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RANDOM ATTRACTORS FOR STOCHASTIC DELAY WAVE EQUATIONS ON $\mathbb{R}^n$ WITH LINEAR MEMORY AND NONLINEAR DAMPING

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Dedicated to Georg Hetzer on occasion of his 75th birthday

Abstract. A non-autonomous stochastic delay wave equation with linear memory and nonlinear damping driven by additive white noise is considered on the unbounded domain $\mathbb{R}^n$. We establish the existence and uniqueness of a random attractor $\mathcal{A}$ that is compact in $C([-h,0];H^1(\mathbb{R}^n)) \times C([-h,0];L^2(\mathbb{R}^n)) \times L^2_\mu(\mathbb{R}^+;H^1(\mathbb{R}^n))$ with $1 \leqslant n \leqslant 3$.

Keywords: Random attractor; stochastic delay wave equation; linear memory; nonlinear damping; pullback asymptotic compactness.

1. Introduction.

Wave equations with delay terms are basic modeling tools in the analysis of oscillatory phenomena including aftereffects, time lags or hereditary characteristics [17, 20, 34, 39], as the deformation of viscoelastic materials [9, 11, 12, 25] or the retarded control of the dynamics of flexible structures [21, 24, 26]. In this paper, we consider the asymptotic behavior of the following non-autonomous stochastic delay wave equation on $\mathbb{R}^n$ with linear memory and nonlinear damping driven by additive white noise:

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$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + J\left(\dfrac{\partial u}{\partial t}\right) - k(0)\Delta u + \lambda u - \displaystyle\int_0^\infty k'(r)\Delta u(t-r)dr \\ +F(x,u) = f(x,u(t-\rho(t))) + g(x,t) + \displaystyle\sum_{j=1}^m h_j(x)\dot{w}_j, \quad t>\tau,\, x\in\mathbb{R}^n, \\ u(t,x) = \phi(t-\tau,x),\ t\leqslant\tau,\, x\in\mathbb{R}^n, \\ \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial \phi}{\partial t}(t-\tau,x),\ t\leqslant\tau,\, x\in\mathbb{R}^n, \end{cases} \tag{1}$$

(Yejuan, we have not said anything about the initial function ϕ. Should we say something?) where $1\leqslant n\leqslant 3$, $\lambda, k(0)>0$ and $k'(0)\leqslant 0$ for every $r\in\mathbb{R}^+$, $\tau\in\mathbb{R}$ is an initial time, ϕ is an initial datum on $(-\infty,0]$, for each $j=1,\dots,m$, $h_j(x)\in L^p(\mathbb{R}^n)\cap H^1(\mathbb{R}^n)$, $\{w_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space which will be specified below, and the other symbols satisfy the following conditions:

(**H1**) There exist two constants β_1, β_2 such that
$$J(0)=0, \quad 0<\beta_1\leqslant J'(v)\leqslant \beta_2<\infty.$$

(**H2**) The memory kernel $\mu := -k'(r)$ satisfies $\mu\in C^1(\mathbb{R}^+)\cap L^1(\mathbb{R}^+)$, $\mu(r)\geqslant 0$, $\mu'(r)+\sigma\mu(r)\leqslant 0$, $\forall r\in\mathbb{R}^+$ and some $\sigma>0$, and we will denote $m_0=\int_0^\infty \mu(r)dr$.

(**H3**) There exist a function $k_1\in L^2(\mathbb{R}^n)$ and a positive constant k_2 such that $f\in C(\mathbb{R}^n\times\mathbb{R};\mathbb{R})$ and $\rho\in C^1(\mathbb{R};[0,h])$ satisfy
$$|f(x,\nu)|^2\leqslant |k_1(x)|^2 + k_2^2|\nu|^2, \quad \forall x\in\mathbb{R}^n,\, \nu\in\mathbb{R},$$
$$|\rho'(t)|\leqslant \rho_* < 1, \quad \forall t\in\mathbb{R},$$
where $h>0$ is a given positive number, which will denote the delay time.

(**H4**) Let $G(x,u)=\int_0^u F(x,s)ds$, where $F(x,\cdot)\in C(\mathbb{R}^n\times\mathbb{R};\mathbb{R})$, and there exist functions $k_3\in L^2(\mathbb{R}^n)$, k_6, $k_8\in L^1(\mathbb{R}^n)$ and positive constants k_4, k_5, k_7 such that
$$|F(x,u)|\leqslant |k_3(x)| + k_4|u|^{p-1}, \quad \forall x\in\mathbb{R}^n,\ u\in\mathbb{R},$$
$$G(x,u)\geqslant k_5|u|^p - k_6(x) \quad \text{and} \quad F(x,u)u\geqslant k_7G(x,u)-k_8(x), \quad \forall x\in\mathbb{R}^n,\ u\in\mathbb{R},$$
where $2\leqslant p<\infty$ if $n=1,2$ and $2\leqslant p<4$ if $n=3$.

(**H5**) The external force $g\in L^2_{loc}(\mathbb{R};L^2(\mathbb{R}^n))$ is such that
$$\int_{-\infty}^t\int_{\mathbb{R}^n} e^{\alpha r}|g(r,x)|^2dxdr<\infty, \quad \forall t\in\mathbb{R},$$
which implies that
$$\lim_{K\to\infty}\int_{-\infty}^t\int_{|x|\geqslant K} e^{\alpha r}|g(r,x)|^2dxdr=0, \quad \forall t\in\mathbb{R},$$
where $\alpha>0$ will be given in Lemma 10.

Deterministic autonomous or non-autonomous damped wave equations with memory or no memory were studied by many authors in regard to global attractors, uniform attractors or pullback attractors, see [2, 3, 4, 5, 7, 8, 14, 16, 19, 25, 27, 29, 30, 36, 41, 42] and the references therein. The random attractors for autonomous or non-autonomous stochastic wave equations with memory or no memory were explored in [10, 13, 18, 22, 28, 31, 33, 38, 40, 43].

The existence of pullback attractors for deterministic damped wave equations with variable delays in bounded domains was initially established in [6], and then extended to the case with non-Lipschitz nonlinearities in [35, 37]. The goal of this paper is to establish the existence and uniqueness of random attractors for the stochastic delay wave equation (1) with linear memory and nonlinear damping on an unbounded domain. Since Sobolev compact embedding is lost for unbounded domain, here we shall present some new uniform estimates on both the tails and the bounded truncations of solutions to prove the asymptotic compactness of system (1). Note that Eq. (1) is a damped wave equation with variable delay, non-autonomous and stochastic forcing terms, so the problem is not only stochastic but non-autonomous as well. Therefore, in order to study the long term behavior of problem (1), we shall use the general theory of attractors for non-compact random dynamical systems introduced in [32].

This paper is organized as follows. In Section 2, we recall some basic concepts and results related to non-autonomous random dynamical systems and global pullback attractors. In Section 3, we define a non-autonomous random dynamical system for (1). Section 4 is devoted to the existence and uniqueness of the pullback attractor.

2. **Preliminaries.**

We recall some basic definitions for non-autonomous random dynamical systems and some results ensuring the existence of random attractors for these systems. The reader is referred to [32] for more details.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and (X, d) be a Polish space with Borel σ-algebra $\mathcal{B}(X)$. Let 2^X be the collection of all subsets of X. The Hausdorff semi-distance between two nonempty subsets A and B of X is defined by

$$d(A,\, B) = \sup\{d(a,\, B) : a \in A\},$$

where $d(a,\, B) = \inf\{d(a,\, b) : b \in B\}$. Denote by $\mathcal{N}_r(A)$ the open r-neighborhood $\{y \in X : d(y, A) < r\}$ of radius $r > 0$ of a subset A of X.

Definition 1. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ be a metric dynamical system. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions are satisfied:*

(1) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X$ *is a* $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$ *measurable mapping;*
(2) $\Phi(0, \tau, \omega, \cdot)$ *is the identity on* X*;*
(3) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \Phi(s, \tau, \omega, \cdot))$*;*
(4) $\Phi(t, \tau, \omega, \cdot) : X \to X$ *is continuous.*

Definition 2. (*See* [32].) *A collection $\mathcal{D}$ of some families of nonempty subsets of X is said to be neighborhood closed if for each $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists a positive number ε depending on D such that the family*

$$\{B(\tau, \omega) : B(\tau, \omega) \text{ is a nonempty subset of } \mathcal{N}_\varepsilon(D(\tau, \omega)), \forall\tau \in \mathbb{R}, \forall\omega \in \Omega\} \tag{2}$$

also belongs to $\mathcal{D}$.

Note that the neighborhood closedness of $\mathcal{D}$ implies for each $D \in \mathcal{D}$,

$$\{\tilde{D}(\tau, \omega) : \tilde{D}(\tau, \omega) \text{ is a nonempty subset of } D(q, \omega), \forall\tau \in \mathbb{R}, \forall\omega \in \Omega\} \in \mathcal{D}. \tag{3}$$

A collection $\mathcal{D}$ satisfying (3) *is said to be inclusion-closed in the literature, see, e.g.,* [15].

Definition 3. (*See* [32].)

(1) *A set-valued mapping $K : \mathbb{R} \times \Omega \to 2^X$ is called measurable with respect to $\mathcal{F}$ in Ω if the value $K(\tau, \omega)$ is a closed nonempty subset of X for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, and the mapping $\omega \in \Omega \to d(x,\, K(\tau, \omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$.*

(2) *Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then K is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and for every $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(B,\tau,\omega) > 0$ such that*

$$\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau,\omega), \ \forall t \geqslant T.$$

If, in addition, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $K(\tau,\omega)$ is a closed nonempty subset of X and K is measurable with respect to the $\mathbb{P}$-completion of $\mathcal{F}$ in Ω, then we say K is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

(3) *Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. Then Φ is said to be $\mathcal{D}$-pullback asymptotically upper-semicompact in X if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence $\Phi(T_n, \tau - T_n, \theta_{-T_n}\omega, x_n)$ has a convergent subsequence in X whenever $T_n \to +\infty$ $(n \to \infty)$, $x_n \in B(\tau - T_n, \theta_{-T_n}\omega)$ with $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.*

Definition 4. *Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor for Φ if it satisfies:*

(1) *For every $\tau \in \mathbb{R}$, $\mathcal{A}(\cdot,\tau) : (\Omega, \mathcal{F}, \mathbb{P})$ is measurable, and $\mathcal{A}(\tau,\omega)$ is compact in X.*

(2) *$\mathcal{A}$ is invariant, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\Phi(t, \tau, \omega, \mathcal{A}(\tau,\omega)) = \mathcal{A}(\tau + t, \theta_t\omega), \ \forall t \geqslant 0.$$

(3) *$\mathcal{A}$ attracts every member of $\mathcal{D}$, that is, for every $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\lim_{t \to +\infty} d(\Phi\left(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)\right), \mathcal{A}(\tau,\omega)) = 0.$$

Definition 5. *Let $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of nonempty subsets of X. For every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let*

$$\Theta(B,\tau,\omega) = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \Phi\left(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)\right)}.$$

Then the family $\{\Theta(B,\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called the Θ-limit set of B and is denoted by $\Theta(B)$.

Theorem 6. (*See* [32].) *Let $\mathcal{D}$ be a neighborhood closed collection of some families of nonempty subsets of X, and let Φ be a continuous cocycle on X over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$. Then Φ has a $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ if and only if Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$. The $\mathcal{D}$-pullback attractor $\mathcal{A}$ is unique and is given, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, by*

$$\mathcal{A}(\tau,\omega) = \Theta(K,\tau,\omega) = \bigcup_{B \in \mathcal{D}} \Theta(B,\tau,\omega). \tag{4}$$

Remark 7. *The concept of neighborhood closedness of $\mathcal{D}$ is used to consider the necessary condition for the existence of a $\mathcal{D}$-pullback attractor. It is obvious that the neighborhood closedness of $\mathcal{D}$ implies the inclusion-closed of $\mathcal{D}$, and we need the concept of inclusion-closed of $\mathcal{D}$ in order to derive a sufficient condition for the existence of such attractor.*

3. Cocycle for a stochastic damped wave equation on $\mathbb{R}^n$.

For the stochastic term in (1), we assume that for each $j = 1, \dots, m$, $\{w_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega = \{\omega = (\omega_1, \omega_2, \dots, \omega_m) \in C(\mathbb{R}, \mathbb{R}^m) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ-algebra generated by the compact-open topology of Ω, and $\mathbb{P}$ is the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then we will identify ω with $W(t)$, i.e.,

$$W(t) = (w_1(t), w_2(t), \ldots, w_m(t)) = \omega(t) \quad \text{for } t \in \mathbb{R}. \tag{5}$$

Define a group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t),\ \omega \in \Omega,\ t \in \mathbb{R}. \tag{6}$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system.

Given $j = 1, \ldots, m$, consider the one-dimensional Ornstein-Uhlenbeck equation

$$dz_j + \alpha z_j dt = dw_j(t). \tag{7}$$

One may easily verify that a solution to (7) is given by

$$z_j(t) = z_j(\theta_t\omega_j) \equiv -\alpha \int_{-\infty}^{0} e^{\alpha\tau}(\theta_t\omega_j)(\tau)d\tau,\ t \in \mathbb{R}.$$

It is known that there exists a θ_t-invariant set $\tilde{\Omega} \subseteq \Omega$ of full $\mathbb{P}$ measure such that $z_j(\theta_t\omega_j)$ is continuous in t for every $\omega \in \tilde{\Omega}$, and the random variable $|z_j(\omega_j)|$ is tempered. Hereafter, we will not distinguish $\tilde{\Omega}$ and Ω, and write $\tilde{\Omega}$ as Ω.

It follows from Proposition 4.3.3 in [1] that there exists a tempered function $r(\omega) > 0$ such that

$$\sum_{j=1}^{m} \left(|z_j(\omega_j)|^2 + |z_j(\omega_j)|^p\right) \leqslant r(\omega), \tag{8}$$

where $r(\omega)$ satisfies, for every $\omega \in \Omega$,

$$r(\theta_t\omega) \leqslant e^{\frac{\alpha}{2}|t|} r(\omega),\ t \in \mathbb{R}. \tag{9}$$

Then (8) and (9) imply that, for every $\omega \in \Omega$,

$$\sum_{j=1}^{m} \left(|z_j(\theta_t\omega_j)|^2 + |z_j(\theta_t\omega_j)|^p\right) \leqslant e^{\frac{\alpha}{2}|t|} r(\omega),\ t \in \mathbb{R}. \tag{10}$$

Putting $z(\theta_t\omega) = \sum_{j=1}^m h_j z_j(\theta_t\omega_j)$, by (7) we have

$$dz + \alpha z dt = \sum_{j=1}^{m} h_j dw_j.$$

Let $\eta(t, x, r) = u(t, x) - u(t - r, x)$, and $\xi = \frac{\partial u}{\partial t} + \delta u$ where $\delta > 0$ will be fixed later. Then (1) can be written in the form of the following equivalent system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \delta u = \xi, \\ \dfrac{\partial \xi}{\partial t} - \delta\xi + J(\xi - \delta u) - \Delta u + (\lambda + \delta^2)u - \displaystyle\int_0^\infty \mu(r)\Delta\eta(r)dr \\ +F(x, u) = f(x, u(t - \rho(t))) + g(x, t) + \displaystyle\sum_{j=1}^{m} h_j(x)\dot{w}_j, \\ \dfrac{\partial \eta}{\partial t} = \xi - \delta u - \dfrac{\partial \eta}{\partial r}, \end{cases} \tag{11}$$

where for simplicity, $\mu(r) = -k'(r)$ and $k(\infty) = 1$. Initial conditions are transformed into

$$\begin{cases} u(t,x) = u_\tau(t-\tau,x) = \phi(t-\tau,x), \ \tau - h \leqslant t \leqslant \tau, \ x \in \mathbb{R}^n, \\ \xi(t,x) = \xi_\tau(t-\tau,x) = \dfrac{\partial \phi(t-\tau,x)}{\partial t} + \delta\phi(t-\tau,x), \ \tau - h \leqslant t \leqslant \tau, \ x \in \mathbb{R}^n, \\ \eta(\tau,x,r) = u(\tau,x) - u(\tau - r,x) = \phi(0,x) - \phi(-r,x), \ r \in \mathbb{R}^+, \ x \in \mathbb{R}^n. \end{cases} \tag{12}$$

