Ulam-Hyers-Rassias stability of neutral stochastic functional differential equations

Tomás Caraballo
Universidad de Sevilla,
Dpto. Ecuaciones Diferenciales y Análisis Numérico,
Facultad de Matemáticas,
c/ Tarfia s/n, 41012-Sevilla (Spain).
caraball@us.es

Lassaad Mchiri
University of Sfax,
Faculty of Sciences of Sfax,
Department of Mathematics, Tunisia
lassaadmchiri0@gmail.com

Mohamed Rhaima
University of Tunis Manar,
Faculty of Sciences of Tunis,
Department of Mathematics, Tunisia.
mohamed.rhaima@fst.rnu.tn

Abstract

In this paper, by using the Gronwall inequality, we show two new results on the Ulam-Hyers and the Ulam-Hyers-Rassias stabilities of neutral stochastic functional differential equations. Two examples illustrating our results are exhibited.

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1 Introduction

The stability problem of functional equations originated from a question of Stanislaw Ulam, posed in 1940, concerning the stability of group of homomorphisms. The question concerning the stability of homomorphisms is as follows:
Let G_1 be a group, and let G_2 be a metric group with a metric d and a positive number ϵ. The problem is to analyze if there exists a positive number δ such that for every $g : G_1 \to G_2$ satisfying the inequality

$$d(g(xy), g(x)g(y)) \leq \delta, \quad \forall x, y \in G_1,$$

there exists a homomorphism $h : G_1 \to G_2$ such that

$$d(g(x), h(x)) \leq \epsilon, \quad \forall x \in G_1.$$

In 1941, Donald H. Hyers [5] gave a partial answer to the question of Ulam in the context of Banach spaces in the case of δ-linear transformations, that is:
Let F_1, F_2 be two Banach spaces and let $g : F_1 \to F_2$ be a linear transformation satisfying

$$\|g(x + y) - g(x) - g(y)\| \leq \delta, \forall x, y \in F_1 \quad \delta > 0.$$

There exists a unique linear transformation $L : F_1 \to F_2$ such that the limit $L(x) = \lim_{n \to +\infty} \frac{g(2^n x)}{2^n}$ exists for each $x \in F_1$ and $\|g(x) - L(x)\| \leq \delta$ for all $x \in F_1$, that was the first significant breakthrough and a step towards more solutions in this area. Since then, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers's theorem (see, for instance, [21], [4], [5], [6], [7], [8], [9], [10], [2], [17], [22], [12], and [3]). In 1978, Rassias [19] provided a generalized solution to the Ulam problem for approximate δ-linear transformations. In [18], Rassias obtained a generalization of the Hyers's theorem. In 1994, Gavruta [1] obtained a generalization of Rassias's Theorem for the unbounded Cauchy difference $g(x + y) - g(x) - g(y)$ and stated the notion of generalized Ulam-Hyers-Rassias stability in the spirit of Rassias approach. To the best of our knowledge, there are a few papers about the Ulam-Hyers and the Ulam-Hyers-Rassias stability of stochastic differential equations in the literature (see [11, 16, 23]). In the literature, neutral stochastic functional differential equations attracted the attention of many researchers (see [13, 14, 15, 20] etc.). Therefore, it is important to generalize the research results of deterministic neutral functional differential equations to the stochastic case. The contents of the paper is as follows. In Section 2, we present some basic results and assumptions. Section 3 is devoted to show some sufficient conditions and assumptions ensuring the Ulam-Hyers and the Ulam-Hyers-Rassias stabilities of the solution of the system. Finally in Section 4, we analyze two examples to illustrate our results.

2 Preliminaries and definitions

Let $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\}$ be a complete probability space with a filtration satisfying the usual conditions, i.e., the filtration is continuous on the right and $\mathcal{F}_0$ contains all $\mathbb{P}$-zero sets. $W(t)$

is an m-dimensional Brownian motion defined on the probability space. Let $\mathcal{L}^2([a,b],\mathbb{R}^n)$ be the family of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\leq t\leq b}$ such that $\int_a^b |f(t)|^2dt < \infty$ a.s. and $\mathcal{M}^2([a,b],\mathbb{R}^n)$ the family of processes $\{f(t)\}_{a\leq t\leq b}$ in $\mathcal{L}^2([a,b],\mathbb{R}^n)$ such that $\mathbb{E}\int_a^b |f(t)|^2dt < \infty$. Let $C([-\tau,0];\mathbb{R}^n)$ denote the family of functions φ from $[-\tau,0]$ to $\mathbb{R}^n$ that are right-continuous and have limits on the left. $C([-\tau,0];\mathbb{R}^n)$ is equipped with the norm $\|\varphi\| = \sup_{-\tau\leq s\leq 0}|\varphi(s)|$ and $|x| = \sqrt{x^Tx}$ for any $x\in\mathbb{R}^n$. Denote $C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ to be the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau,0];\mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau\leq\theta\leq 0\}$. Let $L^2_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$, $t\geq 0$, denote the family of all $\mathcal{F}_t$-measurable, $C([-\tau,0];\mathbb{R}^n)$-valued random variables $\varphi = \{\varphi(\theta) : -\tau\leq\theta\leq 0\}$ such that $\sup_{-\tau\leq\theta\leq 0}\mathbb{E}|\varphi(\theta)|^2 < \infty$.

