Positive Representations of L^1 of a Vector Measure

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Abstract. We characterize the vector measures n on a Banach lattice such that the map $\| \int | \cdot | dn \|$ provides a quasi-norm which is equivalent to the canonical norm $\| \cdot \|_n$ of the space $L_1(n)$ of integrable functions as an specific type of transformations of positive vector measures that we call cone-open transformations. We also prove that a vector measure m on a Banach space X constructed as a cone-open transformation of a positive vector measure can be considered in some sense as a positive vector measure by defining a new order on X.

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1. Introduction and notation

Let $m : \Sigma \to X$ be a (countably additive) vector measure, where X is a Banach lattice. It is said that m is *positive* if $m(A) \in C_X^+$ for every $A \in \Sigma$, where C_X^+ is the positive cone of X. It is well-known that in this case the norm of the space $L^1(m)$ of integrable functions with respect to m can be computed by the formula

$$\|f\|_m = \left\| \int |f| dm \right\|, \quad f \in L^1(m), \tag{1}$$

(see Lemma 1.2 for a short proof of this result). The aim of this paper is to characterize the class of vector measures for which this formula gives an equivalent norm – or at least a quasi-norm – for $L^1(m)$. A problem related to this one has been recently studied in [5, Section 4], where a technique to construct optimal representations of $L^1(m)$ of a positive vector measure m has been introduced. The motivation of the problem is given by the fact that any order continuous Banach function space with a weak unit can be written as $L^1(m)$ of a vector measure m ([3, Theorem 8]). Therefore it seems natural to analyze when there is a representation of $L^1(m)$ using a vector measure n (i.e. $L^1(m) = L^1(n)$) such that the norm can be computed in an easy way using formula (1). In order to develop our analysis, we define and study a particular class of transformations of vector measures defined by means of linear operators preserving certain order relations that we call cone-open transformations. Actually, we prove in Section 2 that the class of vector measures m satisfying that formula (1) gives a quasi-norm for $L^1(m)$ coincides with the class of cone-open transformations of positive vector measures.

It is easy to find examples of vector measures for which the expression given in (1) is not a norm for $L^1(m)$. For instance, if $\nu : \Sigma \to \mathbb{R}$ is a signed measure (with non-trivial positive and negative parts) then there is always a non-zero function $f \in L^1(\nu)$ such that $\int |f| d\nu = 0$, and then the map $f \mapsto \int |f| d\nu$ does not give a norm for $L^1(\nu)$.

Although there are a lot of examples of non-positive vector measures m such that formula (1) gives a norm for $L^1(m)$ (see for instance Example 2.1), we study if in the case that this property holds it is possible to define a suitable order on X such that m can be considered in some sense as a *positive* vector measure. We also answer in the positive this question, although further requirements for the cone-open transformations involved are needed; in particular, we need to introduce the notion of lattice generating operator. This is done in Section 3.

In the rest of this section we give some definitions and basic results. Let X be a real Banach space with dual X^*, (Ω, Σ) a measurable space and $m : \Sigma \longrightarrow X$ a vector measure. For any $x^* \in X^*$ we define $\langle m, x^* \rangle$ to be the scalar measure given by $\langle m, x^* \rangle (A) := \langle m(A), x^* \rangle$, for all $A \in \Sigma$. Following Lewis in [6] we introduce the notion of integrable function with respect to m.

Definition 1.1. A measurable function f is *integrable* with respect to m if

(i) f is integrable with respect to the scalar measure $\langle m, x^* \rangle$, for all $x^* \in X^*$.

