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Gevrey and formal Nilsson solutions of A-hypergeometric systems

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We prove that the space of Gevrey solutions of an A–hypergeometric system along a coordinate subspace is contained in a space of formal Nilsson solutions. Moreover, under some additional condition, both spaces are equal. In the process we prove some other results about formal Nilsson solutions.

0. Introduction

We study formal solutions of A–hypergeometric systems, also known as GKZ–systems, as they were introduced by Gel'fand, Graev, Kapranov and Zelevinsky (see [7] and [8]). They are systems of linear partial differential equations associated with a pair (A, β) where A is a full rank $d \times n$ matrix $A = (a_{ij}) = (a_1 \cdots a_n)$ with $a_j \in \mathbb{Z}^d$ for all $j = 1, \dots, n$ and $\beta \in \mathbb{C}^d$ is a vector of complex parameters. Recall that the toric ideal of A is defined as

$$I_A := \langle \partial^{u_+} - \partial^{u_-} \mid u \in \mathbb{Z}^n,\ Au = 0 \rangle \subseteq \mathbb{C}[\partial_1, \dots, \partial_n]$$

where $(u_+)_j = \max\{u_j, 0\}$ and $(u_-)_j = \max\{-u_j, 0\}$ for $j = 1, \dots, n$. The A–hypergeometric system $H_A(\beta)$ is the left ideal of the Weyl algebra $D = \mathbb{C}[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$ generated by I_A and by the Euler operators $E_i - \beta_i := \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i$ for $i = 1, \dots, d$. The A–hypergeometric D-module is nothing but

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the quotient $M_A(\beta) := D/H_A(\beta)$. It is well known that $M_A(\beta)$ is a holonomic D-module, see [1] and [8]. In [1] and [8], it was also proved that the holonomic rank of $M_A(\beta)$, i.e. the dimension of the space of its holomorphic solutions at a nonsingular point, equals the *normalized volume* of A, denoted by $\text{vol}(A)$ (see (1.1)), when β is generic. Moreover, $M_A(\beta)$ is regular holonomic if and only if I_A is homogeneous (equivalently, $(1, \dots, 1)$ lies in the rowspan of the matrix A), see [10, Ch. II, 6.2, Thm.], [15, Thm. 2.4.11] and [19, Corollary 3.16].

Gevrey series solutions of a holonomic D–module M along a variety Y are closely related with the irregularity sheaf of M along Y defined by Mebkhout [13] and with the so called slopes of M along Y, see [11]. The slopes of $M_A(\beta)$ along a coordinate subspace Y were computed in [19]. The spaces of Gevrey series solutions of $M_A(\beta)$ along Y were described in [4] (see also [6], [5]) for generic enough parameters $\beta \in \mathbb{C}^d$.

On the other hand, there is an algorithm that computes, for any regular holonomic left D–ideal I and a generic vector $w \in \mathbb{R}^n$, a set of *canonical series solutions* of I that belong to certain *Nilsson ring*. These series converge in a certain open set that depends on w and form a basis of holomorphic solutions of I, see [15, chapters 2.5 and 2.6]. In [3] the authors introduced a notion of formal Nilsson solutions of $H_A(\beta)$ in the direction of w, denoted by $\mathcal{N}_w(H_A(\beta))$, and they used it to generalize various results in [15] to the case when $H_A(\beta)$ is not necessarily regular.

In the papers [4] and [3], some of the results assume β to be (very) generic, meaning that it lies outside a certain infinite (but locally finite) collection of affine hyperplanes. In particular, this condition is stronger than β being not *rank jumping*, a condition that only requires to avoid a concrete finite affine subspace arrangement of codimension at least two [12]. The set of rank jumping parameters is $\varepsilon(A) := \{\beta \in \mathbb{C}^d | \text{ rank}(M_A(\beta)) > \text{vol}(A)\}$ and it was computed in [12] in terms of the local cohomology modules of the toric ring $S_A = \mathbb{C}[\partial]/I_A$. In particular they proved that $\varepsilon(A) = \varnothing$ if and only if S_A is Cohen–Macaulay.

In this note we prove, for all $\beta \in \mathbb{C}^d$, that the space of Gevrey series solutions of $M_A(\beta)$ along a coordinate subspace is contained in the space of formal Nilsson solutions of $H_A(\beta)$ in a certain direction, see Theorem 3.3. We also prove that under one additional condition both spaces coincide and that for $\beta \notin \varepsilon(A)$ the dimension of this space is the normalized volume of certain submatrix of A, see Theorem 3.4. Moreover, in Section 2, we provide some additional results about formal Nilsson solutions of $H_A(\beta)$.

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1. Preliminaries

1.1. Notations

Let $A = (a_1 \cdots a_n)$ be a $d \times n$ matrix with columns $a_j \in \mathbb{Z}^d$ such that $\mathbb{Z}A := \sum_{j=1}^n \mathbb{Z}a_j = \mathbb{Z}^d$.

For a given subset $\tau \subseteq \{1, \dots, n\}$, set $\overline{\tau} := \{1, \dots, n\} \setminus \tau$. We shall identify τ with the set of columns of A indexed by τ and write A_τ for the submatrix of A with column set τ. We denote by Δ_τ the convex hull in $\mathbb{R}^d$ of all the columns of A_τ and the origin. We also denote $\text{pos}(\tau) := \sum_{j\in\tau} \mathbb{R}_{\geq 0} a_j$.

We say that a subset $\sigma \subseteq \{1, \dots, n\}$ is a *maximal simplex* if A_σ is an invertible matrix. We associate to a maximal simplex σ an $n \times (n - d)$ matrix B_σ, where its columns are indexed by $\overline{\sigma}$, the j–th column of B_σ has σ–coordinates equal to $-A_\sigma^{-1}a_j$, j–coordinate equal to one and the rest of coordinates equal to zero. In particular, the columns of B_σ form a basis of the kernel of A.

For example, if $\sigma = \{1, \dots, d\}$ then

$$B_\sigma = \begin{pmatrix} -A_\sigma^{-1} a_{d+1} & -A_\sigma^{-1} a_{d+2} & \cdots & -A_\sigma^{-1} a_n \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 1 \end{pmatrix}.$$

Recall that for any subset $\tau \subseteq \{1, \dots, n\}$, the *normalized volume* of A_τ (with respect to the lattice $\mathbb{Z}^d$) is given by:

$$\mathrm{vol}(A_\tau) := d!\mathrm{vol}_{\mathbb{R}^d}(\Delta_\tau) \tag{1.1}$$

where $\mathrm{vol}_{\mathbb{R}^d}(\Delta_\tau)$ denotes the Euclidean volume of $\Delta_\tau \subseteq \mathbb{R}^d$. We also denote $\mathrm{vol}(\tau) := \mathrm{vol}(A_\tau)$. If $\sigma \subseteq \{1, \dots, n\}$ is a maximal simplex, then $\mathrm{vol}(\sigma) = [\mathbb{Z}^d : \mathbb{Z}A_\sigma] = |\det(A_\sigma)|$.

1.2. Regular triangulations

A vector $w \in \mathbb{R}^n$ defines an abstract polyhedral complex T_w with vertex set contained in $\{1, \dots, n\}$ as follows: $\tau \in T_w$ iff there exists a vector $\mathbf{c} \in \mathbb{R}^d$ such that

$$\langle \mathbf{c}, a_j \rangle = w_j \text{ for all } j \in \tau, \tag{1.2}$$

$$\langle \mathbf{c}, a_j \rangle < w_j \text{ for all } j \notin \tau. \tag{1.3}$$

Such a polyhedral complex is called a *regular subdivision* of A if it satisfies $\mathrm{pos}(A) = \cup_{\tau \in T_w} \mathrm{pos}(\tau)$. This happens for example if $w \in \mathbb{R}^n_{>0}$ or if A is *pointed*, i.e. the intersection of $\mathbb{R}^n_{>0}$ with the rowspan of A is nonempty.