Consider the following neutral stochastic functional differential equation for $0\leq t_0 < T$ fixed:

$$d\left[x(t) - G(x_t)\right] = f(t,x_t)dt + g(t,x_t)dW(t), \quad t_0\leq t\leq T, \tag{2.1}$$

with the initial condition

$$x_{t_0} = \xi\in L^2_{\mathcal{F}_{t_0}}([-\tau,0];\mathbb{R}^n), \tag{2.2}$$

and recall that, given $x\in C([t_0,T];\mathbb{R}^n)$, for each $t\in[t_0,T]$ we denote by $x_t(\cdot)$ the function in $C([t_0-\tau,0];\mathbb{R}^n)$ defined as $x_t(\theta) = x(t+\theta)$, $-\tau\leq\theta\leq 0$. We assume that

$$f : [t_0,T]\times C([-\tau,0];\mathbb{R}^n)\longrightarrow\mathbb{R}^n, \quad g : [t_0,T]\times C([-\tau,0];\mathbb{R}^n)\longrightarrow\mathbb{R}^{n\times m},$$
$$G : C([-\tau,0];\mathbb{R}^n)\longrightarrow\mathbb{R}^n.$$

Using the definition of Itô's stochastic differential and integrating the two sides of equation (2.1) from t_0 to t, we have

$$x(t) - G(x_t) = x(t_0) - G(x_{t_0}) + \int_{t_0}^t f(s,x_s)ds + \int_{t_0}^t g(s,x_s)dW(s), \quad t_0\leq t\leq T, \tag{2.3}$$

We will establish some assumptions ensuring the existence and uniqueness of a solution, denoted by $x(t;t_0,\xi)$, for equation (2.1).

$\mathcal{H}_1$: (Uniform Lipschitz condition): Assume that there exists a constant $L > 0$ such that

$$|f(t,\varphi_1) - f(t,\varphi_2)|^2\vee|g(t,\varphi_1) - g(t,\varphi_2)|^2\leq L||\varphi_1-\varphi_2||^2,$$

for all $t\in[t_0,T]$ and φ_1, $\varphi_2\in C([-\tau,0];\mathbb{R}^n)$, where the notation $a\vee b$ define the maximum of a and b.

$\mathcal{H}_2$: (Linear growth condition): Assume that there exists a constant $\alpha > 0$ such that for all $(t,\varphi)\in[t_0,T]\times C([-\tau,0];\mathbb{R}^n)$

$$|f(t,\varphi)|^2\vee|g(t,\varphi)|^2\leq\alpha(1+||\varphi||^2),$$

$\mathcal{H}_3$: Assume that there is a constant $\beta\in[0,\frac{1}{2})$ such that

$$|G(\varphi_1) - G(\varphi_2)|\leq\beta||\varphi_1-\varphi_2||, \tag{2.4}$$

for all φ_1, $\varphi_2\in C([-\tau,0];\mathbb{R}^n)$.

Definition 2.1. An $\mathbb{R}^n$-valued stochastic processes $x(t)$ on $t_0 - \tau \leq t \leq T$ is called a solution to equation (2.1) with initial condition (2.2) if the following conditions are satisfied:

(i) it is continuous and $\{x(t)\}_{t_0 \leq t \leq T}$ is $\mathcal{F}_t$-adapted.

(ii) $\{f(t, x_t)\} \in \mathcal{L}^1([t_0, T], \mathbb{R}^n)$ and $\{g(t, x_t)\} \in \mathcal{L}^2([t_0, T], \mathbb{R}^{n \times m})$.

(iii) $x_{t_0} = \xi$ and (2.3) holds for every $t_0 \leq t \leq T$.

Theorem 2.1. *If assumptions $\mathcal{H}_1 - \mathcal{H}_3$ are satisfied, then there exists a unique solution $x \in \mathcal{M}^2([t_0 - \tau, T], \mathbb{R}^n)$ to equation (2.1) with initial condition (2.2).*

Proof. See [14]. □

3 Main results

In this section, we discuss the Ulam-Hyers and the Ulam-Hyers-Rassias stability of equation (2.1) under the assumptions $\mathcal{H}_1$-$\mathcal{H}_3$.

Definition 3.1. Equation (2.1) is Ulam-Hyers stable with respect to ϵ if there exists a constant $C > 0$ such that for each $\epsilon > 0$ and for each solution $y \in \mathcal{M}^2([t_0 - \tau, T], \mathbb{R}^n)$, with $y_{t_0} = \xi$, of the following inequation:

$$\mathbb{E}|y(t) - G(y_t) - (y(t_0) - G(y_{t_0})) - \int_{t_0}^{t} f(s, y_s) ds - \int_{t_0}^{t} g(s, y_s) dW(s)|^2 \leq \epsilon, \forall t \in [t_0 - \tau, T], \quad (3.1)$$

there exists a solution $x \in \mathcal{M}^2([t_0 - \tau, T], \mathbb{R}^n)$ of (2.1), with $x_{t_0} = \xi$, such that $\mathbb{E}|y(t) - x(t)|^2 \leq C\epsilon$, $\forall t \in [t_0 - \tau, T]$.