To convert the stochastic wave equation to a deterministic one with random parameters, let us consider a new variable given by $v(t,x) = \xi(t,x) - z(\theta_t\omega)$, where $z(\theta_t\omega) = \sum_{j=1}^m h_j z_j(\theta_t\omega_j)$. Then the system (11)-(12) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} + \delta u = v + z(\theta_t\omega), \\ \dfrac{\partial v}{\partial t} - \delta v + J(v - \delta u + z(\theta_t\omega)) - \Delta u + (\lambda + \delta^2)u - \displaystyle\int_0^\infty \mu(r)\Delta\eta(r)dr \\ +F(x,u) = f(x, u(t-\rho(t))) + g(x,t) + (\alpha+\delta)z(\theta_t\omega), \\ \dfrac{\partial \eta}{\partial t} = v - \delta u + z(\theta_t\omega) - \dfrac{\partial \eta}{\partial r}, \end{cases} \tag{13}$$

with initial conditions

$$\begin{cases} u(t,x) = u_\tau(t-\tau,x) = \phi(t-\tau,x), \ \tau - h \leqslant t \leqslant \tau, \ x \in \mathbb{R}^n, \\ v(t,x) = v_\tau(t-\tau,x) = \dfrac{\partial \phi(t-\tau,x)}{\partial t} + \delta\phi(t-\tau,x) - z(\theta_t\omega), \ \tau - h \leqslant t \leqslant \tau, \ x \in \mathbb{R}^n, \\ \eta(\tau,x,r) = u(\tau,x) - u(\tau - r,x) = \phi(0,x) - \phi(-r,x), \ r \in \mathbb{R}^+, \ x \in \mathbb{R}^n. \end{cases} \tag{14}$$

Following Dafermos [11], introduce a Hilbert "history" space $\mathcal{R}_{\mu,1} = L^2_\mu(\mathbb{R}^+; H^1(\mathbb{R}^n))$ with the inner product and norm

$$(\eta,\eta_1)_{\mu,1} = \int_0^\infty \mu(r)(\nabla\eta(r), \nabla\eta_1(r))dr, \quad \forall \eta, \eta_1 \in \mathcal{R}_{\mu,1},$$

$$\|\eta\|^2_{\mu,1} = (\eta,\eta)_{\mu,1} = \int_0^\infty \mu(r)(\nabla\eta(r), \nabla\eta(r))dr, \quad \forall \eta \in \mathcal{R}_{\mu,1},$$

and a new variable $\eta(t,x,r) = u(t,x) - u(t-r,x)$.

Let $(X, \|\cdot\|_X)$ be a Banach space, we denote by C_X the Banach space $C([-h,0];X)$ with the sup-norm, i.e., $\|u\|_{C_X} = \sup_{s\in[-h,0]} \|u(s)\|_X$, for $u \in C_X$. Given $T > \tau$ and $u : [\tau - h, T) \to X$, for each $t \in [\tau, T)$ we denote by u_t the function defined on $[-h,0]$ by the relation $u_t(s) = u(t+s)$, $s \in [-h,0]$.

Let $H = L^2(\mathbb{R}^n)$ with norm $|\cdot|_2$ and inner product $(\cdot,\cdot)$, and let $V = H^1(\mathbb{R}^n)$ with norm $\|\cdot\|$. Denote by $|\cdot|_p$ the norm of $L^p(\mathbb{R}^n)$. Set $E = C_V \times C_H \times \mathcal{R}_{\mu,1}$. In the sequel, C denotes an arbitrary positive constant, which may be different from line to line and even in the same line.

By the standard Galerkin approximation and compactness method, we have the following existence result of solutions for the problem (13)-(14):

Theorem 8. *Suppose that* (**H1**)-(**H4**) *hold true and* $g \in L^2_{loc}(\mathbb{R}; H)$. *Then for each* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and for any* $(u_\tau, v_\tau, \eta(\tau)) \in E$, *there exists a solution* $(u(t), v(t), \eta(t))$ *to problem* (13)-(14), *and*

$$\begin{aligned} &u(\cdot,\tau,\omega,u_\tau) \in C([\tau-h,T];V), \ v(\cdot,\tau,\omega,v_\tau) \in C([\tau-h,T];H), \\ &\eta(\cdot,\tau,\omega,\eta(\tau)) \in C([\tau,T];\mathcal{R}_{\mu,1}), \quad \forall T > \tau. \end{aligned}$$

Proof. We divide the proof into two steps.

Step 1. Taking the inner product in H of the second equation of (13) with v, we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v|_2^2-\delta|v|_2^2+(J(v-\delta u+z(\theta_t\omega)),v)+(\lambda+\delta^2)(u,v)\\&+(\nabla u,\nabla v)-\left(\int_0^\infty\mu(r)\Delta\eta(r)dr,v\right)+(F(x,u),v)\\&=(f(x,u(t-\rho(t))),v)+\big(g(t)+(\alpha+\delta)z(\theta_t\omega),v\big).\end{aligned}\tag{15}$$

Since $v=\frac{\partial u}{\partial t}+\delta u-z(\theta_t\omega)$, it follows from Young's inequality that

$$(\lambda+\delta^2)(u,v)\geqslant\frac{(\lambda+\delta^2)}{2}\frac{d}{dt}|u|_2^2+\frac{\delta(\lambda+\delta^2)}{2}|u|_2^2-C|z(\theta_t\omega)|_2^2,\tag{16}$$

$$(\nabla u,\nabla v)\geqslant\frac{1}{2}\frac{d}{dt}\|u\|^2+\frac{\delta}{2}\|u\|^2-C\|z(\theta_t\omega)\|^2,\tag{17}$$

$$(g(t)+(\alpha+\delta)z(\theta_t\omega),v)\leqslant\frac{\beta_1}{8}|v|_2^2+C|g(t)|_2^2+C|z(\theta_t\omega)|_2^2,\tag{18}$$

and by (**H3**),

$$(f(x,u(t-\rho(t))),v)\leqslant\frac{\beta_1}{8}|v|_2^2+\frac{2|k_1|_2^2}{\beta_1}+\frac{2k_2^2}{\beta_1}|u(t-\rho(t))|_2^2.\tag{19}$$

By Lagrange's mean value theorem and (**H1**), we obtain that

$$\begin{aligned}&(J(v-\delta u+z(\theta_t\omega)),v)=(J'(\zeta)(v-\delta u+z(\theta_t\omega)),v)\\&\geqslant\beta_1|v|_2^2-J'(\zeta)|(\delta u-z(\theta_t\omega),v)|\geqslant\frac{\beta_1}{2}|v|_2^2-\frac{\beta_2^2\delta^2}{\beta_1}|u|_2^2-C|z(\theta_t\omega)|_2^2,\end{aligned}\tag{20}$$

where ζ is between 0 and $v-\delta u+z(\theta_t\omega)$. Integrating by parts and from (**H2**), we have

$$\left(\eta,\frac{\partial\eta}{\partial r}\right)_{\mu,1}=\int_0^\infty\mu(r)\int_{\mathbb{R}^n}\nabla\eta(r)\nabla\frac{\partial\eta(r)}{\partial r}dxdr=-\frac{1}{2}\int_0^\infty\mu'(r)\|\eta(r)\|^2dr\geqslant\frac{\sigma}{2}\|\eta\|_{\mu,1}^2,\tag{21}$$

and thus

$$\begin{aligned}&-\left(\int_0^\infty\mu(r)\Delta\eta(r)dr,v\right)=-\left(\int_0^\infty\mu(r)\Delta\eta(r)dr,\frac{\partial\eta}{\partial t}+\delta u+\frac{\partial\eta}{\partial r}-z(\theta_t\omega)\right)\\&=\frac{1}{2}\frac{d}{dt}\|\eta\|_{\mu,1}^2-\left(\int_0^\infty\mu(r)\Delta\eta(r)dr,\delta u-z(\theta_t\omega)+\frac{\partial\eta}{\partial r}\right)\\&\geqslant\frac{1}{2}\frac{d}{dt}\|\eta\|_{\mu,1}^2+\frac{\sigma}{4}\|\eta\|_{\mu,1}^2-\frac{2\delta^2m_0}{\sigma}\|u\|^2-\frac{2m_0}{\sigma}\|z(\theta_t\omega)\|^2.\end{aligned}\tag{22}$$

Let $\overline{G}(u)=\int_{\mathbb{R}^n}G(x,u)dx$. Then for the last term on the left-hand side of (15), by (**H4**) and Young's inequality, we deduce that

$$\begin{aligned}&(F(x,u),z(\theta_t\omega))\leqslant\int_{\mathbb{R}^n}\left(|k_3(x)|+k_4|u|^{p-1}\right)|z(\theta_t\omega)|dx\\&\leqslant\frac{1}{2}\left(|k_3|_2^2+|z(\theta_t\omega)|_2^2\right)+\frac{k_5\delta k_7}{2}|u|_p^p+C|z(\theta_t\omega)|_p^p\\&\leqslant\frac{1}{2}\left(|k_3|_2^2+|z(\theta_t\omega)|_2^2\right)+\frac{\delta k_7}{2}\overline{G}(u)+C|k_6|_1+C|z(\theta_t\omega)|_p^p,\end{aligned}\tag{23}$$

which implies that

$$
\begin{aligned}
&(F(x,u),v) = (F(x,u), \frac{\partial u}{\partial t} + \delta u - z(\theta_t\omega)) \\
&\geqslant \frac{d}{dt}\overline{G}(u) + \frac{\delta k_7}{2}\overline{G}(u) - |k_8|_1 - \frac{1}{2}\left(|k_3|_2^2 + |z(\theta_t\omega)|_2^2\right) - C|k_6|_1 - C|z(\theta_t\omega)|_p^p.
\end{aligned}
\tag{24}
$$

It follows from (15)-(20), (22) and (24) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|v|_2^2 + \|u\|^2 + (\lambda+\delta^2)|u|_2^2 + \|\eta\|_{\mu,1}^2 + 2\overline{G}(u)\right) + \left(\frac{\beta_1}{4} - \delta\right)|v|_2^2 \\
&+ \left(\frac{\delta(\lambda+\delta^2)}{2} - \frac{\beta_2^2\delta^2}{\beta_1}\right)|u|_2^2 + \left(\frac{\delta}{2} - \frac{2\delta^2 m_0}{\sigma}\right)\|u\|^2 + \frac{\sigma}{4}\|\eta\|_{\mu,1}^2 + \frac{\delta k_7}{2}\overline{G}(u) \\
&\leqslant \frac{2k_2^2}{\beta_1}|u(t-\rho(t))|_2^2 + C|g(t)|_2^2 + C + C(|z(\theta_t\omega)|_2^2 + |z(\theta_t\omega)|_p^p + \|z(\theta_t\omega)\|^2).
\end{aligned}
\tag{25}
$$

Integrating (25) from τ to t, in view of (**H4**), we find that

$$
\begin{aligned}
&|v(t,\tau,\omega,v_\tau)|_2^2 + \|u(t,\tau,\omega,u_\tau)\|^2 + (\lambda+\delta^2)|u(t,\tau,\omega,u_\tau)|_2^2 + \|\eta(t,\tau,\omega,\eta(\tau))\|_{\mu,1}^2 \\
&+ 2k_5|u(t,\tau,\omega,u_\tau)|_p^p + \left(\frac{\beta_1}{2} - 2\delta\right)\int_\tau^t |v(r,\tau,\omega,v_\tau)|_2^2 dr \\
&+ \left(\delta - \frac{4\delta^2 m_0}{\sigma}\right)\int_\tau^t \|u(r,\tau,\omega,u_\tau)\|^2 dr + \frac{\sigma}{2}\int_\tau^t \|\eta(r,\tau,\omega,\eta(\tau))\|_{\mu,1}^2 dr \\
&+ \left(\delta(\lambda+\delta^2) - \frac{2\beta_2^2\delta^2}{\beta_1}\right)\int_\tau^t |u(r,\tau,\omega,u_\tau)|_2^2 dr + \delta k_7 k_5 \int_\tau^t |u(r,\tau,\omega,u_\tau)|_p^p dr \\
&\leqslant |v(\tau,\tau,\omega,v_\tau)|_2^2 + \|u(\tau,\tau,\omega,u_\tau)\|^2 + (\lambda+\delta^2)|u(\tau,\tau,\omega,u_\tau)|_2^2 + C \\
&+ \|\eta(\tau,\tau,\omega,\eta(\tau))\|_{\mu,1}^2 + 2\overline{G}(u(\tau,\tau,\omega,u_\tau)) + C(t-\tau) + C\int_\tau^t |g(r)|_2^2 dr \\
&+ \frac{4k_2^2}{\beta_1}\int_\tau^t |u(r-\rho(r),\tau,\omega,u_\tau)|_2^2 dr + C\int_\tau^t \left(|z(\theta_r\omega)|_2^2 + |z(\theta_r\omega)|_p^p + \|z(\theta_r\omega)\|^2\right) dr.
\end{aligned}
\tag{26}
$$

Let $r' = r - \rho(r)$, where $\rho(r) \in [0,h]$ and $\frac{1}{1-\rho'(r)} \leqslant \frac{1}{1-\rho_*}$ for all $r \in \mathbb{R}$. Therefore,

$$
\begin{aligned}
\int_\tau^t |u(r-\rho(r),\tau,\omega,u_\tau)|_2^2 dr &\leqslant \frac{1}{1-\rho_*}\int_{\tau-h}^t |u(r',\tau,\omega,u_\tau)|_2^2 dr' \\
&\leqslant \frac{1}{1-\rho_*}\left(h\|\phi\|_{C_H}^2 + \int_\tau^t |u(r',\tau,\omega,u_\tau)|_2^2 dr'\right).
\end{aligned}
\tag{27}
$$

Choosing $\delta > 0$ small enough such that

$$
\delta < \min\{\frac{\beta_1}{4}, \frac{\sigma}{4m_0}\} \quad \text{and} \quad \delta(\lambda+\delta^2) - \frac{2\beta_2^2\delta^2}{\beta_1} > 0,
$$

in view of

$$
\begin{aligned}
\overline{G}(u(\tau,\tau,\omega,u_\tau)) &= \int_{\mathbb{R}^n} G(x,u(\tau,\tau,\omega,u_\tau))dx \\
&\leqslant \frac{1}{k_7}\int_{\mathbb{R}^n} F(x,u(\tau,\tau,\omega,u_\tau))u(\tau,\tau,\omega,u_\tau)dx + \frac{1}{k_7}|k_8|_1 \\
&\leqslant C|u(\tau,\tau,\omega,u_\tau)|_2^2 + C|u(\tau,\tau,\omega,u_\tau)|_p^p + C, \qquad (28)
\end{aligned}
$$

then, it follows from (26)-(28) that

$$
\begin{aligned}
&|v(t,\tau,\omega,v_\tau)|_2^2 + \|u(t,\tau,\omega,u_\tau)\|^2 + (\lambda+\delta^2)|u(t,\tau,\omega,u_\tau)|_2^2 + \|\eta(t,\tau,\omega,\eta(\tau))\|_{\mu,1}^2 \\
&+ 2k_5|u(t,\tau,\omega,u_\tau)|_p^p \leqslant |v(\tau,\tau,\omega,v_\tau)|_2^2 + \|u(\tau,\tau,\omega,u_\tau)\|^2 + C|u(\tau,\tau,\omega,u_\tau)|_2^2 \\
&+ \|\eta(\tau,\tau,\omega,\eta(\tau))\|_{\mu,1}^2 + C|u(\tau,\tau,\omega,u_\tau)|_p^p + C + C(t-\tau) + C\|\phi\|_{C_H}^2 \\
&+ C\int_\tau^t |g(r)|_2^2 dr + C\int_\tau^t \left(|z(\theta_r\omega)|_2^2 + |z(\theta_r\omega)|_p^p + \|z(\theta_r\omega)\|^2\right) dr \\
&+ \frac{4k_2^2}{(\lambda+\delta^2)\beta_1(1-\rho_*)}\int_\tau^t \Bigg(|v(r,\tau,\omega,v_\tau)|_2^2 + \|u(r,\tau,\omega,u_\tau)\|^2 \\
&+ (\lambda+\delta^2)|u(r,\tau,\omega,u_\tau)|_2^2 + \|\eta(r,\tau,\omega,\eta(\tau))\|_{\mu,1}^2 + 2k_5|u(r,\tau,\omega,u_\tau)|_p^p\Bigg)dr. \qquad (29)
\end{aligned}
$$