An element $\tau \in T_w$ is called a *facet* of T_w if the rank of A_τ is d. Any regular subdivision is determined by its facets and from now on we will write only $\tau \in T_w$ when τ is a facet of T_w. We say that a regular subdivision T_w of A is a *regular triangulation* of A if all its facets are simplices.

An important case of regular subdivision of A is the following. If $w_j = 1$ for all $j = 1, \dots, n$, a facet of T_w is the same as a facet of Δ_A not containing the origin. This particular regular subdivision of A is denoted by Γ_A.

Notice that for a maximal simplex σ, it is straightforward from (1.2) and (1.3) that

$$\sigma \in T_w \Longleftrightarrow wB_\sigma > 0. \tag{1.4}$$

If T is any regular triangulation of A then the set

$$C(T) := \{w \in \mathbb{R}^n | \, T = T_w\} = \{w \in \mathbb{R}^n | \, wB_\sigma > 0, \, \forall \sigma \in T\} \tag{1.5}$$

is an open nonempty convex rational polyhedral cone. The closures of these cones and their faces form the so called *secondary fan* of A, introduced and studied by Gel'fand, Kapranov and Zelevinsky [9, Chapter 7]. When A is pointed, it is easy to see that the secondary fan is a complete fan, i.e. its support is $\mathbb{R}^n$. In general, it is not necessarily complete but its support contains the orthant $\mathbb{R}^n_{\geq 0}$.

1.3. The A–hypergeometric fan

A vector $w \in \mathbb{R}^n$ defines a partial order on the monomials of the Weyl Algebra D (and also on the monomials in $\mathbb{C}[\partial_1, \ldots, \partial_n]$) by defining the $(-w,w)$–*weight* of $x^\alpha \partial^\gamma \in D$ as the real value $\langle w, \gamma - \alpha \rangle$.

The *initial form* of an element $P = \sum_{\alpha,\gamma \in \mathbb{N}^n} c_{\alpha,\gamma} x^\alpha \partial^\gamma \in D$ with respect to $(-w,w)$, denoted by $\mathrm{in}_{(-w,w)}(P)$, is the sum of the terms $c_{\alpha,\gamma} x^\alpha \partial^\gamma$, with $c_{\alpha,\gamma} \neq 0$, whose $(-w,w)$–weight is maximum. If I is a left D–ideal, its *initial ideal* with respect to $(-w,w)$ is defined as

$$\mathrm{in}_{(-w,w)}(I) := \langle \mathrm{in}_{(-w,w)}(P) |\ P \in I,\ P \neq 0 \rangle.$$

Remark 1.1. When A is pointed, there is a vector $w' \in \mathbb{R}^n_{>0}$ in the rowspan of A and we have that $\mathrm{in}_{w'}(I_A) = I_A$ and $\mathrm{in}_{(-w',w')}(H_A(\beta)) = H_A(\beta)$. Then, for any $w \in \mathbb{R}^n$ we have that $w'' := w' + \epsilon w \in \mathbb{R}^n_{>0}$, $\mathrm{in}_w(I_A) = \mathrm{in}_w(\mathrm{in}_{w'}(I_A)) = \mathrm{in}_{w''}(I_A)$ and $\mathrm{in}_{(-w,w)}(H_A(\beta)) = \mathrm{in}_{(-w,w)}(\mathrm{in}_{(-w',w')}(H_A(\beta))) = \mathrm{in}_{(-w'',w'')}(H_A(\beta))$ for $\epsilon > 0$ small enough, see [15, Lemma 2.1.6].

The Gröbner fan of I_A, see [18, p. 13] (resp. the small Gröbner fan of $H_A(\beta)$, see [15, p. 60]) is a rational polyhedral fan in $\mathbb{R}^n$ whose cones $\mathcal{C}$ satisfy that $\mathrm{in}_w(I_A) = \mathrm{in}_{w'}(I_A)$ (resp. $\mathrm{in}_{(-w,w)}(H_A(\beta)) = \mathrm{in}_{(-w',w')}(H_A(\beta))$) for all $w, w' \in \mathring{\mathcal{C}}$, where $\mathring{\mathcal{C}}$ denotes the relative interior of $\mathcal{C}$. By Remark 1.1, these two fans are also complete fans when A is pointed.

Definition 1.2. The *A–hypergeometric fan* (at β) is the coarsest rational polyhedral fan in $\mathbb{R}^n$ that refines both the Gröbner fan of I_A and the small Gröbner fan of $H_A(\beta)$.

Remark 1.3. We notice that this fan is a refinement of the hypergeometric fan defined in [15, Section 3.3] when I_A is homogeneous and β is generic. By [18, Proposition 8.15] and [2, Corollary 4.4] the A–hypergeometric fan is a refinement of the secondary fan of A.

1.4. Γ–series

Let us denote $\ker_{\mathbb{Z}}(A) := \{u \in \mathbb{Z}^n |\ Au = 0\}$. Following [8], for any vector $v \in \mathbb{C}^n$ such that $Av = \beta$, we consider the Γ–series

$$\varphi_v := \sum_{u \in \ker_{\mathbb{Z}}(A)} \frac{x^{v+u}}{\Gamma(v+u+1)},$$

where $\Gamma(v+u+1) = \prod_{j=1}^n \Gamma(v_j + u_j + 1)$ and Γ is the Euler Gamma function. These series are formally annihilated by $H_A(\beta)$. Moreover, when I_A is homogeneous and $\beta \in \mathbb{C}^d$ is generic, a basis of convergent Γ–series solutions of $M_A(\beta)$ can be constructed by using any regular triangulation of A, see [8]. These Γ–series are handled in [15, Section 3.4] in the following way:

$$\phi_v := \sum_{u \in N_v} \frac{[v]_{u_-}}{[v+u]_{u_+}} x^{v+u} \tag{1.6}$$

where $N_v = \{u \in \ker_{\mathbb{Z}}(A) |\ \forall j = 1, \ldots, n,\ v_j + u_j \in \mathbb{Z}_{<0} \text{ iff } v_j \in \mathbb{Z}_{<0}\}$ and

$$[v]_u = \prod_{j=1}^n v_j(v_j - 1) \cdots (v_j - u_j + 1).$$

Set $\mathrm{nsupp}(v) := \{j \in \{1, \ldots, n\} |\ v_j \in \mathbb{Z}_{<0}\}$ for any $v \in \mathbb{C}^n$.

The series ϕ_v is annihilated by $H_A(\beta)$ if and only if v has *minimal negative support*, i.e. there is no $u \in \ker_{\mathbb{Z}}(A)$ such that $\operatorname{nsupp}(v+u) \subsetneq \operatorname{nsupp}(v)$, see [15, Proposition 3.4.13] whose proof works as well when I_A is not homogeneous.

Remark 1.4. It is easy to check that $\Gamma(v+1)\varphi_v = \phi_v$ when $v \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$. Notice that $\varphi_v = \varphi_{v+u}$ for any $u \in \ker_{\mathbb{Z}}(A)$. Thus, for $u \in N_v$ there is a nonzero scalar $c \in \mathbb{C}$ such that $\phi_v = c \cdot \phi_{v+u}$.