(ii) For every $A \in \Sigma$ there exists an element $\int_A f dm \in X$ such that

$$\left\langle \int_A f dm, x^* \right\rangle = \int_A f d \langle m, x^* \rangle .$$

It is well-known that this definition is equivalent to the one given by Bartle, Dunford and Schwartz in [2]. The space $L^1(m)$ is the Banach lattice (of classes) of integrable functions with respect to m equipped with the norm

$$\|f\|_m := \sup_{x^* \in B_{X^*}} \int |f| \; d|\langle m, x^* \rangle|, \;\; f \in L^1(m).$$

where $|\langle m, x^* \rangle|$ denotes the variation of the measure $\langle m, x^* \rangle$ In fact, $L^1(m)$ is a Köthe function space over any measure of the type $\mu = |\langle m, x^* \rangle|$ that satisfies the Rybakov Theorem (see [7, 1.b.17] for the definition of Köthe function space and [4, IX.2] for the Rybakov Theorem). The order in $L^1(m)$ is the usual μ-almost everywhere order. Note that $L^1(m)$ reduces to the ordinary space of Lebesgue integrable functions if the measure m is scalar.

We will use standard notation of Riesz spaces and Banach lattices which can be found in [1] and [8]. We recall that a *lattice* is a partially ordered set $(X, \leq)$

such that every subset consisting of two elements has supremum and infimum. We denote by $x \vee y$ and $x \wedge y$ the supremum and the infimum of $\{x, y\}$, respectively. On the other hand, $(X, \leq)$ is said to be an *ordered vector space* if X is a vector space and $\leq$ is a partial order compatible with the algebraic structure of X, i.e.,

(i) $x \leq y$ implies $x + z \leq y + z$ for every $x, y, z \in X$, and
(ii) $x \geq 0$ implies $\alpha x \geq 0$ for each real number $\alpha \geq 0$ and for every $x \in X$.

An ordered vector space that is also a lattice is called a *Riesz space*. If $(X, \leq)$ is a Riesz space, we write C_X^+ for its *positive cone*, that is, the set $\{x \in X : x \geq 0\}$. Given $x \in X$ the *positive part* x^+, the *negative part* x^-, and the *absolute value* $|x|$ of x are respectively defined by $x^+ := x \vee 0$, $x^- := (-x) \vee 0$, $|x| := x \vee (-x)$ and they verify $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

A norm in $\|\cdot\|$ on a Riesz space X is a *lattice norm* if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in X$. A Riesz space equipped with a lattice norm is called a *normed Riesz space* and a complete normed Riesz space is called a *Banach lattice*.

We also recall that a *quasi-norm* on a vector space X is any map $\|\cdot\| : X \longrightarrow \mathbb{R}^+$ verifying the following properties:

(i) $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\alpha x\| = |\alpha|\ \|x\|$, for all $\alpha \in \mathbb{R}$ and $x \in X$.
(iii) There exists $M \geq 1$ such that $\|x + y\| \leq M(\|x\| + \|y\|)$, for all $x, y \in X$.

Note that if the constant M is equal to 1 then $\|\cdot\|$ is in fact a norm.

To finish this introductory section and for the purpose of completeness we prove the following result.

Lemma 1.2. *If X is a Banach lattice and $m : \Sigma \longrightarrow X$ is a positive vector measure, then $\|f\|_m = \| \int |f| dm\|, \forall f \in L^1(m)$.*

Proof. Since m is positive and the integration map $I_m : L^1(m) \to X$ is continuous, it is also positive. Therefore, for every $f \in L^1(m)$ the element $\int |f| dm$ belongs to C_X^+ and then

$$\|f\|_m \geq \sup_{x^* \in B_{X^*}} \langle \int |f| dm, x^* \rangle = \left\| \int |f| dm \right\|$$

$$= \sup_{x^* \in B_{X^*}} \langle \int |f| dm, |x^*| \rangle \ = \sup_{x^* \in B_{X^*}} \int |f| d|\langle m, |x^*| \rangle| \geq \|f\|_m,$$

since X^* is also a Banach lattice and thus $\|x^*\| = \|\,|x^*|\,\|$ for every $x^* \in X^*$. $\square$

2. The first representation theorem

Although the positivity of the measure m provides the alternative formula for the norm of $L^1(m)$ given in Lemma 1.2, the converse is not true: there are non-positive measures such that formula (1) also gives the norm $\|\cdot\|_m$. Let us show this with an example.