Definition 3.2. Equation (2.1) is generalized Ulam-Hyers-Rassias stable with respect to $z(\cdot) \in C([t_0 - \tau, T]; \mathbb{R}^n)$ if there exists a constant $M > 0$ such that for each solution $y \in \mathcal{M}^2([t_0 - \tau, T], \mathbb{R}^n)$, with $y_{t_0} = \xi$, satisfying

$$\mathbb{E}|y(t) - G(y_t) - (y(t_0) - G(y_{t_0})) - \int_{t_0}^{t} f(s, y_s) ds - \int_{t_0}^{t} g(s, y_s) dW(s)|^2 \leq z(t), \forall t \in [t_0 - \tau, T], \quad (3.2)$$

there exists a solution $x(t) \in \mathcal{M}^2([t_0 - \tau, T], \mathbb{R}^n)$ of (2.1), with $x_{t_0} = \xi$, such that $\mathbb{E}|y(t) - x(t)|^2 \leq Mz(t)$, $\forall t \in [t_0 - \tau, T]$.

We cite now a technical lemma before proving our main theorems.

Lemma 3.1. *For any x, $y \geq 0$ and $0 < \epsilon < 1$, we have:*

$$(x+y)^2 \leq \frac{x^2}{\epsilon} + \frac{y^2}{1-\epsilon}.$$

Proof. See [14]. □

Theorem 3.2. *Under assumptions $\mathcal{H}_1 - \mathcal{H}_3$, equation (2.1) is Ulam-Hyers stable.*

Proof. By Theorem 2.1, there exists a unique solution $x \in \mathcal{M}^2([t_0-\tau, T], \mathbb{R}^n)$ to equation (2.1) with initial condition (2.2).

Let $y \in \mathcal{M}^2([t_0-\tau, T], \mathbb{R}^n)$, with $y_{t_0} = x_{t_0} = \xi$, be a solution to (3.1), then for $t \in [t_0-\tau, t_0]$, we have $\mathbb{E}|y(t)-x(t)|^2 = 0$.

By the triangle inequality, equation (2.3) and inequality (3.1), we have for $t \in [t_0, T]$

$$\begin{aligned}
&|y(t)-x(t)|^2 \\
&\leq 2\left|y(t) - G(y_t) - (y(t_0) - G(y_{t_0})) - \int_{t_0}^t f(s,y_s)ds - \int_{t_0}^t g(s,y_s)dW(s)\right|^2, \\
&\quad +2\left|(G(y_t)-G(x_t)) + \int_{t_0}^t \big(f(s,y_s)-f(s,x_s)\big)ds + \int_{t_0}^t \big(g(s,y_s)-g(s,x_s)\big)dW(s)\right|^2, \\
&\leq 2L(t) + 2J(t),
\end{aligned}$$

where

$$L(t) = \left|y(t) - G(y_t) - (y(t_0) - G(y_{t_0})) - \int_{t_0}^t f(s,y_s)ds - \int_{t_0}^t g(s,y_s)dW(s)\right|^2$$

and

$$J(t) = \left|(G(y_t)-G(x_t)) + \int_{t_0}^t \big(f(s,y_s)-f(s,x_s)\big)ds + \int_{t_0}^t \big(g(s,y_s)-g(s,x_s)\big)dW(s)\right|^2.$$

We denote by $I(t) = \int_{t_0}^t \big(f(s,y_s)-f(s,x_s)\big)ds + \int_{t_0}^t \big(g(s,y_s)-g(s,x_s)\big)dW(s)$.

Then, $J(t) = |(G(y_t)-G(x_t)) + I(t)|^2$. Thus, by Lemma 3.1, we obtain for any $0 < \beta < \frac{1}{2}$

$$\begin{aligned}
|y(t)-x(t)|^2 &\leq 2L(t) + 2\left|(G(y_t)-G(x_t)) + I(t)\right|^2, \\
&\leq 2L(t) + \frac{2}{\beta}\left|G(y_t)-G(x_t)\right|^2 + \frac{2}{1-\beta}\left|I(t)\right|^2, \\
&\leq 2L(t) + 2\beta\left\|y_t - x_t\right\|^2 + \frac{2}{1-\beta}\left|I(t)\right|^2.
\end{aligned}$$

Therefore, taking supremum and expectation, we have

$$\mathbb{E}\left(\sup_{t_0\le s\le t}|y(s)-x(s)|^2\right)\le 2\epsilon+2\beta\mathbb{E}\left(\sup_{t_0\le s\le t}|y(s)-x(s)|^2\right)+\frac{2}{1-\beta}\mathbb{E}\left(\sup_{t_0\le s\le t}|I(s)|^2\right),$$

which implies

$$\mathbb{E}\left(\sup_{t_0\le s\le t}|y(s)-x(s)|^2\right)\le \frac{2\epsilon}{1-2\beta}+\frac{2}{(1-\beta)(1-2\beta)}\mathbb{E}\left(\sup_{t_0\le s\le t}|I(s)|^2\right).$$