1.5. Gevrey series solutions of A-hypergeometric D-modules

In this section we introduce the notion of Gevrey series and we recall some notations and results from [4]. Let us denote, for a subset $\tau \subseteq \{1,\dots,n\}$, $Y_\tau := \{x_j = 0 | \; j \in \overline{\tau}\}$ and x_τ for the set of variables x_j with $j \in \tau$. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on $X = \mathbb{C}^n$ and by $\mathcal{O}_{\widehat{X|Y_\tau}}$ the sheaf of formal series along Y_τ. A germ of $\mathcal{O}_{\widehat{X|Y_\tau}}$ at $p \in Y_\tau$ can be written as

$$f = \sum_{\alpha \in \mathbb{N}^{\overline{\tau}}} f_\alpha(x_\tau) x_{\overline{\tau}}^{\alpha} \in \mathcal{O}_{\widehat{X|Y_\tau},p} \subseteq \mathbb{C}\{x_\tau - p_\tau\}[[x_{\overline{\tau}}]]$$

where $f_\alpha(x_\tau) \in \mathcal{O}_{Y_\tau}(U)$ for certain nonempty relatively open subset $U \subseteq Y_\tau$, $p \in U$. A formal series $f = \sum_{\alpha \in \mathbb{N}^{\overline{\tau}}} f_\alpha(x_\tau) x_{\overline{\tau}}^{\alpha} \in \mathcal{O}_{\widehat{X|Y_\tau},p}$ is said to be *Gevrey* of order $s \in \mathbb{R}$ along Y_τ at $p \in Y_\tau$ if the series

$$\sum_{\alpha \in \mathbb{N}^{\overline{\tau}}} \frac{f_\alpha(x_\tau)}{(\prod_{j \in \overline{\tau}} \alpha_j!)^{s-1}} x_{\overline{\tau}}^{\alpha}$$

is convergent at p.

Since $M_A(\beta)$ is a holonomic D–module, any of its formal solutions along Y_τ is Gevrey of some order. We denote by $Hom_D(M_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau},p})$ the space of all Gevrey solutions of $M_A(\beta)$ along Y_τ at $p \in Y_\tau$.

Given a maximal simplex σ and a vector $\mathbf{k} = (k_i)_{i \notin \sigma} \in \mathbb{N}^{\overline{\sigma}}$ we denote by $v_\sigma^{\mathbf{k}} \in \mathbb{C}^n$ the vector with σ–coordinates equal to $A_\sigma^{-1}(\beta - A_{\overline{\sigma}}\mathbf{k})$ and $\overline{\sigma}$–coordinates equal to $\mathbf{k}$. Let $\Omega_\sigma \subseteq \mathbb{N}^{\overline{\sigma}}$ be a set of representatives for the different classes with respect to the following equivalence relation in $\mathbb{N}^{\overline{\sigma}}$: we say that $\mathbf{k} \sim \mathbf{k}'$ if and only if $A_{\overline{\sigma}}\mathbf{k} - A_{\overline{\sigma}}\mathbf{k}' \in \mathbb{Z}A_\sigma$. Thus, Ω_σ is a set of cardinality $\operatorname{vol}(\sigma) = [\mathbb{Z}^d : \mathbb{Z}A_\sigma]$.

When β is generic, the space of Gevrey solutions of $M_A(\beta)$ along Y_τ is explicitly described in [4].

Theorem 1.5. [4, Theorem 6.7 and Remark 6.8] *If $T(\tau)$ is a regular triangulation of A_τ that refines Γ_{A_τ} and $\beta \in \mathbb{C}^d$ is generic enough, the set $\{\phi_{v_\sigma^{\mathbf{k}}} : \; \sigma \in T(\tau), \; \mathbf{k} \in \Omega_\sigma\}$ is a basis of the space of Gevrey series solutions of $M_A(\beta)$ along Y_τ at any point p of a certain nonempty relatively open set $\mathcal{W}_{T(\tau)} \subseteq Y_\tau$. In particular,* $\dim_{\mathbb{C}}(Hom_D(M_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau},p})) = \operatorname{vol}(\tau)$.

For more precise statements, including the Gevrey order of these series and its relation with the so called *slopes* of $M_A(\beta)$ along Y_τ, see [4]. The slopes of $M_A(\beta)$ along Y_τ were described in [19].

Notice that if $\tau = A$ then $Y_\tau = \mathbb{C}^n$, $\mathcal{O}_{\widehat{X|Y_\tau}} = \mathcal{O}_X$, and Theorem 1.5 gives a basis of holomorphic functions of $M_A(\beta)$ at any point of $\mathcal{W}_{T(\tau)}$ when $\beta \in \mathbb{C}^d$ is generic. Such a basis was first described in [14] (see also [8] when I_A is homogeneous).

The following result is the first part of [4, Theorem 6.2].

Theorem 1.6. *If p is a generic point of Y_τ then, for all $\beta \in \mathbb{C}^d$,*

$$\dim_{\mathbb{C}}(Hom_D(M_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau},p})) \geq \operatorname{vol}(\tau).$$

1.6. Formal Nilsson solutions of A-hypergeometric D-modules

We recall here some definitions and results from [3], see also [15] when I_A is homogeneous. In the former paper the authors write the following results in terms of a regular triangulation of the matrix $\rho(A) := (\widetilde{a}_0\ \widetilde{a}_1 \cdots \widetilde{a}_n)$, that is constructed from A by adding a first column of zeroes and then a first row of ones. It follows from the definition of regular subdivision (see (1.2) and (1.3)) that for a subset $\sigma \subseteq \{1,\ldots,n\}$, we have that $\sigma \in T_w$ if and only if $\{0\} \cup \sigma \in T_{(0,w)}$.

Given a cone $\mathcal{C} \subseteq \mathbb{R}^n$, the *dual cone* of $\mathcal{C}$, denoted by $\mathcal{C}^*$, is a closed cone consisting of vectors $u \in \mathbb{R}^n$ such that $\langle w, u\rangle \geq 0$ for all $w \in \mathcal{C}$ and all $u \in \mathcal{C}^*$. If $\mathcal{C}$ is full dimensional, then the cone $\mathcal{C}^*$ is *strongly convex* (i.e. it doesn't contain non trivial linear subspaces) and $\langle w, u\rangle > 0$ for all $w \in \mathring{\mathcal{C}}$ and all nonzero $u \in \mathcal{C}^*$

We say that $w \in \mathbb{R}^n_{>0}$ is a *weight vector* (for $H_A(\beta)$) if it belongs to the interior of a full dimensional cone of the A–hypergeometric fan.

For a weight vector $w \in \mathbb{R}^n$ we denote by $\mathcal{C}_w$ the interior of the (full dimensional) cone in the A–hypergeometric fan such that $w \in \mathcal{C}_w$. Notice that $\mathcal{C}_w \subseteq C(T_w)$.

Definition 1.7. [3, Definition 2.6] Let w be a weight vector for $H_A(\beta)$. Write $\log(x) = (\log x_1, \ldots, \log x_n)$. A *basic Nilsson solution of $H_A(\beta)$ in the direction of* w is a series of the form

$$\phi = x^v \sum_{u \in C} x^u p_u(\log(x)), \tag{1.7}$$

where $v \in \mathbb{C}^n$, that satisfies

i) ϕ is annihilated by the partial differential operators of $H_A(\beta)$;
ii) C is contained in $\ker_{\mathbb{Z}}(A) \cap \mathcal{C}^*$ for some strongly convex open cone $\mathcal{C} \subseteq \mathcal{C}_w$ such that $w \in \mathcal{C}$;
iii) the p_u are nonzero polynomials and there exists $K \in \mathbb{Z}$ such that $\deg(p_u) \leq K$ for all $u \in C$;
iv) $0 \in C$.