Example 2.1. Consider the Lebesgue measure space $([0,1],\Sigma,\mu)$ and the vector measure $\tau:\Sigma\to\ell_2$ given by

$$\tau(A):=\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\mu(A_n\cap A)e_n,\quad A\in\Sigma,$$

where $\{A_n : n=1,2,\dots\}$ is a non-trivial measurable partition of $[0,1]$ and $\{e_n : n=1,2,\dots\}$ is the canonical basis of ℓ_2. A direct calculation shows that, although the measure is clearly not positive, $\|\int|\cdot|d\tau\|=\|\cdot\|_\tau$.

The purpose of this section is to describe the class of measures n yielding a quasi-norm $\|\int|\cdot|dn\|$ equivalent to $\|\cdot\|_n$. To this end, we introduce the notion of cone-open transformation of a measure.

Definition 2.2. Let X be a normed Riesz space and Y a normed space. A linear and continuous operator $S:X\longrightarrow Y$ is called a *cone-open* operator (resp. a *cone-isometry*) if there exists $K\geq 0$ such that $\|S(x)\|\geq K\|x\|$, $\forall x\in C_X^+$ (resp. $\|S(x)\|=\|x\|$, $\forall x\in C_X^+$).

We also introduce the dual notion in the following sense: let X and Y be two normed spaces and assume that X is also a Riesz space. A linear operator $S:X\longrightarrow Y$ is said to be *cone-continuous* if there exists $Q\geq 0$ such that $\|S(x)\|\leq Q\|x\|$, $\forall x\in C_X^+$.

However, whenever X is a normed Riesz space, it is easy to see that if S is cone-continuous then S is in fact continuous (and hence both concepts coincide). To see this, take an element x of X. Then

$$\|S(x)\|=\|S(x^+-x^-)\|\leq\|S(x^+)\|+\|S(x^-)\|\leq 2Q\cdot\max\{\|x^+\|,\|x^-\|\}\leq 2Q\cdot\|x\|,$$

since $|x^+|\leq|x|$, $|x^-|\leq|x|$ and $\|\cdot\|$ is a lattice norm.

Definition 2.3. Given a vector measure $m:\Sigma\longrightarrow X$ and a cone-open operator $S:X\longrightarrow Y$, we will say that $n:=S\circ m$ is a *cone-open transformation* of the measure m.

Proposition 2.4. *Let X be a Banach lattice and Y a Banach space. If $m:\Sigma\longrightarrow X$ is a vector measure and $S:X\longrightarrow Y$ is a linear and continuous operator, then $n:=S\circ m$ is also a vector measure, $L^1(m)\subset L^1(n)$, and $\int_A f dn=S(\int_A f dm)$, $\forall f\in L^1(m)$, $\forall A\in\Sigma$.*

Moreover, if S is cone-open (resp. cone-isometry) and m is positive then n and m are equivalent vector measures. Thus, $\|\int|\cdot|dn\|$ is a quasi-norm (resp. norm) on $L^1(m)$.

Proof. The first part of the proposition is well-known. Given $f\in L^1(m)$ and $y^*\in Y^*$, it is clear that $\langle m,S^*(y^*)\rangle=\langle n,y^*\rangle$. Since f is integrable with respect to $\langle m,S^*(y^*)\rangle$ we have that f is integrable with respect to $\langle n,y^*\rangle$ for all $y^*\in Y^*$, and from

$$\langle S\left(\left(\int_A fdm\right), y^*\right\rangle = \left\langle\left(\int_A fdm\right), S^*(y^*)\right\rangle = \int_A fd\langle m, S^*(y^*)\rangle = \int_A fd\langle n, y^*\rangle,$$

for every $y^* \in Y^*$, we conclude that $f \in L^1(n)$ with $\int_A fdn = S(\int_A fdm)$.