By Gronwall's lemma, we deduce that

$$
\begin{aligned}
&|v(t,\tau,\omega,v_\tau)|_2^2 + \|u(t,\tau,\omega,u_\tau)\|^2 + (\lambda+\delta^2)|u(t,\tau,\omega,u_\tau)|_2^2 + \|\eta(t,\tau,\omega,\eta(\tau))\|_{\mu,1}^2 \\
&+ 2k_5|u(t,\tau,\omega,u_\tau)|_p^p \leqslant e^{C(t-\tau)}\left(|v(\tau,\tau,\omega,v_\tau)|_2^2 + \|u(\tau,\tau,\omega,u_\tau)\|^2 + C|u(\tau,\tau,\omega,u_\tau)|_2^2\right) \\
&+ e^{C(t-\tau)}\left(\|\eta(\tau,\tau,\omega,\eta(\tau))\|_{\mu,1}^2 + C|u(\tau,\tau,\omega,u_\tau)|_p^p + C\|\phi\|_{C_H}^2 + C(t-\tau+1)\right) \\
&+ Ce^{C(t-\tau)}\int_\tau^t |g(r)|_2^2 dr + Ce^{C(t-\tau)}\int_\tau^t (|z(\theta_r\omega)|_2^2 + |z(\theta_r\omega)|_p^p + \|z(\theta_r\omega)\|^2)dr. \qquad (30)
\end{aligned}
$$

Step 2. We consider the Dirichlet problem in a bounded domain

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \delta u = v + z(\theta_t\omega),\ t>\tau,\ x\in\Omega_K, \\
\dfrac{\partial v}{\partial t} - \delta v + J(v - \delta u + z(\theta_t\omega)) - \Delta u + (\lambda+\delta^2)u - \displaystyle\int_0^\infty \mu(r)\Delta\eta(r)dr \\
+F(x,u) = f(x,u(t-\rho(t))) + g(x,t) + (\alpha+\delta)z(\theta_t\omega),\ t>\tau,\ x\in\Omega_K, \\
\dfrac{\partial \eta}{\partial t} = v - \delta u + z(\theta_t\omega) - \dfrac{\partial\eta}{\partial r},\ t>\tau,\ x\in\Omega_K,
\end{cases} \qquad (31)
$$

with initial and boundary conditions

$$
u|_{\partial\Omega_K} = v|_{\partial\Omega_K} = \eta|_{\partial\Omega_K} = 0,\ t>\tau,
$$

$$
\begin{cases}
u(t,x) = u_\tau(t-\tau,x) = \phi_K(t-\tau,x),\ \tau-h\leqslant t\leqslant\tau,\ x\in\Omega_K, \\
v(t,x) = v_\tau(t-\tau,x) = \dfrac{\partial\phi_K(t-\tau,x)}{\partial t} + \delta\phi_K(t-\tau,x) - z(\theta_t\omega)\psi_K(|x|), \\
\tau-h\leqslant t\leqslant\tau,\ x\in\Omega_K, \\
\eta(\tau,x,r) = \phi_K(0,x) - \phi_K(-r,x),\ r\in\mathbb{R}^+,\ x\in\Omega_K,
\end{cases} \qquad (32)
$$

where $\Omega_K = B(0,K)$ is the open ball of radius $K \geqslant 1$ centered at 0, $\phi_K(t,x) = \phi(t,x)\psi_K(|x|)$ for each $t \in (-\infty, 0]$, and ψ_K is a smooth function satisfying

$$\psi_K(\xi) = \begin{cases} 1, \text{ if } 0 \leqslant \xi \leqslant K-1, \\ 0 \leqslant \psi_K(\xi) \leqslant 1, \text{ if } K-1 \leqslant \xi \leqslant K, \\ 0, \text{ if } \xi > K. \end{cases}$$

Let $H_K = L^2(\Omega_K)$ and $V_K = H_0^1(\Omega_K)$. Note that there exists a family of eigenfunctions $\{e_n\}_{n=1}^\infty$ of $-\Delta$, which is the orthonormal basis of H_K. We consider the subspace V_{Km} of V_K spanned by e_1, e_2, …, e_m, and the projector $P_m : H_K \to V_{Km}$ defined as

$$P_m u = \sum_{i=1}^m (u, e_i) e_i, \; u \in H_K.$$

Let $u_m(t) = \sum\limits_{i=1}^m \gamma_{mi}(t) e_i$ be a solution of the ordinary functional differential system

$$\begin{cases} \dfrac{du_m}{dt} + \delta u_m = v_m + P_m z(\theta_t\omega), \\ \dfrac{dv_m}{dt} - \delta v_m + P_m J(v_m - \delta u_m + P_m z(\theta_t\omega)) - \Delta u_m + (\lambda + \delta^2) u_m - \displaystyle\int_0^\infty \mu(r) \Delta \eta_m(r) dr \\ +P_m F(x, u_m) = P_m f(x, u_m(t-\rho(t))) + P_m g(x,t) + (\alpha+\delta) P_m z(\theta_t\omega), \\ \dfrac{d\eta_m}{dt} = v_m - \delta u_m + P_m z(\theta_t\omega) - \dfrac{\partial \eta_m}{\partial r}, \end{cases} \tag{33}$$

with initial conditions

$$\begin{cases} u_m(t,x) = P_m \phi_K(t-\tau, x), \; \tau - h \leqslant t \leqslant \tau, \\ v_m(t,x) = \dfrac{dP_m\phi_K(t-\tau,x)}{dt} + \delta P_m \phi_K(t-\tau,x) - P_m z(\theta_t\omega)\psi_k(|x|), \; \tau - h \leqslant t \leqslant \tau, \\ \eta_m(\tau, x, r) = P_m\phi_K(0,x) - P_m \phi_K(-r,x), \; r \in \mathbb{R}^+. \end{cases} \tag{34}$$

Then it follows from (30) that

$$\{u_m(t)\} \text{ is bounded in } L^\infty(\tau - h, T; V_K),$$
$$\{v_m(t)\} \text{ is bounded in } L^\infty(\tau - h, T; H_K),$$

and

$$\{\eta_m(t)\} \text{ is bounded in } L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V_K)).$$

Analogous to the proof of Theorem 3.1 in [8] Section XV.3 and the argument in [30] Sections II.4 and IV.4.4, in view of the continuity of $z(\theta_t\omega)$ in t for every $\omega \in \Omega$, by a standard argument we obtain the existence of weak solutions

$$u(\cdot, \tau, \omega, u_\tau) \in C([\tau - h, T]; V_K), \; v(\cdot, \tau, \omega, v_\tau) \in C([\tau - h, T]; H_K),$$
$$\eta(\cdot, \tau, \omega, \eta(\tau)) \in C([\tau, T]; L^2_\mu(\mathbb{R}^+; V_K)).$$

Let $\{\Omega_{K_j}\}$ be a sequence of bounded subdomains of $\mathbb{R}^n$ and $\Omega_{K_j} \to \mathbb{R}^n$ as $K_j \to \infty$. By the similar approximation argument of Theorem 5 in [23], we have the existence of weak solutions $(u(t), v(t), \eta(t))$ associated to problem (13)-(14), and

$$u(\cdot, \tau, \omega, u_\tau) \in C([\tau - h, T]; V), \; v(\cdot, \tau, \omega, v_\tau) \in C([\tau - h, T]; H),$$
$$\eta(\cdot, \tau, \omega, \eta(\tau)) \in C([\tau, T]; \mathcal{R}_{\mu,1}), \quad \forall T > \tau,$$

so its proof is omitted here. □

In order to obtain the uniqueness and continuous dependence of solutions, we also need the following conditions:

(**H6**) There exists $L > 0$ such that

$$|f(x,u) - f(x,v)| \leqslant L|u-v|, \forall x \in \mathbb{R}, u, v \in \mathbb{R}.$$

(**H7**) There exists a $L_1 > 0$ such that

$$|F(x,u) - F(x,v)| \leqslant L_1(1 + |u|^{p-2} + |v|^{p-2})|u-v|, \forall x \in \mathbb{R}, u, v \in \mathbb{R}.$$

Theorem 9. *In addition to the hypotheses in Theorem 8, suppose that* (**H6**)-(**H7**) *hold true. Then there exists a unique solution to the problem* (13)-(14)*, and the solutions continuously depend on the initial data in* E *for any* $\tau \leqslant t$ *and* $\omega \in \Omega$.

Proof. Assume that $(u'_\tau, v'_\tau, \eta'(\tau))$, $(u^*_\tau, v^*_\tau, \eta^*(\tau)) \in E$, and consider the solutions $(u'(\cdot), v'(\cdot), \eta'(\cdot))$, $(u^*(\cdot), v^*(\cdot), \eta^*(\cdot))$ to (13) corresponding to the initial data $(u'_\tau, v'_\tau, \eta'(\tau))$ and $(u^*_\tau, v^*_\tau, \eta^*(\tau))$, respectively. Let $\tilde{u} = u' - u^*$, $\tilde{v} = v' - v^*$, $\tilde{\eta} = \eta' - \eta^*$, $\tilde{\phi} = \phi' - \phi^*$. Then we have from (13) that

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} + \delta\tilde{u} = \tilde{v}, \\ \dfrac{\partial \tilde{v}}{\partial t} - \delta\tilde{v} + J(v' - \delta u' + z(\theta_t\omega)) - J(v^* - \delta u^* + z(\theta_t\omega)) - \Delta\tilde{u} \\ +(\lambda+\delta^2)\tilde{u} - \displaystyle\int_0^\infty \mu(r)\Delta\tilde{\eta}(r)dr + F(x,u') - F(x,u^*) \\ = f(x, u'(t-\rho(t))) - f(x, u^*(t-\rho(t))), \\ \dfrac{\partial \tilde{\eta}}{\partial t} = \tilde{v} - \delta\tilde{u} - \dfrac{\partial \tilde{\eta}}{\partial r}, \end{cases} \tag{35}$$

with initial conditions

$$\begin{cases} \tilde{u}(t,x) = \tilde{u}_\tau(t-\tau,x) = \tilde{\phi}(t-\tau,x), \ \tau - h \leqslant t \leqslant \tau, \ x \in \mathbb{R}^n, \\ \tilde{v}(t,x) = \tilde{v}_\tau(t-\tau,x) = \dfrac{\partial\tilde{\phi}(t-\tau,x)}{\partial t} + \delta\tilde{\phi}(t-\tau,x), \ \tau - h \leqslant t \leqslant \tau, \ x \in \mathbb{R}^n, \\ \tilde{\eta}(\tau,x,r) = \tilde{u}(\tau,x) - \tilde{u}(\tau - r,x) = \tilde{\phi}(0,x) - \tilde{\phi}(-r,x), \ r \in \mathbb{R}^+, \ x \in \mathbb{R}^n. \end{cases} \tag{36}$$

Take the inner product in H of the second equation of (35) with $\tilde{v}$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\tilde{v}|_2^2 - \delta|\tilde{v}|_2^2 + (J(v' - \delta u' + z(\theta_t\omega)) - J(v^* - \delta u^* + z(\theta_t\omega)), \tilde{v}) \\ &+ (\nabla\tilde{u}, \nabla\tilde{v}) + (\lambda+\delta^2)(\tilde{u},\tilde{v}) - \left(\int_0^\infty \mu(r)\Delta\tilde{\eta}(r)dr, \tilde{v}\right) + (F(x,u') - F(x,u^*), \tilde{v}) \\ &= (f(x,u'(t-\rho(t))) - f(x,u^*(t-\rho(t))), \tilde{v}). \end{aligned} \tag{37}$$

Since $\tilde{v} = \frac{\partial\tilde{u}}{\partial t} + \delta\tilde{u}$, we deduce that

$$(\lambda+\delta^2)(\tilde{u},\tilde{v}) = \frac{(\lambda+\delta^2)}{2}\frac{d}{dt}|\tilde{u}|_2^2 + \delta(\lambda+\delta^2)|\tilde{u}|_2^2, \tag{38}$$

$$(\nabla\tilde{u}, \nabla\tilde{v}) = \frac{1}{2}\frac{d}{dt}\|\tilde{u}\|^2 + \delta\|\tilde{u}\|^2. \tag{39}$$

By Hölder inequality and (**H6**)-(**H7**),

$$(f(x,u'(t-\rho(t))) - f(x,u^*(t-\rho(t))), \tilde{v}) \leqslant \frac{\beta_1}{8}|\tilde{v}|_2^2 + C|\tilde{u}(t-\rho(t))|_2^2, \tag{40}$$

$$\begin{aligned}
&(F(x,u^*)-F(x,u'),\tilde{v}) \leqslant \frac{\beta_1}{8}|\tilde{v}|_2^2 + C|F(x,u^*)-F(x,u')|_2^2 \\
&\leqslant \frac{\beta_1}{8}|\tilde{v}|_2^2 + C\int_{\mathbb{R}^n}(1+|u'|^{2p-4}+|u^*|^{2p-4})|\tilde{u}|_2^2 \\
&\leqslant \frac{\beta_1}{8}|\tilde{v}|_2^2 + C\|\tilde{u}\|^2, \qquad (41)
\end{aligned}$$

where we have used (30) in the last inequality. By the similar arguments in (20)-(22), we find that

$$\begin{aligned}
&(J(v'-\delta u'+z(\theta_t\omega)) - J(v^*-\delta u^*+z(\theta_t\omega)),\tilde{v}) = \Big(J'(\hat{\zeta})(\tilde{v}-\delta\tilde{u}),\tilde{v}\Big) \\
&\geqslant \beta_1|\tilde{v}|_2^2 - \delta\beta_2|(\tilde{u},\tilde{v})| \geqslant \frac{\beta_1}{2}|\tilde{v}|_2^2 - \frac{\beta_2^2\delta^2}{2\beta_1}|\tilde{u}|_2^2, \qquad (42)
\end{aligned}$$

where $\hat{\zeta}$ is between $v'-\delta u'+z(\theta_t\omega)$ and $v^*-\delta u^*+z(\theta_t\omega)$, and

$$\begin{aligned}
&-\left(\int_0^\infty \mu(r)\Delta\tilde{\eta}(r)dr,\tilde{v}\right) = -\left(\int_0^\infty \mu(r)\Delta\tilde{\eta}(r)dr, \frac{\partial\tilde{\eta}}{\partial t}+\delta\tilde{u}+\frac{\partial\tilde{\eta}}{\partial r}\right) \\
&= \frac{1}{2}\frac{d}{dt}\|\tilde{\eta}\|_{\mu,1}^2 + \frac{\sigma}{2}\|\tilde{\eta}\|_{\mu,1}^2 - \left(\int_0^\infty \mu(r)\Delta\eta(r)dr,\delta\tilde{u}\right) \\
&\geqslant \frac{1}{2}\frac{d}{dt}\|\tilde{\eta}\|_{\mu,1}^2 + \frac{\sigma}{4}\|\tilde{\eta}\|_{\mu,1}^2 - \frac{\delta^2 m_0}{\sigma}\|\tilde{u}\|^2. \qquad (43)
\end{aligned}$$