The set $\operatorname{supp}(\phi) = \{v + u |\ u \in C\}$ is called the *support* of ϕ.

The $\mathbb{C}$–linear span of all basic Nilsson solutions of $H_A(\beta)$ in the direction of w is called the *space of formal Nilsson solutions of $H_A(\beta)$ in the direction of* w and it is denoted by $\mathcal{N}_w(H_A(\beta))$.

We define the w–weight of a term $x^v p(\log(x))$, where $v \in \mathbb{C}^n$ and p is a polynomial, as the real value $\operatorname{Re}(\langle w, v\rangle)$. For a series ϕ consisting in a (possibly infinite) sum of terms, we say that it has an initial form if there exists the minimum for the set of w–weights of all its nonzero terms. In this case, its initial form in the direction of w, denoted by $\operatorname{in}_w(\phi)$, consists in the sum of all the terms of ϕ with minimum w–weight.

Remark 1.8. Notice that if a series ϕ as in (1.7) satisfies all conditions in Definition 1.7 we have $\operatorname{in}_w(\phi) = x^v p_0(\log(x))$.

Proposition 1.9. *[3, Proposition 2.11] Let $w \in \mathbb{R}^n$ be a weight vector for $H_A(\beta)$, then* $\dim_{\mathbb{C}}(\mathcal{N}_w(H_A(\beta))) \leq \operatorname{rank}(\operatorname{in}_{(-w,w)}(H_A(\beta)))$.

We recall that a vector $v \in \mathbb{C}^n$ is called an *exponent* of $H_A(\beta)$ with respect to w if x^v is a solution of $\operatorname{in}_{(-w,w)}(H_A(\beta))$.

Theorem 1.10. [3, Theorem 4.8] *If β is generic and w is a weight vector for $H_A(\beta)$ then the set*

$$\{\phi_v |\ v \text{ is an exponent of } H_A(\beta) \text{ with respect to } w\}$$

is a basis of $\mathcal{N}_w(H_A(\beta))$*, where* ϕ_v *is defined in* (1.6)*.*

Theorem 1.11. *[3, Corollaries 4.9 and 4.11] If* β *is generic and* w *is a weight vector for* $H_A(\beta)$ *then*

$$\dim_{\mathbb{C}}(\mathcal{N}_w(H_A(\beta)) = \operatorname{rank}(\operatorname{in}_{(-w,w)}(H_A(\beta))) = \deg(\operatorname{in}_w(I_A)) = \sum_{\sigma\in T_w}\operatorname{vol}(\sigma). \tag{1.8}$$

Let $w \in \mathbb{R}^n_{>0}$ be a weight vector for $H_A(\beta)$. We say that w is a *perturbation* of a vector $w_0 \in \mathbb{R}^n$ if there exists a full dimensional cone $\mathcal{C}$ of the A–hypergeometric fan such that $w_0 \in \mathcal{C}$ and $w \in \mathring{\mathcal{C}}$.

We notice that a weight vector w is a perturbation of $(1,\dots,1)$ if and only if the regular triangulation T_w is a refinement of Γ_A.

Theorem 1.12. *[3, Theorem 6.4] Assume that* A *is pointed. If* w *is a perturbation of* $(1,\dots,1)$ *then, for all* $\beta\in\mathbb{C}^d$*,*

$$\dim_{\mathbb{C}}(\mathcal{N}_w(H_A(\beta))) = \operatorname{rank}(\operatorname{in}_{(-w,w)}(H_A(\beta)) = \operatorname{rank}(M_A(\beta)). \tag{1.9}$$

More precisely, $\mathcal{N}_w(H_A(\beta))$ *is the space of convergent series solutions of* $M_A(\beta)$ *at any point in a certain nonempty open set* $\mathcal{U}_w \subseteq \mathbb{C}^n$*.*

2. Some remarks on formal Nilsson solutions of $H_A(\beta)$

In this section we provide some additional results about $\mathcal{N}_w(H_A(\beta))$.

Lemma 2.1. *For any* $\beta\in\mathbb{C}^d$ *and any weight vector* w*,*

$$\dim_{\mathbb{C}}(\mathcal{N}_w(H_A(\beta)) \geq \sum_{\sigma\in T_w}\operatorname{vol}(\sigma). \tag{2.1}$$

Proof. By Theorem 1.11 equality holds in (2.1) when β is generic and, by Theorem 1.10, there is a basis of $\mathcal{N}_w(H_A(\beta))$ that consists of the set of series ϕ_v, see (1.6), for v varying in the set of exponents of $H_A(\beta)$ with respect to w. In this situation, when β is not generic, we can apply the same procedure as in the proof of [15, Theorem 3.5.1] and obtain a set of linearly independent formal Nilsson solutions of $H_A(\beta)$ in the direction of w. The cardinality of this set is the rightmost quantity in (2.1). □

Corollary 2.2. *If* w *is a weight vector, then* (1.8) *holds for any* $\beta\in\mathbb{C}^d\setminus\varepsilon(A)$*.*

Proof. By [19, Theorem 4.28] and [2, Lemma 3.1], $\operatorname{rank}(\operatorname{in}_{(-w,w)}(H_A(\beta)))$ is constant for $\beta\in\mathbb{C}^d\setminus\varepsilon(A)$ and hence the second equality in (1.8) holds in this case too. The first equality in (1.8) now follows from Lemma 2.1 and Proposition 1.9. □

The following result states that the basis given in Theorem 1.10 only depends, up to multiplication of their elements by nonzero scalars, on the regular triangulation T_w and not on the cone $\mathcal{C}_w \subseteq C(T_w)$.

Proposition 2.3. *If* $\beta\in\mathbb{C}^d$ *is generic and* w *is a weight vector, then the set*

$$\mathcal{B}_w(\beta) := \{\phi_{v_\sigma^{\mathbf{k}}} :\ \sigma\in T_w,\ \mathbf{k}\in\Omega_\sigma\}$$

is a basis of $\mathcal{N}_w(H_A(\beta))$*.*

Proof. By the assumption on β, the difference between two vectors in the set $\{v_\sigma^{\mathbf{k}} | \ \sigma \in T_w, \ \mathbf{k} \in \Omega_\sigma\}$ is not an integer vector. Thus, the series in $\mathcal{B}_w(\beta)$ have pairwise disjoint supports, hence they are linearly independent. Moreover, $\operatorname{nsupp}(v_\sigma^{\mathbf{k}}) = \varnothing$, which implies that $\phi_{v_\sigma^{\mathbf{k}}}$ is annihilated by $H_A(\beta)$, see [15, Proposition 3.4.13].