On the other hand, by the Hölder inequality, the Burkhölder–Davis–Gundy inequality and the Lipschitz condition, one has

$$\begin{aligned}
\mathbb{E}\left(\sup_{t_0\le s\le t}|I(s)|^2\right) &\le 2\mathbb{E}\left(\left|\int_{t_0}^t f(s,y_s)-f(s,x_s)ds\right|^2\right)+2\mathbb{E}\left(\left|\int_{t_0}^t g(s,y_s)-g(s,x_s)dW(s)\right|^2\right)\\
&\le 2(t-t_0)\mathbb{E}\left(\int_{t_0}^t |f(s,y_s)-f(s,x_s)|^2\,ds\right)\\
&\quad +8\mathbb{E}\left(\int_{t_0}^t |g(s,y_s)-g(s,x_s)|^2\,ds\right)\\
&\le 2L(T-t_0)\int_{t_0}^t \mathbb{E}\,\|y_s-x_s\|^2\,ds+8L\int_{t_0}^t \mathbb{E}\,\|y_s-x_s\|^2\,ds\\
&= 2L(T-t_0+4)\int_{t_0}^t \mathbb{E}\,\|y_s-x_s\|^2\,ds\\
&\le 2L(T-t_0+4)\int_{t_0}^t \mathbb{E}\left(\sup_{t_0\le r\le s}|y(r)-x(r)|^2\right)ds.
\end{aligned}$$

Therefore,

$$E\left(\sup_{t_0\le s\le t}|y(s)-x(s)|^2\right)\le \frac{2\epsilon}{1-2\beta}+\frac{4L(T-t_0+4)}{(1-\beta)(1-2\beta)}\int_{t_0}^t \mathbb{E}\left(\sup_{t_0\le r\le s}|y(r)-x(r)|^2\right)ds.$$

The Gronwall inequality implies now, for all $t\in[t_0,T]$ and $0<\beta<\frac{1}{2}$,

$$\begin{aligned}
E\left(\sup_{t_0\le s\le t}|y(s)-x(s)|^2\right) &\le \frac{2\epsilon}{1-2\beta}\exp\left(\frac{4L(T-t_0+4)}{(1-\beta)(1-2\beta)}(T-t_0)\right)\\
&\le \frac{2\epsilon}{1-2\beta}\exp\left(\frac{4LT(T+4)}{(1-\beta)(1-2\beta)}\right),
\end{aligned}$$

which implies that

$$\mathbb{E}\,|y(t)-x(t)|^2\le M\epsilon,$$

where $M=\frac{2}{1-2\beta}\exp\left(\frac{4LT(T+4)}{(1-\beta)(1-2\beta)}\right)$, $\forall t\in[t_0-\tau,T]$. The proof is complete. □

Theorem 3.3. *Assume that hypotheses $\mathcal{H}_2$ and $\mathcal{H}_3$ are satisfied and that there exists $\psi \in L^1([t_0-\tau,+\infty),\mathbb{R}_+)$ such that, for all φ_1, $\varphi_2 \in C([-\tau,0],\mathbb{R}^n)$ and $t \in [t_0,T]$,*

$$|f(t,\varphi_1)-f(t,\varphi_2)|^2 \vee |g(t,\varphi_1)-g(t,\varphi_2)|^2 \leq \psi(t)\,\|\varphi_1-\varphi_2\|^2 .$$

Then,

(a) Equation (2.1) has a unique solution x which belongs to $\mathcal{M}^2([t_0-\tau,T],\mathbb{R}^n)$.

(b) Equation (2.1) is generalized Ulam-Hyers-Rassias stable with respect to nondecreasing functions $z(\cdot) \in C([t_0-\tau,T];\mathbb{R}^n)$.

Proof. (a) Uniqueness: Let $x^1(t)$ and $x^2(t)$ be two solutions of equation (2.1). Using Lemma 2.4 in [14, page 204], both solutions belong to $\mathcal{M}^2([t_0-\tau,T],\mathbb{R}^n)$. Moreover, we have

$$x^1(t)-x^2(t) = G(x_t^1)-G(x_t^2)+K(t),$$

where

$$K(t)=\int_{t_0}^{t}\big(f(s,x_s^1)-f(s,x_s^2)\big)ds+\int_{t_0}^{t}\big(g(s,x_s^1)-g(s,x_s^2)\big)dW(s).$$

By Lemma 3.1 and assumption $\mathcal{H}_3$,

$$\begin{aligned}\left|x^1(t)-x^2(t)\right|^2 &= \left|G(x_t^1)-G(x_t^2)+K(t)\right|^2\\ &\leq \frac{1}{\beta}\left|G(x_t^1)-G(x_t^2)\right|^2+\frac{1}{1-\beta}\left|K(t)\right|^2\\ &\leq \beta\left\|x_t^1-x_t^2\right\|^2+\frac{1}{1-\beta}\left|K(t)\right|^2 .\end{aligned}$$

Then,

$$\mathbb{E}\left(\sup_{t_0\leq s\leq t}\left|x^1(s)-x^2(s)\right|^2\right) \leq \beta\mathbb{E}\left(\sup_{t_0\leq s\leq t}\left|x^1(s)-x^2(s)\right|^2\right)+\frac{1}{1-\beta}\mathbb{E}\left(\sup_{t_0\leq s\leq t}|K(s)|^2\right).$$

It follows that

$$\mathbb{E}\left(\sup_{t_0\leq s\leq t}\left|x^1(s)-x^2(s)\right|^2\right) \leq \frac{1}{(1-\beta)^2}\mathbb{E}\left(\sup_{t_0\leq s\leq t}|K(s)|^2\right).$$

On the other hand,

$$\begin{aligned}\mathbb{E}\left(\sup_{t_0\leq s\leq t}|K(s)|^2\right) &\leq 2\,(T-t_0+4)\int_{t_0}^{t}\psi(s)\mathbb{E}\left\|x_s^1-x_s^2\right\|^2 ds\\ &\leq 2\,(T-t_0+4)\int_{t_0}^{t}\psi(s)\mathbb{E}\left(\sup_{t_0\leq r\leq s}\left|x^1(r)-x^2(r)\right|^2\right)ds.\end{aligned}$$