Inserting (38)-(43) into (37) yields that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|\tilde{v}|_2^2 + \|\tilde{u}\|^2 + (\lambda+\delta^2)|\tilde{u}|_2^2 + \|\tilde{\eta}\|_{\mu,1}^2\right) + \left(\frac{\beta_1}{4}-\delta\right)|\tilde{v}|_2^2 \\
&+ \left(\delta(\lambda+\delta^2) - \frac{\beta_2^2\delta^2}{2\beta_1}\right)|\tilde{u}|_2^2 + \left(\delta - \frac{\delta^2 m_0}{\sigma}\right)\|\tilde{u}\|^2 + \frac{\sigma}{4}\|\tilde{\eta}\|_{\mu,1}^2 \\
&\leqslant C\|\tilde{u}\|^2 + C|\tilde{u}(t-\rho(t))|_2^2. \qquad (44)
\end{aligned}$$

By the selection of δ in Theorem 8, we obtain

$$\frac{\beta_1}{4}-\delta>0,\ \delta(\lambda+\delta^2)-\frac{\beta_2^2\delta^2}{2\beta_1}>0, \delta-\frac{\delta^2 m_0}{\sigma}>0.$$

Integrating (44) from t to τ, and arguing as in (27), we deduced that

$$\begin{aligned}
&|\tilde{v}(t,\tau,\omega,\tilde{v}_\tau)|_2^2 + \|\tilde{u}(t,\tau,\omega,\tilde{u}_\tau)\|^2 + (\lambda+\delta^2)|\tilde{u}(t,\tau,\omega,\tilde{u}_\tau)|_2^2 + \|\tilde{\eta}(t,\tau,\omega,\tilde{\eta}(\tau))\|_{\mu,1}^2 \\
&\leqslant |\tilde{v}(\tau,\tau,\omega,\tilde{v}_\tau)|_2^2 + \|\tilde{u}(\tau,\tau,\omega,\tilde{u}_\tau)\|^2 + \|\tilde{\eta}(\tau,\tau,\omega,\tilde{\eta}(\tau))\|_{\mu,1}^2 \\
&+ C\|\tilde{\phi}\|_{C_H}^2 + C\int_\tau^t \Big(|\tilde{v}(r,\tau,\omega,\tilde{v}_\tau)|_2^2 + \|\tilde{u}(r,\tau,\omega,\tilde{u}_\tau)\|^2 \\
&+ (\lambda+\delta^2)|\tilde{u}(r,\tau,\omega,\tilde{u}_\tau)|_2^2 + \|\tilde{\eta}(r,\tau,\omega,\tilde{\eta}(\tau))\|_{\mu,1}^2\Big)dr. \qquad (45)
\end{aligned}$$

The Gronwall lemma implies that

$$\begin{aligned}
&|\tilde{v}(t,\tau,\omega,\tilde{v}_\tau)|_2^2 + \|\tilde{u}(t,\tau,\omega,\tilde{u}_\tau)\|^2 + (\lambda+\delta^2)|\tilde{u}(t,\tau,\omega,\tilde{u}_\tau)|_2^2 + \|\tilde{\eta}(t,\tau,\omega,\tilde{\eta}(\tau))\|_{\mu,1}^2 \\
&\leqslant e^{C(t-\tau)}\left(|\tilde{v}(\tau,\tau,\omega,\tilde{v}_\tau)|_2^2 + \|\tilde{u}(\tau,\tau,\omega,\tilde{u}_\tau)\|^2 + C\|\tilde{\phi}\|_{C_H}^2 + \|\tilde{\eta}(\tau,\tau,\omega,\tilde{\eta}(\tau))\|_{\mu,1}^2\right), \qquad (46)
\end{aligned}$$

and thus the assertion of this theorem follows immediately. □

Thanks to Theorems 8-9 and Lemma 10, we define a continuous cocycle Φ for problem (12)-(13) which is given by $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E$ and for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $(u_\tau, v_\tau, \eta(\tau)) \in E$,

$$\Phi(t, \tau, \omega, (u_\tau, v_\tau, \eta(\tau)) = \Big\{ \big(u_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, u_\tau), v_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, v_\tau), \eta(t+\tau, \tau, \theta_{-\tau}\omega, \eta(\tau)) \big) \Big\},$$

where $u_{t+\tau}$ and $v_{t+\tau}$ are defined for $\theta \in [-h, 0]$ as $u_{t+\tau}(\theta) = u(t+\tau+\theta)$ and $v_{t+\tau}(\theta) = v(t+\tau+\theta)$ respectively, and $v_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, v_\tau) = \xi_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \xi_\tau) - z(\theta_{t+\cdot}\omega)$ with $v_\tau(\cdot) = \xi_\tau(\cdot) - z(\theta.\omega)$.

4. **Uniform estimates of solutions.** In this section, we derive uniform estimates on the solutions of (13)-(14) defined on $\mathbb{R}^n$ for the purpose of proving the existence of a pullback absorbing set of the random dynamical system. In particular, we will show uniform estimates on the tail parts of the solutions for large space variables when time is sufficiently large in order to prove the pullback asymptotic compactness of continuous cocycle associated with the equation on the unbounded domain $\mathbb{R}^n$.

Assume that $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a family of bounded nonempty subsets of E satisfying, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and each $\eta \in \{\alpha, \alpha_1\}$,

$$\lim_{t \to -\infty} e^{\eta t} \sup_{(\varphi, \psi, \eta) \in D(\tau+t, \theta_t \omega)} \left(\|\varphi\|^2_{C_H} + \|\varphi\|^2_{C_V} + \|\varphi\|^p_{C_{L^p(\mathbb{R}^n)}} + \|\psi\|^2_{C_H} + \|\eta\|^2_{\mu,1} \right) = 0, \tag{47}$$

where α and α_1 will be given in Lemma 10 and Theorem 13, respectively. Denote by $\mathcal{D}$ the collection of all families of bounded nonempty subsets of E which fulfill condition (47), i.e.,

$$\mathcal{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies } (47)\}.$$

Obviously $\mathcal{D}$ is neighborhood closed.

Lemma 10. *In addition to the assumptions* (**H1**)-(**H7**)*, assume that there exists a positive constant* α *such that*

$$\alpha < \frac{\sigma}{2}, \tag{48}$$

$$\frac{2\alpha}{k_7} < \delta < \frac{\beta_1}{4} - \frac{\alpha}{2}, \tag{49}$$

$$-\frac{4m_0}{\sigma}\delta^2 + \delta - \alpha > 0, \tag{50}$$

and

$$(\delta - \alpha)(\lambda + \delta^2) - \frac{2\beta_2^2 \delta^2}{\beta_1} - \frac{4k_2^2 e^{\alpha h}}{\beta_1(1-\rho_*)} > 0. \tag{51}$$

Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the solution of (13)-(14) *with ω replaced by $\theta_{-\tau}\omega$ satisfies for all $t \geqslant h$,*

$$\begin{aligned}
&\|v_\tau\|_{C_H}^2 + \|u_\tau\|_{C_H}^2 + \|u_\tau\|_{C_V}^2 + \|u_\tau\|_{C_{L^p(\mathbb{R}^n)}}^p + \|\eta(\tau)\|_{\mu,1}^2 \\
&\leqslant Ce^{-\alpha t}\left(|v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_2^2 + \|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\right) \\
&+ Ce^{-\alpha t}\left(|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_2^2 + |u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p\right) \\
&+ Ce^{-\alpha t}\|\eta(\tau-t,\tau-t,\theta_{-\tau}\omega,\eta(\tau-t))\|_{\mu,1}^2 + C + Ce^{-\alpha\tau}\int_{-\infty}^{\tau} e^{\alpha r}|g(r)|_2^2 dr \\
&+ Ce^{-\alpha t}\|\phi\|_{C_H}^2 + C\int_{-\infty}^{0} e^{\alpha r}\sum_{j=1}^{m}(|z_j(\theta_r\omega_j)|^2 + |z_j(\theta_r\omega_j)|^p)dr,
\end{aligned}$$

where C is a positive constant independent of τ and ω.

Proof. It follows from (25) that

$$\begin{aligned}
&\frac{d}{dt}\left(e^{\alpha t}\left(|v|_2^2 + \|u\|^2 + (\lambda+\delta^2)|u|_2^2 + \|\eta\|_{\mu,1}^2 + 2\overline{G}(u)\right)\right) + \left(\frac{\beta_1}{2} - 2\delta - \alpha\right)e^{\alpha t}|v|_2^2 \\
&+ \left(\delta - \frac{4\delta^2 m_0}{\sigma} - \alpha\right)e^{\alpha t}\|u\|^2 + \left((\delta-\alpha)(\lambda+\delta^2) - \frac{2\beta_2^2\delta^2}{\beta_1}\right)e^{\alpha t}|u|_2^2 \\
&+ \left(\frac{\sigma}{2} - \alpha\right)e^{\alpha t}\|\eta\|_{\mu,1}^2 + (\delta k_7 - 2\alpha)e^{\alpha t}\overline{G}(u) \leqslant \frac{4k_2^2}{\beta_1}e^{\alpha t}|u(t-\rho(t))|_2^2 \\
&+ Ce^{\alpha t}|g(t)|_2^2 + Ce^{\alpha t} + Ce^{\alpha t}(|z(\theta_t\omega)|_2^2 + |z(\theta_t\omega)|_p^p + \|z(\theta_t\omega)\|^2). \qquad (52)
\end{aligned}$$

Given $t \geqslant 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\tau - t \leqslant T \leqslant \tau$, integrating (52) over $(\tau - t, T)$, in view of (**H4**), we obtain that

$$
\begin{aligned}
&|v(T, \tau-t, \omega, v_{\tau-t})|_2^2 + \|u(T, \tau-t, \omega, u_{\tau-t})\|^2 + (\lambda+\delta^2)|u(T, \tau-t, \omega, u_{\tau-t})|_2^2 \\
&+ \|\eta(T, \tau-t, \omega, \eta(\tau-t))\|_{\mu,1}^2 + 2k_5|(u(T, \tau-t, \omega, u_{\tau-t})|_p^p \\
&+ \left(\frac{\beta_1}{2} - 2\delta - \alpha\right) e^{-\alpha T} \int_{\tau-t}^{T} e^{\alpha r} |v(r, \tau-t, \omega, v_{\tau-t})|_2^2 dr \\
&+ \left(\delta - \frac{4\delta^2 m_0}{\sigma} - \alpha\right) e^{-\alpha T} \int_{\tau-t}^{T} e^{\alpha r} \|u(r, \tau-t, \omega, u_{\tau-t})\|^2 dr \\
&+ \left((\delta-\alpha)(\lambda+\delta^2) - \frac{2\beta_2^2\delta^2}{\beta_1}\right) e^{-\alpha T} \int_{\tau-t}^{T} e^{\alpha r} |u(r, \tau-t, \omega, u_{\tau-t})|_2^2 dr \\
&+ \left(\frac{\sigma}{2} - \alpha\right) e^{-\alpha T} \int_{\tau-t}^{T} e^{\alpha r} \|\eta(r, \tau-t, \omega, \eta(\tau-t))\|_{\mu,1}^2 dr \\
&+ k_5(\delta k_7 - 2\alpha) e^{-\alpha T} \int_{\tau-t}^{T} e^{\alpha r} |u(r, \tau-t, \omega, u_{\tau-t})|_p^p dr \\
&\leqslant e^{-\alpha(T-\tau+t)} \left(|v(\tau-t, \tau-t, \omega, v_{\tau-t})|_2^2 + \|u(\tau-t, \tau-t, \omega, u_{\tau-t})\|^2\right) \\
&+ e^{-\alpha(T-\tau+t)} \left((\lambda+\delta^2)|u(\tau-t, \tau-t, \omega, u_{\tau-t})|_2^2 + \|\eta(\tau-t, \tau-t, \omega, \eta(\tau-t))\|_{\mu,1}^2\right) \\
&+ 2e^{-\alpha(T-\tau+t)} \overline{G}(u(\tau-t, \tau-t, \omega, u_{\tau-t})) + C \\
&+ \frac{4k_2^2}{\beta_1} e^{-\alpha T} \int_{\tau-t}^{T} e^{\alpha r} |u(r-\rho(r), \tau-t, \omega, u_{\tau-t})|_2^2 dr + Ce^{-\alpha T} \int_{\tau-t}^{T} e^{\alpha r} |g(r)|_2^2 dr \\
&+ Ce^{-\alpha T} \int_{\tau-t}^{T} e^{\alpha r} (|z(\theta_r\omega)|_2^2 + |z(\theta_r\omega)|_p^p + \|z(\theta_r\omega)\|^2) dr.
\end{aligned} \tag{53}
$$

Let $r' = r - \rho(r)$, where $\rho(r) \in [0, h]$ and $\frac{1}{1-\rho'(r)} \leqslant \frac{1}{1-\rho_*}$ for all $r \in \mathbb{R}$. Therefore,

$$
\begin{aligned}
&\int_{\tau-t}^{T} e^{\alpha r} |u(r-\rho(r), \tau-t, \omega, u_{\tau-t})|_2^2 dr \\
&\leqslant \frac{e^{\alpha h}}{1-\rho_*} \int_{\tau-t-h}^{T} e^{\alpha r'} |u(r', \tau-t, \omega, u_{\tau-t})|_2^2 dr' \\
&\leqslant \frac{e^{\alpha h}}{1-\rho_*} \left(\int_{\tau-t-h}^{\tau-t} e^{\alpha r'} |u(r', \tau-t, \omega, u_{\tau-t})|_2^2 dr' + \int_{\tau-t}^{T} e^{\alpha r'} |u(r', \tau-t, \omega, u_{\tau-t})|_2^2 dr'\right) \\
&\leqslant \frac{e^{\alpha(\tau-t)} e^{\alpha h} \|\phi\|_{C_H}^2}{\alpha(1-\rho_*)} + \frac{e^{\alpha h}}{1-\rho_*} \int_{\tau-t}^{T} e^{\alpha r'} |u(r', \tau-t, \omega, u_{\tau-t})|_2^2 dr'.
\end{aligned} \tag{54}
$$

By slightly modifying the argument of (28) and replacing ω by $\theta_{-\tau}\omega$, we have from (**H5**), (48)-(51) and (53)-(54) that

$$\begin{aligned}
&|v(T,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_2^2+\|u(T,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+(\lambda+\delta^2)|u(T,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_2^2\\
&+\|\eta(T,\tau-t,\theta_{-\tau}\omega,\eta(\tau-t))\|_{\mu,1}^2+2k_5|u(T,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p\\
&\leqslant e^{-\alpha(T-\tau+t)}\left(|v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_2^2+\|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\right)\\
&+e^{-\alpha(T-\tau+t)}\left(C|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_2^2+\|\eta(\tau-t,\tau-t,\theta_{-\tau}\omega,\eta(\tau-t))\|_{\mu,1}^2\right)\\
&+Ce^{-\alpha(T-\tau+t)}|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p+Ce^{-\alpha T}\int_{-\infty}^{T}e^{\alpha r}|g(r)|_2^2dr+C\\
&+Ce^{-\alpha(T-\tau+t)}\|\phi\|_{C_H}^2+Ce^{-\alpha T}\int_{\tau-t}^{T}e^{\alpha r}\left(|z(\theta_{r-\tau}\omega)|_2^2+|z(\theta_{r-\tau}\omega)|_p^p+\|z(\theta_{r-\tau}\omega)\|^2\right)dr.
\end{aligned}\tag{55}$$

Note that $z(\theta_t\omega)=\sum_{j=1}^m h_jz_j(\theta_t\omega_j)$ and $h_j\in L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)\cap H^1(\mathbb{R}^n)$. Hence, we deduce that for every $\omega\in\Omega$,

$$\begin{aligned}
&e^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}(|z(\theta_{r-\tau}\omega)|_2^2+|z(\theta_{r-\tau}\omega)|_p^p+\|z(\theta_{r-\tau}\omega)\|^2)dr\\
&\leqslant\int_{-t}^{0}e^{\alpha r'}(|z(\theta_{r'}\omega)|_2^2+|z(\theta_{r'}\omega)|_p^p+\|z(\theta_{r'}\omega)\|^2)dr'\\
&\leqslant C\int_{-\infty}^{0}e^{\alpha r'}\sum_{j=1}^{m}\left(|z_j(\theta_{r'}\omega_j)|^2+|z_j(\theta_{r'}\omega_j)|^p\right)dr'.
\end{aligned}\tag{56}$$