Note that $m(A) = 0$ always implies that $n(A) = 0$ and, if S is cone-open and m is positive, the converse is also true. Moreover, in this case, it is clear that $\| \int |f|dn\| = 0$ if and only if $f = 0$, $\| \int |\alpha f|dn\| = |\alpha| \ \| \int |f|dn\|$ and, since there exist $K, Q \geq 0$ such that $K\|x\| \leq \|S(x)\| \leq Q\|x\|$ for all $x \in C_X^+$, we deduce that

$$\begin{aligned}\left\|\int |f+g|dn\right\| &= \left\| S\left(\int |f+g|dm\right)\right\| \leq Q\left\|\int |f+g|dm\right\| \\ &\leq Q\left(\left\|\int |f|dm\right\| + \left\|\int |g|dm\right\|\right) \leq \frac{Q}{K}\left(\left\|S\left(\int |f|dm\right)\right\| + \left\|S\left(\int |g|dm\right)\right\|\right) \\ &\leq \frac{Q}{K}\left(\left\|\int |f|dn\right\| + \left\|\int |g|dn\right\|\right).\end{aligned}$$

Therefore $\| \int |\cdot|dm\|$ is a quasi-norm (and in fact a norm if and only if $Q = K$, that is, if S is a cone-isometry). □

Proposition 2.4 guarantees that every cone-open transformation of a positive measure yield a quasi-norm $\| \int |\cdot|dn\|$ on $L^1(m)$. It is a natural matter to study the relation of these quasi-norms with the canonical norm on $L^1(n)$.

To finish this section we characterize vector measures for which the function $f \mapsto \int |f|dm$ gives an equivalent expression for the norm of $L^1(m)$.

Proposition 2.5. *Let $m : \Sigma \longrightarrow X$ be a positive vector measure, $S : X \longrightarrow Y$ be a cone-open operator and $n := S \circ m$. Then:*

(i) *The quasi-norm $\| \int |\cdot|dn\|$ is equivalent to the norm $\| \int |\cdot|dm\|$ on $L^1(m)$.*
(ii) *$\|\cdot\|_n$ and $\|\cdot\|_m$ are equivalent norms on $L^1(m)$. In particular, $L^1(n) = L^1(m)$.*

Proof. Since S is continuous and cone-open, there exist $K, Q \geq 0$ such that $K\|x\| \leq \|S(x)\| \leq Q\|x\|$, $\forall x \in C_X^+$. Applying these inequalities to $x = \int |f|dm \in C_X^+$, for each $f \in L^1(m)$ it follows that $\| \int |\cdot|dn \|$ and $\| \int |\cdot|dm \|$ are equivalent on $L_1(m)$.

To prove (ii), we will work with the following equivalent norms of L^1 (see [3]):

$$\|\cdot\|_n \sim \sup_{A\in\Sigma}\left\|\int_A \cdot\, dn\right\| \quad \text{and} \quad \|\cdot\|_m \sim \sup_{A\in\Sigma}\left\|\int_A \cdot\, dm\right\|$$

Thus, there exist convenient constants K', K'', Q', Q'' such that

$$\|f\|_n \leq Q' \sup_{A\in\Sigma}\left\|\int_A fdn\right\| \leq Q' \sup_{A\in\Sigma}\left\|S(\int_A fdm)\right\| \leq Q'' \sup_{A\in\Sigma}\left\|\int_A fdm\right\| \leq Q''\|f\|_m$$

and

$$\begin{aligned}\|f\|_m &\le K' \sup_{A\in\Sigma}\left\|\int_A f dm\right\| \;\le\; K'\left(\sup_{A\in\Sigma}\left\|\int_A f^+ dm\right\| + \sup_{A\in\Sigma}\left\|\int_A f^- dm\right\|\right) \;\le \\ &\le K''\left(\sup_{A\in\Sigma}\left\|\int_A f^+ dn\right\| + \sup_{A\in\Sigma}\left\|\int_A f^- dn\right\|\right) \;\le\; 2K'' \sup_{A\in\Sigma}\left\|\int_A f dn\right\| \\ &\le 2K''\|f\|_n\end{aligned}$$

which yield the equivalence between both norms. □

Using these results, we will prove that the cone-open transformations of positive measures are precisely the measures n for which the map $\|\int|\cdot|dn\|$ is a quasi-norm describing the topology of $L^1(n)$, that is