Thus,

$$\mathbb{E}\left(\sup_{t_0\le s\le t}\left|x^1(s)-x^2(s)\right|^2\right)\le\frac{2(T-t_0+4)}{(1-\beta)^2}\int_{t_0}^{t}\psi(s)\mathbb{E}\left(\sup_{t_0\le r\le s}\left|x^1(r)-x^2(r)\right|^2\right)ds$$

By the Gronwall inequality,

$$\mathbb{E}\left(\sup_{t_0\le t\le T}\left|x^1(t)-x^2(t)\right|^2\right)=0,$$

which implies that $x^1(t)=x^2(t)$ for $t_0\le t\le T$, and therefore for all $t_0-\tau\le t\le T$, almost surely.

Existence: The proof of the existence is similar to [14], but we take

$$\delta=\beta+\frac{2(T-t_0+4)\lambda}{1-\beta},$$

where λ is a positive constant such that $\displaystyle\int_{t_0-\tau}^{+\infty}\psi(t)dt\le\lambda$.

(b) Using equation (2.3), $x(t)$ is a solution of equation (2.1). By condition (3.2), for $z\in C([t_0-\tau,T],\mathbb{R}^n)$, we have

$$\mathbb{E}\left|y(t)-G(y_t)-(y(t_0)-G(y_{t_0}))-\int_{t_0}^{t}f(s,y_s)ds-\int_{t_0}^{t}g(s,y_s)dW(s)\right|^2\le z(t),\quad\forall t\in[t_0-\tau,T].$$

Moreover, we know that, for $t\in[t_0-\tau,t_0]$, $\mathbb{E}\left|y(t)-x(t)\right|^2=0$. Proceeding as in the previous theorem when $t\in[t_0,T]$, we have

$$\left|y(t)-x(t)\right|^2\le 2L(t)+2J(t).$$

Then, for any $0<\beta<\frac{1}{2}$ and $t\in[t_0,T]$

$$\left|y(t)-x(t)\right|^2\le 2L(t)+2\beta\left\|y_t-x_t\right\|^2+\frac{2}{1-\beta}\left|I(t)\right|^2.$$

Thus, taking again supremum and expectation, we have

$$\mathbb{E}\left(\sup_{t_0\le s\le t}\left|y(s)-x(s)\right|^2\right)\le 2z(t)+2\beta\mathbb{E}\left(\sup_{t_0\le s\le t}\left|y(s)-x(s)\right|^2\right)+\frac{2}{1-\beta}\mathbb{E}\left(\sup_{t_0\le s\le t}\left|I(s)\right|^2\right).$$

Therefore,

$$\mathbb{E}\left(\sup_{t_0\le s\le t}\left|y(s)-x(s)\right|^2\right)\le\frac{2}{1-2\beta}z(t)+\frac{2}{(1-\beta)(1-2\beta)}\mathbb{E}\left(\sup_{t_0\le s\le t}\left|I(s)\right|^2\right).$$

On the other hand, we can see that

$$\begin{aligned}
\mathbb{E}\left(\sup_{t_0\le s\le t}|I(s)|^2\right) &\le 2(T-t_0+4)\int_{t_0}^{t}\psi(s)\mathbb{E}\left\|y_s-x_s\right\|^2 ds\\
&\le 2(T-t_0+4)\int_{t_0}^{t}\psi(s)\mathbb{E}\left(\sup_{t_0\le r\le s}|y(r)-x(r)|^2\right) ds\\
&\le 2(T+4)\int_{t_0}^{t}\psi(s)\mathbb{E}\left(\sup_{t_0\le r\le s}|y(r)-x(r)|^2\right) ds.
\end{aligned}$$

Then,

$$\mathbb{E}\left(\sup_{t_0\le s\le t}|y(s)-x(s)|^2\right)\le \frac{2}{1-2\beta}z(t)+\frac{4(T+4)}{(1-\beta)(1-2\beta)}\int_{t_0}^{t}\psi(s)\mathbb{E}\left(\sup_{t_0\le r\le s}|y(r)-x(r)|^2\right) ds.$$

Using the Gronwall lemma,

$$\begin{aligned}
\mathbb{E}\left(\sup_{t_0\le s\le t}|y(s)-x(s)|^2\right) &\le \frac{2}{1-2\beta}z(t)\exp\left(\frac{4(T+4)}{(1-\beta)(1-2\beta)}\int_{t_0}^{t}\psi(s)ds\right)\\
&\le \frac{2}{1-2\beta}z(t)\exp\left(\frac{4(T+4)}{(1-\beta)(1-2\beta)}\int_{-\tau}^{+\infty}\psi(s)ds\right)\\
&\le Cz(t),
\end{aligned}$$

for all $t\in[t_0,T]$, where $C=\dfrac{2}{1-2\beta}\exp\left(\lambda\dfrac{4(T+4)}{(1-\beta)(1-2\beta)}\right)$.

Finally, for all $t\in[t_0-\tau,T]$, we have

$$\mathbb{E}\left(|y(t)-x(t)|^2\right)\le Cz(t).$$

The proof is complete. □

4 Examples

In this section we will show two examples to illustrate the applicability and interest of our abstract results.