Replacing T by $\tau+s$ in (55) where $s\in[-h,0]$, in view of (56), we find that for all $t\geqslant h$,

$$\begin{aligned}
&|v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_2^2+\|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_2^2\\
&+|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p+\|\eta(\tau+s,\tau-t,\theta_{-\tau}\omega,\eta(\tau-t))\|_{\mu,1}^2\\
&\leqslant Ce^{-\alpha t}\left(|v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_2^2+\|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\right)\\
&+Ce^{-\alpha t}\left(|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_2^2+|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p\right)\\
&+Ce^{-\alpha t}\|\eta(\tau-t,\tau-t,\theta_{-\tau}\omega,\eta(\tau-t))\|_{\mu,1}^2+C+Ce^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}|g(r)|_2^2dr\\
&+Ce^{-\alpha t}\|\phi\|_{C_H}^2+C\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^{m}\left(|z_j(\theta_r\omega_j)|^2+|z_j(\theta_r\omega_j)|^p\right)dr.
\end{aligned}\tag{57}$$

Hence, from (57) we obtain that for all $t \geqslant h$,

$$\begin{aligned}
&\|v_\tau\|^2_{C_H} + \|u_\tau\|^2_{C_H} + \|u_\tau\|^2_{C_V} + \|u_\tau\|^p_{C_{L^p(\mathbb{R}^n)}} + \|\eta(\tau)\|^2_{\mu,1} \\
&\leqslant Ce^{-\alpha t}\left(|v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2_2 + \|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\right) \\
&+ Ce^{-\alpha t}\left(|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2_2 + |u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^p_p\right) \\
&+ Ce^{-\alpha t}\|\eta(\tau-t,\tau-t,\theta_{-\tau}\omega,\eta(\tau-t))\|^2_{\mu,1} + C + Ce^{-\alpha\tau}\int_{-\infty}^{\tau} e^{\alpha r}|g(r)|^2_2 dr \\
&+ Ce^{-\alpha t}\|\phi\|^2_{C_H} + C\int_{-\infty}^{0} e^{\alpha r}\sum_{j=1}^{m}(|z_j(\theta_r\omega_j)|^2 + |z_j(\theta_r\omega_j)|^p)dr. \qquad (58)
\end{aligned}$$

This completes the proof. □

Lemma 11. *Assume that the hypotheses in Lemma 10 hold. Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then for any $\varepsilon > 0$, there exist $T_1 = T_1(\varepsilon,\tau,\omega,B) \leqslant h$ and $K_1 = K_1(\varepsilon,\tau,\omega) > 0$ such that for all $t \geqslant T_1$ and $K \geqslant K_1$, the solution of (13)-(14) with ω replaced by $\theta_{-\tau}\omega$ satisfies*

$$\begin{aligned}
&\sup_{s\in[-h,0]}\int_{\Omega_K^c}\left(|\nabla u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2 + |v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\right)dx \\
&+ \sup_{s\in[-h,0]}\int_{\Omega_K^c}\left(|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2 + |u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^p\right)dx \\
&+ \int_0^\infty \mu(r)\int_{\Omega_K^c}|\nabla\eta(\tau,\tau-t,\theta_{-\tau}\omega,\eta(\tau-t),r)|^2 dxdr \leqslant \varepsilon,
\end{aligned}$$

where $\Omega_K^c = \{x \in \mathbb{R}^n \mid |x| \geqslant K\}$ and $(u_{\tau-t}, v_{\tau-t}, \eta(\tau-t)) \in B(\tau-t,\theta_{-t}\omega)$.

Proof. We choose a smooth function $\mathfrak{t}$ such that $0 \leqslant \mathfrak{t}(s) \leqslant 1$ for any $s \in \mathbb{R}^+$ with

$$\mathfrak{t}(s) = 0 \text{ for } 0 \leqslant s \leqslant 1 \text{ and } \mathfrak{t}(s) = 1 \text{ for } s \geqslant 2,$$

and there exist positive constants C_0, C_0' such that $|\mathfrak{t}'(s)| \leqslant C_0$ and $\mathfrak{t}''(s) \leqslant C_0'$ for any $s \in \mathbb{R}^+$. Multiplying the second equation of (13) by $\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)v$ and then integrating over $\mathbb{R}^n$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|v|^2dx - \delta\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|v|^2dx - \int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)v\Delta udx \\
&+ (\lambda+\delta^2)\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)uvdx + \int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)J(v-\delta u + z(\theta_t\omega))vdx \\
&- \int_0^\infty\mu(r)\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)\Delta\eta vdxdr + \int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)F(x,u)vdx \qquad (59)\\
&= \int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)f(x,u(t-\rho(t)))vdx + \int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)\left(g(t) + (\alpha+\delta)z(\theta_t\omega)\right)vdx.
\end{aligned}$$

We now estimate the terms in (59) as follows. First, by Young's inequality, we obtain

$$
\begin{aligned}
&-\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) v\Delta u dx = \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \nabla u \nabla v dx + \int_{\mathbb{R}^n} \frac{2x}{K^2}\mathfrak{t}'\left(\frac{|x|^2}{K^2}\right) \nabla u v dx \\
&\geqslant \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \nabla u \left(\nabla \frac{\partial u}{\partial t} + \delta \nabla u - \nabla z(\theta_t\omega)\right) dx - \frac{2\sqrt{2}}{K}\int_{K\leqslant|x|\leqslant\sqrt{2}K} \mathfrak{t}'\left(\frac{|x|^2}{K^2}\right) |\nabla u||v| dx \\
&\geqslant \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |\nabla u|^2 dx + \frac{\delta}{2}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |\nabla u|^2 dx \\
&- C\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |\nabla z(\theta_t\omega)|^2 dx - \frac{C}{K}(\|u\|^2 + |v|_2^2). \qquad (60)
\end{aligned}
$$

Similar to the argument of (21) and (22), we derive that

$$
\begin{aligned}
&-\int_0^\infty \mu(r)\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \Delta\eta v dx dr \\
&= \int_0^\infty \mu(r)\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \nabla\eta\nabla v dx dr + \int_0^\infty \mu(r)\int_{\mathbb{R}^n} \frac{2x}{K^2}\mathfrak{t}'\left(\frac{|x|^2}{K^2}\right) \nabla\eta v dx dr \\
&= \int_0^\infty \mu(r)\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \nabla\eta\nabla\left(\frac{\partial\eta}{\partial t} + \delta u + \frac{\partial\eta}{\partial r} - z(\theta_t\omega)\right) dx dr \\
&+ \int_0^\infty \mu(r)\int_{\mathbb{R}^n} \frac{2x}{K^2}\mathfrak{t}'\left(\frac{|x|^2}{K^2}\right) \nabla\eta v dx dr \qquad (61) \\
&\geqslant \frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(r)\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |\nabla\eta|^2 dx dr + \frac{\sigma}{4}\int_0^\infty \mu(r)\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |\nabla\eta|^2 dx dr \\
&- \frac{2\delta^2 m_0}{\sigma}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |\nabla u|^2 dx - \frac{2m_0}{\sigma}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |\nabla z(\theta_t\omega)|^2 dx - \frac{C}{K}\left(\|\eta\|_{\mu,1}^2 + |v|_2^2\right).
\end{aligned}
$$

By Lagrange's mean value theorem and (**H1**), similar to (20), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) J(v - \delta u + z(\theta_t\omega)) v dx \geqslant \beta_1 \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |v|^2 dx \\
&- \beta_2\delta\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |u||v| dx - \beta_2\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |z(\theta_t\omega)||v| dx \\
&\geqslant \frac{\beta_1}{2}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |v|^2 dx - \frac{\beta_2^2\delta^2}{\beta_1}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |u|^2 dx \\
&- C\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |z(\theta_t\omega)|^2 dx, \qquad (62)
\end{aligned}
$$

and by Young's inequality,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) uv dx = \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) u\left(\frac{\partial u}{\partial t} + \delta u - z(\theta_t\omega)\right) dx \\
&\geqslant \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |u|^2 dx + \frac{\delta}{2}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |u|^2 dx \\
&- C\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |z(\theta_t\omega)|^2 dx. \qquad (63)
\end{aligned}
$$

For the last term on the left-hand side of (59), by (**H4**) and Young's inequality, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) F(x,u) z(\theta_t\omega) dx \leqslant \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left(|k_3(x)| + k_4|u|^{p-1}\right) |z(\theta_t\omega)| dx \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left(|k_3(x)|^2 + |z(\theta_t\omega)|^2\right) dx \\
&+ \frac{k_5\delta k_7}{2}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |u|^p dx + \frac{k_4^p}{2k_5\delta k_7}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |z(\theta_t\omega)|^p dx \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left(|k_3|^2 + |z(\theta_t\omega)|^2\right) dx + \frac{\delta k_7}{2}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) G(x,u) dx \\
&+ \frac{\delta k_7}{2}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |k_6(x)| dx + \frac{k_4^p}{2k_5\delta k_7}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |z(\theta_t\omega)|^p dx,
\end{aligned} \tag{64}$$

and thus

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) F(x,u) v dx = \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) F(x,u) \left(\frac{\partial u}{\partial t} + \delta u - z(\theta_t\omega)\right) dx \\
&\geqslant \frac{d}{dt}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) G(x,u) dx + \frac{\delta k_7}{2}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) G(x,u) dx \\
&- C\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left(|k_8(x)| + |k_3(x)|^2 + |k_6(x)|\right) dx \\
&- C\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left(|z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^p\right) dx.
\end{aligned} \tag{65}$$

By Young's inequality once more and (**H3**), the terms on the right-hand side of (59) are bounded by

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left(g(t) + (\alpha+\delta) z(\theta_t\omega)\right) v dx \leqslant \frac{\beta_1}{8}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |v|^2 dx \\
&+ C\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |g(t)|^2 dx + C\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |z(\theta_t\omega)|^2 dx,
\end{aligned} \tag{66}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) f(x, u(t-\rho(t))) v dx \leqslant \frac{\beta_1}{8}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |v|^2 dx \\
&+ \frac{2}{\beta_1}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |k_1(x)|^2 dx + \frac{2k_2^2}{\beta_1}\int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |u(t-\rho(t))|^2 dx.
\end{aligned} \tag{67}$$

Then it follows from (59)-(63) and (65)-(67) that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)\big(|v|^2+|\nabla u|^2+(\lambda+\delta^2)|u|^2+2G(x,u)\big)\,dx\\
&+\frac{d}{dt}\int_0^\infty\mu(r)\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|\nabla\eta|^2dxdr+\left(\frac{\beta_1}{2}-2\delta\right)\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|v|^2dx\\
&+\left(\delta-\frac{4\delta^2m_0}{\sigma}\right)\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|\nabla u|^2dx+\left(\delta(\lambda+\delta^2)-\frac{2\beta_2^2\delta^2}{\beta_1}\right)\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|u|^2dx\\
&+\frac{\sigma}{2}\int_0^\infty\mu(r)\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|\nabla\eta|^2dxdr+\delta k_7\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)G(x,u)dx\\
&\leqslant\frac{C}{K}\left(|v|_2^2+\|u\|^2+\|\eta\|_{\mu,1}^2\right)+\frac{4k_2^2}{\beta_1}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|u(t-\rho(t))|^2dx\\
&+C\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|g(t)|^2dx+C\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)\big(|k_1(x)|^2+|k_3(x)|^2+|k_6(x)|+|k_8(x)|\big)\,dx\\
&+C\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)\big(|z(\theta_t\omega)|^2+|z(\theta_t\omega)|^p+|\nabla z(\theta_t\omega)|^2\big)\,dx.
\end{aligned}\tag{68}$$

Hence,

$$\begin{aligned}
&\frac{d}{dt}\left(e^{\alpha t}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)\big(|v|^2+|\nabla u|^2+(\lambda+\delta^2)|u|^2+2G(x,u)\big)\,dx\right)\\
&+\frac{d}{dt}\left(e^{\alpha t}\int_0^\infty\mu(r)\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|\nabla\eta|^2dxdr\right)\\
&+\left(\frac{\beta_1}{2}-2\delta-\alpha\right)e^{\alpha t}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|v|^2dx\\
&+\left(\delta-\frac{4\delta^2m_0}{\sigma}-\alpha\right)e^{\alpha t}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|\nabla u|^2dx\\
&+\left((\delta-\alpha)(\lambda+\delta^2)-\frac{2\beta_2^2\delta^2}{\beta_1}\right)e^{\alpha t}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|u|^2dx\\
&+\left(\frac{\sigma}{2}-\alpha\right)e^{\alpha t}\int_0^\infty\mu(r)\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|\nabla\eta|^2dxdr\\
&+(\delta k_7-2\alpha)\,e^{\alpha t}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)G(x,u)dx\\
&\leqslant\frac{C}{K}e^{\alpha t}\left(|v|_2^2+\|u\|^2+\|\eta\|_{\mu,1}^2\right)+\frac{4k_2^2}{\beta_1}e^{\alpha t}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|u(t-\rho(t))|^2dx\\
&+Ce^{\alpha t}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|g(t)|^2dx\\
&+Ce^{\alpha t}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)\big(|k_1(x)|^2+|k_3(x)|^2+|k_6(x)|+|k_8(x)|\big)\,dx\\
&+Ce^{\alpha t}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)\big(|z(\theta_t\omega)|^2+|z(\theta_t\omega)|^p+|\nabla z(\theta_t\omega)|^2\big)\,dx.
\end{aligned}\tag{69}$$

Given $t \geqslant 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\tau - t \leqslant T \leqslant \tau$, integrating the inequality (69) over $(\tau - t, T)$ and replacing ω by $\theta_{-\tau}\omega$, by (**H4**), (48)-(51) and the similar argument of (28) and (54), we deduce that

$$
\begin{aligned}
& e^{\alpha T} \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left(|v(T, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})|^2 + |\nabla u(T, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^2\right) dx \\
& + e^{\alpha T} \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left((\lambda+\delta^2)|u(T, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^2 + 2k_5|u(T, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^p\right) dx \\
& + e^{\alpha T} \int_0^\infty \mu(r) \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |\nabla \eta(T, \tau-t, \theta_{-\tau}\omega, \eta(\tau-t), r)|^2 dx dr \\
\leqslant\; & e^{\alpha(\tau-t)} \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left(|v(\tau-t, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})|^2 + |\nabla u(\tau-t, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^2\right) dx \\
& + C e^{\alpha(\tau-t)} \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left(|u(\tau-t, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^2 + |u(\tau-t, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^p\right) dx \\
& + e^{\alpha(\tau-t)} \int_0^\infty \mu(r) \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |\nabla \eta(\tau-t, \tau-t, \theta_{-\tau}\omega, \eta(\tau-t), r)|^2 dx dr \\
& + C e^{\alpha(\tau-t)} \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left(|k_3|^2 + |k_8|\right) dx + \frac{C}{K} \int_{\tau-t}^{T} e^{\alpha r} |v(r, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})|_2^2 dr \\
& + \frac{C}{K} \int_{\tau-t}^{T} e^{\alpha r} \left(\|u(r, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 + \|\eta(r, \tau-t, \theta_{-\tau}\omega, \eta(\tau-t))\|_{\mu,1}^2\right) dr \\
& + C e^{\alpha(\tau-t)} \|\phi\|_{C_H}^2 + C \int_{\tau-t}^{T} e^{\alpha r} \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |g(r)|^2 dx dr \\
& + C e^{\alpha T} \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left(|k_1(x)|^2 + |k_3(x)|^2 + |k_6(x)| + |k_8(x)|\right) dx \\
& + C \int_{\tau-t}^{T} e^{\alpha r} \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) (|z(\theta_{r-\tau}\omega)|^2 + |z(\theta_{r-\tau}\omega)|^p + |\nabla z(\theta_{r-\tau}\omega)|^2) dx dr. \qquad (70)
\end{aligned}
$$