On the other hand, the support of $\phi_{v_\sigma^{\mathbf{k}}}$ is the set

$$\operatorname{supp}(\phi_{v_\sigma^{\mathbf{k}}}) = \{v_\sigma^{\mathbf{k}} + (B_\sigma \mathbf{m})^t | \ \mathbf{m} \in \mathbb{Z}^{\overline{\sigma}}, \ B_\sigma \mathbf{m} \in \mathbb{Z}^n \text{ and } \mathbf{k} + \mathbf{m} \in \mathbb{N}^{\overline{\sigma}}\}.$$

Notice that $v_\sigma^{\mathbf{k}} + (B_\sigma \mathbf{m})^t = v_\sigma^{\mathbf{0}} + (B_\sigma(\mathbf{k}+\mathbf{m}))^t$ where $\mathbf{k}+\mathbf{m} \in \mathbb{N}^n$. Thus, since $wB_\sigma > 0$ for any $\sigma \in T_w$, we have that $\operatorname{Re}(\langle w, v' \rangle) \geq \operatorname{Re}(\langle w, v_\sigma^{\mathbf{0}} \rangle)$ for all $v' \in \operatorname{supp}(\phi_{v_\sigma^{\mathbf{k}}})$. It follows that there exists the initial form $\operatorname{in}_w(\phi_{v_\sigma^{\mathbf{k}}})$ and that it consists in a finite sum of terms. By the proof of [15, Theorem 2.5.5] we also have that $\operatorname{in}_w(\phi_{v_\sigma^{\mathbf{k}}})$ is a solution of $\operatorname{in}_{(-w,w)}(H_A(\beta))$, but a basis of its solutions is given by the set of monomials x^v for v varying in the set of exponents of $H_A(\beta)$ (see Theorems 1.10 and 1.11). It follows that there exists an exponent $\widetilde{v}_\sigma^{\mathbf{k}}$ of $H_A(\beta)$ with respect to w such that $\widetilde{v}_\sigma^{\mathbf{k}} \in \operatorname{supp}(\phi_{v_\sigma^{\mathbf{k}}})$, hence $\phi_{v_\sigma^{\mathbf{k}}} = c_{\sigma,\mathbf{k}} \cdot \phi_{\widetilde{v}_\sigma^{\mathbf{k}}}$ for some nonzero scalar $c_{\sigma,\mathbf{k}} \in \mathbb{C}$, see Remark 1.4. This implies that $\mathcal{B}_w(\beta)$ is a basis of $\mathcal{N}_w(H_A(\beta))$ in this case, by Theorem 1.10. □

Corollary 2.4. *If w, w' are weight vectors for $H_A(\beta)$, then we have the following:*

i) *If β is generic, $\mathcal{N}_w(H_A(\beta)) = \mathcal{N}_{w'}(H_A(\beta))$ if and only if $T_w = T_{w'}$.*
ii) *For all $\beta \in \mathbb{C}^d \setminus \varepsilon(A)$, $\mathcal{N}_w(H_A(\beta)) = \mathcal{N}_{w'}(H_A(\beta))$ if $\mathcal{C}_w = \mathcal{C}_{w'}$.*
iii) *For all $\beta \in \mathbb{C}^d \setminus \varepsilon(A)$, the cone $\mathcal{C}$ in Definition 1.7 can be chosen to be $\mathcal{C}_w$ for any basic Nilsson solution of $H_A(\beta)$ in the direction of w.*

Proof. Proposition 2.3 directly implies i). Let us prove ii). We can take a basis $\mathcal{B} = \{\phi_1, \ldots, \phi_r\}$ of $\mathcal{N}_w(H_A(\beta))$ so that each $\phi_i = x^{v^{(i)}} \sum_{u \in C_i} p_u^{(i)}(\log(x))$ is a basic Nilsson solution of $H_A(\beta)$ in the direction of w. Thus, for $i = 1, \ldots, r$, there exists a strongly convex open cone $\mathcal{C}_i$ as in condition ii) of Definition 1.7, i.e. $w \in \mathcal{C}_i \subseteq \mathcal{C}_w$ and $C_i \subseteq \mathcal{C}_i^* \cap \ker_{\mathbb{Z}}(A)$. Then the strongly convex open cone $\mathcal{C} := \cap_{k=1}^r \mathcal{C}_k \subseteq \mathcal{C}_w$ satisfies that condition for all $i = 1, \ldots, r$, since $\mathcal{C} \subseteq \mathcal{C}_i$ implies $\mathcal{C}_i^* \subseteq \mathcal{C}^*$. It follows that for all $w' \in \mathcal{C}$, the series ϕ_i are also basic Nilsson solutions in the direction of w' and, by Remark 1.8, $\operatorname{in}_{w'}(\phi_i) = x^{v^{(i)}} p_0^{(i)}(\log(x))$. This implies that $\mathcal{N}_w(H_A(\beta)) \subseteq \mathcal{N}_{w'}(H_A(\beta))$, for all $w' \in \mathcal{C}$. But this last inclusion must be an equality since both spaces have the same dimension, see Corollary 2.2. It follows that the space $\mathcal{N}_w(H_A(\beta))$, its subset of basic Nilsson solutions ϕ_i and their initial forms $\operatorname{in}_w(\phi_i)$ are locally constant with respect to the weight vector w, hence they are constant in the whole open cone $\mathcal{C}_w$ of the A–hypergeometric fan at β. This proves ii). For iii), notice that, the fact that $\operatorname{in}_{w'}(\phi_i) = x^{v^{(i)}} p_0^{(i)}(\log(x))$ for all $w' \in \mathcal{C}_w$ implies that $\langle w', u \rangle \geq 0$ for all $u \in C_i$ and all $w' \in \mathcal{C}_w$. Thus, $C_i \subseteq \mathcal{C}_w^*$ for all $i = 1, \ldots, r$, which proves the result. □

Remark 2.5. If β is generic and w, w' are weight vectors such that $T_w = T_{w'}$, it may happen that $\operatorname{in}_w(\phi) \neq \operatorname{in}_{w'}(\phi)$ for some series $\phi \in \mathcal{B}_w(\beta) = \mathcal{B}_{w'}(\beta)$ (in which case $\operatorname{in}_{(-w,w)}(H_A(\beta)) \neq \operatorname{in}_{(-w',w')}(H_A(\beta))$). For example, let us consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

and a generic $\beta \in \mathbb{C}^3$. The maximal simplex $\sigma = \{1, 4\}$ defines a regular triangulation of A induced by any of the weight vectors $w^{(1)} = (1, 2, 5, 1)$ and $w^{(2)} = (1, 5, 2, 1)$. We can choose Ω_σ so that the series ϕ_v with $v = (\beta_1 - (\beta_2 + 2)/3, 1, 0, (\beta_2 - 1)/3)$, see (1.6), belongs to $\mathcal{B}_{w^{(1)}}(\beta) = \mathcal{B}_{w^{(2)}}(\beta)$. Notice that

$$N_v = \{u = (-(2m_2 + m_3)/3, m_2, m_3, -(m_2 + 2m_3)/3) \in \mathbb{Z}^4 | \ m_2 \geq -1, \ m_3 \geq 0 \}.$$

Thus, $\mathrm{in}_{w^{(1)}}(\phi_v) = x^v \neq \mathrm{in}_{w^{(2)}}(\phi_v)$ since $v + (0, -1, 2, -1) \in \mathrm{supp}(\phi_v) = v + N_v$ has smaller $w^{(2)}$-weight than v.

The following result improves [3, Corollary 6.9].

Corollary 2.6. *If w is a weight vector, we have, for all $\beta \in \mathbb{C}^d$, that*

$$\dim_{\mathbb{C}}(\{\phi \in \mathcal{N}_w(H_A(\beta)) \mid \phi \text{ is convergent}\}) \geq \sum_{\sigma \in T_w^0} \mathrm{vol}(\sigma) \tag{2.2}$$

where $T_w^0 = \{\sigma \in T_w \mid \sigma \subseteq \eta$ for some $\eta \in \Gamma_A\}$. Moreover, equality holds in (2.2) if β is generic.