Theorem 2.6. *Let Y be a normed space and $n : \Sigma \longrightarrow Y$ be a vector measure. The following statements are equivalent:*

(i) *The function $\|\int|\cdot|dn\|$ is a quasi-norm equivalent to (resp. a norm coinciding with) the norm $\|\cdot\|_n$ on $L^1(n)$.*

(ii) *There exist a Banach lattice X, a positive vector measure $m : \Sigma \longrightarrow X$ and a cone-open operator (resp. cone-isometry) $S : X \longrightarrow Y$ such that $n = S \circ m$.*

Moreover, in this case, $L^1(n)$ and $L^1(m)$ are isomorphic Banach lattices.

Proof. (i) ⇒ (ii) Setting $X := L^1(n)$, $m(A) := \chi_A, \forall A \in \Sigma$ and $S(f) := \int f\, dn$, $\forall f \in X$ we have a positive vector measure $m : \Sigma \longrightarrow X$ and a cone-open operator (resp. cone-isometry) $S : X \longrightarrow Y$ such that $n = S \circ m$.

(ii) ⇒ (i) By proposition 2.5 and lemma 1.2 we conclude that $\|\cdot\|_n \sim \|\cdot\|_m = \|\int|\cdot|dm\| \sim \|\int|\cdot|dn\|$ on $L^1(m) = L^1(n)$. (The equivalence is an equality if S is an isometry.) □

3. The second representation theorem

In this section we study if it is possible to define an order on the image of the vector measure m in such a way that if the expression $\|\int|\cdot|dm\|$ is a quasi-norm for $L^1(m)$, then m can be considered, in a sense, as a positive vector measure. This happens for instance in Example 2.1, where τ is positive whenever the new order $(\lambda_i)_{i=1}^\infty \le (\eta_i)_{i=1}^\infty$ iff $\lambda_i \le \eta_i$ for $i = 2, 4, \ldots$ and $\lambda_i \ge \eta_i$ for $i = 1, 3, 5, \ldots$ is considered in ℓ_2. Therefore, a natural question arises: is this in general true?, i.e. is it always possible to define a new order on the space such that m is positive with respect to this order?

We will see that cone-open transformations of positive vector measures are closely related to positive vector measures since cone-open operators induce a natural order on its range which is partially compatible with the normed space structure in the following precise sense:

Proposition 3.1. *Let X be a Banach lattice, let $S : X \longrightarrow Y$ be a cone-open operator and let Z be the range $rg(S)$ of S. The relation $z_1 \leq_S z_2 \Leftrightarrow \exists x \in C_X^+ : z_2 - z_1 = S(x)$ defines an order in Z under which Z is an ordered vector space.*

Moreover, there exists $M \geq 1$ such that if $z_1, z_2 \in Z$, $z_1 \geq_S 0$, $z_2 \geq_S 0$ and $z_1 \leq_S z_2$ then $\|z_1\| \leq M\|z_2\|$ (and $M = 1$ if and only if S is an cone-isometry).

Proof. Since $z - z = 0 = S(0), \forall z \in Z$, the relation $\leq_S$ is reflexive. Given $z_1 \leq_S z_2$ and $z_2 \leq_S z_3$ there exist $x_1, x_2 \in C_X^+$ such that $z_2 - z_1 = S(x_1)$ and $z_3 - z_2 = S(x_2)$. Thus, we have $z_3 - z_1 = S(x_1 + x_2)$ with $x_1 + x_2 \in C_X^+$, that is, $z_1 \leq_S z_3$, and consequently the relation $\leq_S$ is transitive. To prove that $\leq_S$ is also anti-symmetric, assume that $z_1 \leq_S z_2$ and $z_2 \leq_S z_1$. Thus, there exist $x_1, x_2 \in C_X^+$ such that $z_2 - z_1 = S(x_1)$ and $z_1 - z_2 = S(x_2)$ which implies that $S(x_1 + x_2) = 0$. Since S is cone-open we conclude that $x_1 = x_2 = 0$ and hence $z_1 = z_2$. The compatibility of this order with the vector space structure of Z follows directly from the definition of the order and the linearity of S.

