proof of these results will be given in Section 3.

We assume the following conditions on Ω, p, $a = a(x, s, \xi)$ and $F = F(x, s)$:

• For Ω and p we assume:

$$\Omega \text{ is a bounded open set in } \mathbb{R}^N, \ N \geq 1. \tag{2.2}$$

$$p \in (1, \infty). \tag{2.3}$$

Moreover, we define $p^* \in (1, \infty]$, $p' \in (1, \infty)$ and $(p^*)' \in [1, \infty)$ by

$$\begin{cases} p^* = \dfrac{Np}{N - p} & \text{if } p < N \\ p^* \ \text{ is a fixed number in } (1, \infty) & \text{if } p = N \\ p^* = +\infty & \text{if } p > N, \end{cases} \tag{2.4}$$

$$p' = \frac{p}{p - 1}, \qquad (p^*)' = \frac{p^*}{p^* - 1}. \tag{2.5}$$

• For the function $a : \Omega \times [0, \infty) \times \mathbb{R}^N \to \mathbb{R}^N$ we assume:

∘ The function a is a Carathéodory function, i.e.

$$\begin{cases} a(., s, \xi) \ \text{ is measurable in } \Omega, \ \forall (s, \xi) \in [0, \infty) \times \mathbb{R}^N \\ a(x, ., .) \ \text{ is continuous in } [0, \infty) \times \mathbb{R}^N, \ \text{a.e. } x \in \Omega. \end{cases} \tag{2.6}$$

∘ There exist α, γ, a_0 such that

$$\begin{cases} \alpha > 0, \ \gamma \geq 0, \ a_0 \in L^{(p^*)'}(\Omega), \ a_0 \geq 0 \\ a(x, s, \xi) \cdot \xi \geq \alpha |\xi|^p - \gamma s^p - a_0(x) s, \quad \forall (s, \xi) \in [0, \infty) \times \mathbb{R}^N, \ \text{a.e. } x \in \Omega. \end{cases} \tag{2.7}$$

∘ There exist β, b such that

$$\begin{cases} \beta > 0,\ b \in L^p(\Omega),\ b \geq 0, \\ |a(x,s,\xi)| \leq \beta\big(|\xi| + s + b(x)\big)^{p-1}, \quad \forall\, (s,\xi) \in [0,\infty) \times \mathbb{R}^N,\ \text{a.e. } x \in \Omega. \end{cases} \tag{2.8}$$

∘ The function a is monotone in the second variable, i.e.

$$\big(a(x,s,\xi) - a(x,s,\eta)\big) \cdot (\xi - \eta) \geq 0, \quad \forall s \in [0,\infty),\ \forall \xi, \eta \in \mathbb{R}^N,\ \text{ a.e. } x \in \Omega. \tag{2.9}$$

Moreover, we define λ by

$$\lambda = \liminf_{R\to\infty} \min \left\{ \frac{R}{\int_\Omega |v|^p dx} : v \in W_0^{1,p}(\Omega) \setminus \{0\},\ \int_\Omega \big(a(x,v,\nabla v)\cdot \nabla v + \gamma |v|^p\big) dx = R \right\}. \tag{2.10}$$

• For the function $F : \Omega \times [0,\infty) \to \mathbb{R} \cup \{+\infty\}$, we assume:
∘ The function F is a Carathéodory function, i.e.

$$F(.,s) \text{ is measurable in } \Omega,\ \forall s \in [0,\infty), \quad F(x,.) \text{ is continuous in } [0,\infty),\ \text{a.e. } x \in \Omega. \tag{2.11}$$

∘

$$F(x,0) \geq 0, \quad \text{a.e. } x \in \Omega. \tag{2.12}$$

∘ There exists $\nu \geq 0$ such that for every $\delta > 0$, there exists k_δ satisfying

$$k_\delta \in L^{(p^*)'}(\Omega),\ k_\delta \geq 0, \qquad F(x,s) \leq k_\delta(x) + \nu s^{p-1},\ \forall s \geq \delta,\ \text{a.e. } x \in \Omega. \tag{2.13}$$

∘ The constants γ in (2.7), ν in (2.13), and λ defined by (2.10) are related by

$$\gamma + \nu < \lambda. \tag{2.14}$$

∘

$$\begin{aligned} &\text{If } p \leq N, \quad \begin{cases} \exists \tilde{\nu} \geq 0,\ \tilde{k} \in L^{(p^*)'}(\Omega),\ \tilde{k} \geq 0 \\ F(x,s) \geq -\tilde{k}(x) - \tilde{\nu} s^{p^*},\ \forall s \geq 0,\ \text{a.e. } x \in \Omega \end{cases} \\ &\text{If } p > N, \quad \begin{cases} \forall m \in \mathbb{N},\ \exists \tilde{k}_m \in L^1(\Omega),\ \tilde{k}_m \geq 0, \\ F(x,s) \geq -\tilde{k}_m(x),\ \forall s \in [0,m],\ \text{a.e. } x \in \Omega. \end{cases} \end{aligned} \tag{2.15}$$

Remark 2.1. The minimum in (2.10) is well defined, thanks to (2.7).

If a does not depend on s and satisfies the homogeneity condition

$$a(x,t\xi) = t^{p-1} a(x,\xi), \quad \forall \xi \in \mathbb{R}^N,\ \forall t \geq 0,\ \text{ a.e. } x \in \Omega,$$

then λ agrees with the first eigenvalue of the operator $v \in W_0^{1,p}(\Omega) \mapsto -\operatorname{div} a(x,\nabla v)$, defined by

$$\lambda = \min_{\substack{v \in W_0^{1,p}(\Omega) \\ v \neq 0}} \frac{\int_\Omega a(x,\nabla v)\cdot\nabla v\, dx}{\int_\Omega |v|^p dx}.$$

Moreover, if λ_p denotes the first eigenvalue of the p-Laplacian operator, then, we have

$$\alpha \lambda_p \leq \lambda. \tag{2.16}$$

This inequality can be proved as follows: We take a sequence R_n tending to infinity and a sequence $u_n \in H_0^1(\Omega)$ such that

$$R_n = \int_\Omega \big(a(x,u_n,\nabla u_n)\cdot\nabla u_n + \gamma |u_n|^p\big) dx, \quad \lim_{n\to\infty} \frac{R_n}{\int_\Omega |u_n|^p dx} = \lambda. \tag{2.17}$$

In particular $\|u_n\|_{L^p(\Omega)}$ tends to infinity while the first equality, (2.7) and the definition of λ_p gives

$$\lambda \geq \limsup_{n\to\infty} \frac{\int_\Omega (\alpha|\nabla u_n|^p - a_0|u_n|)dx}{\int_\Omega |u_n|^p dx} \geq \alpha\lambda_p.$$

From (2.16) we deduce that a sufficient condition to have (2.14) is to assume

$$\gamma + \nu < \alpha\lambda_p.$$

Remark 2.2. Condition (2.7) on a implies

$$a(x,0,0) = 0, \quad \text{a.e. } x \in \Omega. \tag{2.18}$$

In fact, taking in (2.7) $s = 0$, replacing ξ by $t\xi$ with $t > 0$, and dividing by t, we get

$$a(x,0,t\xi)\cdot\xi \geq \alpha t^{p-1}|\xi|^p, \quad \forall \xi \in \mathbb{R}^N,\ \forall t > 0,\ \text{a.e. } x \in \Omega.$$

Letting t tend to zero, this provides

$$a(x,0,0)\cdot\xi \geq 0, \quad \forall \xi \in \mathbb{R}^N, \ \text{a.e. } x \in \Omega,$$

and then (2.18).

The definition of nonnegative solution for the singular problem (2.1) is as follows

Definition 2.3. For a bounded open set $\Omega \subset \mathbb{R}^N$ and functions a, F satisfying (2.6), ..., (2.9), (2.11), ..., (2.15), we say that $u : \Omega \to \mathbb{R}$ is a nonnegative solution of (2.1) if it satisfies

$$u \in L^{p^*}(\Omega) \tag{2.19}$$

$$u \geq 0 \ \text{ a.e. in } \Omega \tag{2.20}$$

$$(u-\delta)^+ \in W_0^{1,p}(\Omega), \quad \forall \delta > 0 \tag{2.21}$$

$$\varphi\nabla u \in L^p(\Omega)^N, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \tag{2.22}$$

$$F(x,u)^+\varphi^p \in L^1(\Omega), \quad \forall \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\ \varphi \geq 0 \tag{2.23}$$

$$\begin{cases} \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla\big(h(u)\varphi^p\big)\,dx = \int_\Omega F(x,u)h(u)\varphi^p\,dx \\ \forall \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\ \varphi \geq 0,\ \forall h \in W^{1,\infty}(0,\infty). \end{cases} \tag{2.24}$$

Remark 2.4. Assumption (2.15) on F and (2.19) implies that $F(x,u)^-$ belongs to $L^1(\Omega)$ for every nonnegative solution u of (2.1). By (2.23), this proves

$$F(x,u)\varphi^p \in L^1(\Omega), \quad \forall\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\ \varphi \geq 0. \tag{2.25}$$

Moreover, using

$$\nabla(h(u)\varphi^p) = h'(u)\nabla u\,\varphi^p + p\,h(u)\varphi^{p-1}\nabla\varphi,$$

and assumptions (2.8), (2.22), we also have that $a(x,\nabla u)\nabla(h(u)\varphi)$ is in $L^1(\Omega)$ for every φ and h in the conditions of (2.24). Thus, the two integrals which appear in (2.24) have a sense.

Assumption (2.21) in combination with (2.20) gives

$$0 \leq u < \delta \ \text{ on } \partial\Omega, \quad \forall\delta > 0,$$

which gives a sense to the boundary condition $u = 0$ on $\partial\Omega$.

The following proposition shows that in (2.24) we can enlarge the set of test functions by introducing

$$W=\Big\{w\in W^{1,p}(\Omega),\quad \exists\,\varphi\in W^{1,p}_0(\Omega)\cap L^\infty(\Omega),\ \varphi\ge 0 \text{ a.e. in } \Omega,\\ |w|\le\varphi^p\ \text{ a.e. in } \Omega,\ \frac{|\nabla w|}{\varphi^{p-1}}\chi_{\{\varphi\neq 0\}}\in L^p(\Omega)\Big\}. \tag{2.26}$$

Observe that contrarily to the test functions used in (2.24), this set does not depend on the solution u. We have preferred to give the definition of solution taking test functions of the form $h(u)\varphi^p$ with $\varphi\in W^{1,p}_0(\Omega)\cap L^\infty(\Omega)$, $\varphi\in W^{1,p}_0(\Omega)$, $h\in W^{1,\infty}(0,\infty)$, instead of functions in W, in order to compare with the definition of solution given in [13] and [14].

Proposition 2.5. *The space W is a vectorial subspace of $W^{1,p}_0(\Omega)$. It contains the space of functions in $W^{1,p}(\Omega)\cap L^\infty(\Omega)$ with compact support in Ω and satisfies*

$$wv\in W,\quad \forall\, w\in W,\ \forall\, v\in W^{1,p}(\Omega)\cap L^\infty(\Omega). \tag{2.27}$$

If a function $u:\Omega\to\mathbb{R}$ is a nonnegative solution of (2.1) in the sense established in (2.3), then it also satisfies

$$F(x,u)=0,\ \ a.e.\ in\ \{u=0\} \tag{2.28}$$

$$|\nabla u|^{p-1}|\nabla w|\in L^1(\Omega),\quad \forall\, w\in W \tag{2.29}$$

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla w\,dx=\int_\Omega F(x,u)w\,dx,\ \ \forall\, w\in W. \tag{2.30}$$

Remark 2.6. Proposition 2.5 proves in particular that every nonnegative solution of (2.1) in the sense given by Definition 2.3 is a solution in the distribution sense, i.e. we have

$$-\operatorname{div} a(x,u,\nabla u)=F(x,u)\ \text{ in } \mathcal{D}'(\Omega). \tag{2.31}$$

By (2.22), we also know that u is in $W^{1,p}_{loc}(\Omega)$ and then, for every open set ω strictly contained in Ω, we have that $\operatorname{div} a(x,u,\nabla u)$ is in $W^{-1,p'}(\omega)$ while by (2.23) $F(x,u)$ is in $L^1(\omega)$. It is well known that then (2.31) implies that $F(x,u)v$ belongs to $L^1(\Omega)$ for every $v\in W^{1,p}(\Omega)$ with compact support and

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla v\,dx=\int_\Omega F(x,u)v\,dx,\quad \forall\, v\in W^{1,p}(\Omega)\ \text{ with compact support in } \Omega. \tag{2.32}$$

The following theorem provides a stability result for the nonnegative solutions of (2.1) when the right-hand side varies.

Theorem 2.7. *We consider a bounded open set $\Omega\subset\mathbb{R}^N$, a function $a:\Omega\times[0,\infty)\times\mathbb{R}^N\to\mathbb{R}^N$, satisfying (2.6), (2.7), (2.8) and (2.9), and a sequence of functions $F_n:\Omega\times\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ such that (2.11), (2.12), (2.13) (2.14) and (2.15) hold, with constants $\nu,\tilde\nu$ and functions k_δ, $\tilde k$ and $\tilde k_m$ independent of n. We also assume the existence of $F:\Omega\times\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ such that*

$$For\ a.e.\ x\in\Omega,\ we\ have:\ t_n\ge 0,\ t_n\to t\Longrightarrow F_n(x,t_n)\to F(x,t). \tag{2.33}$$

Then, if u_n is a sequence of nonnegative solutions of

$$\begin{cases}-\operatorname{div} a(x,u_n,\nabla u_n)=F_n(x,u_n)\ \ in\ \Omega\\ u_n=0\ \ in\ \Omega,\end{cases} \tag{2.34}$$

there exists a nonnegative solution u of (2.1) such that for a subsequence of n, still denoted by n, we have

$$u_n\to u\ \ in\ L^q(\Omega),\ \ \forall\, q<p^* \tag{2.35}$$

$$u_n\rightharpoonup u\ \ in\ L^{p^*}(\Omega) \tag{2.36}$$

$$(u_n-\delta)^+\rightharpoonup(u-\delta)^+\ \ in\ W^{1,p}_0(\Omega),\ \ \forall\,\delta>0. \tag{2.37}$$

Thanks to Theorem 2.7 we will deduce the existence of nonnegative solutions for (2.1) which is given in the following theorem.