Proof. Assume first that $\beta \in \mathbb{C}^d$ is generic. Thus, $\mathcal{B}_w(\beta)$ is a basis of $\mathcal{N}_w(H_A(\beta))$ by Proposition 2.3. On the other hand, it follows from [4, Theorem 3.11] and the definition of B_σ, that $\phi_{v_\sigma^{\mathbf{k}}}$ is convergent if and only if $(1, \ldots, 1)B_\sigma \geq 0$, that holds if and only if σ is contained in a facet of Γ_A. Thus, $\mathcal{B}_w^0(\beta) := \{\phi_{v_\sigma^{\mathbf{k}}} : \sigma \in T_w^0,\ \mathbf{k} \in \Omega_\sigma\}$ is a linearly independent set of convergent formal Nilsson solutions of $H_A(\beta)$ in the direction of w. The equality in (2.2) for generic β follows from the fact that the series in $\mathcal{B}_w(\beta) \setminus \mathcal{B}_w^0(\beta)$ are all divergent and have pairwise disjoint supports, so no linear combination of them can be convergent. If $\beta \in \mathbb{C}^d$ is not assumed to be generic, we can consider a generic parameter β' and apply to the set $\mathcal{B}_w^0(\beta + \epsilon\beta')$, with $\epsilon \in \mathbb{C}$ such that $|\epsilon|$ is small enough, the same method as in [15, Theorem 3.5.1] to get the desired lower bound in (2.2). □

Lemma 2.7. [16, Lemma 5.3] *Let $p(y) \in \mathbb{C}[y]$, $v \in \mathbb{C}^n$ and $\nu \in \mathbb{N}^n$, then*

$$\partial^\nu[x^v p(\log(x))] = x^{v-\nu} \sum_{0 \leq \nu' \leq \nu} \lambda_{\nu'} \partial^{\nu-\nu'}[p](\log(x))$$

where the sum is over $\nu' \in \mathbb{N}^n$ such that $\nu'_j \leq \nu_j$ for all j, and $\lambda_{\nu'} \in \mathbb{C}$. In particular,

$$\lambda_\nu = [v]_\nu = \prod_{j=1}^n v_j(v_j - 1) \cdots (v_j - \nu_j + 1).$$

The following lemma guarantees that in the third condition of Definition 1.7 we can assume that the constant K is independent of β.

Lemma 2.8. *Let w be a weight vector. Then for any basic Nilsson solution ϕ of $H_A(\beta)$ in the direction of w as in (1.7) we have that*

$$\deg(p_u) \leq (n+1)(2^{2(d+1)}\mathrm{vol}(A) - 1)$$

for all $u \in C$.

Proof. Assume first that I_A is homogeneous. In this case $M_{A_\tau}(\beta - A_{\overline{\tau}}\alpha)$ is regular holonomic [10] and $\deg(p_u) \leq n(2^{2d}\mathrm{vol}(A) - 1)$ by [15, Theorem 2.5.14 and Corollary 4.1.2].

If I_A is not homogeneous, we can consider the $(d+1) \times (n+1)$ matrix $\rho(A)$ as defined at the beginning of Subsection 1.6. Notice that ϕ is also a basic Nilsson solution of $H_A(\beta)$ in the direction of w' for all $w' \in \mathcal{C}$, where $\mathcal{C}$ is an open cone as in condition ii) in Definition 1.7. In particular, we can assume without loss of genericity that w is generic. This implies that $(0, w) + \lambda(1, \ldots, 1) \in \mathbb{R}^{n+1}$ is generic if $\lambda > 0$ is generic. Since $(1, \ldots, 1)$ belongs to the rowspan of $\rho(A)$, we have that $(0, w)$ is a weight vector for $H_{\rho(A)}(\beta_0, \beta)$, where $\beta_0 \in \mathbb{C}$ (see also [3, Remark 2.5]).

Assume that $\beta_0 \in \mathbb{C}$ is sufficiently generic and consider the following series, see [3, Definition 3.16],

$$\rho(\phi) := \sum_{u\in C} \partial_0^{|u|}[x_0^{\beta_0-|v|}x^{v+u}\widehat{p}_u(\log(x_0),\dots,\log(x_n))]$$

where $|u| := \sum_{j=1}^n u_j$, ∂_0^{-k} is defined in [3, Definition 3.13] when $k > 0$, and $\widehat{p}_u \in \mathbb{C}[y_0,\dots,y_n]$ is defined from p_u as in [3, (3.2)]. We remark here that $\deg(p_u) \leq \deg(\widehat{p}_u)$ because $\widehat{p}_u(0,y_1,\dots,y_n) = p_u(y_1,\dots,y_n)$.

By Lemma 2.7 and [3, Lemma 3.12 and Definition 3.13], we can write

$$\rho(\phi) = \sum_{u\in C} x_0^{\beta_0-|v|-|u|}x^{v+u}h_u(\log(x_0),\dots,\log(x_n))$$

where $h_u(y) \in \mathbb{C}[y_0,\dots,y_n]$. By Lemma 2.7, if $|u| \geq 0$, then h_u equals $[\beta_0 - |v|]_{|u|} \cdot \widehat{p}_u$ plus other polynomial of degree smaller than $\deg(\widehat{p}_u)$. This implies that $\deg(h_u) = \deg(\widehat{p}_u)$ when $|u| \geq 0$ because β_0 is generic. For $|u| < 0$ it is also true that $\deg(h_u) = \deg(\widehat{p}_u)$ by [3, Lemma 3.12 and Definition 3.13].

On the other hand, by [3, Proposition 3.17] the series $\rho(\phi)$ is a basic Nilsson solution in the direction of $(0,w)$ of the hypergeometric system $H_{\rho(A)}(\beta_0,\beta)$ and since $I_{\rho(A)}$ is homogeneous, we have that $\deg(h_u) \leq (n+1)(2^{2(d+1)}\mathrm{vol}(\rho(A))-1)$, where $\mathrm{vol}(\rho(A)) = \mathrm{vol}(A)$. Thus,

$$\deg(p_u) \leq \deg(\widehat{p}_u) = \deg(h_u) \leq (n+1)(2^{2(d+1)}\mathrm{vol}(A)-1). \quad \square$$

3. Gevrey versus formal Nilsson solutions of $H_A(\beta)$

Let $\tau \subseteq \{1,\dots,n\}$ be a subset such that A_τ is pointed and $\mathrm{rank}(A_\tau) = d$. In this section we prove, for all $\beta \in \mathbb{C}^d$, that the space of Gevrey solutions of $M_A(\beta)$ along a coordinate subspace Y_τ is contained in the space of formal Nilsson solutions of $H_A(\beta)$ in a certain direction, see Theorem 3.3. If we further assume that $\mathrm{pos}(A) = \mathrm{pos}(A_\tau)$, we also prove that both spaces are the same and, for $\beta \in \mathbb{C}^d \setminus \varepsilon(A)$, its corresponding dimension is $\mathrm{vol}(\tau)$, see Theorem 3.4.

The following result follows from [3, Lemma 3.6] (see also [15, Lemma 4.1.3]), [17, (3.2)] (see also [15, (3.13)]) and [18, Corollary 8.4].