Let ε_0 be given arbitrarily. Since $(u_{\tau-t}, v_{\tau-t}, \eta(\tau-t)) \in D(\tau-t, \theta_{-t}\omega)$ and $D \in \mathcal{D}$, we can choose t sufficiently large such that

$$
\begin{aligned}
& C e^{-\alpha t} \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) \left(|\nabla u(\tau-t, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^2 + |u(\tau-t, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^2\right) dx \\
& + C e^{-\alpha t} \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |u(\tau-t, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^p dx \\
& + C e^{-\alpha t} \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |v(\tau-t, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})|^2 dx + C e^{-\alpha t} \|\phi\|_{C_H}^2 \\
& + C e^{-\alpha t} \int_0^\infty \mu(r) \int_{\mathbb{R}^n} \mathfrak{t}\left(\frac{|x|^2}{K^2}\right) |\nabla \eta(\tau-t, \tau-t, \theta_{-\tau}\omega, \eta(\tau-t), r)|^2 dx dr \\
& \leqslant C e^{-\alpha t} \left(\|u(\tau-t, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 + |u(\tau-t, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|_2^2 + \|\phi\|_{C_H}^2\right) \\
& + C e^{-\alpha t} |u(\tau-t, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|_p^p + C e^{-\alpha t} |v(\tau-t, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})|_2^2 \\
& + C e^{-\alpha t} \|\eta(\tau-t, \tau-t, \theta_{-\tau}\omega, \eta(\tau-t))\|_{\mu,1}^2 \leqslant C\varepsilon_0. \qquad (71)
\end{aligned}
$$

Note that $z(\theta_t\omega)=\sum_{j=1}^m h_j z_j(\theta_t\omega_j)$ and $h_j\in L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)\cap H^1(\mathbb{R}^n)$. Hence, there is $R_1=R_1(\varepsilon_0,\omega)$ such that for all $K\geqslant R_1$ and $j=1,2,\dots,m$,

$$\int_{|x|\geqslant K}(|h_j(x)|^2+|h_j(x)|^p+|\nabla h_j(x)|^2)dx\leqslant\frac{\varepsilon_0}{r(\omega)},\tag{72}$$

where $r(\omega)$ is the tempered function in (8). By (8)-(9) and (72), we have for all $K\geqslant R_1$,

$$\begin{aligned}&Ce^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)\left(|z(\theta_{r-\tau}\omega)|^2+|z(\theta_{r-\tau}\omega)|^p+|\nabla z(\theta_{r-\tau}\omega)|^2\right)dxdr\\&\leqslant C\int_{-t}^{0}e^{\alpha r'}\int_{|x|\geqslant K}(|z(\theta_{r'}\omega)|^2+|z(\theta_{r'}\omega)|^p+|\nabla z(\theta_{r'}\omega)|^2)dxdr'\\&\leqslant C\int_{-\infty}^{0}e^{\alpha r'}\int_{|x|\geqslant K}\sum_{j=1}^m\left(|h_j(x)|^2+|\nabla h_j(x)|^2\right)|z_j(\theta_{r'}\omega_j)|^2dxdr'\\&+C\int_{-\infty}^{0}e^{\alpha r'}\int_{|x|\geqslant K}\sum_{j=1}^m|h_j(x)|^p|z_j(\theta_{r'}\omega_j)|^pdxdr'\\&\leqslant\frac{C\varepsilon_0}{r(\omega)}\int_{-\infty}^{0}e^{\alpha r'}\sum_{j=1}^m\left(|z_j(\theta_{r'}\omega_j)|^2+|z_j(\theta_{r'}\omega_j)|^p\right)dr'\\&\leqslant\frac{C\varepsilon_0}{r(\omega)}\int_{-\infty}^{0}e^{\alpha r'}r(\theta_{r'}\omega)dr'\leqslant C\varepsilon_0.\end{aligned}\tag{73}$$

By the assumption (**H5**), in view of $k_1,\ k_3\in L^2(\mathbb{R}^n)$ and $k_6,\ k_8\in L^1(\mathbb{R}^n)$, we can choose K sufficiently large such that

$$\begin{aligned}&C\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)\left(|k_1(x)|^2+|k_3(x)|^2+|k_6(x)|+|k_8(x)|\right)dx\\&+Ce^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|g(r)|^2dxdr\leqslant C\varepsilon_0.\end{aligned}\tag{74}$$

By (55), (56) and (71), we can take K and t large enough such that

$$\begin{aligned}
&\frac{C}{K}e^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}\Big(|v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_2^2+\|u(r,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\Big)dr\\
&+\frac{C}{K}e^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}\|\eta(r,\tau-t,\theta_{-\tau}\omega,\eta(\tau-t))\|_{\mu,1}^2dr\\
&\leqslant\frac{C}{K}\Big(e^{-\alpha t}\left(|v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|_2^2+\|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\right)\\
&+e^{-\alpha t}\left(|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_2^2+|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p\right)\\
&+e^{-\alpha t}\|\eta(\tau-t,\tau-t,\theta_{-\tau}\omega,\eta(\tau-t))\|_{\mu,1}^2+1+e^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}|g(r)|_2^2dr\\
&+e^{-\alpha t}\|\phi\|_{C_H}^2+\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^{m}(|z_j(\theta_r\omega_j)|^2+|z_j(\theta_r\omega_j)|^p)dr\Big)\\
&\leqslant C\varepsilon_0\left(1+e^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}|g(r)|_2^2dr+\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^{m}(|z_j(\theta_r\omega_j)|^2+|z_j(\theta_r\omega_j)|^p)dr\right). \qquad (75)
\end{aligned}$$

Replacing T by $\tau+s$ in (70) where $s\in[-h,0]$, in view of (71) and (73)-(75), when t and K are sufficiently large, we obtain

$$\begin{aligned}
&\sup_{s\in[-h,0]}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)\left(|v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2+|\nabla u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2\right)dx\\
&+\sup_{s\in[-h,0]}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2dx\\
&+\sup_{s\in[-h,0]}\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^pdx\\
&+\int_0^{\infty}\mu(r)\int_{\mathbb{R}^n}\mathfrak{t}\left(\frac{|x|^2}{K^2}\right)|\nabla\eta(\tau,\tau-t,\theta_{-\tau}\omega,\eta(\tau-t),r)|^2dxdr \qquad (76)\\
&\leqslant C\varepsilon_0\left(1+e^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}|g(r)|_2^2dr+\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^{m}(|z_j(\theta_r\omega_j)|^2+|z_j(\theta_r\omega_j)|^p)dr\right).
\end{aligned}$$

Thus Lemma 11 follows from (76) by choosing ε_0 appropriately for a given $\varepsilon>0$. □

5. **Existence of pullback attractors.** In this section, we prove the existence of a pullback attractor for the continuous cocycle Φ associated with the system (13)-(14) on $\mathbb{R}^n$. First we apply Lemma 10 to present the existence of a pullback absorbing set in $\mathcal{D}$.

Lemma 12. *Assume that the hypotheses in Lemma 10 hold. Then there exists $G=\{G(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ such that K is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ, that is, for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $B=\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, there exists $T_2=T_2(\tau,\omega,B)>0$ such that for all $t\geqslant T_2$,*

$$\Phi(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega))\subseteq G(\tau,\omega).$$

Proof. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let $G(\tau, \omega) = \{(\varphi, \psi, \eta) \in E : \|\varphi\|^2_{C_V} + \|\varphi\|^2_{C_H} + \|\varphi\|^p_{C_{L^p(\mathbb{R}^n)}} + \|\psi\|^2_{C_H} + \|\eta\|^2_{\mu,1} \leqslant L(\tau, \omega)\}$, where

$$L(\tau, \omega) = C + Ce^{-\alpha\tau} \int_{-\infty}^{\tau} e^{\alpha r} |g(r)|_2^2 dr + C \int_{-\infty}^{0} e^{\alpha r} \sum_{j=1}^{m} (|z_j(\theta_r \omega_j)|^2 + |z_j(\theta_r \omega_j)|^p) dr, \tag{77}$$

where $L(\tau, \omega)$ is the constant given by the right-hand side of (58). Then for each $\tau \in \mathbb{R}$, $L(\tau, \cdot) : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable, and

$$\lim_{r \to -\infty} e^{\alpha r} L(\tau + r, \theta_r \omega) = 0. \tag{78}$$

Therefore, $G = \{G(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ belongs to $\mathcal{D}$. By Lemma 10, $G = \{G(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a closed measurable $\mathcal{D}$-pullback absorbing set in $\mathcal{D}$ for Φ. □

We are now in a position to state and prove our main result: the existence of a pullback attractor for Φ in $\mathcal{D}$.

Theorem 13. *Assume that the hypotheses in Lemma 10 hold. Then the continuous cocycle Φ associated with problem* (13)-(14) *possesses a unique $\mathcal{D}$-pullback attractor $\mathcal{A} \in \mathcal{D}$ in E.*

Proof. Let $\tau \in \mathbb{R}, \omega \in \Omega$ be given arbitrarily. By Lemmas 10 and 12, for any $T \geqslant \tau - t$ with $t \geqslant 0$ let

$$\begin{aligned} &\Phi(T - \tau + t, \tau - t, \theta_{-t}\omega, (u_{\tau-t}, v_{\tau-t}, \eta(\tau - t))) = (u_T(\cdot, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}), \\ &\quad v_T(\cdot, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}), \eta(T, \tau - t, \theta_{-\tau}\omega, \eta(\tau - t))) \end{aligned}$$

where $(u(\cdot), v(\cdot), \eta(\cdot))$ is a solution of (13)-(14) satisfying the energy equation (15) with $(u_{\tau-t}, v_{\tau-t}, \eta(\tau - t)) \in G(\tau - t, \theta_{-t}\omega)$, and $G = \{G(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ in E.

Given a positive integer R, let $\Omega_{2R} = \{x \in \mathbb{R}^n : |x| \leqslant 2R\}$, and we define the following new variables

$$\begin{aligned} \widehat{u}(T, x) &= \left(1 - \mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right) u(T, x), \\ \widehat{v}(T, x) &= \left(1 - \mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right) v(T, x), \\ \widehat{\eta}(T, x) &= \left(1 - \mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right) \eta(T, x), \end{aligned}$$

where $\mathfrak{t}$ is the cutoff function defined in the proof of Lemma 11. Multiplying Eqs. (13) and (14) by $1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)$, we obtain

$$\begin{cases} \dfrac{\partial \widehat{u}}{\partial T}+\delta\widehat{u}=\widehat{v}+\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)z(\theta_T\omega), \\ \dfrac{\partial \widehat{v}}{\partial T}-\delta\widehat{v}+\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)J\left(v-\delta u+z(\theta_T\omega)\right)-\Delta\widehat{u}+(\lambda+\delta^2)\widehat{u} \\ -\displaystyle\int_0^\infty \mu(r)\Delta\widehat{\eta}(r)dr+\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)F(x,u) \\ =\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)f(x,u(T-\rho(T)))+\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)g(x,T) \\ +(\alpha+\delta)\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)z(\theta_T\omega)+2\nabla\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\nabla u+u\Delta\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right) \\ +2\displaystyle\int_0^\infty \mu(r)\nabla\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\nabla\eta(r)dr+\int_0^\infty \mu(r)\Delta\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\eta(r)dr, \\ \dfrac{\partial \widehat{\eta}}{\partial T}=\widehat{v}-\delta\widehat{u}+\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)z(\theta_T\omega)-\dfrac{\partial \widehat{\eta}}{\partial r}, \end{cases} \tag{79}$$

with initial conditions

$$\begin{cases} \widehat{u}(T,x)=\widehat{u}_{\tau-t}\left(T-\tau+t,x\right)=\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)\phi(T-\tau+t,x), \\ T\in[\tau-t-h,\tau-t],\ x\in\Omega_{2R}, \\ \widehat{v}(T,x)=\widehat{v}_{\tau-t}(T-\tau+t,x)=\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)\dfrac{\partial\phi(T-\tau+t,x)}{\partial T} \\ +\delta\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)\phi(T-\tau+t,x)-\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)z(\theta_T\omega), \\ T\in[\tau-t-h,\tau-t],\ x\in\Omega_{2R}, \\ \widehat{\eta}(\tau-t,x,r)=\widehat{u}(\tau-t,x)-\widehat{u}(\tau-t-r,x) \\ =\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)\phi(0,x)-\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)\phi(-r,x),\ r\in\mathbb{R}^+,\ x\in\Omega_{2R}, \end{cases} \tag{80}$$

and boundary conditions

$$\begin{cases} \widehat{u}(T,x)=0, & T\in(\tau-t,\infty),\ |x|=2R, \\ \widehat{v}(T,x)=0, & T\in(\tau-t,\infty),\ |x|=2R, \\ \widehat{\eta}(T,x)=0, & T\in(\tau-t,\infty),\ |x|=2R. \end{cases} \tag{81}$$

Let $\widehat{u}=u_1+u_2,\ \widehat{v}=v_1+v_2,\ \widehat{\eta}=\eta_1+\eta_2$. Then we can decompose Eqs. (79)-(81) as follows:

$$\begin{cases} \dfrac{\partial u_2}{\partial T}+\delta u_2=v_2, \\ \dfrac{\partial v_2}{\partial T}-\delta v_2-\Delta u_2+(\lambda+\delta^2)u_2+\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)J\left(v-\delta u+z(\theta_T\omega)\right) \\ -\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)J\left(v_1-\delta u_1+z(\theta_T\omega)\right)-\displaystyle\int_0^\infty \mu(r)\Delta\eta_2(r)dr=0, \\ \dfrac{\partial \eta_2}{\partial T}=v_2-\delta u_2-\dfrac{\partial \eta_2}{\partial r}, \end{cases} \tag{82}$$

with initial conditions

$$\begin{cases} u_2(T,x) = u_{2,\tau-t}\,(T-\tau+t,x) = \left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)\phi(T-\tau+t,x), \\ \ T\in[\tau-t-h,\tau-t],\ x\in\Omega_{2R}, \\ v_2(T,x) = v_{2,\tau-t}(T-\tau+t,x) = \left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)\frac{\partial\phi(T-\tau+t,x)}{\partial T} \\ +\delta\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)\phi(T-\tau+t,x) - \left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)z(\theta_T\omega), \\ \ T\in[\tau-t-h,\tau-t],\ x\in\Omega_{2R}, \\ \eta_2(\tau-t,x,r) = u_2(\tau-t,x)-u_2(\tau-t-r,x) \\ = \left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)\phi(0,x) - \left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)\phi(-r,x),\ r\in\mathbb{R}^+,\ x\in\Omega_{2R}, \end{cases} \tag{83}$$

and boundary conditions

$$\begin{cases} u_2(T,x)=0, \quad T\in(\tau-t,\infty),\ |x|=2R, \\ v_2(T,x)=0, \quad T\in(\tau-t,\infty),\ |x|=2R, \\ \eta_2(T,x)=0, \quad T\in(\tau-t,\infty),\ |x|=2R, \end{cases} \tag{84}$$

and

$$\begin{cases} \frac{\partial u_1}{\partial T}+\delta u_1 = v_1+\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)z(\theta_T\omega), \\ \frac{\partial v_1}{\partial T}-\delta v_1+\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)J\left(v_1-\delta u_1+z(\theta_T\omega)\right)-\Delta u_1+(\lambda+\delta^2)u_1 \\ -\int_0^\infty \mu(r)\Delta\eta_1(r)dr+\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)F(x,u) \\ = \left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)f(x,u(T-\rho(T)))+\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)g(x,T) \\ +(\alpha+\delta)\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)z(\theta_T\omega)+2\nabla\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\nabla u+u\Delta\mathfrak{t}\left(\frac{|x|^2}{R^2}\right) \\ +2\int_0^\infty \mu(r)\nabla\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\nabla\eta(r)dr+\int_0^\infty \mu(r)\Delta\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\eta(r)dr, \\ \frac{\partial\eta_1}{\partial T} = v_1-\delta u_1+\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)z(\theta_T\omega)-\frac{\partial\eta_1}{\partial r}, \end{cases} \tag{85}$$

with initial conditions

$$\begin{cases} u_1(T,x) = u_{1,\tau-t}(T-\tau+t,x) = 0, \quad T\in[\tau-t-h,\tau-t],\ x\in\Omega_{2R}, \\ v_1(T,x) = v_{1,\tau-t}(T-\tau+t,x) = 0, \quad T\in[\tau-t-h,\tau-t],\ x\in\Omega_{2R}, \\ \eta_1(\tau-t,x,r)=0, \quad r\in\mathbb{R}^+,\ x\in\Omega_{2R}, \end{cases} \tag{86}$$

and boundary conditions

$$\begin{cases} u_1(T,x)=0, \quad T\in(\tau-t,\infty),\ |x|=2R, \\ v_1(T,x)=0, \quad T\in(\tau-t,\infty),\ |x|=2R, \\ \eta_1(T,x)=0, \quad T\in(\tau-t,\infty),\ |x|=2R. \end{cases} \tag{87}$$