Theorem 2.8. *Let Ω be a bounded open set of $\mathbb{R}^N$. Then, for every function $a : \Omega \times [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ which satisfies (2.6), (2.7), (2.8) and (2.9), and every function F which satisfies (2.11), (2.12), (2.13), (2.14) and (2.15), there exists a nonnegative solution of problem (2.1) in the sense of Definition 2.3.*

We finish this section with the following comparison result for the nonnegative solutions of (2.1) when the operator $a = a(x,s,\xi)$ does not depend on s. It proves in particular the uniqueness of nonnegative solutions when a is strictly monotone in ξ and $F = F(x,s)$ is nondecreasing in s. In the case when a depends on s it is also possible to extend some uniqueness results for pseudomonotone problems (see e.g. [2], [4], [5]), but the corresponding proofs are more complicated.

Theorem 2.9. *Let Ω be a bounded open set of $\mathbb{R}^N$ and let $a : \Omega \times [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ be independent on s and satisfying (2.6), (2.7), (2.8) and*

$$(a(x,\xi_1) - a(x,\xi_2)) \cdot (\xi_1 - \xi_2) > 0, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^N, \ a.e.\ x \in \Omega. \tag{2.38}$$

We consider two functions F_1 and F_2 which satisfy (2.11), (2.12), (2.13) and (2.15). We also assume

$$\exists i \in \{1,2\} \ \text{ such that } F_i(x,\cdot) \text{ is nonincreasing}, \ a.e.\ x \in \Omega \tag{2.39}$$

$$F_1(x,s) \le F_2(x,s), \ \ \forall s \in [0,\infty), \ a.e.\ x \in \Omega. \tag{2.40}$$

Then, if u_1, u_2 are nonnegative solutions of

$$\begin{cases} -\operatorname{div} a(x,\nabla u_i) = F_i(x,u_i) \ \text{ in } \Omega \\ u = 0 \ \text{ in } \Omega, \end{cases} \qquad i = 1,2, \tag{2.41}$$

we have

$$u_1 \le u_2 \ a.e.\ in\ \Omega. \tag{2.42}$$

3. Proof of the stability, existence and uniqueness results

Let us prove in this section the different results exposed in Section 2 relative to the existence and properties of the nonnegative solutions for the singular problem (2.1).

Proof of Proposition 2.5. In order to show that W is contained in $W_0^{1,p}(\Omega)$, we consider $w \in W$ and $\varphi \in W_0^{1,p}(\Omega)$ such that

$$w \in W, \ \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \ \ \varphi \ge 0, \quad |w| \le \varphi^p \text{ a.e. in } \Omega, \ \frac{|\nabla w|}{\varphi^{p-1}} \chi_{\{w \ne 0\}} \in L^p(\Omega). \tag{3.1}$$

Then, taking $\varphi_n \in W^{1,p}(\Omega)$, $\varphi_n \ge 0$, with compact support which converges to φ in $W_0^{1,p}(\Omega)$ and is bounded in $L^\infty(\Omega)$, it is immediate to show that the sequence

$$w_n = \left[w \vee (-\varphi_n^p)\right] \wedge \varphi_n^p \in W_0^{1,p}(\Omega),$$

converges to w in $W^{1,p}(\Omega)$. This proves that w is in $W_0^{1,p}(\Omega)$.

Now, we consider $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support. Let us prove that w is in W. It is enough to observe that for every $\varphi \in C_c^1(\Omega)$ with $\varphi \ge \|w\|_{L^\infty(\Omega)}$ in $\operatorname{supp} w$, the functions w and φ satisfy (3.1).

Let us prove that W is stable by addition. For this purpose, we take w_1, φ_1 and w_2, φ_2 which are related as in (3.1), then taking into account that for $i = 1,2$ $\nabla w_i = 0$ a.e. in $\{w_i = 0\} \subset \{\varphi_i = 0\}$ it is simple to check that $w_1 + w_2$, $\varphi_1 + \varphi_2$ also satisfy (3.1).

To show (2.27), we take $w \in W$ and $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, which we can assume not identically zero, then we observe that for φ satisfying (3.1), we have

$$|wv| \le \left(\varphi \|v\|_{L^\infty(\Omega)}^{\frac{1}{p}}\right)^p, \quad \frac{|\nabla(wv)|}{\|v\|_{L^\infty(\Omega)}^{\frac{p-1}{p}} \varphi^{p-1}} \chi_{\{\varphi\neq 0\}} \le \left(\frac{|\nabla w|}{\varphi^{p-1}} \|v\|_{L^\infty(\Omega)}^{\frac{1}{p}} + \frac{|\nabla v|\varphi}{\|v\|_{L^\infty(\Omega)}^{\frac{p-1}{p}}} \right) \chi_{\{\varphi\neq 0\}} \in L^p(\Omega).$$

Assume now u a nonnegative solution of (2.1) in the sense of Definition 2.3 and consider a function $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\varphi \ge 0$ a.e. in Ω. For $S_\varepsilon : [0, \infty) \to \mathbb{R}$ defined by

$$S_\varepsilon(s) = \left(1 - \frac{s}{\varepsilon}\right) \vee 0, \quad \forall s \ge 0, \tag{3.2}$$

we take $h = S_\varepsilon$ in (2.24) to deduce

$$\begin{aligned} &-\frac{1}{\varepsilon} \int_{\{u<\varepsilon\}} a(x,u,\nabla u) \cdot \nabla u \, \varphi^p dx + p \int_\Omega a(x,u,\nabla u) \cdot \nabla \varphi \, S_\varepsilon(u) \varphi^{p-1} dx \\ &= \int_\Omega F(x,u) S_\varepsilon(u) \varphi^p dx. \end{aligned} \tag{3.3}$$

In the first term of this equality we use (2.7) and (2.18) to deduce

$$\frac{1}{\varepsilon} \int_{\{u<\varepsilon\}} a(x,u,\nabla u) \cdot \nabla u \, \varphi^p dx \ge \frac{\alpha}{\varepsilon} \int_{\{0<u<\varepsilon\}} |\nabla u|^p \varphi^p dx - \int_{\{0<u<\varepsilon\}} (\gamma \varepsilon^{p-1} + a_0) \varphi^p dx$$

and then

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\{0<u<\varepsilon\}} a(x,u,\nabla u) \cdot \nabla u \, \varphi^p dx \ge \alpha \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\{0<u<\varepsilon\}} |\nabla u|^p \varphi^p dx.$$

For the second term, we observe that (2.8), (2.22) and Hölder's inequality imply that the function $a(x,u,\nabla u) \cdot \nabla\varphi \, \varphi^{p-1}$ belongs to $L^1(\Omega)$. Using then that $S_\varepsilon(u)$ converges a.e. to $\chi_{\{u=0\}}$ and (2.18), and the Lebesgue dominated convergence theorem we deduce

$$\lim_{\varepsilon\to 0} \int_\Omega a(x,u,\nabla u) \cdot \nabla \varphi \, S_\varepsilon(u) \varphi^{p-1} dx = 0. \tag{3.4}$$

For the third term in (3.3), we use (2.25), which allows us to apply Lebesgue dominated convergence theorem to get

$$\lim_{\varepsilon\to 0} \int_\Omega F(x,u) S_\varepsilon(u) \varphi^p dx = \int_{\{u=0\}} F(x,0) \varphi^p dx.$$

Therefore, taking the limit in (3.5) when ε tends to zero we obtain

$$0 \ge \int_{\{u=0\}} F(x,0) \varphi^p dx + \alpha \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\{0<u<\varepsilon\}} |\nabla u|^p \varphi^p dx,$$

which by (2.12) shows (2.28) and

$$\lim_{\varepsilon\to\infty} \frac{1}{\varepsilon} \int_{\{0<u<\varepsilon\}} |\nabla u|^p \varphi^p \, dx = 0, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \ \varphi \ge 0. \tag{3.5}$$

Let us now prove (2.29) and (2.30), for $w \in W$. Decomposing $w = w^+ + w^-$, where both w^+ and w^- belong to W, we can assume w nonnegative. We take φ such that (3.1) holds. Using

$$|\nabla u|^{p-1} |\nabla w| \le \left(|\nabla u|^{p-1} \varphi^{p-1}\right) \left(\frac{|\nabla w|}{\varphi^{p-1}} \chi_{\{\varphi\neq 0\}}\right), \quad \text{a.e. in } \Omega$$

and taking into account (2.22) and (3.1) we conclude that (2.29) is a simple consequence of Hölder's inequality.

For $\varepsilon \in (0,1)$, we define $Z_\varepsilon \in W^{1,\infty}(0,\infty)$ by

$$Z_\varepsilon(s) = \left(\frac{s}{\varepsilon} - 1\right)^+ \wedge 1. \tag{3.6}$$

Then, for $\delta > 0$, we use (2.24) with

$$\varphi = \left(\left(\frac{2s}{\varepsilon} - 1\right)^+ \wedge 1\right)(w+\delta)^{\frac{1}{p}} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad h = Z_\varepsilon$$

Observing that

$$h(u)\varphi^p = Z_\varepsilon(u)(w+\delta),$$

we get

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla w\, Z_\varepsilon(u)\,dx + \frac{1}{\varepsilon}\int_{\{\varepsilon<u<2\varepsilon\}} a(x,u,\nabla u)\cdot\nabla u\,(w+\delta)\,dx$$
$$= \int_\Omega F(x,u)\,(w+\delta)Z_\varepsilon(u)\,dx.$$

Since $Z_\varepsilon(u)$ vanishes on the set $\{u < \varepsilon\}$ and (2.21), (2.13) and (2.15) hold, we can pass to the limit when δ tends to zero, to get

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla w\, Z_\varepsilon(u)\,dx + \frac{1}{\varepsilon}\int_{\{\varepsilon<s<2\varepsilon\}} a(x,u,\nabla u)\cdot\nabla u\, w\,dx = \int_\Omega F(x,u)\, wZ_\varepsilon(u)\,dx.$$

Thanks to (2.25), (2.8), (2.29), (3.1), (3.5) and Lebesgue dominated convergence theorem we can pass to the limit in this equality to deduce (2.30). □

Proof of Theorem 2.7. The first part of the theorem is devoted to obtain some a priori estimates for u_n.

Taking into account the definition of λ given by (2.10) and (2.14), we can fix in the proof $R_0, \varepsilon > 0$ such that

$$\begin{cases} (\nu+\gamma)\displaystyle\int_\Omega |v|^p dx \le (1-\varepsilon)\int_\Omega \big(a(x,v,\nabla v)\cdot\nabla v + \gamma|v|^p\big)\,dx \\ \forall\, v \in W_0^{1,p}(\Omega) \text{ with } \displaystyle\int_\Omega \big(a(x,v,\nabla v)\cdot\nabla v + \gamma|v|^p\big)\,dx \ge R_0. \end{cases} \tag{3.7}$$

For $\delta, k > 0$, we take $(u_n-\delta)^+ \wedge k \in W$, as test function in (2.34) which gives

$$\int_{\{k+\delta>u_n>\delta\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx = \int_\Omega F_n(x,u_n)\big((u_n-\delta)^+\wedge k\big)dx.$$

Here, thanks to (2.21), (2.15) and (2.23), we have that $F_n(x,u_n)(u_n-\delta)^+$ belongs to $L^1(\Omega)$. Thus, we can pass to the limit when k tends to infinity to get

$$\int_{\{u_n>\delta\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx = \int_\Omega F_n(x,u_n)(u_n-\delta)^+dx.$$

Taking into account that for every $t > 0$, there exists $C_t > 0$ such that

$$x^p \le (1+t)(x-\delta)^p + C_t\delta^p, \quad \forall\, x > \delta > 0, \tag{3.8}$$

and defining the Sobolev constant C_Ω by (C_Ω does not depend on Ω if $p < N$)

$$\|v\|_{L^{p^*}(\Omega)} \le C_\Omega\|\nabla v\|_{L^p(\Omega)}, \quad \forall\, v \in W_0^{1,p}(\Omega), \tag{3.9}$$

we have for every $t > 0$

$$\begin{aligned}&\int_{\{u_n>\delta\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx = \int_\Omega F_n(x,u_n)(u_n-\delta)^+\,dx\\ &\le \int_\Omega \left(k_\delta + \nu u_n^{p-1}\right)(u_n-\delta)^+dx \le \frac{1}{p't^{p'}}\|k_\delta\|^{p'}_{L^{(p^*)'}(\Omega)} + \frac{t^pC^p_\Omega}{p}\int_{\{u_n>\delta\}}|\nabla u_n|^p dx\\ &+\nu(1+t)\int_{\{u_n>\delta\}}(u_n-\delta)^p dx + \nu C_t\delta^p|\Omega|.\end{aligned} \tag{3.10}$$

If

$$\int_{\{u_n>\delta\}} \left(a(x,u_n,\nabla u_n)\cdot\nabla u_n + \gamma u_n^p\right)dx \le R_0, \tag{3.11}$$

then, thanks to (2.7) we easily get

$$(u_n-\delta)^+ \text{ is bounded in } W^{1,p}_0(\Omega). \tag{3.12}$$

If (3.11) does not hold, using (3.7) with $v = (u_n-\delta)^+$ we deduce from (3.10)

$$\begin{aligned}&\left(1-(1-\varepsilon)(1+t)\right)\int_{\{u_n>\delta\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx\\ &\le \frac{1}{p't^{p'}}\|k_\delta\|^{p'}_{L^{(p^*)'}(\Omega)} + \frac{t^pC^p_\Omega}{p}\int_{\{u_n>\delta\}}|\nabla u_n|^p dx + \nu C_t\delta^p|\Omega|,\end{aligned}$$

which using (2.7) and taking t small enough to have

$$\alpha\left(1-(1-\varepsilon)(1+t)\right) - \frac{t^pC^p_\Omega}{p} > 0,$$

proves that (3.12) is also true when (3.11) does not hold.