Example 1: Consider the following neutral stochastic functional differential system for each $\epsilon>0$ and for $t\in[t_0-\tau,T]$

$$\begin{cases}
d\left[x(t)-G(x_t)\right]=f(t,x_t)dt+g(t,x_t)dW(t),\\
\mathbb{E}|x(t)-G(x_t)-(x(t_0)-G(x_{t_0}))-\int_{t_0}^{t}f(s,x_s)ds-\int_{t_0}^{t}g(s,x_s)dW(s)|^2\le\epsilon,\\
x_{t_0}=\xi,
\end{cases}\tag{4.1}$$

where

$$\begin{aligned}
\xi &\in L^2_{\mathcal{F}_{t_0}}([-\tau,0];\mathbb{R}), x(t)\in\mathcal{M}^2([t_0-\tau,T],\mathbb{R})\\
G(\varphi) &= \beta\varphi(-\tau), \quad \varphi\in C([-\tau,0];\mathbb{R})\\
f(t,\varphi) &= \frac{e^{-t}}{\sqrt{1+t^2}}\varphi(0)+\frac{\cos(t)}{\sqrt{1+t^2}}\varphi(-\tau), \quad \varphi\in C([-\tau,0];\mathbb{R})\\
g(t,\varphi) &= \frac{\sin(t)}{\sqrt{1+t^2}}\varphi(-\tau), \quad \varphi\in C([-\tau,0];\mathbb{R}).
\end{aligned}$$

Here $\tau>0$ and $\beta\in[0,\frac{1}{2})$. Then, replacing now φ by the segment of a solution x_t we have

$$\begin{aligned}
G(x_t) &= \beta x(t-\tau),\\
f(t,x_t) &= \frac{e^{-t}}{\sqrt{1+t^2}}x(t)+\frac{\cos(t)}{\sqrt{1+t^2}}x(t-\tau),\\
g(t,x_t) &= \frac{\sin(t)}{\sqrt{1+t^2}}x(t-\tau).
\end{aligned}$$

We will prove that equation (4.1) is Ulam-Hyers stable. Let $\varphi,\psi\in C([-\tau,0];\mathbb{R})$, then

$$\begin{aligned}
|f(t,\varphi)-f(t,\psi)|^2 &= \left|\frac{e^{-t}}{\sqrt{1+t^2}}\left(\phi(0)-\psi(0)\right)+\frac{\cos(t)}{\sqrt{1+t^2}}\left(\varphi(-\tau)-\psi(-\tau)\right)\right|^2\\
&\le \frac{2}{1+t^2}\left|\phi(0)-\psi(0)\right|^2+\frac{2}{1+t^2}\left|\varphi(-\tau)-\psi(-\tau)\right|^2\\
&\le 4\left\|\varphi-\psi\right\|^2,
\end{aligned}$$

and

$$\begin{aligned}
|g(t,\varphi)-g(t,\psi)|^2 &= \frac{\sin^2(t)}{1+t^2}\left|\varphi(-\tau)-\psi(-\tau)\right|^2\\
&\le \left\|\varphi-\psi\right\|^2.
\end{aligned}$$

Hence, the uniform Lipschitz condition is satisfied. Moreover,

$$\begin{aligned}
|f(t,\varphi)|^2 &= \left|\frac{e^{-t}}{\sqrt{1+t^2}}\varphi(0)+\frac{\cos(t)}{\sqrt{1+t^2}}\varphi(-\tau)\right|^2\\
&\le \frac{2}{1+t^2}|\varphi(0)|^2+\frac{2}{1+t^2}|\varphi(-\tau)|^2\\
&\le 4\left(1+\|\varphi\|^2\right)
\end{aligned}$$

and

$$|g(t,\varphi)|^2=\frac{\sin^2(t)}{1+t^2}|\varphi(-\tau)|^2\le 1+\|\varphi\|^2.$$

Consequently, the linear growth condition holds true. Moreover, it is easy to verify that, for $\beta \in [0, \frac{1}{2})$, G satisfies hypothesis $\mathcal{H}_3$. Therefore, by Theorem 3.2, equation (4.1) is Ulam-Hyers stable.

Example 2: Consider the following neutral stochastic functional differential system for $t \in [t_0 - \tau, T]$:

$$\begin{cases} d\left[x(t) - G(x_t)\right] = f(t, x_t)dt + g(t, x_t)dW(t), \\ \\ \mathbb{E}|x(t) - G(x_t) - (x(t_0) - G(x_{t_0})) - \int_{t_0}^t f(s, x_s)ds - \int_{t_0}^t g(s, x_s)dW(s)|^2 \leq z(t), \\ \\ x_{t_0} = \xi, \end{cases} \tag{4.2}$$

where $x \in \mathcal{M}^2([t_0-\tau, T], \mathbb{R})$, $W(t)$ is a one dimensional Brownian motion, $z(\cdot) \in C([t_0-\tau, T]; \mathbb{R}^n)$ is a nondecreasing function and $f, g : \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ are defined by

$$f(t, \varphi) = \int_{-\tau}^0 e^{-(t+\theta)} \varphi(\theta) d\theta, \quad \varphi \in C([-\tau, 0]; \mathbb{R}),$$

$$g(t, \varphi) = \int_{-\tau}^0 \frac{e^{-t}}{\sqrt{1+t^2}} \sin(\theta) \varphi(\theta) d\theta, \quad \varphi \in C([-\tau, 0]; \mathbb{R}),$$