Lemma 3.1. *If w is a weight vector and v is an exponent of $H_A(\beta)$ with respect to w, there exists $\sigma \in T_w$ such that $v_j \in \mathbb{N}$ for all $j \notin \sigma$.*

Lemma 3.2. *For any regular triangulation $T(\tau)$ of A_τ there exists a regular triangulation T of A such that $T(\tau) \subseteq T$. In particular, if $\mathrm{pos}(A) = \mathrm{pos}(A_\tau)$ then $T(\tau) = T$.*

Proof. By definition of regular triangulation, there is a weight vector $w(\tau) \in \mathbb{R}^\tau$ such that $T(\tau) = T_{w(\tau)}$. Then choose another generic vector $w(\overline{\tau}) \in \mathbb{R}^{\overline{\tau}}_{>0}$, and consider $w \in \mathbb{R}^n$ to be a vector with τ–coordinates $\epsilon w(\tau)$, with $\epsilon > 0$ small enough, and $\overline{\tau}$–coordinates $w(\overline{\tau})$. Since $\epsilon w(\tau)$, $w(\overline{\tau})$ and $\epsilon > 0$ can be chosen to be generic, it follows that w induces a regular triangulation $T := T_w$ of A. By using the definition of regular triangulation, see conditions (1.2) and (1.3) (but substitute τ there by a maximal simplex σ), it is easy to check that $T(\tau) \subseteq T$ if $\epsilon > 0$ is small enough. $\square$

Let $T(\tau)$ be a regular triangulation of A_τ refining Γ_{A_τ} and w a weight vector for $H_A(\beta)$ chosen as in the proof of Lemma 3.2. Thus, by the assumption on $T(\tau)$ we have that $w(\tau)$ is a perturbation of $(1,\dots,1) \in \mathbb{R}^\tau$. Recall that w, $w(\tau)$ and $w(\overline{\tau})$ are all chosen to be generic.

Theorem 3.3. *If A_τ is pointed and $\mathrm{rank}(A_\tau) = d$, any Gevrey solution of $M_A(\beta)$ along Y_τ (at $p \in \mathcal{U}_{w(\tau)}$) can be written as a formal Nilsson solution of $H_A(\beta)$ in the direction of w.*

Proof. Let $f = \sum_{\alpha\in\mathbb{N}^{\overline{\tau}}} f_\alpha(x_\tau)x_{\overline{\tau}}^{\alpha} \in \mathcal{O}_{\widehat{X|Y},p}$ be any Gevrey series solution of $M_A(\beta)$. By [4, Lemma 6.11] we have that $f_\alpha(x_\tau)$ is a holomorphic solution of $M_{A_\tau}(\beta - A_{\overline{\tau}}\alpha)$ at p. Thus, by Theorem 1.12, each $f_\alpha(x_\tau)$ can be written as an element of $\mathcal{N}_{w(\tau)}(H_{A_\tau}(\beta - A_{\overline{\tau}}\alpha))$, i.e. a finite linear combination of series of the form (1.7) in the variables x_τ convergent at p. In particular, f can be rewritten as a sum of terms $x^\gamma q_\gamma(\log x_\tau)$ where $\gamma \in \mathbb{C}^n$ and $q_\gamma \in \mathbb{C}[x_\tau]$ are polynomials with degree $\deg(q_\gamma) \leq (n+1)2^{2(d+1)}\mathrm{vol}(A_\tau)$, see Lemma 2.8. Moreover, notice that the result of applying a monomial $x^\lambda\partial^\mu$ in the Weyl Algebra D to $x^\gamma\log(x)^\nu$ is of the form $x^{\gamma-\mu+\lambda}g(\log(x))$ for some polynomial g. This implies that any subseries of f of the form $\sum_{\gamma\in(v+\mathbb{Z}^n)} x^\gamma q_\gamma(\log x_\tau)$, for some $v \in \mathbb{C}^n$, is still annihilated by $H_A(\beta)$ and hence defines a Gevrey solution of $M_A(\beta)$. Since the dimension of the space of Gevrey series solutions is finite, f is a finite sum of such subseries, say $F_1,\dots,F_r$. Since F_k is annihilated by the Euler operators of A we have that $A\gamma = \beta$ for all $\gamma \in \mathbb{C}^n$ such that $q_\gamma \neq 0$. Thus, we can write each F_k in the form

$$F_k = x^v \sum_{u\in\ker_{\mathbb{Z}}(A)} x^u p_u(\log(x_\tau))$$

where $v \in \mathbb{C}^n$ and $p_u = q_{v+u}$. It is enough to prove that a Gevrey solution F_k of $M_A(\beta)$ is a formal Nilsson solution of $H_A(\beta)$ in the direction of w and we can assume for simplicity that the original $f = \sum_{\alpha\in\mathbb{N}^{\overline{\tau}}} f_\alpha(x_\tau)x_{\overline{\tau}}^{\alpha} \in \mathcal{O}_{\widehat{X|Y},p}$ can also be written in this form. Notice that we have shown that f can be written as in (1.7) satisfying conditions i) and iii) in Definition 1.7. We need to prove that the support of f is of the form $v' + C$ for some $v' \in v + \ker_{\mathbb{Z}}(A)$ and C satisfying conditions ii) and iv) in Definition 1.7.

Notice that for all $\alpha \in \mathbb{N}^{\overline{\tau}}$, we have

$$f_\alpha(x_\tau) = \sum_u x^{(v+u)_\tau} p_u(\log(x_\tau))$$

where the sum is over the vectors $u \in \ker_{\mathbb{Z}}(A)$ such that $(v+u)_{\overline{\tau}} = \alpha$ (in particular, u_τ varies in a translate of $\ker_{\mathbb{Z}}(A_\tau)$).

Let us see that f has an initial form with respect to w. It is enough to find a lower bound for the w-weights of all terms $x^{v+u}p_u(\log(x_\tau))$ of f with $p_u \neq 0$.

By [15, Theorem 2.5.5], the initial form of $f_\alpha \in \mathcal{N}_{w(\tau)}(H_{A_\tau}(\beta - A_{\overline{\tau}}\alpha))$ with respect to $w(\tau)$ is a solution of $\mathrm{in}_{(-w(\tau),w(\tau))}(H_{A_\tau}(\beta - A_{\overline{\tau}}\alpha))$. Since $w(\tau)$ is generic, $\mathrm{in}_{w(\tau)}(f_\alpha) = x_\tau^{\widetilde{v}_\tau} p_{u'}(\log(x_\tau))$ for some $\widetilde{v}_\tau \in \mathbb{C}^\tau$ and $u' \in \ker_{\mathbb{Z}}(A)$. By [15, Theorems 2.3.3(2), 2.3.9, 2.3.11], $\widetilde{v}_\tau$ is an exponent of $H_{A_\tau}(\beta - A_{\overline{\tau}}\alpha)$ with respect to $w(\tau)$. Thus, by Lemma 3.1 there is a maximal simplex $\sigma \in T_{w(\tau)}$ such that $\widetilde{v}_j \in \mathbb{N}$ for all $j \in \tau \setminus \sigma$.

Let us denote by $\widetilde{v} \in \mathbb{C}^n$ the vector with $\overline{\tau}$-coordinates $\widetilde{v}_{\overline{\tau}} = \alpha$ and τ-coordinates equal to $\widetilde{v}_\tau$. Notice that $\widetilde{v}_{\overline{\sigma}} \in \mathbb{N}^{\overline{\sigma}}$ and since $A\widetilde{v} = \beta$ we have that $\widetilde{v}_\sigma = A_\sigma^{-1}(\beta - A_{\overline{\sigma}}\widetilde{v}_{\overline{\sigma}})$. We can write

$$\widetilde{v} = \beta^\sigma + B_\sigma \widetilde{v}_{\overline{\sigma}}$$

where B_σ was defined in Subsection 1.1 and $\beta^\sigma \in \mathbb{C}^n$ denotes the vector whose σ-coordinates agree with $A_\sigma^{-1}\beta$ and whose other coordinates are zero.