Observe that decomposing u_n as $u_n = (u_n \wedge 1) + (u_n-1)^+$, estimates (3.12) and (3.9) also imply

$$u_n \text{ is bounded in } L^{p^*}(\Omega). \tag{3.13}$$

We have obtained an estimate for ∇u_n on the set $\{u_n > \delta\}$. Let us now obtain an estimate on the set $\{u_n < \delta\}$. For this purpose, we take $(\delta-u_n)^+\varphi^p$ with $\varphi \in W^{1,p}_0(\Omega)\cap L^\infty(\Omega)$, $\varphi \ge 0$ a.e. in Ω, as test function in (2.34). We get

$$\begin{aligned}&-\int_{\{u_n<\delta\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,\varphi^p dx + p\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi\,\varphi^{p-1}(\delta-u_n)^+dx\\ &=\int_\Omega F_n(x,u_n)(\delta-u_n)^+\varphi^p\,dx.\end{aligned} \tag{3.14}$$

Taking into account (2.7), (2.8) and $\nabla u_n = 0$ a.e. in $\{u_n = 0\}$, we deduce

$$\begin{aligned}&\frac{\alpha}{\delta}\int_{\{u_n<\delta\}}|\nabla u_n|^p\varphi^p dx + \int_\Omega F_n^+(x,u_n)\left(1-\frac{u_n}{\delta}\right)^+\varphi^p dx\\ &\le \int_{\{u_n<\delta\}}(\gamma\delta^{p-1}+a_0)\varphi^p dx + \beta\int_{\{u_n<\delta\}}\left(|\nabla u_n|+\delta+b\right)^{p-1}|\nabla\varphi|\varphi^{p-1}dx\\ &+\int_{\{u_n<\delta\}} F_n^-(x,u_n)\varphi^p\,dx,\end{aligned} \tag{3.15}$$

which using (2.15) easily implies the existence of $C > 0$ such that

$$\frac{1}{\delta}\int_{\{u_n<\delta\}} |\nabla u_n|^p \varphi^p dx + \int_\Omega F_n^+(x,u_n)\Big(1-\frac{u_n}{\delta}\Big)^+ \varphi^p dx \leq C\left(\|\varphi\|^p_{L^\infty(\Omega)} + \|\nabla\varphi\|^p_{L^p(\Omega)^N}\right)$$
$$\forall \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\ \varphi \geq 0 \text{ a.e. in } \Omega, \quad \forall \delta \in (0,1). \tag{3.16}$$

From this inequality, (3.12) and (2.13) applied to F_n, we deduce in particular

$$|\nabla u_n|\varphi \ \text{ bounded in } L^p(\Omega),\ F_n(x,u_n)^+\varphi^p \text{ bounded in } L^1(\Omega),\ \forall \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega). \tag{3.17}$$

Taking into account (3.12), (3.13), (2.33), Rellich-Kondrachov's compactness theorem and Fatou's lemma we deduce the existence of a subsequence of u_n, still denoted by u_n, and a function u such that

$$\begin{cases} u \in L^{p^*}(\Omega), \quad u \geq 0 \text{ a.e. in } \Omega, \quad (u-\delta)^+ \in W_0^{1,p}(\Omega),\ \forall \delta > 0 \\ |\nabla u|\varphi \in L^p(\Omega), \quad F(x,u)^+\varphi \in L^1(\Omega), \quad \forall \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \end{cases} \tag{3.18}$$

and such that (2.35), (2.36) and (2.37) hold.

Returning now to (3.15) and using (2.15) applied to F_n, (2.33), (2.12) and the Rellich-Kondrachov compactness theorem we deduce

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\left(\frac{1}{\delta}\int_{\{u_n<\delta\}} |\nabla u_n|^p\varphi^p dx + \int_\Omega F_n^+(x,u_n)\Big(1-\frac{u_n}{\delta}\Big)^+\varphi^p dx\right) = 0, \tag{3.19}$$

for every $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\varphi \geq 0$.

Let us now prove that u is a nonnegative solution of (2.1). By (3.18), it just remains to show (2.24). Even more, let us prove (2.30).

Taking into account (3.17) and (2.8), we can assume the existence of $\sigma \in L^{p'}_{loc}(\Omega)^N$, such that

$$a(x,u_n,\nabla u_n)\,\varphi^{p-1} \rightharpoonup \sigma\,\varphi^{p-1} \ \text{ in } L^{p'}(\Omega)^N, \quad \forall \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\ \varphi \geq 0. \tag{3.20}$$

Taking $w \in W$ as test function in (2.34), we get

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla w\,dx = \int_\Omega F_n(x,u_n)\,w\,dx. \tag{3.21}$$

In the left-hand side of this equality we use

$$a(x,u_n,\nabla u_n)\cdot\nabla w = \left(a(x,u_n,\nabla u_n)\varphi^{p-1}\right)\cdot\left(\frac{\nabla w}{\varphi^{p-1}}\right)\chi_{\{\varphi\neq 0\}},$$

with $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ associated to w following Definition (2.26) of W. Thanks to (3.20) we can then pass to the limit when n tends to infinity, to get

$$\lim_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla w\,dx = \int_\Omega \sigma\cdot\nabla w\,dx. \tag{3.22}$$

In the right-hand side of (3.21), we use (2.36), (2.37), (3.19) and assumptions (2.13), (2.15) applied to F_n to also obtain

$$\lim_{n\to\infty}\int_\Omega F_n(x,u_n)\,w\,dx = \int_\Omega F(x,u)\,w\,dx.$$

Therefore, we have shown

$$\int_\Omega \sigma\cdot\nabla w\,dx = \int_\Omega F(x,u)w\,dx, \quad \forall w \in W. \tag{3.23}$$

To finish, let us prove

$$\sigma = a(x, u, \nabla u) \text{ a.e. in } \Omega, \tag{3.24}$$

which combined with (3.18) and (3.23) will prove that u is a nonnegative solution of (2.1).

Taking $m > 0$ and $(u_n \wedge m)\varphi$, with $\varphi \in C_c^\infty(\Omega)$ as test function in (2.34), we have

$$\int_{\{u_n<m\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n \, \varphi \, dx + \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla \varphi \, (u_n \wedge m) \, dx$$
$$= \int_\Omega F_n(x, u_n)(u_n \wedge m) \, \varphi \, dx,$$

where we can pass to the limit thanks to (2.35), (2.36), (2.37), (2.33) and (3.19) to get

$$\lim_{n\to\infty} \int_{\{u_n<m\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n \, \varphi \, dx + \int_\Omega \sigma \cdot \nabla \varphi \, (u \wedge m) \, dx = \int_\Omega F(x, u)(u \wedge m) \, \varphi \, dx.$$

On the other hand, using $w = (u \wedge m)\varphi$ in (3.23), we have

$$\int_{\{u<m\}} \sigma \cdot \nabla u \, \varphi \, dx + \int_\Omega \sigma \cdot \nabla \varphi \, (u \wedge m) \, dx = \int_\Omega F(x, u)(u \wedge m) \, \varphi \, dx.$$

Thus, we have proved

$$\lim_{n\to\infty} \int_{\{u_n<m\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n \, \varphi \, dx = \int_{\{u<m\}} \sigma \cdot \nabla u \, \varphi \, dx, \quad \forall m > 0.$$

This equality allows us to use the Minty rule (see e.g. [17], [18]) to deduce (3.24). □

Proof of Theorem 2.8. For every $n \in \mathbb{N}$, we define $F_n : \Omega \times [0, \infty) \to \mathbb{R}$ by

$$F_n(x, s) = [F(x, s) \vee (-n)] \wedge n, \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

Then, we extend F_n to $\Omega \times \mathbb{R}$ and a to $\Omega \times \mathbb{R} \times \mathbb{R}^N$ by taking

$$F_n(x, s) = F_n(x, 0), \quad a(x, s, \xi) = a(x, 0, \xi), \quad \forall (s, \xi) \in (-\infty, 0) \times \mathbb{R}^N, \text{ a.e. } x \in \Omega.$$

Taking into account that F_n is a Carathéodory function and $|F_n(x, s)| \le n$ a simple application of the Schauder fixed point theorem and the theory of monotone operators provides a solution u_n of

$$\begin{cases} -\operatorname{div} a(x, u_n, \nabla u_n) = F_n(x, u_n) \text{ in } \Omega \\ u_n \in W_0^{1,p}(\Omega). \end{cases} \tag{3.25}$$

Multiplying (3.25) by $-u_n^-$, we have

$$\int_{\{u_n<0\}} a(x, 0, \nabla u_n) \cdot \nabla u_n \, dx = \int_{\{u_n<0\}} F_n(x, 0) u_n \, dx,$$

which using (2.7) and (2.12) proves $u_n \ge 0$, i.e. u_n is a nonnegative solution of (3.25). Applying Theorem 2.7 to this sequence we deduce the existence of a subsequence of u_n which converges to a nonnegative solution of (2.1). □

Proof of Theorem 2.9. For $\delta, k > 0$, we consider the function

$$z = \left[(u_1 - u_2 - \delta)^+ \wedge k\right]^2,$$

which satisfies

$$\nabla z = 2\big(u_1 - u_2 - \delta\big)\nabla(u_1 - u_2)\chi_{\{u_2+\delta<u_1<k+u_2+\delta\}}$$

and therefore

$$|\nabla z| \le 2\big((u_1-\delta)^+ \wedge k\big)\big(|\nabla u_1| + |\nabla u_2|\big) \text{ a.e. in } \Omega.$$

By u_1, u_2 nonnegative solutions of (2.41) and properties (2.21) and (2.22) of the nonnegative solutions, we get that z is a nonnegative function of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and therefore z^p belongs to the space W defined by (2.26). Taking z^p as test function in the difference of the equations satisfied by u_1 and u_2, we get

$$\begin{aligned} &2p \int\limits_{\{u_2+\delta<u_1<k+u_2+\delta\}} (a(x,\nabla u_1) - a(x,\nabla u_2)) \cdot \nabla(u_1-u_2)\left(u_1-u_2-\delta\right)^{2p-1} dx \\ &= \int\limits_\Omega \big(F_1(x,u_1) - F_2(x,u_2)\big)\big[(u_1-u_2-\delta)^+ \wedge k\big]^{2p} dx. \end{aligned} \tag{3.26}$$

Let us first assume $F_1(x,.)$ nonincreasing, then using $F_1(x,u_2) \le F_2(x,u_2)$ a.e. in Ω, we have

$$\begin{aligned} &2p \int\limits_{\{u_2+\delta<u_1<k+u_2+\delta\}} (a(x,\nabla u_1) - a(x,\nabla u_2)) \cdot \nabla(u_1-u_2)\left(u_1-u_2-\delta\right)^{2p-1} dx \\ &\le \int\limits_\Omega \big(F_1(x,u_1) - F_1(x,u_2)\big)\big[(u_1-u_2-\delta)^+ \wedge k\big]^{2p} dx. \end{aligned}$$

Here, we observe that

$$u_1 \le u_2 \Longrightarrow \big[(u_1-u_2-\delta)^+ \wedge k\big] = 0,$$

while by F_1 nonincreassing we have

$$u_1 \ge u_2 \Longrightarrow F_1(x,u_1) - F_1(x,u_2) \le 0,$$

Therefore, the right-hand side of (3.26) is nonpositive which, taking into account (2.38), proves

$$\nabla u_1 = \nabla u_2 \text{ a.e. in } \{u_2+\delta < u_1 < k+u_2+\delta\}, \quad \forall\delta>0,\ \forall k>0,$$

and then

$$\nabla u_1 = \nabla u_2 \text{ a.e. in } \{u_2 < u_1\},$$

or equivalently, $\nabla(u_1-u_2)^+ = 0$ a.e. in Ω. This proves that $(u_1-u_2)^+$ is constant in every connected component of Ω, which combined with $(u_i-\delta)^+ \in W_0^{1,p}(\Omega)$ for every $\delta>0$, $i=1,2$, proves $u_1 \le u_2$ a.e. in Ω.

Assume now $F_2(x,.)$ nonincreasing. Using $F_1(x,u_1) \le F_2(x,u_1)$ in (3.26) we get

$$\begin{aligned} &2p \int\limits_{\{u_2+\delta<u_1<k+u_2+\delta\}} (a(x,\nabla u_1) - a(x,\nabla u_2)) \cdot \nabla(u_1-u_2)\left(u_1-\delta-u_2\right)^{2p-1} dx \\ &\le \int\limits_\Omega \big(F_2(x,u_1) - F_2(x,u_2)\big)\big[(u_1-u_2-\delta)^+ \wedge k\big]^{2p} dx. \end{aligned}$$

This allows to repeat the above reasoning to deduce again $u_1 \le u_2$ a.e. in Ω. □

4. Nonexistence of a solution taking negative values when F is too large near $s=0$

In Section 2, we have proved the existence of a nonnegative solution for problem (2.1) when the function $F = F(x,s)$ on the right-hand side can take the value plus infinity at $s=0$. In the present section we prove that if $F(x,s)$ is bigger than $\tau/|s|$, $\tau>0$, when s is close to zero, then every solution of the semilinear problem is necessarily nonnegative. This result is a consequence of the following lemma.