Finally, given $z_1, z_2 \in Z$, $z_1 \geq_S 0$, $z_2 \geq_S 0$ and $z_1 \leq_S z_2$, there exist $x_1, x_2, x_3 \in C_X^+$ such that $z_1 = S(x_1)$, $z_2 = S(x_2)$ and $z_2 - z_1 = S(x_3)$. In addition, there exist $K, Q \geq 0$ such that $K\|x\| \leq \|S(x)\| \leq Q\|x\|$, $\forall x \in C_X^+$. Therefore

$$\|z_2\| = \|S(x_3 + x_1)\| \geq K\|x_3 + x_1\| \geq K\|x_1\| \geq \frac{K}{Q}\|S(x_1)\| = \frac{K}{Q}\|z_1\|$$

and thus $M := \frac{Q}{K}$ verifies the required property. □

Definition 3.2. Given $M \geq 1$, a normed space X with norm $\|\cdot\|$ is called an M-*normed Riesz space* if there exists an order $\leq$ such that X is a Riesz space and $\|x_1\| \leq M\|x_2\|$ holds for every $x_1, x_2 \in X$ with $x_1 \geq 0$, $x_2 \geq 0$ and $x_1 \leq x_2$.

Definition 3.3. A vector measure $n : \Sigma \longrightarrow Y$ on a normed space Y is called *pseudo-positive* if there exists an order $\leq$ in Y such that Y is an M-normed Riesz space and n is positive for this order.

Definition 3.4. Let X be a normed Riesz space and let Y be a normed space. Let $S : X \longrightarrow Y$ be a cone-open operator and let $\leq_S$ be the order induced by S in $Z := rg(S)$. We will say that S is *lattice generating* if for all $z_1, z_2 \in Z$ there exist $x_1, x_2 \in X$ such that

(i) $z_1 = S(x_1)$, $z_2 = S(x_2)$, and
(ii) $\forall z \in Z : \ z \geq_S z_1, \ z \geq_S z_2, \ \exists h \in X : \ S(h) = z, \ h \geq x_1 \vee x_2.$

Example 3.5. A in a sense canonical example of a lattice generating cone-open transformation is the integration map $f \mapsto I_\mu(f) = \int f d\mu$ with respect to a probability measure μ, $I_\mu : L^1(\mu) \to \mathbb{R}$. To see this is enough to consider, for every element $r \in \mathbb{R}$, the function $r \cdot \chi_\Omega$. Taking these functions as the associated elements $x_1, x_2 \in L^1(\mu)$ to $z_1 = r_1, z_2 = r_2$, $r_1, r_2 \in \mathbb{R}$ in Definition 3.4, it is clear that (ii) in this definition is satisfied for $h := \max\{r_1, r_2\} \cdot \chi_\Omega$.

Other simple example is given by the identity map $S = Id : X \to X$ in any Banach lattice X. In this case, $S(x_1) = x_1$, $S(x_2) = x_2$ and clearly (ii) is satisfied for $h := x_1 \vee x_2$.

Corollary 3.6. *The range of every cone-open and lattice generating operator is an M-normed Riesz space.*

Therefore every cone-open and lattice generating transformation of a positive vector measure is a pseudo-positive vector measure.

Proof. It is sufficient to prove that $z_1 \vee_S z_2$ is well defined by $S(x_1 \vee x_2)$ for every $z_1, z_2 \in rg(S)$ independently of x_1, x_2 satisfying (i) and (ii) of Definition 3.4.