and $G : C\left([-\tau, 0], \mathbb{R}\right) \to \mathbb{R}$ is a linear operator defined by

$$G(\varphi) = \frac{1}{\tau} \int_{-\tau}^0 u(\theta) \varphi(\theta) d\theta,$$

where $\tau > 0$, $u \in C\left([-\tau, 0], \mathbb{R}\right)$, $\xi \in L^2_{\mathcal{F}_{t_0}}([-\tau, 0]; \mathbb{R})$, and $\|u\| < \frac{1}{2}$.
Then, for $\varphi \in C\left([-\tau, 0], \mathbb{R}\right)$, we have

$$\begin{aligned} f(t, \varphi) &= \int_{-\tau}^0 e^{-(t+\theta)} \varphi(\theta) d\theta, \\ g(t, \varphi) &= \int_{-\tau}^0 \frac{e^{-t}}{\sqrt{1+t^2}} \sin(\theta) \varphi(\theta) d\theta. \end{aligned}$$

For all $\varphi_1, \varphi_2 \in C\left([-\tau, 0], \mathbb{R}\right)$, we have

$$\begin{aligned} |f(t, \varphi_1) - f(t, \varphi_2)| &= \left| \int_{-\tau}^0 e^{-(t+\theta)} \left(\varphi_1(\theta) - \varphi_2(\theta)\right) d\theta \right| \\ &\leq \int_{-\tau}^0 e^{-(t+\theta)} \left|\varphi_1(\theta) - \varphi_2(\theta)\right| d\theta \\ &\leq \|\varphi_1 - \varphi_2\| \int_{-\tau}^0 e^{-(t+\theta)} d\theta \\ &= \|\varphi_1 - \varphi_2\| \left(e^{\tau} - 1\right) e^{-t}. \end{aligned}$$

Therefore,

$$|f(t,\varphi_1) - f(t,\varphi_2)|^2 \leq (e^{\tau} - 1)^2 e^{-2t} \|\varphi_1 - \varphi_2\|^2 .$$

On the other hand,

$$\begin{aligned}
|g(t,\varphi_1) - g(t,\varphi_2)| &= \left| \int_{-\tau}^{0} \frac{e^{-t}}{\sqrt{1+t^2}} \sin(\theta) \left(\varphi_1(\theta) - \varphi_2(\theta)\right) d\theta \right| \\
&\leq \frac{e^{-t}}{\sqrt{1+t^2}} \int_{-\tau}^{0} |\sin(\theta)| \, |\varphi_1(\theta) - \varphi_2(\theta)| \, d\theta \\
&\leq \frac{e^{-t}}{\sqrt{1+t^2}} \|\varphi_1 - \varphi_2\| \int_{-\tau}^{0} d\theta \\
&= \frac{\tau e^{-t}}{\sqrt{1+t^2}} \|\varphi_1 - \varphi_2\| .
\end{aligned}$$

Therefore,

$$|g(t,\varphi_1) - g(t,\varphi_2)|^2 \leq \frac{\tau^2}{1+t^2} e^{-2t} \|\varphi_1 - \varphi_2\|^2 .$$

Then,

$$|f(t,\varphi_1) - f(t,\varphi_2)|^2 \vee |g(t,\varphi_1) - g(t,\varphi_2)|^2 \leq \psi(t) \|\varphi_1 - \varphi_2\|^2,$$

where $\psi(t) = \max\left(\psi_1(t), \psi_2(t)\right)$, with $\psi_1(t) = (e^{\tau} - 1)^2 e^{-2t} \in L^1\left([-\tau, +\infty[, \mathbb{R}_+\right)$ and $\psi_2(t) = \frac{\tau^2}{1+t^2} e^{-2t} \in L^1\left([-\tau, +\infty[, \mathbb{R}_+\right)$.

For all $(t,\varphi) \in [-\tau, T] \times C\left([-\tau, 0], \mathbb{R}\right)$, we have

$$\begin{aligned}
|f(t,\varphi)| &= \left| \int_{-\tau}^{0} e^{-(t+\theta)} \varphi(\theta) d\theta \right| \\
&\leq e^{\tau} \|\varphi\| \int_{-\tau}^{0} e^{-\theta} d\theta \\
&= e^{\tau} \left(e^{\tau} - 1\right) \|\varphi\| .
\end{aligned}$$

And this implies that

$$\begin{aligned}
|f(t,\varphi)|^2 &\leq e^{2\tau} \left(e^{\tau} - 1\right)^2 \|\varphi\|^2 \\
&\leq e^{2\tau} \left(e^{\tau} - 1\right)^2 \left(1 + \|\varphi\|^2\right) .
\end{aligned}$$

Moreover,

$$\begin{aligned}
|g(t,\varphi)| &= \left| \int_{-\tau}^{0} \frac{e^{-t}}{\sqrt{1+t^2}} \sin(\theta)\varphi(\theta) d\theta \right| \\
&\leq \frac{e^{-t}}{\sqrt{1+t^2}} \|\varphi\| \int_{-\tau}^{0} d\theta \\
&= \frac{\tau e^{-t}}{\sqrt{1+t^2}} \|\varphi\| \\
&\leq \tau e^{\tau} \|\varphi\| .
\end{aligned}$$

Then,

$$\begin{aligned}
|g(t,\varphi)|^2 &\leq \tau^2 e^{2\tau} \|\varphi\|^2 \\
&\leq \tau^2 e^{2\tau} \left(1 + \|\varphi\|^2\right).
\end{aligned}$$

Therefore,

$$|f(t,\varphi)|^2 \vee |g(t,\varphi)|^2 \leq \alpha(1 + ||\varphi||^2),$$

where $\alpha = \max\left(e^{2\tau}\left(e^{\tau}-1\right)^2, \tau^2 e^{2\tau}\right)$.