Lemma 4.1. *Let Ω be a bounded open set of $\mathbb{R}^N$, $p > 1$, and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a function which satisfies*

$$\begin{cases} a(.,s,\xi) \text{ is measurable in } \Omega, \ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N \\ a(x,.,.) \text{ is continuous in } \mathbb{R} \times \mathbb{R}^N, \ a.e.\ x \in \Omega. \end{cases} \tag{4.1}$$

∘ *There exist α, γ, a_0 such that*

$$\begin{cases} \alpha > 0, \ \gamma \geq 0, \ a_0 \in L^{(p^*)'}(\Omega), \ a_0 \geq 0 \\ a(x,s,\xi) \cdot \xi \geq \alpha |\xi|^p - \gamma |s|^p - a_0(x)|s|, \quad \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N, \ a.e.\ x \in \Omega. \end{cases} \tag{4.2}$$

∘ *There exist β, b such that*

$$\begin{cases} \beta > 0, \ b \in L^p(\Omega), \ b \geq 0, \\ |a(x,s,\xi)| \leq \beta\big(|\xi| + |s| + b(x)\big)^{p-1}, \quad \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N, \ a.e.\ x \in \Omega. \end{cases} \tag{4.3}$$

Let u and H be two measurable functions in Ω such that

$$(u+\delta)^- \in W_0^{1,p}(\Omega), \quad \forall \delta > 0 \tag{4.4}$$

$$H \in L^1(\{u < -\delta\}), \quad \forall \delta > 0 \tag{4.5}$$

$$\exists \tau, \delta_0 > 0 \text{ such that } H \geq \frac{\tau}{|u|} \ \text{ a.e. in } \{-\delta_0 < u < 0\} \tag{4.6}$$

$$\begin{cases} \displaystyle\int_\Omega a(x,u,\nabla u) \cdot \nabla v \, dx = \int_\Omega H v \, dx \\ \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \ \text{ such that } \exists \delta > 0 \text{ with } v = 0 \text{ in } \{u > -\delta\}. \end{cases} \tag{4.7}$$

Then, we have

$$u \geq 0 \ \text{ a.e. in } \Omega. \tag{4.8}$$

The previous Lemma implies that any solution of problem (2.1) cannot take negative values when $F(x,s) \geq \tau/|s|$, $\tau > 0$, for s small. More exactly, one has:

Theorem 4.2. *Let Ω be a bounded open set of $\mathbb{R}^N$ and $p > 1$. We consider a function $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ which satisfies (4.1), (4.2) and (4.3) and a function $F : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, which satisfies*

$$F(\cdot,s) \text{ is measurable in } \Omega, \ \forall s \in \mathbb{R}, \ \ F(x,\cdot) \text{ is continuous in } \mathbb{R} \setminus \{0\}, \ a.e.\ x \in \Omega \tag{4.9}$$

$$\exists \delta_0, \tau > 0, \ \text{ with } F(x,s) \geq \frac{\tau}{|s|}, \quad \forall s \in (-\delta_0, 0), \tag{4.10}$$

and

$$\begin{cases} \text{if } p \leq N, & \begin{cases} \exists \nu_\delta \geq 0, \ k_\delta \in L^1(\Omega), \ k_\delta \geq 0, \ \forall \delta > 0 \\ |F(x,s)| \leq k_\delta(x) + \nu_\delta |s|^{p^*}, \ \forall s \text{ with } |s| > \delta, \ a.e.\ x \in \Omega \end{cases} \\ \text{if } p > N, & \begin{cases} \exists k_{m,\delta} \in L^1(\Omega), \ k_{m,\delta} \geq 0, \ \forall m \in \mathbb{N}, \ \forall \delta > 0 \\ |F(x,s)| \leq k_{m,\delta}(x), \ \forall s \text{ with } \delta < |s| < m, \ a.e.\ x \in \Omega. \end{cases} \end{cases} \tag{4.11}$$

Then, any measurable function $u : \Omega \to \mathbb{R}$ which is a solution of

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = F(x,u) \ \text{ in } \Omega \setminus \{u = 0\} \\ u \geq 0 \ \text{ on } \partial\Omega, \end{cases} \tag{4.12}$$

in the following sense:

$$\begin{cases} (u+\delta)^- \in W_0^{1,p}(\Omega), \quad \forall \delta > 0 \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot \nabla v\,dx = \int_\Omega F(x,u)v\,dx \\ \forall v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \ \textit{such that } \exists \delta > 0 \textit{ with } v = 0 \textit{ in } \{|u| < \delta\}, \end{cases} \tag{4.13}$$

satisfies

$$u \geq 0 \ \textit{a.e. in } \Omega. \tag{4.14}$$

Remark 4.3. Problem (4.12) is non standard since the equation takes place in the set $\Omega \setminus \{u = 0\}$ which depends on the solution u. In general this set is not an open set since u has not reason to be continuous.

The fact that the equation takes place in $\Omega \setminus \{u = 0\}$ is reflected in the (mathematically correct) formulation (4.13) where the test functions v have to vanish on the set $\{u = 0\}$.

Remark 4.4. Definition 2.3 in Section 2 was concerned with nonnegative solutions. Thus, the functions $a = a(x, s, \xi)$ and $F = F(x, s)$ were only defined for s nonnegative. It is clear that Definition 2.3 could be extended in a natural way to the case where F is defined for $s \in \mathbb{R}$ and where the solution can take negative values. In this new setting, any such solution to (2.1) would have to be nonnegative in view of Theorem 4.2 when the function F satisfies assumption (4.10), reinforcing in this case the uniqueness result of Theorem 2.9.

Remark 4.5. If in Lemma 4.1 we replace assumptions (4.4), (4.5), (4.6) and (4.7) by

$$(u-\delta)^+ \in W_0^{1,p}(\Omega), \quad \forall \delta > 0, \quad H \in L^1(\{u > \delta\}), \quad \forall \delta > 0,$$

$$\exists \tau, \delta_0 > 0 \ \text{ such that } H \leq -\frac{\tau}{|u|} \ \text{ a.e. in } \{0 < u < \delta\}$$

$$\begin{cases} \displaystyle\int_\Omega a(x,u,\nabla u)\cdot \nabla v = \int_\Omega Hv\,dx \\ \forall v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \ \text{ such that } \exists \delta > 0 \text{ with } v = 0 \text{ in } \{u < \delta\}. \end{cases}$$

Then, instead of (4.8) we have that $u \leq 0$ a.e. in Ω. The proof of this result just follows by applying Lemma 4.1 to the functions $\tilde{a}$, $\tilde{u}$ and $\tilde{H}$ defined by

$$\tilde{a}(x,s,\xi) = -a(x,-s,-\xi), \quad \tilde{u} = -u, \quad \tilde{H} = -H.$$

Using this result one can also modify Theorem 4.2 to prove that the solution u of a nonlinear problem with a singular right-hand side term $F(x, u)$ such that

$$F(x,u) \leq -\frac{\tau}{u}, \ \text{ if } 0 < u < \delta, \ \text{ for some } \delta > 0,$$

which is nonpositive on the boundary is necessarily nonpositive in the whole of Ω.

The results stated in Lemma 4.1 and Theorem 4.2 are based on the fact that the sign of u on the boundary is known. When no boundary conditions are given it is still possible to show Lemma 4.6 below. It proves that the solution of a nonlinear problem with a singular term, which is sufficiently large for u close to zero, cannot change sign.

Lemma 4.6. *Assume $\Omega \subset \mathbb{R}^N$, open, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfying (4.1), (4.2) and (4.3). We assume there exist $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $H \in L^1_{\mathrm{loc}}(\Omega)$ such that there exist $\tau, \delta_0 > 0$ satisfying one of the following conditions:*

$$\begin{cases} |H| \geq \dfrac{\tau}{|u|} \ \textit{ a.e. in } \{0 < u < \delta_0\} \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot \nabla v\,dx = \int_\Omega Hv\,dx \\ \forall v \in W^{1,p}(\Omega)\cap L^\infty(\Omega), \ \text{spt}(v) \subset \Omega \textit{ compact}, \exists \delta > 0 \textit{ with } v = 0 \textit{ in } \{v < \delta\}, \end{cases} \tag{4.15}$$

or

$$\begin{cases} |H| \geq \dfrac{\tau}{|u|} \ \text{ a.e. in } \{-\delta_0 < u < 0\} \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla v\,dx = \int_\Omega H v\,dx \\ \forall v \in W^{1,p}(\Omega)\cap L^\infty(\Omega),\ \operatorname{spt}(v)\subset\Omega \text{ compact}, \exists\delta>0 \text{ with } v=0 \text{ in } \{v>-\delta\}. \end{cases} \tag{4.16}$$

Then, one of the following assertions holds:

$$u \geq 0 \ \text{ a.e. in } \Omega \ \text{ or } \ u \leq 0 \ \text{ a.e. in } \Omega. \tag{4.17}$$

Proof of Lemma 4.1. For $\varepsilon < \delta_0/2$, we take

$$v_\varepsilon := \Big(1+\frac{u}{\varepsilon}\Big)^- \wedge 1 \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \quad \forall \varepsilon > 0, \tag{4.18}$$

as test function in (4.7). We have

$$-\frac{1}{\varepsilon}\int_{\{-2\varepsilon<u<-\varepsilon\}} a(x,u,\nabla u)\cdot\nabla u\,dx = \int_\Omega H v_\varepsilon\,dx = \int_{\{u\leq-\delta_0\}} H\,dx + \int_{\{-\delta_0<u\}} H v_\varepsilon\,dx,$$

and thus

$$-\int_{\{u\leq-\delta_0\}} H\,dx = \frac{1}{\varepsilon}\int_{\{-2\varepsilon<u<-\varepsilon\}} a(x,u,\nabla u)\cdot\nabla u\,dx + \int_{\{-\delta_0<u\}} H v_\varepsilon\,dx. \tag{4.19}$$

By (4.6), we know that $H \geq 0$ a.e. in $\{-\delta_0 < u < 0\}$. This allows us to use the monotone convergence theorem in the last term, which combined with (4.2), proves

$$\limsup_{\varepsilon\to0} \frac{\alpha}{\varepsilon}\int_{\{-2\varepsilon<u<-\varepsilon\}} |\nabla u|^p dx + \int_{\{-\delta_0<u<0\}} H\,dx \leq -\int_{\{u\leq-\delta_0\}} H\,dx,$$

and thus, taking into account (4.5), we get

$$H \in L^1(\{u<0\}), \qquad \limsup_{\varepsilon\to0} \frac{1}{\varepsilon}\int_{\{-2\varepsilon<u<-\varepsilon\}} |\nabla u|^p dx \leq -\frac{1}{\alpha}\int_{\{u<0\}} H\,dx. \tag{4.20}$$

Now, we observe that thanks to (4.6), for $0 < 2\varepsilon < \delta_0$, we have

$$\frac{1}{2\varepsilon}\big|\{x\in\Omega:\ -2\varepsilon<u<-\varepsilon\}\big| \leq \int_{\{-2\varepsilon<u<-\varepsilon\}} \frac{1}{|u|}\,dx \leq \frac{1}{\tau}\int_{\{-2\varepsilon<u<-\varepsilon\}} H\,dx,$$

and then, by the first assertion in (4.20)

$$\lim_{\varepsilon\to0} \frac{1}{\varepsilon}\big|\{x\in\Omega:\ -2\varepsilon<u<-\varepsilon\}\big| = 0. \tag{4.21}$$

Let us now take $\Psi \in C_c^\infty(\mathbb{R}^N)^N$. Since v_ε defined by (4.18) belongs to $W_0^{1,p}(\Omega)$, we have

$$\int_\Omega v_\varepsilon \operatorname{div}\Psi\,dx = -\int_\Omega \nabla v_\varepsilon\cdot\Psi\,dx = \frac{1}{\varepsilon}\int_{\{-2\varepsilon<u<-\varepsilon\}} \nabla u\cdot\Psi\,dx. \tag{4.22}$$

Taking into account (4.20) and (4.21), we have on the first hand

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_{\{-2\varepsilon<u<-\varepsilon\}}|\nabla u||\Psi|\,dx$$

$$\leq \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\int\limits_{\{-2\varepsilon<u<-\varepsilon\}}|\nabla u|^p\,dx\right)^{\frac{1}{p}}\left(\frac{1}{\varepsilon}\Big|\{x\in\omega:\ -2\varepsilon<u<-\varepsilon\}\Big|\right)^{\frac{1}{p'}}\|\Psi\|_{L^\infty(\Omega)^N}=0.$$

On the other hand, using the definition (4.18) of v_ε and Lebesgue's dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega}v_\varepsilon\operatorname{div}\Psi\,dx=\int\limits_{\{u<0\}}\operatorname{div}\Psi\,dx.$$

Thus, (4.22) provides

$$0=\int\limits_{\Omega}\chi_{\{u<0\}}\operatorname{div}\Psi\,dx,\quad \forall\,\Psi\in C_c^\infty(\mathbb{R}^N)^N\Longrightarrow\nabla\chi_{\{u<0\}}=0\ \text{ in }\mathcal{D}'(\mathbb{R}^N).\tag{4.23}$$

This proves that $\chi_{\{u<0\}}$ is a constant function in $\mathbb{R}^N$, but $\{x\in\Omega:\ u(x)<0\}\subset\Omega$ implies that $\chi_{\{u<0\}}$ vanishes outside Ω. Therefore $\chi_{\{u<0\}}=0$ a.e. in $\mathbb{R}^N$. $\square$

Proof of Theorem 4.2. It is easily deduced from Lemma 4.1 with $H(x)=F(x,u(x))$. Namely, we observe that (4.13) implies (4.4) and (4.7), while Sobolev's embedding theorem and (4.11) provide (4.5). Finally, assumption (4.6) follows from (4.10). $\square$

Proof of Lemma 4.6. To fix ideas we assume in the following that (4.15) holds. The other case is completely similar. The proof follows the ideas of Lemma 4.1.