Let us check it. Let x_1, x_2 and x_1', x_2' two couples associated to the elements z_1 and z_2 satisfying (i) and (ii) above. Then $S(x_1 \vee x_2) = S(x_1' \vee x_2')$.

To see this, let us define $z = S(x_1 \vee x_2)$ and $z' = S(x_1' \vee x_2')$, and let us prove that $z = z'$. Since clearly $S(x_1' \vee x_2') \geq_S z_1 = S(x_1') = S(x_1)$ and $S(x_1' \vee x_2') \geq_S z_2 = S(x_2') = S(x_2)$, we have that there is an element h' such that $S(h') = z'$ and $h' \geq x_1 \vee x_2$. Therefore, the positive element $r := h' - x_1 \vee x_2$ satisfies that $S(h') - S(x_1 \vee x_2) = S(r)$, and then $z' \geq_S z$.

The same argument can be given for proving that $z \geq_S z'$, that implies that $z = z'$ as a consequence of the fact that S is cone-open and then $\leq_S$ is an order relation by Proposition 3.1. □

In the opposite direction we prove in Theorem 3.7 that pseudo-positive measures are precisely this particular type of cone-open transformations of positive vector measures, that is:

Theorem 3.7. *If $n : \Sigma \longrightarrow Y$ is a pseudo-positive vector measure then n can be written as a cone-open and lattice generating transformation of a positive vector measure (into a normed Riesz space).*

This result can be deduced from the following facts:

(i) If Y is an M-normed Riesz space with norm $\|\cdot\|$ then

$$|||y||| := ||(|y|)||, \ \forall y \in Y$$

defines a quasi-norm on Y such that $||y|| \leq 2M|||y|||$ for all $y \in Y$.

(ii) If Y is an M-normed Riesz space with norm $\|\cdot\|$ then

$$|||y|||_0 := \inf\{|||z||| : |y| \leq |z|\}, \ \forall y \in Y$$

is a lattice norm on Y which is equivalent to $|||\cdot|||$. In fact,

$$|||y|||_0 \leq |||y||| \leq M|||y|||_0 \text{ for all } y \in Y$$

(iii) If Y is an M-normed Riesz space with norm $\|\cdot\|$ then the identity map $Id : (Y, |||\cdot|||_0) \longrightarrow (Y, \|\cdot\|)$ is a (continuous) cone-open operator.

(iv) If Y is an M-normed Riesz space, X is a normed space, $T : Y \longrightarrow X$ is a linear and cone-continuous operator and $\sum_n y_n$ is a convergent series of positive terms $y_n \in Y$, then $\sum_n T(y_n)$ is also a convergent series in X and

$$T\left(\sum_{n=1}^{\infty} y_n\right) = \sum_{n=1}^{\infty} T(y_n)$$

(v) If $n : \Sigma \longrightarrow Y$ is a pseudo-positive vector measure and $T : Y \longrightarrow X$ is cone-continuous then $\tilde{n} := T \circ n$ is a vector measure into X.

Proof. (i) It is clear that $|||y||| = 0$ if and only if $y = 0$ and that $|||\alpha y||| = |\alpha|\ |||y|||$ for all $\alpha \in \mathbb{R}$ and for all $y \in Y$. Moreover, given $x, y \in Y$ we have

$$|||x+y||| = \|(|x+y|)\| \leq M(\|(|x|)\| + \|(|y|)\|) = M(|||x||| + |||y|||)$$

and since $y^+ \leq |y|,\ y^- \leq |y|$ it follows that $\|y^+\| \leq M|||y|||,\ \|y^-\| \leq M|||y|||$ and hence

$$\|y\| = \|y^+ - y^-\| \leq \|y^+\| + \|y^-\| \leq 2M|||y|||.$$

(ii) Let us check that $|||\cdot|||_0$ is a norm. Only the triangle inequality is not obvious. Now take $\epsilon > 0$ and consider two elements $x, y \in Y$. Then there are $z_x, z_y \in Y$ such that $|x| \leq |z_x|$ and $|y| \leq |z_y|$, and $|||z_x||| \leq |||x|||_0 + \frac{\epsilon}{2}$ and $|||z_y||| \leq |||y|||_0 + \frac{\epsilon}{2}$. Then $|x+y| \leq |z_x| + |z_y|$, and so

$$\begin{aligned} |||x+y|||_0 &\leq |||\,|z_x| + |z_y|\,||| = \|\,|z_x| + |z_y|\,\| \\ &\leq |||z_x||| + |||z_y||| \\ &\leq |||x|||_0 + |||y|||_0 + \epsilon. \end{aligned}$$