Step 1. Uniform estimates on the solutions of Eqs. (82)-(84). Taking the inner product in H of the second equation of (82) with v_2, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dT}|v_2|_2^2-\delta|v_2|_2^2+\left(\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)\left(J\left(v-\delta u+z(\theta_T\omega)\right)-J\left(v_1-\delta u_1+z(\theta_T\omega)\right),v_2\right)\right.\\&+(\nabla u_2,\nabla v_2)+(\lambda+\delta^2)(u_2,v_2)-\left(\int_0^\infty\mu(r)\Delta\eta_2(r)dr,v_2\right)=0.\end{aligned}\tag{88}$$

Arguing as in (16)-(17), (20) and (22), in view of $v_2=\frac{\partial u_2}{\partial T}+\delta u_2$ and $v_2=\frac{\partial\eta_2}{\partial T}+\delta u_2+\frac{\partial\eta_2}{\partial r}$, In a similar way as in Theorem 8 and by Eq. (82), we deduce that

$$\begin{aligned}&\left(\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)\left(J\left(v-\delta u+z(\theta_T\omega)\right)-J\left(v_1-\delta u_1+z(\theta_T\omega)\right)\right),v_2\right)\\&=\left(\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)J'(\zeta)(v-v_1-\delta(u-u_1)),v_2\right)\geqslant\frac{\beta_1}{2}|v_2|_2^2-\frac{\beta_2^2\delta^2}{2\beta_1}|u_2|_2^2,\end{aligned}\tag{89}$$

where ζ is between $v-\delta u+z(\theta_T\omega)$ and $v_1-\delta u_1+z(\theta_T\omega)$, and

$$\begin{aligned}&-\left(\int_0^\infty\mu(r)\Delta\eta_2(r)dr,v_2\right)=-\left(\int_0^\infty\mu(r)\Delta\eta_2(r)dr,\frac{\partial\eta_2}{\partial T}+\delta u_2+\frac{\partial\eta_2}{\partial r}\right)\\&\geqslant\frac{1}{2}\frac{d}{dT}\|\eta_2\|_{\mu,1}^2+\frac{\sigma}{4}\|\eta_2\|_{\mu,1}^2-\frac{\delta^2m_0}{\sigma}\|u_2\|^2,\end{aligned}\tag{90}$$

$$(\nabla u_2,\nabla v_2)=\left(\nabla u_2,\nabla\frac{\partial u_2}{\partial T}+\delta\nabla u_2\right)=\frac{1}{2}\frac{d}{dT}\|u_2\|^2+\delta\|u_2\|^2,\tag{91}$$

$$(u_2,v_2)=\left(u_2,\frac{\partial u_2}{\partial T}+\delta u_2\right)=\frac{1}{2}\frac{d}{dT}|u_2|_2^2+\delta|u_2|_2^2.\tag{92}$$

Let $\alpha_1:=\min\left\{\beta_1-2\delta,2\delta-\frac{2\delta^2m_0}{\sigma},2\delta-\frac{\beta_2^2\delta^2}{\beta_1(\lambda+\delta^2)},\frac{\sigma}{2}\right\}>0$. Then it follows from (88)-(92) that

$$\begin{aligned}&\frac{d}{dT}\left(|v_2|_2^2+\|u_2\|^2+(\lambda+\delta^2)|u_2|_2^2+\|\eta_2\|_{\mu,1}^2\right)\\&+\alpha_1\left(|v_2|_2^2+\|u_2|^2+(\lambda+\delta^2)|u_2|_2^2+\|\eta_2\|_{\mu,1}^2\right)\leqslant0.\end{aligned}\tag{93}$$

Given $t\geqslant0$, $\tau\in\mathbb{R}$ and $\omega\in\Omega$, using the Gronwall inequality to (93), and replacing ω by $\theta_{-\tau}\omega$, we obtain that for all $T\in[\tau-t,\tau]$

$$\begin{aligned}&|v_2(T,\tau-t,\theta_{-\tau}\omega,v_{2,\tau-t})|_2^2+\|u_2(T,\tau-t,\theta_{-\tau}\omega,u_{2,\tau-t})\|^2\\&+(\lambda+\delta^2)|u_2(T,\tau-t,\theta_{-\tau}\omega,u_{2,\tau-t})|_2^2+\|\eta_2(T,\tau-t,\theta_{-\tau}\omega,\eta_2(\tau-t))\|_{\mu,1}^2\\&\leqslant e^{-\alpha_1(T-\tau+t)}\left(|v_2(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{2,\tau-t)})|_2^2+\|u_2(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{2,\tau-t})\|^2\right)\\&+(\lambda+\delta^2)e^{-\alpha_1(T-\tau+t)}|u_2(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{2\,\tau-t})|_2^2\\&+e^{-\alpha_1(T-\tau+t)}\|\eta_2(\tau-t,\tau-t,\theta_{-\tau}\omega,\eta_2(\tau-t))\|_{\mu,1}^2.\end{aligned}\tag{94}$$

Replacing T by $\tau+s$ in (94) where $s\in[-h,0]$, we have that for all $t\geqslant h$,

$$\begin{aligned}
&\|v_{2\tau}\|_{C_H}^2+\|u_{2\tau}\|_{C_V}^2+\|u_{2\tau}\|_{C_H}^2+\|\eta_2(\tau)\|_{\mu,1}^2\\
&=\sup_{s\in[-h,0]}|v_2(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{2,\tau-t})|_2^2+\sup_{s\in[-h,0]}\|u_2(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{2,\tau-t})\|^2\\
&+\sup_{s\in[-h,0]}|u_2(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{2,\tau-t})|_2^2+\|\eta_2(\tau,\tau-t,\theta_{-\tau}\omega,\eta_2(\tau-t))\|_{\mu,1}^2\\
&\leqslant Ce^{-\alpha_1 t}\left(|v_2(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{2,\tau-t})|_2^2+\|u_2(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{2,\tau-t})\|^2\right) \qquad (95)\\
&+Ce^{-\alpha_1 t}\left(|u_2(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{2,\tau-t})|_2^2+\|\eta_2(\tau-t,\tau-t,\theta_{-\tau}\omega,\eta_2(\tau-t))\|_{\mu,1}^2\right).
\end{aligned}$$

Step 2. Uniform estimates on the solutions of Eqs. (85)-(87). We consider a couple of solutions $(u^1(t),v^1(t),\eta^1(t))$ and $(u^2(t),v^2(t),\eta^2(t))$ of system (13) with initial data $(u^1_\tau,v^1_\tau,\eta^1(\tau))$ and $(u^2_\tau,v^2_\tau,\eta^2(\tau))$, respectively. Let $\overline{u}(T)=u^1_1(T)-u^2_1(T)$, $\overline{v}(T)=v^1_1(T)-v^2_1(T)$, $\overline{\eta}(T)=\eta^1_1(T)-\eta^2_1(T)$. Then it follows from Eqs. (85)-(87) that $(\overline{u}(T),\overline{v}(T),\overline{\eta}(T))$ satisfies

$$\begin{cases}
\dfrac{\partial\overline{u}}{\partial T}+\delta\overline{u}=\overline{v},\\
\dfrac{\partial\overline{v}}{\partial T}-\delta\overline{v}-\Delta\overline{u}+(\lambda+\delta^2)\overline{u}+\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)J\left(v^1_1-\delta u^1_1+z(\theta_T\omega)\right)\\
-\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)J\left(v^2_1-\delta u^2_1+z(\theta_T\omega)\right)-\displaystyle\int_0^\infty\mu(r)\Delta\overline{\eta}(r)dr\\
+\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)F(x,u^1)-\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)F(x,u^2)\\
=\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)f(x,u^1(T-\rho(T)))-\left(1-\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)\right)f(x,u^2(T-\rho(T)))\\
+2\nabla\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)(\nabla u^1-\nabla u^2)+\Delta\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)(u^1-u^2)\\
+2\displaystyle\int_0^\infty\mu(r)\nabla\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)(\nabla\eta^1(r)-\nabla\eta^2(r))dr+\int_0^\infty\mu(r)\Delta\mathfrak{t}\left(\dfrac{|x|^2}{R^2}\right)(\eta^1(r)-\eta^2(r))dr,\\
\dfrac{\partial\overline{\eta}}{\partial T}=\overline{v}-\delta\overline{u}-\dfrac{\partial\overline{\eta}}{\partial r},
\end{cases} \qquad (96)$$

with initial conditions

$$\begin{cases}
\overline{u}(T,x)=\overline{u}_{\tau-t}(T-\tau+t,x)=0,\quad T\in[\tau-t-h,\tau-t],\ x\in\Omega_{2R},\\
\overline{v}(T,x)=\overline{v}_{\tau-t}(T-\tau+t,x)=0,\quad T\in[\tau-t-h,\tau-t],\ x\in\Omega_{2R},\\
\overline{\eta}(\tau-t,x,r)=0,\quad r\in\mathbb{R}^+,\ x\in\Omega_{2R},
\end{cases} \qquad (97)$$

and boundary conditions

$$\begin{cases}
\overline{u}(T,x)=0,\quad T\in(\tau-t,\infty),\ |x|=2R,\\
\overline{v}(T,x)=0,\quad T\in(\tau-t,\infty),\ |x|=2R,\\
\overline{\eta}(T,x)=0,\quad T\in(\tau-t,\infty),\ |x|=2R.
\end{cases} \qquad (98)$$

Taking the inner product in H of the second equation of (96) with $\overline{v}$, in a similar way as above we have

$$
\begin{aligned}
&\frac{d}{dT}\left(|\overline{v}|_2^2+\|\overline{u}\|^2+(\lambda+\delta^2)|\overline{u}|_2^2+\|\overline{\eta}\|_{\mu,1}^2\right)+\alpha_1\left(|\overline{v}|_2^2+\|\overline{u}\|^2+(\lambda+\delta^2)|\overline{u}|_2^2+\|\overline{\eta}\|_{\mu,1}^2\right)\\
&\leqslant 2\left|\left(\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)\left(F(x,u^1)-F(x,u^2)\right),\overline{v}\right)\right|\\
&+2\left|\left(\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)\left(f(x,u^1(T-\rho(T)))-f(x,u^2(T-\rho(T)))\right),\overline{v}\right)\right|\\
&+4\left|\left(\nabla\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)(\nabla u^1-\nabla u^2),\overline{v}\right)\right|+2\left|\left(\Delta\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)(u^1-u^2),\overline{v}\right)\right|\\
&+4\left|\left(\int_0^\infty\mu(r)\nabla\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)(\nabla\eta^1(r)-\nabla\eta^2(r))dr,\overline{v}\right)\right|\\
&+2\left|\left(\int_0^\infty\mu(r)\Delta\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)(\eta^1(r)-\eta^2(r))dr,\overline{v}\right)\right|, \qquad (99)
\end{aligned}
$$

where $\alpha_1>0$ is given in (93). By a simple computation, we find that

$$
\Delta\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)=\mathfrak{t}''\left(\frac{|x|^2}{R^2}\right)\frac{4x^2}{R^4}+\mathfrak{t}'\left(\frac{|x|^2}{R^2}\right)\frac{2x}{R^2}, \qquad (100)
$$

and

$$
\nabla\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)=\mathfrak{t}'\left(\frac{|x|^2}{R^2}\right)\frac{2x}{R^2}. \qquad (101)
$$

Given $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $t\geqslant 0$, integrating (99) over $(\tau-t,T)$ with $T\in[\tau-t,\tau]$ and replacing ω by $\theta_{-\tau}\omega$, in view of (99)-(101) and $|\mathfrak{t}'(s)|\leqslant C_0$, $|\mathfrak{t}''(s)|\leqslant C_0'$ for any $s\in\mathbb{R}^+$, we deduce that

$$
\begin{aligned}
&|\overline{v}(T,\tau-t,\theta_{-\tau}\omega,\overline{v}_{\tau-t})|_2^2+\|\ \overline{u}(T,\tau-t,\theta_{-\tau}\omega,\overline{u}_{\tau-t})\ \|^2\\
&+|\overline{u}(T,\tau-t,\theta_{-\tau}\omega,\overline{u}_{\tau-t})|_2^2+\|\overline{\eta}(T,\tau-t,\theta_{-\tau}\omega,\overline{\eta}(\tau-t))\|_{\mu,1}^2\\
&\leqslant C\left\|\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)\big(F(x,u^1(r,\tau-t,\theta_{-\tau}\omega,u^1_{\tau-t}))\right. \qquad (102)\\
&\qquad\left.-F(x,u^2(r,\tau-t,\theta_{-\tau}\omega,u^2_{\tau-t}))\big)\right\|_{L^2(\Omega_{2R}\times[\tau-t,\tau])}\\
&\times\|\overline{v}(t,\tau-t,\theta_{-\tau}\omega,\overline{v}_{\tau-t})\|_{L^2(\Omega_{2R}\times[\tau-t,\tau])}\\
&+C\left\|\left(1-\mathfrak{t}\left(\frac{|x|^2}{R^2}\right)\right)\big(f(x,u^1(r-\rho(r),\tau-t,\theta_{-\tau}\omega,u^1_{\tau-t}))\right.\\
&\left.-f(x,u^2(r-\rho(r),\tau-t,\theta_{-\tau}\omega,u^1_{\tau-t}))\big)\right\|_{L^2(\Omega_{2R}\times[\tau-t,\tau])}\\
&\times\|\overline{v}(t,\tau-t,\theta_{-\tau}\omega,\overline{v}_{\tau-t})\|_{L^2(\Omega_{2R}\times[\tau-t,\tau])}\\
&+\frac{C}{R}\Big(\big\|\nabla u^1(r,\tau-t,\theta_{-\tau}\omega,u^1_{\tau-t}))\big\|_{L^2(\Omega_{2R}\times[\tau-t,\tau])}+\big\|\nabla u^2(r,\tau-t,\theta_{-\tau}\omega,u^2_{\tau-t}))\big\|_{L^2(\Omega_{2R}\times[\tau-t,\tau])}\Big)\\
&\times\|\overline{v}(t,\tau-t,\theta_{-\tau}\omega,\overline{v}_{\tau-t})\|_{L^2(\Omega_{2R}\times[\tau-t,\tau])}\\
&+\frac{C}{R}\Big(\big\|\eta^1(r,\tau-t,\theta_{-\tau}\omega,\eta^1_{\tau-t})\big\|_{L^2(\tau-t,\tau;L^2_\mu(\mathbb{R}^+;H_0^1(\Omega_{2R})))}\\
&+\big\|\eta^2(r,\tau-t,\theta_{-\tau}\omega,\eta^2_{\tau-t}))\big\|_{L^2(\tau-t,\tau;L^2_\mu(\mathbb{R}^+;H_0^1(\Omega_{2R})))}\\
&+\|\overline{v}(t,\tau-t,\theta_{-\tau}\omega,\overline{v}_{\tau-t})\|_{L^2(\Omega_{2R}\times[\tau-t,\tau])}\Big), \qquad (103)
\end{aligned}
$$

where we have used the Sobolev embedding $H_0^1(\Omega_{2R})\hookrightarrow L^p(\Omega_{2R})$.