We take $\varphi\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$. Then, for $\varepsilon>0$, we define

$$v_\varepsilon:=\left[\left(\frac{u}{\varepsilon}-1\right)^+\wedge 1\right].$$

Taking $v=v_\varepsilon\varphi$ in (4.15), we get

$$\frac{1}{\varepsilon}\int\limits_{\{\varepsilon<u<2\varepsilon\}}a(x,u,\nabla u)\cdot\nabla u\,\varphi\,dx+\int\limits_{\Omega}a(x,u,\nabla u)\cdot\nabla\varphi\,v_\varepsilon\,dx=\int\limits_{\Omega}Hv_\varepsilon\varphi\,dx.$$

Then, using Lebesgue dominated convergence theorem we can pass to the limit when ε tends to zero to deduce

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_{\{\varepsilon<u<2\varepsilon\}}a(x,u,\nabla u)\cdot\nabla u\,\varphi\,dx=\int\limits_{\{0<u\}}\big(H\varphi-a(x,u,\nabla u)\cdot\nabla\varphi\big)dx,$$

which combined with (4.2) shows

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_{\{\varepsilon<u<2\varepsilon\}}|\nabla u|^p\varphi\,dx<+\infty.\tag{4.24}$$

On the other hand, for $\varepsilon<\delta_0/2$, we have

$$\frac{\tau}{\varepsilon}\int\limits_{\{\varepsilon<u<2\varepsilon\}}\varphi\,dx\leq\int\limits_{\{\varepsilon<u<2\varepsilon\}}|H|\varphi\,dx,$$

and so,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_{\{\varepsilon<u<2\varepsilon\}}\varphi\,dx=0.\tag{4.25}$$

From (4.24) and (4.25) we then deduce similarly to (4.23) (observe that now the support of Ψ is contained in Ω).

$$\int_{\{u>0\}} \operatorname{div} \Psi \, dx = 0, \quad \forall \Psi \in C_c^\infty(\Omega)^N.$$

This shows that the distributional gradient of $\chi_{\{u>0\}}$ is zero and so that $\chi_{\{u>0\}}$ is constant a.e. in Ω, i.e. $u \geq 0$ a.e. in Ω or $u \leq 0$ a.e. in Ω. $\square$

5. A one-dimensional example of a singular equation with many solutions which change sign

In Section 2 we have proved the existence of a nonnegative solution for the semilinear problem (2.1) where $F = F(x, s)$ is nonnegative at $s = 0$ and can take the value $+\infty$. Later, in Section 4, we have shown that if $F(x, s)$ is bigger than $\tau/|s|$, $\tau > 0$, when s is negative and close to zero, then every solution of (2.1) is nonnegative. If contrarily, the right-hand side F tends to $+\infty$ when s tends to 0 with a speed of order $1/|s|^\gamma$, $0 < \gamma < 1$, then some examples in [9] and [10] show that a nonpositive solution can also exist. This proves that condition (4.10) in Theorem 4.2 is optimal in the sense that it cannot be relaxed by

$$F(x, s) \geq \frac{\tau}{|s|^\gamma} \text{ a.e. in } \{\delta_0 < s < 0\}, \quad \text{with } \delta_0, \tau > 0, \ \gamma < 1.$$

In the present section we explore the simple example in dimension $N = 1$ given by

$$\begin{cases} -u'' = \dfrac{f}{|u|^\gamma} - g \ \text{ in } (0, L) \\ u(0) = u(L) = 0, \end{cases} \tag{5.1}$$

with L, f, γ positive constants and $g \in \mathbb{R}$. We will describe all the possible solutions.

Theorems 2.8 and 2.9 applied to (5.1) show the existence and the uniqueness of a nonnegative solution for (5.1), for every $f, \gamma > 0$ and $g \in \mathbb{R}$. For $\gamma \geq 1$, Theorem 4.2 shows that every solution of the problem is nonnegative. So, the solution given by Theorem 2.8 is the unique solution to this problem. Also, if $\gamma < 1$ and $g \leq 0$, the right-hand side in problem (5.1) is nonnegative and then, the classical weak maximum principle implies that every solution has also to be nonnegative. Thus, we will assume in what follows

$$f, g > 0, \quad 0 < \gamma < 1.$$

We first observe that a change of scale in the variables x and u allows us to reduce the number of parameters. Namely, for $r, t > 0$ to be chosen, we define the new unknown function w by

$$w(x) = r u(tx), \quad x \in (0, L/t). \tag{5.2}$$

The differential equation in (5.1) is then transformed into

$$-w'' = \frac{f r^{\gamma+1} t^2}{|w|^\gamma} - g r t^2 \ \text{ in } (0, L/t).$$

Taking

$$r = \left(\frac{g}{f}\right)^{\frac{1}{\gamma}}, \quad t = \frac{f^{\frac{1}{2\gamma}}}{g^{\frac{\gamma+1}{2\gamma}}}, \quad l = \frac{L}{t}, \tag{5.3}$$

problem (5.1) reduces to

$$\begin{cases} -w'' = \dfrac{1}{|w|^\gamma} - 1 \ \text{ in } (0, l) \\ w(0) = w(l) = 0. \end{cases} \tag{5.4}$$

The main results of this section are Theorems 5.3 and 5.7 below. They show in particular the existence of solutions to problem (5.1) which take negative values and even change its sign when l is big enough. Moreover the number of such solutions increases as l increases.

5.1. Statements of the results concerning problem (5.4)

In the introduction to this section we were speaking about solutions of problem (5.4) and (5.1), but we did not define what we understand by a solution of these singular problems. The definition that we will use is the following one:

Definition 5.1. Let γ be such that

$$0 < \gamma < 1.$$

A measurable function $w : (0, l) \to \mathbb{R}$ is a solution of (5.4) if it satisfies

$$w \in L^1(0, l) \tag{5.5}$$

$$(|w| - \delta)^+ \in W_0^{1,1}(0, l), \quad \forall \delta > 0 \tag{5.6}$$

$$\frac{1}{|w|^\gamma} \in L^1_{loc}(0, l) \tag{5.7}$$

$$-w'' = \frac{1}{|w|^\gamma} - 1 \text{ in } \mathcal{D}'(0, l). \tag{5.8}$$

This definition could seem to be weaker than the one given by Definition 2.3 for nonnegative solutions, but this is not the case because of the following Proposition.

Proposition 5.2. *Let w be a solution of (5.4) in the sense of Definition 5.1. Then one has*

$$w \in W^{2,1}(0, l) \cap H_0^1(0, l), \tag{5.9}$$

and then in particular

$$\frac{1}{|w|^\gamma} \in L^1(0, l), \quad w \in C^1([0, l]). \tag{5.10}$$

Moreover, defining $c \geq 0$ by

$$c = |w'(0)|^2, \tag{5.11}$$

one has

$$|w'|^2 = b(w) + c \text{ in } [0, l], \tag{5.12}$$

where $b : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$b(s) = -\frac{2}{1-\gamma}\frac{s}{|s|^\gamma} + 2s, \quad \forall s \in \mathbb{R}, \tag{5.13}$$

where $b(0)$ is understood as zero.

Finally, when $c \neq 0$ one has

$$w \in W^{2,q}(0, l), \ \frac{1}{|w|^\gamma} \in L^q(0, l), \quad \forall q < \frac{1}{\gamma}. \tag{5.14}$$

In the case where $\gamma = 1/2$, we are able to describe all the solutions of problem (5.4) in the sense of Definition 5.1. This is the most simple case of the main result of this section

Theorem 5.3. *Let γ be such that*

$$\gamma = \frac{1}{2}.$$

Define T_0 by

$$T_0 = 2\sqrt{2}\pi.$$

Then, the set of the solutions of problem (5.4) in the sense of Definition 5.1 is the union of the following branches:

1. *For every $l > 0$, there exists 1 solution of problem (5.4) which is positive in $(0, l)$. This solution is the unique nonnegative solution of (5.4).*
2. *For $l = T_0$, there exists 1 solution of problem (5.4) which is negative in $(0, l)$. This solution is the unique negative solution of (5.4).*
3. *For every $l > T_0$, there exist 4 solutions of problem (5.4) which are as follows: the first one is negative in $(0, l)$, and is the unique solution (strictly) negative on $(0, l)$; the 3 other ones change sign at every zero and are negative on 1 nonempty subinterval of $(0, l)$ and positive on 1 or 2 nonempty subintervals of $(0, l)$.*
4. *For $l = 2T_0$, there exists 1 solution of problem (5.4) which is negative in $(0, T_0) \cup (T_0, 2T_0)$ but which vanishes at T_0.*
5. *For every $l > 2T_0$, there exist 4 solutions to problem (5.4) which change sign at every zero and are negative on 2 nonempty subintervals of $(0, l)$ and positive on 1, 2 or 3 nonempty subintervals of $(0, l)$.*
6. *Similarly, for every $k \in \mathbb{N}$, with $k \geq 3$, for every $l = kT_0$, there exists 1 solution of problem (5.4) which is negative on $\bigcup_{j=0}^{k-1}(jT_0, (j+1)T_0)$ but which vanishes at $t = jT_0$, $j \in \{1, ..., k-1\}$.*
7. *Similarly, for every $k \in \mathbb{N}$, with $k \geq 3$, for every $l > kT_0$, there exist 4 solutions to problem (5.4) which change sign at every zero and are negative on k nonempty subintervals of $(0, l)$ and positive on $k-1$, k or $k+1$ nonempty subintervals of $(0, l)$.*

Therefore when $\gamma = 1/2$, problem (5.4) has exactly

$$1 + 4k \text{ solutions if } kT_0 < l < (k+1)T_0, \quad \forall k \geq 0, \tag{5.15}$$

$$4k - 2 \text{ solutions if } l = kT_0, \quad \forall k \geq 1. \tag{5.16}$$

Remark 5.4. Let us denote by B_k^j, with $k \in \mathbb{N}$, $k \geq 1$, and $j = 1, 2, 3, 4$, the 4 functions which to any $l > k$ associate the 4 solutions of problem (5.4) in $(0, l)$, which are negative on k subintervals of $(0, l)$, described in the latest point of Theorem 5.3, or more exactly their extensions by zero to $(0, \infty)$ of these 4 solutions. Since these 4 solutions are uniquely defined by formulas (5.38) and (5.42) below, it is not difficult to prove that the functions B_k^j are continuous on $(kT_0, +\infty)$ with values in $L^1(0, \infty)$ or even in $W_0^{1,p}(0, \infty)$ for any $p < \infty$. This proves that in the case where $\gamma = 1/2$, the set of all the solutions of (5.4) coincide with a set made of a countable number of bundles of 4 continuous branches originating at every point $l = kT_0$, with $k \geq 1$, to which one has to add the branch originating at $l = 0$ consisting of the nonnegative solution. A representation of these branches with the type of solutions corresponding to each branch is given in Fig. 2.

Remark 5.5. As already said in the introduction of this section, problems (5.1) and (5.4) are equivalent through the change of (independent and dependent) variables (5.2). In the case $\gamma = 1/2$, since (5.3) implies that

$$g = \left(\frac{fl}{L}\right)^{\frac{2}{3}},$$

setting

$$M = \left(\frac{fT_0}{L}\right)^{\frac{2}{3}},$$

Theorem 5.3 implies that problem (5.1) has a unique solution (which is positive) when $0 < g < M$, and a unique negative solution when $g \geq M$. Moreover at every value $g = Mk^{\frac{2}{3}}$, with $k \in \mathbb{N}$, a bundle of 4 branches of solutions appears, so that problem (5.1) has exactly $1 + 4k$ solutions for $Mk^{\frac{2}{3}} < g < M(k+1)^{\frac{2}{3}}$, $k \geq 0$ and $4k - 2$ solutions for $g = Mk^{\frac{2}{3}}$, $k \geq 1$.

Theorem 5.3 refers to the case $\gamma = 1/2$. In the case $\gamma \in (0, 1/2) \cup (1/2, 1)$ we do not know the exact number of solutions of problem (5.4) and the structure of such solutions. However we can still show some related results. For this purpose consider the function b defined by (5.13), whose graph is given in Fig. 1. The following proposition gives some properties of this function b we will need later to state and prove the results corresponding to the existence of solutions for problem (5.4).

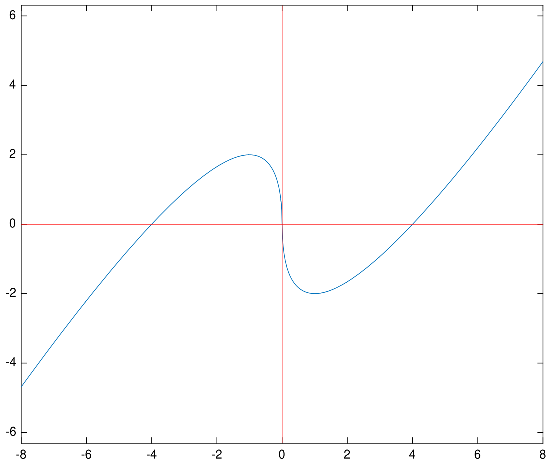

Fig. 1. Graph of the function b.

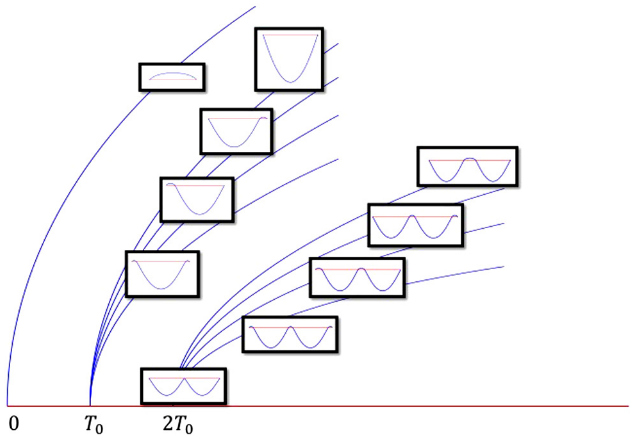

Fig. 2. Branches of solutions for $\gamma = 1/2$.

Proposition 5.6. *For $\gamma \in (0, 1)$, the function b defined by (5.13) satisfies*

$$\begin{cases} b \text{ is strictly increasing in } (-\infty, -1) \text{ and } (1, \infty), \quad b \text{ is strictly decreasing in } (-1, 1) \\ \lim_{s\to-\infty} b(s) = -\infty, \quad b(-1) = \dfrac{2\gamma}{1-\gamma}, \quad b(1) = -\dfrac{2\gamma}{1-\gamma}, \quad \lim_{s\to\infty} b(s) = \infty \\ b \text{ vanishes at } s = -\dfrac{1}{(1-\gamma)^{\frac{1}{\gamma}}}, \; s = 0, \; s = \dfrac{1}{(1-\gamma)^{\frac{1}{\gamma}}}. \end{cases} \tag{5.17}$$

- *For $0 \le c < 2\gamma/(1-\gamma)$, the equation $b(s) + c = 0$ has three solutions $z_1(c)$, $z_2(c)$ and $z_3(c)$ with $z_1(c) < -1$, $0 \le z_2(c) < 1$, $1 < z_3(c)$.*
- *For $c = 2\gamma/(1-\gamma)$ the equation $b(s) + c = 0$ has two solutions $z_1(2\gamma/(1-\gamma)) < -1$ and $z_2(2\gamma/(1-\gamma)) = 1$.*
- *For $c > 2\gamma/(1-\gamma)$, equation $b(s) + c = 0$ has a unique solution $z_1(c)$ and $z_1(c) < 1$.*

We define $T^- : [0, \infty) \to \mathbb{R}$ and $T^+ : [0, 2\gamma/(1-\gamma)) \to \mathbb{R}$ by

$$T^-(c) = 2 \int_{z_1(c)}^{0} \frac{dt}{\sqrt{b(t) + c}}, \qquad \forall c \in [0, \infty). \tag{5.18}$$

$$T^{+}(c)=2\int_{0}^{z_2(c)}\frac{dt}{\sqrt{b(t)+c}},\quad \forall c\in\Big[0,\frac{2\gamma}{1-\gamma}\Big). \tag{5.19}$$

Then, we have

$$T^{-}\in C^0([0,\infty)),\quad T^{*}:=\min_{c\geq 0}T^{-}(c)>0,\quad \lim_{c\to\infty}T^{-}(c)=\infty \tag{5.20}$$

$$T^{+}\in C^0\Big(\Big[0,\frac{2\gamma}{1-\gamma}\Big)\Big),\ T^{+}\ \textit{is strictly increasing},\ T^{+}(0)=0,\ \lim_{c\to\frac{2\gamma}{1-\gamma}}T^{+}(c)=\infty. \tag{5.21}$$

Using the function T^- defined by (5.18) we can now state our main result about the existence of solution for problem (5.8) for $\gamma\in(0,1)$.