For all $\varphi_1, \varphi_2 \in C\left([-\tau,0],\mathbb{R}\right)$,

$$\begin{aligned}
|G(\varphi_1) - G(\varphi_2)| &= \frac{1}{\tau} \left| \int_{-\tau}^{0} u(\theta)\left(\varphi_1(\theta) - \varphi_2(\theta)\right) d\theta \right| \\
&\leq \|u\| \, \|\varphi_1 - \varphi_2\| .
\end{aligned}$$

Therefore, all the assumptions of Theorem 3.3 are satisfied. Then equation (4.2) has a unique solution $x(t) \in \mathcal{M}^2\left([t_0-\tau,T],\mathbb{R}\right)$ and the Ulam-Hyers-Rassias stability with respect to the nondecreasing $z(\cdot) \in C([t_0-\tau,T];\mathbb{R}^n)$ is fulfilled.

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References

[1] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* 184 (1994), 431-436.

[2] P. Gavruta, L. Gavruta, A new method for the generalized Hyers-Ulam-Rassias stability, *Int. J. Nonlinear Anal. Appl.* 1(2)(2010), 11-18.

[3] Y. Guo, X. B. Shu, Y. Li, F. Xu, 2019. The existence and HyersUlam stability of solution for an impulsive RiemannLiouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$.*Boundary Value Problems*, 2019(1), p.59.

[4] D. H. Hyers, G. Isac, TH. M. Rassias, *Stability of functional equation in several variables*, Birkhäuser, Basel, 1998.

[5] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.* 27 (1941), 222-224.

[6] K. D. Kucche and P. U. Shikhare, Ulam Stabilities for Nonlinear Volterra Delay Integro-differential Equations, *Journal of Contemporary Mathematical Analysis.* 54 (5) (2019), 27-43.

[7] K. D. Kucche and P. U. Shikhare, UlamHyers Stability of Integrodifferential Equations in Banach Spaces via Pachpattes Inequality, *Asian-European Journal of Mathematics,* 11(2) (2018), 1850062 (19 pages).

[8] K. D. Kucche and P. U. Shikhare, Ulam Stabilities via Pachpattes Inequality for Volterra-Fredholm Delay Integrodifferential Equations in Banach Spaces, *Note di Matematica,* 38 (1) (2018), 67-82.

[9] K. D. Kucche, S. T. Sutar, On Existence and Stability Results for Nonlinear Fractional Delay Differential Equations, *Boletim da Sociedade Paranaense de Matemtica,* 36 (4) (2018), 55-77.

[10] K. D. Kucche and P. U. Shikhare, Ulam stabilities for nonlinear Volterra-Fredholm delay integrodifferential equations, *Int. J. Nonlinear Anal. Appl,* 9(2) (2018), 145-159.

[11] S. Li, L. Shu, X. B. Shu, F. Xu, Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays, *Stochastics An International Journal of Probability and Stochastic Processes*, Volume 90, 2018, Issue 5, 663-681.

[12] T. Li, A. Zada, Connections between Hyers-Ulam stability and uniform exponential stability of discrete evolution families of bounded linear operators over Banach spaces, *Adv. Differ. Equ.* 2016, Article ID 153 (2016).

[13] X. Mao. Exponential Stability in Mean Square of Neutral Stochastic Differential Functional Equations, *Systems and Control Letters* (1995), 26: 245-251.

[14] X. Mao, *Stochastic Differential Equations and Applications,* Ellis Horwood, Chichester, U.K, 1997.

[15] X. Mao, Razumikhin Type Theorems on Exponential Stability of Neutral Stochastic Functional Differential Equations, *SIAM J. Math. Anal.* 28(2) (1997), 389-401.

[16] N. P. N. Ngoc, Ulam-Hyers-Rassias stability of a nonlinear Stochastic Integral Equation of Volterra Type, *Differential Equations And Applications*, Volume 9, Number 2 (2017), 183-193.

[17] P. Niu, X. B. Shu, Y. Li, The existence and Hyers-Ulam stability for second order random impulsive differential equations, *Dynamic Systems and Applications,* 28(3), (2019) pp.673-690.

[18] Th. M. Rassias, On a modified Hyers-Ulam sequence, *J. Math. Anal. Appl.* 158(1991),106-113.

[19] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* 72 (1978), 297-300.

[20] L. Tan, W. Jin, Y. Suo, Stability in distribution of neutral stochastic functional differential equations, *Stat. Probab. Lett.* 107 (2015), 27-36.

[21] C. Tunc, E. Bicer, Hyers-Ulam-Rassias stability for a first order functional differential equation, *J. Math. and Fundamental Sciences* 47 (2) (2015) 143-153.

[22] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ, New York, 1960.

[23] X. Zhao, Mean square Hyers-Ulam stability of stochastic differential equations driven by Brownian motion, *Advances in Difference Equations*, (2016), p. 271.