Step 3. $\mathcal{D}$-pullback asymptotically upper-semicompact in E. Thanks to Lemma 11, we see that for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and any $\varepsilon>0$, there exist $R_0=R_0(\varepsilon,\tau,\omega,G)>0$ and $T_0=T_0(\varepsilon,\tau,\omega,G)\leqslant h$ such that for any solution $\big(u_\tau(\cdot,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}),v_\tau(\cdot,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}),\eta(\tau,\tau-t,\theta_{-\tau}\omega,\eta(\tau-t))\big)\in\Phi\big(t,\tau-t,\theta_{-t}\omega,G(\tau-t,\theta_{-t}\omega)\big)$,

$$
\begin{aligned}
&\sup_{s\in[-h,0]}\int_{\Omega^c_{R_0}}\big(|\nabla u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2+|v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\big)dx\\
&\quad+\sup_{s\in[-h,0]}\int_{\Omega^c_{R_0}}\big(|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2+|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^p\big)\,dx\\
&\quad+\int_0^\infty\mu(r)\int_{\Omega^c_{R_0}}|\nabla\eta(\tau,\tau-t,\theta_{-\tau}\omega,\eta(\tau-t),r)|^2dxdr\leqslant\varepsilon,\quad\forall t\geqslant T_0, \qquad (104)
\end{aligned}
$$

and by $G\in\mathcal{D}$, (47), (95) and the Sobolev embedding $H_0^1(\Omega_{2R})\hookrightarrow L^p(\Omega_{2R})$, we have

$$
\|v_{2\tau}\|^2_{C_H}+\|u_{2\tau}\|^2_{C_V}+\|u_{2\tau}\|^2_{C_H}+\|u_{2\tau}\|^2_{C_{L^p(\mathbb{R}^n)}}+\|\eta_2(\tau)\|^2_{\mu,1}\leqslant\varepsilon,\quad\forall t\geqslant T_0. \qquad (105)
$$

In order to prove the $\mathcal{D}$-pullback asymptotic compactness of Φ, let $B=\{B(\tau,\omega):\tau\in\mathbb{R}^n,\omega\in\Omega\}\in\mathcal{D}$, sequence $t^n\to\infty$ $(n\to\infty)$ and $(u^n_\tau(\cdot,\tau-t^n,\theta_{-\tau}\omega,u^n_{\tau-t^n}),v^n_\tau(\cdot,\tau-t^n,\theta_{-\tau}\omega,v^n_{\tau-t^n}),\eta^n(\tau,\tau-t^n,\theta_{-\tau}\omega,\eta^n(\tau-t^n)))\in\Phi(t^n,\tau-t^n,\theta_{-t^n}\omega,B(\tau-t^n,\theta_{-t^n}\omega))$ be given arbitrarily. Since G is a

$\mathcal{D}$-pullback absorbing for Φ, we obtain that there exists $N > 0$ such that for all $n \geqslant N$,

$$\begin{aligned}&\Phi(t^n, \tau - t^n, \theta_{-t^n}\omega, B(\tau - t^n, \theta_{-t^n}\omega))\\&= \Phi(T_0, \tau - T_0, \theta_{-T_0}\omega, \Phi(t^n - T_0, \tau - t^n, \theta_{-t^n}\omega, B(\tau - t^n, \theta_{-t^n}\omega))\\&\subset \Phi(T_0, \tau - T_0, \theta_{-T_0}\omega, G(\tau - T_0, \theta_{-T_0}\omega)).\end{aligned}$$

This implies that $(u^n_\tau(\cdot, \tau - t^n, \theta_{-\tau}\omega, u^n_{\tau-t^n}), v^n_\tau(\cdot, \tau - t^n, \theta_{-\tau}\omega, v^n_{\tau-t^n}), \eta^n(\tau, \tau - t^n, \theta_{-\tau}\omega, \eta^n(\tau - t^n))) \in \Phi(T_0, \tau-T_0, \theta_{-T_0}\omega, G(\tau-T_0, \theta_{-T_0}\omega))$. Hence for each n, there exists a solution $(\tilde{u}(\cdot), \tilde{v}(\cdot), \tilde{\eta}(\cdot))$ of (13) satisfying the energy equation (15) with $(\tilde{u}^n_{\tau-T_0}, \tilde{v}^n_{\tau-T_0}, \tilde{\eta}^n(\tau - T_0)) \in G(\tau - T_0, \theta_{-T_0}\omega)$ such that

$$(\tilde{u}^n_\tau(\cdot), \tilde{v}^n_\tau(\cdot), \tilde{\eta}^n(\tau)) = (u^n_\tau(\cdot), v^n_\tau(\cdot), \eta^n(\tau)). \tag{106}$$

Then it follows from Lemma 10 that

$$\{\tilde{u}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0})\} \text{ is bounded in } L^\infty(\tau - T_0 - h, \tau; H^1(\mathbb{R}^n)), \tag{107}$$

$$\{\tilde{v}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{v}^n_{\tau-T_0})\} \text{ is bounded in } L^\infty(\tau - T_0 - h, \tau; L^2(\mathbb{R}^n)), \tag{108}$$

$$\{\tilde{\eta}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{\eta}^n(\tau - T_0))\} \text{ is bounded in } L^\infty(\tau - T_0, \tau; L^2_\mu(\mathbb{R}^+; H^1(\mathbb{R}^n))). \tag{109}$$

Since $\frac{\partial \tilde{u}}{\partial T} = \tilde{v} - \delta\tilde{u} + z(\theta_T\omega)$, in view of the continuity of $z(\theta_T\omega)$ and (8)-(10), we obtain that for every $\omega \in \Omega$,

$$\left\{\frac{\partial \tilde{u}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0})}{\partial T}\right\} \text{ is bounded in } L^\infty(\tau - T_0 - h, \tau; L^2(\mathbb{R}^n)). \tag{110}$$

Hence, there exists $R_1 > R_0$ such that for all $n, m > N$,

$$\begin{aligned}&\frac{C}{R_1}\bigg(\left\|\nabla\tilde{u}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0})\right\|_{L^2(\Omega_{2R_1}\times[\tau-T_0,\tau])}\\&+ \left\|\nabla\tilde{u}^m(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0})\right\|_{L^2(\Omega_{2R_1}\times[\tau-T_0,\tau])}\bigg)\\&\times \left\|\tilde{v}^n_1(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{v}^n_{1(\tau-T_0)}) - \tilde{v}^m_1(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{v}^m_{1(\tau-T_0)})\right\|_{L^2(\Omega_{2R_1}\times[\tau-T_0,\tau])}\\&+ \frac{C}{R_1}\left\|(\tilde{\eta}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{\eta}^n(\tau - T_0))\right\|^2_{L^2(\tau-T_0,\tau;L^2_\mu(\mathbb{R}^+;H^1_0(\Omega_{2R_1})))}\\&+ \frac{C}{R_1}\left\|(\tilde{\eta}^m(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{\eta}^m(\tau - T_0))\right\|^2_{L^2(\tau-T_0,\tau;L^2_\mu(\mathbb{R}^+;H^1_0(\Omega_{2R_1})))}\\&+ \frac{C}{R_1}\left\|\tilde{v}^n_1(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{v}^n_{1,\tau-T_0}) - \tilde{v}^m_1(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{v}^m_{1,\tau-T_0})\right\|^2_{L^2(\Omega_{2R_1}\times[\tau-T_0,\tau])} < \varepsilon,\end{aligned} \tag{111}$$

thanks to the continuous embedding $H^1_0(\Omega_{2R_1}) \hookrightarrow L^2(\Omega_{2R_1})$. Recall that

$$L^\infty(\tau - T_0 - h, \tau; H^1_0(\Omega_{2R_1})) \cap L^\infty(\tau - T_0 - h, \tau; L^2(\Omega_{2R_1})) \hookrightarrow L^m(\tau - T_0 - h, \tau; L^s(\Omega_{2R_1})),$$

compactly for any $1 < m < \infty$, $1 \leqslant s < \infty$ if $n = 1, 2$ and $1 \leqslant s < 6$ if $n = 3$. Hence, without loss of generality, by (107) and (110) we have

$$u^n(T, \tau - T_0, \theta_{-\tau}\omega, u^n_{\tau-T_0}) \to u(T, \tau - T_0, \theta_{-\tau}\omega, u_{\tau-T_0}) \quad \text{in } L^r(\Omega_{2R_1} \times [\tau - T_0 - h, \tau]) \tag{112}$$

for any $1 < m < \infty$, $1 \leqslant r < \infty$ if $n = 1, 2$ and $1 \leqslant r < 6$ if $n = 3$. In view of **(H6)-(H7)** and Hölder's inequality, we deduce from (107) and (112) that

$$\begin{aligned}
&\bigg\| \left(1 - \mathfrak{t}\left(\frac{|x|^2}{R_1^2}\right)\right) F(x, \tilde{u}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0})) \\
&- \left(1 - \mathfrak{t}\left(\frac{|x|^2}{R_1^2}\right)\right) F(x, \tilde{u}(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}_{\tau-T_0}))\bigg\|_{L^2(\Omega_{2R_1} \times [\tau - T_0, \tau])} \\
&\leqslant C \int_{\tau-T_0}^{\tau} \int_{\Omega_{2R_1}} \left(1 + |\tilde{u}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0})|^{2p-4} + |\tilde{u}(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}_{\tau-T_0})|^{2p-4}\right) \\
&\left|\tilde{u}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0}) - \tilde{u}(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}_{\tau-T_0})\right|^2 dx dT \\
&\leqslant C|\tilde{u}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0}) - \tilde{u}(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}_{\tau-T_0})|^2_{L^{r'}(\Omega_{2R_1} \times [\tau - T_0 - h, \tau])} \to 0
\end{aligned} \tag{113}$$

as $n \to \infty$ for some $1 \leqslant r' < \infty$ if $n = 1, 2$ and $1 \leqslant r' < 6$ if $n = 3$, and

$$\begin{aligned}
&\bigg\| \left(1 - \mathfrak{t}\left(\frac{|x|^2}{R_1^2}\right)\right) f(x, \tilde{u}^n(T - \rho(T), \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0})) \\
&- \left(1 - \mathfrak{t}\left(\frac{|x|^2}{R_1^2}\right)\right) f(x, \tilde{u}(T - \rho(T), \tau - T_0, \theta_{-\tau}\omega, \tilde{u}_{\tau-T_0}))\bigg\|_{L^2(\Omega_{2R_1} \times [\tau - T_0, \tau])} \\
&\leqslant C \int_{\tau-T_0}^{\tau} \int_{\Omega_{2R_1}} \left|\tilde{u}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0}) - \tilde{u}(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}_{\tau-T_0})\right|^2 dx dT \\
&\leqslant C\|\tilde{u}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0}) - \tilde{u}(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}_{\tau-T_0})\|^2_{L^2(\Omega_{2R_1} \times [\tau - T_0 - h, \tau])} \to 0
\end{aligned} \tag{114}$$

as $n \to \infty$. Observe that the sequence $\{\tilde{v}_1^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{v}^n_{1,\tau-T_0})\}$ is bounded in $L^2(\Omega_{2R_1} \times [\tau - T_0, \tau])$. Thus, there exists $N_1 > N$ such that for all $n, m \geqslant N_1$,

$$\begin{aligned}
C&\bigg\| \left(1 - \mathfrak{t}\left(\frac{|x|^2}{R_1^2}\right)\right) (F(x, \tilde{u}^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0})) \\
&- F(x, \tilde{u}^m(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^m_{\tau-T_0})))\bigg\|_{L^2(\Omega_{2R_1} \times [\tau - T_0, \tau])} \\
&\times \left\|\tilde{v}_1^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{v}^n_{1,\tau-T_0}) - \tilde{v}_1^m(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{v}^m_{1,\tau-T_0})\right\|_{L^2(\Omega_{2R_1} \times [\tau - T_0, \tau])} < C\varepsilon,
\end{aligned} \tag{115}$$

and

$$\begin{aligned}
C&\bigg\| \left(1 - \mathfrak{t}\left(\frac{|x|^2}{R_1^2}\right)\right) (f(x, \tilde{u}^n(T - \rho(T), \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^n_{\tau-T_0})) \\
&- f(x, \tilde{u}^m(T - \rho(T), \tau - T_0, \theta_{-\tau}\omega, \tilde{u}^m_{\tau-T_0})))\bigg\|_{L^2(\Omega_{R_1} \times [\tau - T_0, \tau])} \\
&\times \left\|\tilde{v}_1^n(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{v}^n_{1,\tau-T_0}) - \tilde{v}_1^m(T, \tau - T_0, \theta_{-\tau}\omega, \tilde{v}^m_{1,\tau-T_0})\right\|_{L^2(\Omega_{2R_1} \times [\tau - T_0, \tau])} < C\varepsilon.
\end{aligned} \tag{116}$$

Combing (115)-(116) with (102) and (111) together, we deduce that for all $n, m \geqslant N_1$,

$$\begin{aligned}
&\|\tilde{v}_{1\tau}^n - \tilde{v}_{1\tau}^m\|_{C_{L^2(\Omega_{2R_1})}}^2 + \| \tilde{u}_{1\tau}^n - \tilde{u}_{1\tau}^m \|_{C_{L^2(\Omega_{2R_1})}}^2 + \| \tilde{u}_{1\tau}^n - \tilde{u}_{1\tau}^m \|_{C_{H_0^1(\Omega_{2R_1})}}^2 \\
&+ \|\tilde{\eta}_1^n(\tau) - \tilde{\eta}_1^m(\tau)\|_{L_\mu^2(\mathbb{R}^+; H_0^1(\Omega_{2R_1}))}^2 \\
&= \sup_{s\in[-h,0]} \Bigg(\left\|\tilde{v}_1^n(\tau+s, \tau-T_0, \theta_{-\tau}\omega, \tilde{v}_{1,\tau-T_0}^n) - \tilde{v}_1^m(\tau+s, \tau-T_0, \theta_{-\tau}\omega, \tilde{v}_{1,\tau-T_0}^m)\right\|_{L^2(\Omega_{2R_1})}^2 \\
&+ \left\|\tilde{u}_1^n(\tau+s, \tau-T_0, \theta_{-\tau}\omega, \tilde{u}_{1,\tau-T_0}^n) - \tilde{u}_1^m(\tau+s, \tau-T_0, \theta_{-\tau}\omega, \tilde{u}_{1,\tau-T_0}^m)\right\|_{L^2(\Omega_{2R_1})}^2 \\
&+ \left\|\tilde{u}_1^n(\tau+s, \tau-T_0, \theta_{-\tau}\omega, \tilde{u}_{1,\tau-T_0}^n) - \tilde{u}_1^m(\tau+s, \tau-T_0, \theta_{-\tau}\omega, \tilde{u}_{1,\tau-T_0}^m)\right\|_{H_0^1(\Omega_{2R_1})}^2 \Bigg) \\
&+ \|\tilde{\eta}_1^n(\tau, \tau-T_0, \theta_{-\tau}\omega, \tilde{\eta}_1^n(\tau-T_0)) - \tilde{\eta}_1^m(\tau, \tau-T_0, \theta_{-\tau}\omega, \tilde{\eta}_1^m(\tau-T_0))\|_{L_\mu^2(\mathbb{R}^+; H_0^1(\Omega_{2R_1}))}^2 \\
&< C\varepsilon. \qquad (117)
\end{aligned}$$

This together with (104)-(106) implies that for all $n, m \geqslant N_1$ and every $\omega \in \Omega$,

$$\|v_\tau^n - v_\tau^m\|_{C_H}^2 + \| u_\tau^n - u_\tau^m \|_{C_V}^2 + \|\eta^n(\tau) - \eta^m(\tau)\|_{\mu,1}^2 < C\varepsilon, \qquad (118)$$

and thus the sequences $\{(u_\tau^n(\cdot, \tau-t^n, \theta_{-\tau}\omega, u_{\tau-t^n}^n), v_\tau^n(\cdot, \tau-t^n, \theta_{-\tau}\omega, v_{\tau-t^n}^n), \eta^n(\tau, \tau-t^n, \theta_{-\tau}\omega, \eta^n(\tau - t^n)))\}_{n=1}^\infty$ is precompact in E. The proof of this theorem is complete. □

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