Theorem 5.7. *Let γ be in $(0,1)$. Define T^* by (5.20) and T_0 by*

$$T_0=T^{-}(0). \tag{5.22}$$

Then, we have:

If T^- is strictly increasing in $[0,\infty)$, then all the results in the statement of Theorem 5.3 still hold true.

If T^- is not necessarily strictly increasing, we have:

1. *For every $l>0$, there exists 1 solution of problem (5.4) which is positive in $(0,l)$. This solution is the unique nonnegative solution of (5.4).*
2. *For every $l<kT^*$, $k\geq 1$, there is not any solution of problem (5.4) which is negative on at least k nonempty open subintervals of $(0,l)$ and vanishes on the boundary of these intervals.*
3. *For every $l\geq T^*$, there exists at least 1 solution of problem (5.4) which is negative in $(0,l)$.*
4. *For every $l=kT_0$, $k\geq 1$, there exists 1 solution of problem (5.4) which is negative on $\bigcup_{j=0}^{k-1}(jT_0,(j+1)T_0)$ but which vanishes at $t=jT_0$, $j\in\{1,\dots,k-1\}$.*
5. *For every $l>kT_0$, $k\geq 1$, there exist at least 4 solutions of problem (5.4) which are negative on exactly k disjoint open subintervals of $(0,l)$, they vanish on the boundary of these subintervals and are positive on the rest of $(0,l)$, which is composed by $k-1$, k or $k+1$ disjoint open subintervals.*

In particular, for $\gamma\in(0,l)$, problem (5.4) has at least

$$1+4k\ \textit{solutions if}\ kT_0<l<(k+1)T_0,\quad \forall k\geq 0, \tag{5.23}$$

$$4k-2\ \textit{solutions if}\ l=kT_0,\quad \forall k\geq 1. \tag{5.24}$$

Remark 5.8. By Theorem 5.7, the number of solutions of (5.4) agrees with (5.15) or (5.16), when the function T^- is strictly increasing. In the case where $\gamma=1/2$ this follows from the next proposition which provides an explicit expression for T^-. In the general case we do not know when this is true or not. A numerical computation provides Fig. 5 showing the graph for T^- for several values on γ. It seems to indicate the existence of $\tilde\gamma\leq 1/2$, close to $1/2$, such that T^- is not strictly increasing for $\gamma\in(0,\tilde\gamma)$, while it is strictly increasing for $\gamma\geq\tilde\gamma$.

Proposition 5.9. *Assume $\gamma=\frac{1}{2}$. Then, the functions z_1 and z_2 defined by Proposition 5.6 are given by*

$$z_1(c)=-\Big(2+\frac{c}{2}+\sqrt{4+2c}\Big),\ \forall c\geq 0,\qquad z_2(c)=2-\frac{c}{2}-\sqrt{4-2c},\ \forall c\in[0,2]. \tag{5.25}$$

The functions T^- and T^+ defined by (5.18) are given by

$$T^{-}(c)=2\sqrt{2}\left(\pi-2\arctan\frac{\sqrt{-z_1(c)-2\sqrt{-z_1(c)}}}{\sqrt{-z_1(c)}}+\sqrt{-z_1(c)-2\sqrt{-z_1(c)}}\right),\quad \forall c\geq 0, \tag{5.26}$$

$$T^{+}(c)=2\sqrt{2}\left(2\,\mathrm{arctanh}\frac{\sqrt{z_2(c)}}{\sqrt{2\sqrt{z_2(c)}-z_2(c)}}-\sqrt{2\sqrt{z_2(c)}-z_2(c)}\right),\quad \forall s\in[0,2). \tag{5.27}$$

The function T^+ is a strictly increasing function in $[0,\infty)$.

5.2. Proof of the results corresponding to the one-dimensional example

The results exposed above are essentially a consequence of Theorem 5.10 below. First of stating and proving this result, let us show Propositions 5.2 and 5.6.

Proof of Proposition 5.2. We start by observing that if w is a solution of (5.4) in the sense of Definition 5.1, then, (5.7) and (5.8) imply that w is in $W^{2,1}_{loc}(0,l)$, and therefore w' is in $C^1(0,l)$. Multiplying equation (5.8) by w', we get

$$-\left(\frac{|w'|^2}{2}\right)' = \frac{w'}{|w|^\gamma} - w' \text{ in } (0,l),$$

and thus, there exists a constant $c \in \mathbb{R}$ such that

$$|w'|^2 = -\frac{2}{1-\gamma}\frac{w}{|w|^\gamma} + 2w + c \text{ in } (0,l). \tag{5.28}$$

Since $w \in L^1(0,l)$, we deduce from this equality that $|w'|^2$ belongs to $L^1(0,l)$ and therefore $w \in H^1(0,l)$. Then, (5.6) implies that w belongs to $H^1_0(0,l)$. In particular, w belongs to $C^0([0,l])$ and then (5.28) shows that $|w'|$ is in $C^0([0,l])$.

Integrating equation (5.8) in $(\delta, l-\delta)$, with $\delta > 0$, we also have

$$\int_\delta^{l-\delta} \frac{dx}{|w|^\gamma} = -w'(l-\delta) + w'(\delta) + l - 2\delta.$$

Since $|w'|$ in $C^0([0,l])$ implies w' in $L^\infty(0,l)$, the monotone convergence theorem implies that

$$\frac{1}{|w|^\gamma} \in L^1(0,l), \tag{5.29}$$

and then (5.8) proves that w is in $W^{2,1}(0,l)$. In particular, this means that w' is in $C^0([0,l])$ and by (5.28) that (5.12) holds.

If $c \neq 0$, then, using that $w \in C^1([0,l])$ and that $|w'(s)| = c^2$ for every $s \in [0,l]$ such that $w(s) = 0$, we deduce that $1/|w|^\gamma$ belongs to $L^q(0,l)$ for every $q < 1/\gamma$ which combined with (5.4) shows (5.14). □

Proof of Proposition 5.6. Statement (5.17) and the results about the number and position of the zeros of the function b follow immediately by studying the sign of the derivative of b.

The continuity of the functions T^- and T^+ defined by (5.18) and (5.19) is simple to check. Moreover, the definition of T^+ and $z_2(0) = 0$ imply $T^+(0) = 0$. Since for $c = 2\gamma/(1-\gamma)$, we have $z_2(c) = 1$ and $b(1) + c = b'(1) = 0$, we get

$$\lim_{c \to \frac{2\gamma}{1-\gamma}} T^+(c) = 2 \lim_{c \to \frac{2\gamma}{1-\gamma}} \int_0^{z_2(c)} \frac{dt}{\sqrt{b(t)+c}} = \infty. \tag{5.30}$$

In order to prove that T^+ is strictly increasing, we recall that Theorems 2.8 and 2.9 show that problem (5.1) has a unique nonnegative solution for every $l \geq 0$. Taking into account Theorem 5.10 below (see (5.41)) this solution is necessarily obtained as the restriction to $[0,l]$ of a periodic function of period $T^-(c) + T^+(c)$, with $l = T^+(c)$ for some $c \in [0, 2\gamma/(1-\gamma))$. Moreover, by (5.42), it is given by

$$w(x) = \begin{cases} G^{-1}(x) & \text{if } x \in \left[0, \dfrac{l}{2}\right] \\ G^{-1}(l-x) & \text{if } x \in \left[\dfrac{l}{2}, l\right], \end{cases}$$

with G defined by (5.36). This provides the implicit definition of w

$$\begin{cases} x = \displaystyle\int_0^{w(x)} \frac{dt}{\sqrt{b(t)+c}} & \text{if } x \in \left[0, \frac{l}{2}\right] \\ x = \displaystyle\int_0^{z_2} \frac{dt}{\sqrt{b(t)+c}} + \int_{w(x)}^{z_2} \frac{dt}{\sqrt{b(t)+c}} & \text{if } x \in \left[\frac{l}{2}, l\right]. \end{cases}$$

The uniqueness of w implies then that for every $l \in [0,\infty)$, there exists a unique $c \in [0, 2\gamma/(1-\gamma))$, which satisfies $T^+(c) = l$. Combined with $T^+(0) = 0$, (5.30) and T continuous, this shows that T^+ is strictly increasing.

In order to study the values of T^-, we consider the function R defined by

$$R(z) = T^-(z_1^{-1}(-z)), \quad \forall z \in (z_1^{-1}(0), \infty), \tag{5.31}$$

which is obtained by writing the function T^- in the variable $z = -z_1(c)$. Then, recalling definition (5.13) of b and that $z_1(c)$ is the unique negative root of $b + c$, we get that R is given by

$$R(z) = 2\int_0^z \frac{dr}{\sqrt{\frac{2}{1-\gamma}\left(r^{1-\gamma} - z^{1-\gamma}\right) + 2(z-r)}}, \quad \forall z \geq \frac{1}{(1-\gamma)^{\frac{1}{\gamma}}}. \tag{5.32}$$

Now, for $z \in (0,\infty)$ we define $\psi : [0,\infty) \to \mathbb{R}$ by

$$\psi(r) = \frac{2}{1-\gamma}\left(r^{1-\gamma} - z^{1-\gamma}\right) + \frac{2}{z^\gamma}(z-r), \quad \forall r > 0.$$

Then, using

$$\psi(z) = 0, \quad \psi'(r) = 2\left(\frac{1}{r^\gamma} - \frac{1}{z^\gamma}\right) \geq 0, \quad \forall r \in (0, z],$$

we deduce that $\psi(r) \leq 0$ for every $r \in [0, z]$, which implies

$$\frac{2}{1-\gamma}\left(r^{1-\gamma} - z^{1-\gamma}\right) + 2(z-r) \leq 2\left(1 - \frac{1}{z^\gamma}\right)(z-r), \quad \forall r \in [0, z],$$

and then, for every $z > 1$, we have

$$\int_0^z \frac{dr}{\sqrt{\frac{2}{1-\gamma}\left(r^{1-\gamma} - z^{1-\gamma}\right) + 2(z-r)}} \geq \frac{1}{\sqrt{2\left(1-\frac{1}{z^\gamma}\right)}} \int_0^z \frac{dr}{\sqrt{z-r}} = \sqrt{\frac{2z^{\gamma+1}}{z^\gamma - 1}}.$$

Taking into account (5.32), the function $z \to z^{\gamma+1}/(z^\gamma - 1)$ increasing for $z > (1+\gamma)^{\frac{1}{\gamma}}$ and $(1-\gamma)^{-\frac{1}{\gamma}} > (1+\gamma)^{\frac{1}{\gamma}}$ for $0 < \gamma < 1$, we deduce

$$R(z) \geq \sqrt{\frac{2}{\gamma(1-\gamma)^{\frac{1}{\gamma}}}}, \quad \forall z \geq \frac{1}{(1-\gamma)^{\frac{1}{\gamma}}}, \qquad \lim_{z\to\infty} R(z) = \infty.$$

Returning to the variable $c = z_1^{-1}(-z)$, we have then proved

$$T^-(c) \geq \sqrt{\frac{2}{\gamma(1-\gamma)^{\frac{1}{\gamma}}}}, \quad \forall c \geq 0, \qquad \lim_{c\to\infty} T^-(c) = \infty. \quad \square \tag{5.33}$$

Theorem 5.10. *Assume that w is a solution of (5.4) and define c by (5.11).*

• *If $0 \leq c < 2\gamma/(1-\gamma)$, then w is the restriction to $[0,l]$ of a function in $W^{2,1}_{\rm loc}(\mathbb{R})$ ($W^{2,q}_{\rm loc}(\mathbb{R})$, for every $q < 1/\gamma$ if $c > 0$), still denoted by w, which is a solution of*

$$-w'' = \frac{1}{|w|^\gamma} - 1 \ \ in \ \ \mathbb{R}, \tag{5.34}$$

periodic of period

$$T(c) := T^-(c) + T^+(c), \tag{5.35}$$

with T^-, T^+ defined by (5.18), (5.19). Defining G by

$$G(s) = \int_0^s \frac{dt}{\sqrt{b(t)+c}}, \qquad \forall s \in [z_1(s), z_2(s)], \tag{5.36}$$

we have that one of the two following conditions hold:
a)

$$l = kT(c), \ k \geq 1 \ \ or \ \ l = kT(c) + T^-(c), \ k \geq 0 \tag{5.37}$$

and for every $j \in \mathbb{Z}$, we have

$$w(x) = \begin{cases} G^{-1}\big(jT(c) - x\big) & if\, x \in \Big[jT(c) - \dfrac{T^+(c)}{2}, jT(c) + \dfrac{T^-(c)}{2}\Big] \\ G^{-1}\big(x - (jT(c) + T^-(c))\big) & if\, x \in \Big[jT(c) + \dfrac{T^-(c)}{2}, (j+1)T(c) - \dfrac{T^+(c)}{2}\Big]. \end{cases} \tag{5.38}$$

$$\begin{cases} w\big(jT(c)\big) = w\big(jT(c) + T^-(c)\big) = 0 \\ w < 0 \ in \ \big(jT(c), jT(c) + T^-(c)\big), \ \ w > 0 \ in \ \big(jT(c) + T^-(c), (j+1)T(c)\big) \end{cases} \tag{5.39}$$

$$\begin{cases} w \ is \ strictly \ decreasing \ in \ \Big[jT(c) - \dfrac{T^+(c)}{2}, jT(c) + \dfrac{T^-(c)}{2}\Big] \\ w \ is \ strictly \ increasing \ in \ \Big[jT(c) + \dfrac{T^-(c)}{2}, (j+1)T(c) - \dfrac{T^+(c)}{2}\Big] \\ w\Big(jT(c) + \dfrac{T^-(c)}{2}\Big) = z_1(c), \quad w\Big((j+1)T(c) - \dfrac{T^+(c)}{2}\Big) = z_2(c) \\ w\Big(jT(c) + \dfrac{T^-(c)}{2} - r\Big) = w\Big(jT(c) + \dfrac{T^-(c)}{2} + r\Big), \ \forall r \in \Big[0, \dfrac{T(c)}{2}\Big], \end{cases} \tag{5.40}$$

b)

$$l = kT(c), \ k \geq 1 \ \ or \ \ l = kT(c) + T^+(c), \ k \geq 0 \tag{5.41}$$

$$w(x) = z(x - T^+(c)) \ \ with\ z \ satisfying\ (5.38),\ (5.39)\ and\ (5.40). \tag{5.42}$$