Moreover, if $|x| \leq |y|$ then for any $z \in Y$ such that $|y| \leq |z|$ in particular we have $|x| \leq |z|$ and hence $|||x|||_0 \leq |||z|||$. Thus $|||x|||_0 \leq \inf\{|||z||| : |y| \leq |z|\} = |||y|||_0$, which proves that $|||\cdot|||_0$ is in fact a lattice norm.

The equivalence between $|||\cdot|||$ and $|||\cdot|||_0$ is easy to check. On the one hand it is evident that $|||y|||_0 \leq |||y|||$ for all $y \in Y$, and on the other hand given any $z \in Y$ with $|y| \leq |z|$ it follows that $|||y||| \leq M|||z|||$ and hence $|||y||| \leq M|||y|||_0$ for all $y \in Y$.

(iii) $Id : (Y, |||\cdot|||_0) \longrightarrow (Y, \|\cdot\|)$ is a cone-open operator since for all $y \in Y$,

$$\|Id(y)\| = \|y\| \leq 2M|||y||| \leq 2M^2|||y|||_0$$

and for all $y \in Y$ with $y \geq 0$,

$$\|Id(y)\| = \|y\| = |||y||| \geq |||y|||_0.$$

(iv) Since T is cone-continuous, there exists $Q \geq 0$ such that $\|T(y)\| \leq Q\|y\|$, for all $y \geq 0$. Thus, given $n \in \mathbb{N}$ we have

$$\left\| T\left(\sum_{k=1}^{\infty} y_k\right) - \sum_{k=1}^{n} T(y_k) \right\| = \left\| \sum_{k=n+1}^{\infty} y_k \right\|$$

which converges to 0 when n goes to infinity.

(v) $\tilde{n}$ is countably additive since for all pairwise disjoint $A_i \in \Sigma$, $n(A_i) \geq 0$,

$$\tilde{n}\left(\bigcup_{i=1}^{\infty} A_i\right) = T \circ n\left(\bigcup_{i=1}^{\infty} A_i\right) = T\left(\sum_{i=1}^{\infty} n(A_i)\right) = \sum_{i=1}^{\infty} T \circ n(A_i) = \sum_{i=1}^{\infty} \tilde{n}(A_i).$$

$\square$

Proof (Proof of Theorem 3.7). We define S to be $Id : (Y, |||\cdot|||_0) \longrightarrow (Y, \|\cdot\|)$ which is a cone-open operator by (iii) and hence $T := Id^{-1}$ is cone-continuous. Then, from (iv) we deduce that $\tilde{n} := T \circ n$ is a measure. Moreover it is clear that $\tilde{n}$ is positive and, since S generates the same order on Y, we conclude that $S \circ \tilde{n}$ is a cone-open and lattice generating transformation of a positive vector measure and $n = S \circ \tilde{n}$. □

We can summarize the main results of this paper in the following

Corollary 3.8. *Let Y be a normed space and $n : \Sigma \longrightarrow Y$ be a vector measure. The following sentences are equivalent:*

(i) *The map $\|\int |\cdot| dn\|$ is a quasi-norm equivalent to the norm $\|\cdot\|_n$ on $L^1(n)$.*
(ii) *n is a cone-open transformation of a positive vector measure.*

Moreover, the following stronger statements are also equivalent:

(iii) *n is a cone-open and lattice generating transformation of a positive vector measure*
(iv) *n is a pseudo-positive vector measure.*

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