• *If $c \geq 2\gamma/(1-\gamma)$, then*

$$l = T^-(c). \tag{5.43}$$

Defining G by (5.36) in $[z_1(c), 1]$, we have that w is the restriction to $[0, l]$ of a function in $W^{2,q}_{loc}(\mathbb{R})$ for every $q < 1/\gamma$ solution of (5.34) defined by

$$w(x) = \begin{cases} G^{-1}(-x) & if\, x \in \Big(-\infty, \dfrac{l}{2}\Big), \\ w(x) = G^{-1}\big(x + l\big) & if\, x \in \Big[\dfrac{l}{2}, \infty\Big). \end{cases} \tag{5.44}$$

Moreover, w satisfies

$$w > 0 \ \ in\, \mathbb{R} \setminus (0, l), \quad w < 0 \ \ in\, (0, l) \tag{5.45}$$

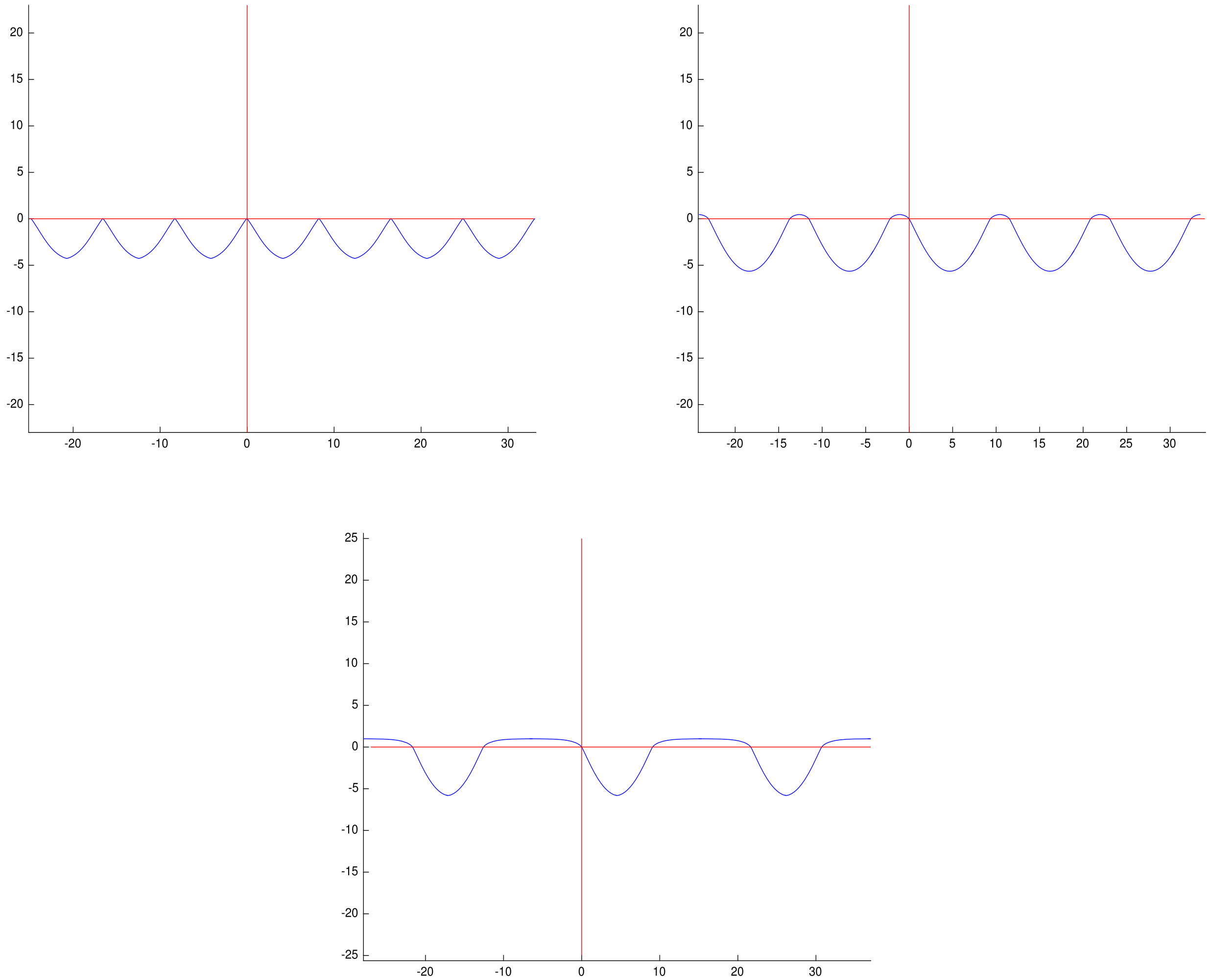

Fig. 3. The cases $\gamma = \frac{1}{2}$, $c = 0$, $c = 1.8$, $c = 1.9999$.

$$\begin{cases} w \text{ is strictly decreasing in } \Big(-\infty, \dfrac{l}{2}\Big), \quad w \text{ is strictly increasing in } \Big(\dfrac{l}{2}, \infty\Big) \\ w\Big(\dfrac{l}{2}\Big) = z_1(c), \quad w\Big(\dfrac{l}{2} - r\Big) = w\Big(\dfrac{l}{2} + r\Big), \quad \forall r \in (0, \infty) \\ \lim\limits_{x \to \pm\infty} w(x) = \begin{cases} 1 & \text{if } c = \frac{2\gamma}{1-\gamma} \\ +\infty & \text{if } c > \frac{2\gamma}{1-\gamma} \end{cases} \end{cases} \tag{5.46}$$

Reciprocally, for every $c \in \mathbb{R}$ and l given by (5.37) or (5.41) if $c < 2\gamma/(1-\gamma)$ and (5.43) if $c \geq 2\gamma/(1-\gamma)$, the above expressions provide a solution of (5.4).

For $\gamma = 1/2$, the graph of the different types of solutions given by Theorem 5.10 is represented in Figs. 3 and 4.

Proof of Theorem 5.10. We distinguish the different cases depending on the value of c:

Case 1: $0 \leq c < 2\gamma/(1-\gamma)$.

By (5.12) and Proposition 5.6 we know that $w([0, l])$ is contained either in $[z_1(c), z_2(c)]$ or $[z_3(c), \infty)$, but since $z_3(c) > 0$ and $w(0) = w(l) = 0$, the last possibility cannot hold true. Thus, we have

$$w(x) \in [z_1(c), z_2(c)], \quad \forall x \in [0, l].$$

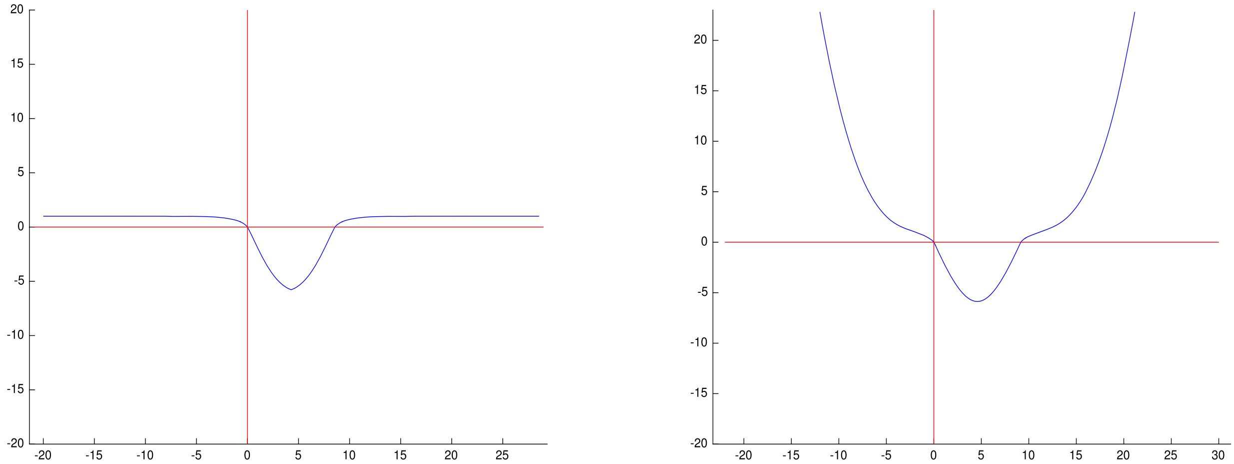

Fig. 4. The cases $\gamma = \frac{1}{2}$, $c = 2$, $c = 2.1$.

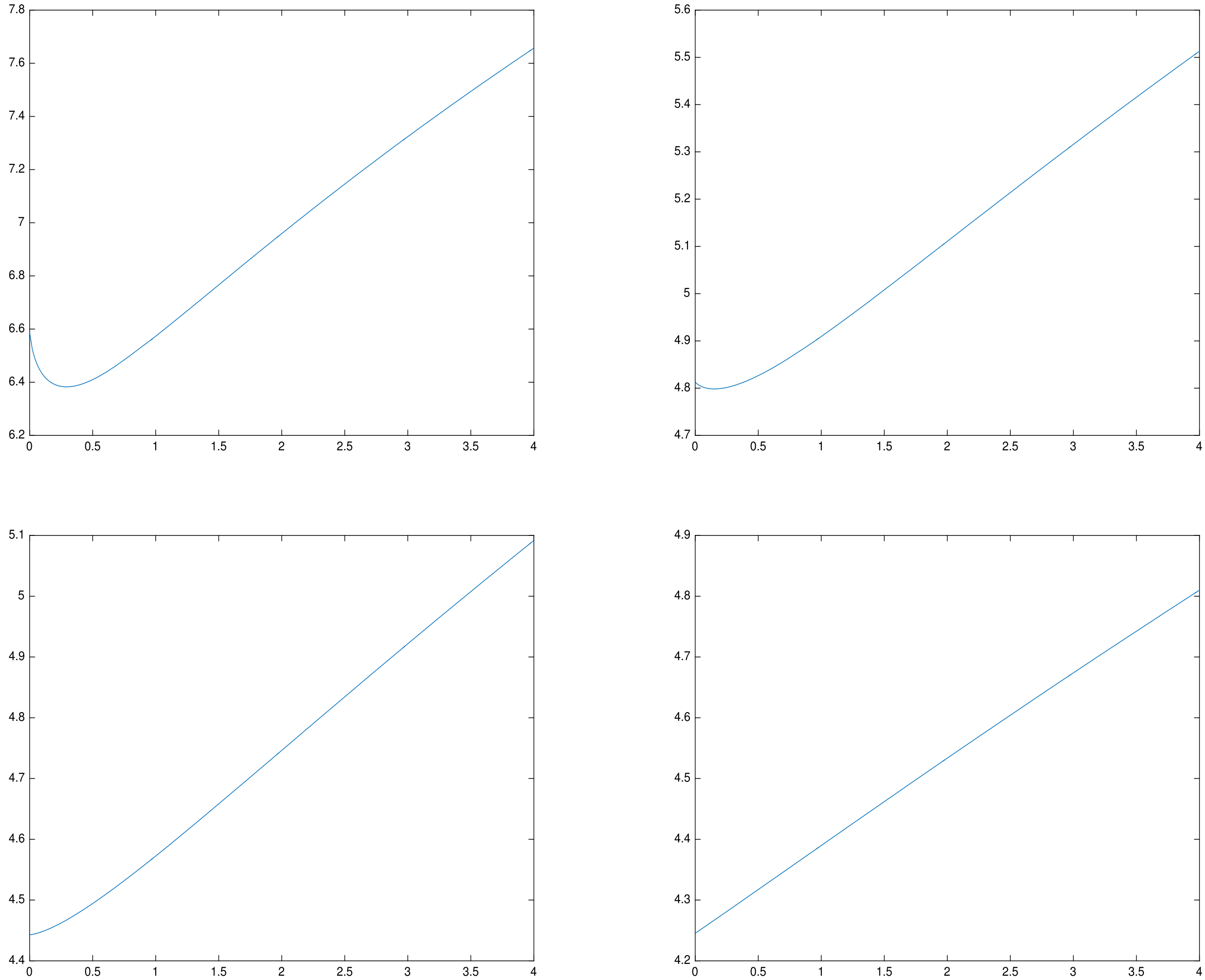

Fig. 5. The graph of T^- for $\gamma = 0.2$, $\gamma = 0.4$, $\gamma = 0.5$, $\gamma = 0.7$

From $w(0)=w(l)=0$, there exists $r\in(0,l)$ such that $w'(r)=0$, which combined with (5.12) implies $w(r)=z_1(c)$ or $w(r)=z_2(c)$. To fix ideas we assume

$$\exists\, r\in(0,l)\ \text{ with }\ w(r)=z_1(c), \tag{5.47}$$

the case $w(r)=z_2(c)$ is similar. Using (5.4) and $z_1(c)<-1$, we have

$$w'(r)=0,\qquad w''(r)>0. \tag{5.48}$$

This implies that for x close to r, $w'(x)$ is negative if $x<r$ and positive if $x>r$. Taking into account (5.12), we then have

$$\frac{w'}{\sqrt{b(w)+c}}=\begin{cases}-1 & \text{in the biggest interval } (\alpha,r) \text{ such that } b(w)+c>0 \ \text{ in } (\alpha,r)\\ 1 & \text{in the biggest interval } (r,\beta) \text{ such that } b(w)+c>0 \ \text{ in } (r,\beta).\end{cases} \tag{5.49}$$

Extending w as the solution of the differential equation (5.49) in the case $\alpha=0$ or $\beta=l$, we can assume

$$w(\alpha)=w(\beta)=z_2(c). \tag{5.50}$$

Using that $b'(z_1(c)),b'(z_2(c))\neq 0$ we get that the function $s\to 1/\sqrt{b(s)+c}$ is integrable in $[z_1(c),z_2(c)]$. Then, integrating in $[x,r]$ or $[r,x]$ in (5.49), and defining $G:[z_1(c),z_2(c)]\to\mathbb{R}$ by (5.36), we get

$$G(w(x))=\begin{cases} r-x-\dfrac{T^-(c)}{2} & \text{if } x\in\left[r-\dfrac{T(c)}{2},r\right]\\[2mm] x-r-\dfrac{T^-(c)}{2} & \text{if } x\in\left[r,r+\dfrac{T(c)}{2}\right],\end{cases} \tag{5.51}$$

with $T^-(c)$ and $T(c)$ defined by (5.18), (5.35), i.e.

$$\begin{cases} T^-(c)=2\displaystyle\int_{z_1(c)}^{0}\frac{dt}{\sqrt{b(t)+c}}=-2G(z_1(c))\\[4mm] T(c)=2\displaystyle\int_{z_1(c)}^{z_2(c)}\frac{dt}{\sqrt{b(t)+c}}=2\big(G(z_2(c))-G(z_1(c))\big).\end{cases} \tag{5.52}$$

Observe that

$$G(s)<0\ \text{ if } s\in[z_1(c),0),\qquad G(s)>0\ \text{ if } s\in(0,z_2(c)],$$

implies

$$w<0\ \text{ in } \left(r-\frac{T^-(c)}{2},r+\frac{T^-(c)}{2}\right),\quad w>0\ \text{ in } \left(r-\frac{T(c)}{2},r-\frac{T^-(c)}{2}\right)\bigcup\left(r+\frac{T^-(c)}{2},r+\frac{T(c)}{2}\right).$$

Equation (5.51) provides the function w in the interval $[\alpha,\beta]=[r-T(c)/2,r+T(c)/2]$. Recalling (5.48) we can now extend the definition of w to $[r-T(c),r+T(c)]$. Continuing with this process and taking into account that by assumption $w(0)=0$ we conclude that w is periodic of period $T(c)$ and its is given by formula (5.38) if $w'(0)\le 0$ or by (5.42) if $w'(0)>0$. The properties of w stated in (5.40) easily follow from (5.38). The fact that l must satisfy (5.37) or (5.41) just follows from $w(l)=0$.

Case 2: $c=2\gamma/(1-\gamma)$.

Using (5.28) and then that $b+c$ is nonnegative on the range of w, we deduce by Proposition 5.6 that $w([0,l])$ is contained in $[z_1,+\infty)$. By (5.28) and $w(0)=0$, we get that in a neighborhood of zero, the function w is the solution of one of the two following Cauchy's problems

$$\begin{cases} w'=\sqrt{b(w)+c}\\ w(0)=0\end{cases}\qquad\text{or}\qquad\begin{cases} w'=-\sqrt{b(w)+c}\\ w(0)=0\end{cases} \tag{5.53}$$

In the first case, defining G by (5.36), and using that the unique point bigger than zero, where $b + c$ vanishes is 1, we deduce that for every $\delta \in (0, l)$ such that in the interval $(0, \delta)$, the function w is increasing and less than 1, we have

$$G(w(x)) = x \quad \text{in } (0, \delta), \tag{5.54}$$

but for $c = 2\gamma/(1-\gamma)$, we have $b(1) = b'(1) = 0$, and thus

$$G(1) = \int_0^1 \frac{dt}{\sqrt{b(t) + c}} = \infty. \tag{5.55}$$

Then (5.54) would provide the expression of G in the whole interval $[0, l]$. However the function constructed in this way is positive in $(0, \infty)$ in contradiction with $w(l) = 0$. This proves that only the second possibility in (5.53) can hold true. But since $w(0) = w(l) = 0$, there exists $r \in (0, l)$ such that $w'(r) = 0$. Since we know $w'(0) \leq 0$, by (5.28) and denoting z_1 the unique negative zero of $b + c$, we must have $w(r) = z_1$. Now, we can repeat the reasoning in the case $c < 2\gamma/(1-\gamma)$ when we assumed (5.48). This shows that (an extension of) w satisfies (5.51) where now, thanks to (5.55), we have

$$T = 2\int_{z_1(c)}^1 \frac{dt}{\sqrt{b(t) + c}} = \infty.$$

Then, (5.48) provides the expression of w in the whole of $\mathbb{R}$. Since $w(l) = 0$, we get by symmetry that $r = l/2$ and we easily conclude (5.43), (5.44), (5.45) and (5.46).

Case 3: $c > 2\gamma/(1-\gamma)$.

Now, the unique zero of the function $b + c$ is $z_1(c)$ and by (5.28), we have that $w([0, l])$ is contained in $[1, \infty)$. Taking into account $w(0) = w(l) = 0$, we deduce as above the existence of $r \in (0, l)$ such that $w'(r) = 0$, $w(r) = z_1 < -1$ and then that (5.51) holds with the difference that now, there is not any point z_2 bigger than z_1 such that $b(z_2) + c = 0$. Therefore T must be defined as

$$T = 2\int_{z_1(c)}^{\infty} \frac{dt}{\sqrt{b(t) + c}} = \infty.$$

Statement (5.51) then provides an expression of (an extension of) w in the whole of $\mathbb{R}$. By $w(0) = w(l) = 0$ we conclude again by symmetry that (5.43), (5.44), (5.45) and (5.46) hold where now the limit of w at infinity is $+\infty$ and not 1. □

Let us now prove Theorem 5.7. Observe that Theorem 5.3 follows from this result and Proposition 5.9. Thus, the proof of Theorem 5.3 will not explicitly given.

Proof of Theorem 5.7. By Theorems 2.7 and 2.9, we know that problem (5.4) has a unique positive solution for $\gamma \in (0, 1)$.

If $l < T^*$, then we have $T^-(c) > l$ for every $c \geq 0$, but since $w(0) = w(l) = 0$, $w < 0$ in $(0, l)$, we get that Theorem 5.10 (see (5.39)), (5.43)) implies $l = T^-(c)$. Therefore, it cannot exist a negative solution of (5.4) in $(0, l)$ for $l < T^*$. Analogously, for $l < kT^*$ it cannot exist a solution of (5.4) which is negative in k nonempty open subintervals of $(0, l)$ and vanishes on the boundary.

Assume $l \geq T^*$. Since T^- is continuous and tends to infinity as infinity, we have that for every $l \geq T^*$, there exists $c \geq 0$ such that $T^-(c) = l$. Defining then w by (5.38) if $c < 2\gamma/(1-\gamma)$ or (5.44) if $c \geq 2\gamma/(1-\gamma)$ we deduce the existence of a negative solution of (5.4). If T^- is strictly increasing then $T^* = T_0$ and equation $T^-(c) = l$ has a unique solution, and so there is a unique negative solution for problem (5.4).

Assume $l > T_0$. By Proposition 5.6, the function $T^- + T^+$ is continuous in $[0, 2\gamma/(1-\gamma))$ and satisfies

$$\min_{c \in [0, 2\gamma/(1-\gamma))} (T^-(c) + T^+(c)) \geq T^-(0) + T^+(0) = T_0, \qquad \lim_{c \to \frac{2\gamma}{1-\gamma}} (T^-(c) + T^+(c)) = +\infty,$$

therefore, there exists $c \in (0, 2\gamma/(1-\gamma))$ such that $T^-(c) + T^+(c) = l$. Equation (5.38) then provides a solution of problem (5.4) which is negative in $(0, T^-(c))$ and positive in $(T^-(c), l)$, with $0 < T^-(c) < l$, while (5.42) provides a solution which is positive in $(0, T^+(c))$ and negative in $(T^+(c), l)$. On the other hand, using also that $T^- + 2T^+$ is continuous in $[0, 2\gamma/(1-\gamma))$ and

$$\min_{c\in[0,2\gamma/(1-\gamma))} (T^-(c) + 2T^+(c)) \geq T^-(0) + 2T^+(0) = T_0, \qquad \lim_{c\to\frac{2\gamma}{1-\gamma}} (T^-(c) + 2T^+(c)) = +\infty,$$

we can also find another number $c \in (0, 2\gamma/(1-\gamma))$ such that $T^-(c) + 2T^+(c) = l$. Equation (5.38) then provides a solution of problem (5.4) which is negative in $(0, T^-(c)) \cup (T(c), T(c) + T^-(c))$ and positive in $(T^-(c), T(c))$, with $0 < T^-(c) < T(c) + T^-(c) < l$. Adding the branch consisting of the negative solution in $(0, l)$ found above, this proves the existence of at least 4 solutions of (5.4) which are negative in a nonempty interval of $(0, l)$ and positive in zero, one or two nonempty intervals of $(0, l)$.

In the case $l > kT_0$, $k \geq 2$ a similar reasoning provides 4 solutions which are negative in k nonempty intervals of $(0, l)$ and positive in $k-1$, k or $k+1$ intervals.

For $l = kT_0$, $k \geq 2$, equation (5.38) with $c = 0$ provides a solution which is nonpositive in $(0, l)$ but vanishes on jT_0 for $j = 1, \ldots, k-1$. Observe that by Theorem 5.10 and $T^+(c) > 0$ for every $c > 0$, we get that this is the unique solution satisfying this property.

If the function T^- is strictly increasing, then the function $mT^- + nT^+$ is strictly increasing for every $m, n \in \mathbb{N}$, with $m + n \geq 1$ and then it is injective. Therefore the solutions described above are the only ones which are negative in k nonempty intervals of $(0, l)$. $\square$

Proof of Proposition 5.9. We recall that $z_1(c) < z_2(c)$ are the smallest solutions of the equation $b(s) + c = 0$, with b given by (5.13). For $\gamma = 1/2$ this equation reduces to

$$4\sqrt{-s} + 2s + c = 0 \ \text{ if } s < 0, \qquad -4\sqrt{s} + 2s + c = 0 \ \text{ if } s > 0,$$

whose resolution provides (5.25).

Let us now compute T^-, i.e. the integral in (5.18), which written as function of

$$\eta = \sqrt{-z_1(c)} \iff c = -4\eta + 2\eta^2, \tag{5.56}$$

reads as

$$-2\int_{-\eta^2}^{0} \frac{dt}{\sqrt{2t + 4\sqrt{-t} + 2\eta^2 - 4\eta}},$$

or, using the change of variables $t = -p^2 \iff p = \sqrt{-t}$, as

$$\int_0^\eta \frac{4p\,dp}{\sqrt{-2p^2 + 4p + 2\eta^2 - 4\eta}} = 2\sqrt{2}\int_0^\eta \frac{p\,dp}{\sqrt{\eta^2 - p^2 - 2(\eta - p)}}.$$

Using a second change of variables

$$\sqrt{\eta^2 - p^2 - 2(\eta - p)} = r(\eta - p) \Longrightarrow p = \frac{r^2\eta - \eta + 2}{r^2 + 1} \Longrightarrow dp = \frac{4r(\eta - 1)}{(r^2 + 1)^2}dr,$$

we can transform this integral into a rational integral. Namely, denoting

$$r_1 = \frac{\sqrt{\eta^2 - 2\eta}}{\eta}, \tag{5.57}$$

we have

$$T^-(c) = 4\sqrt{2}\int_{r_1}^{\infty} \frac{r^2\eta - \eta + 2}{(r^2 + 1)^2}dr = 8\sqrt{2}(\eta - 1)\int_{r_1}^{\infty} \frac{r^2}{(r^2 + 1)^2}dr + 4\sqrt{2}(2 - \eta)\int_{r_1}^{\infty} \frac{dr}{r^2 + 1},$$

where a primitive of $r^2/(1+r^2)^2$ is given by

$$\int \frac{r^2}{(r^2+1)^2}dr = -\frac{1}{2}\int r\frac{d}{dr}\left(\frac{1}{r^2+1}\right)dr = -\frac{r}{2(r^2+1)} + \frac{1}{2}\arctan r.$$

Using (5.57), we then get

$$T^-(c) = 2\sqrt{2}\pi - 4\sqrt{2}\arctan\frac{\sqrt{\eta^2-2\eta}}{\eta} + 2\sqrt{2\eta^2-4\eta}. \tag{5.58}$$

By (5.56), this provides expression (5.26) for T^-.

Let us now show that T^- is strictly increasing in c. By (5.58) and $\eta = \sqrt{-z_1(c)}$ strictly increasing in c, we just need to show that the function

$$\phi(\eta) = -2\sqrt{2}\arctan\frac{\sqrt{\eta^2-2\eta}}{\eta} + \sqrt{2\eta^2-4\eta}$$

is strictly increasing in $\left[\sqrt{-z_1(0)}, \infty\right) = [2,\infty)$. This just follows from

$$\phi'(\eta) = \frac{(2\eta-1)(\eta-2)}{(\eta-1)\sqrt{2\eta^2-4\eta}} \geq 0, \quad \forall \eta \in (2,\infty). \quad \square$$

Declaration of competing interest

There is no competing interest.

Acknowledgements

This work has been partially supported by the projects MTM2017-83583-P of the "Ministerio de Economía, industria y competitividad" of Spain.

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