Moreover, by Lemma 3.2, $\sigma \in T_w$. Thus, by (1.4) and the fact that $\widetilde{v}_{\overline{\sigma}} \in \mathbb{N}^{\overline{\sigma}}$, we have that

$$\mathrm{Re}(\langle w, \widetilde{v}\rangle) = \mathrm{Re}(\langle w, \beta^\sigma\rangle) + \langle w, B_\sigma\widetilde{v}_{\overline{\sigma}}\rangle = \mathrm{Re}(\langle w, \beta^\sigma\rangle) + \langle wB_\sigma, \widetilde{v}_{\overline{\sigma}}\rangle \geq \mathrm{Re}(\langle w, \beta^\sigma\rangle)$$

where $\mathrm{Re}(\langle w, \widetilde{v}\rangle)$ is the w-weight of $\mathrm{in}_w(f_\alpha(x_\tau)x_{\overline{\tau}}^{\alpha}) = \mathrm{in}_{w(\tau)}(f_\alpha(x_\tau))x_{\overline{\tau}}^{\alpha}$.

Then the minimum of the finite set

$$\{\mathrm{Re}(\langle w, \beta^\sigma\rangle) \mid \sigma \in T_{w(\tau)}\}$$

is a lower bound for the w-weights of all the terms of f. Thus, f has an initial form with respect to w and $\mathrm{in}_w(f)$ must be a term $x^{v+u'}p_{u'}(\log(x_\tau))$ for some $u' \in \ker_{\mathbb{Z}}(A)$ because w is generic. We may assume for simplicity that $\mathrm{in}_w(f) = x^v p_0(\log(x_\tau))$ since $v+\ker_{\mathbb{Z}}(A) = v+u'+\ker_{\mathbb{Z}}(A)$. This implies that $x^v p_0(\log(x_\tau))$ is a solution of $\mathrm{in}_{(-w,w)}(H_A(\beta))$ by the same argument as in the proof of [15, Theorem 2.5.5], hence v is an exponent of $H_A(\beta)$ with respect to w. The set $C := \{u \in \ker_{\mathbb{Z}}(A) |\ p_u \neq 0\}$ satisfies condition iv) in Definition 1.7.

Let $w' \in \mathcal{C}_w$ be generic and such that w'_τ is a perturbation of $w(\tau)$. Then the previous argument works as well for w' instead of w and we have that f has also an initial form with respect to w' of the form $\mathrm{in}_{w'}(f) = x^{v+u'}p_{u'}(\log(x_\tau))$ and that $v+u'$ is an exponent of $H_A(\beta)$ with respect to w'. Since the set of exponents of $H_A(\beta)$ with respect to w' is finite and constant for all $w' \in \mathcal{C}_w$, we can find an open cone $\mathcal{C}' \subseteq \mathcal{C}_w$ such that $w \in \mathcal{C}'$ and $\mathrm{in}_{w'}(\phi) = x^v p_0(\log(x_\tau)$ for all $w' \in \mathcal{C}'$. Hence, we have $\langle w', u\rangle > 0$ for all $w' \in \mathcal{C}'$ and all $u \in C \setminus \{0\}$. Thus, f satisfies condition ii) in Definition 1.7. $\square$

The following result provides a partial converse to Theorem 3.3.

Theorem 3.4. *If A is pointed, $\mathrm{pos}(A_\tau) = \mathrm{pos}(A)$ and $p \in \mathcal{U}_{w(\tau)} \subseteq Y_\tau$, we have*

$$\dim_{\mathbb{C}}(Hom_D(M_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau,p}})) = \dim_{\mathbb{C}}(\mathcal{N}_w(H_A(\beta)))$$

for all $\beta \in \mathbb{C}^d$. More precisely, $\mathcal{N}_w(H_A(\beta))$ is the space of Gevrey series solutions of $M_A(\beta)$ along Y_τ at any point $p \in \mathcal{U}_{w(\tau)}$. Moreover, for $\beta \in \mathbb{C}^d \setminus \varepsilon(A)$, the dimension of this space is $\mathrm{vol}(\tau)$.

Proof. By Theorem 3.3, it is enough to show that any basic Nilsson series ϕ in the direction of w as in (1.7) is a Gevrey series along Y_τ at any point $p \in \mathcal{U}_{w(\tau)} \subseteq Y$.

Since $\mathrm{pos}(A_\tau) = \mathrm{pos}(A)$ we have that $T(\tau) = T_w$ by the construction of w, see the proof of Lemma 3.2 and the subsequent paragraph.

By [3, Lemma 2.10] we have that v is an exponent of $H_A(\beta)$ with respect to w. Thus, there exists $\sigma \in T_w = T(\tau)$ such that $v_j \in \mathbb{N}$ for all $j \notin \sigma$, see Lemma 3.1. In particular, we have $v_{\overline{\tau}} \in \mathbb{N}^{\overline{\tau}}$. Moreover, by [16, Proposition 5.4] (whose proof is also valid when I_A is not necessarily homogeneous) we have that $p_u(y) \in \mathbb{C}[y_j :\ j \in \mathrm{vert}(T_w)]$ for all $u \in C$, where $\mathrm{vert}(T_w)$ is the set of vertices of T_w. In particular, $p_u \in \mathbb{C}[y_j :\ j \in \tau]$. Thus, we can define $p_u(y_\tau) := p_u(y)$ and write

$$\phi = \sum_{u\in C} x^{v+u} p_u(\log(x_\tau)) = \sum_{\alpha\in\mathbb{Z}^n} f_\alpha(x_\tau) x_{\overline{\tau}}^{\alpha}$$

where

$$f_\alpha(x_\tau) = \sum_{u\in C,\ (v+u)_{\overline{\tau}}=\alpha} x^{(v+u)_\tau} p_u(\log(x_\tau))$$

is annihilated by $H_{A_\tau}(\beta - A_{\overline{\tau}}\alpha)$, see the proof of [4, Lemma 6.11].

We need to prove that $f_\alpha = 0$ if $\alpha \notin \mathbb{N}^n$. That is, we need to show that $v_j + u_j \geq 0$ for all $j \notin \tau$ and $u \in C$, where $v + C = \mathrm{supp}(\phi)$. Since $\mathrm{in}_w(\phi) = x^v p_0(\log(x_\tau))$, we know that $\langle w, u\rangle > 0$ for all $u \in C \setminus \{0\}$. Assume to the contrary that there exist $u \in C$ and $j \in \overline{\tau}$ such that $v_j + u_j < 0$ and choose such a vector u so that $\langle w, u\rangle$ is minimal. Since $j \in \overline{\tau}$ and $\mathrm{pos}(A) = \mathrm{pos}(A_\tau) = \cup_{\sigma\in T(\tau)} \mathrm{pos}(\sigma)$, there exist $m_\sigma \in \mathbb{N}^\sigma$ and $m_j \in \mathbb{N}$ with $m_j \geq 1$ such that $m_j a_j = \sum_{i\in\sigma} m_i a_i$. Then $u(m) := -m_j e_j + \sum_{i\in\sigma} m_i e_i \in \ker_{\mathbb{Z}}(A)$, where e_ℓ denotes the ℓ-th vector of the standard basis of $\mathbb{R}^n$. Notice that $P = \partial^{u(m)_+} - \partial^{u(m)_-} = \partial_\sigma^{m_\sigma} - \partial_j^{m_j} \in H_A(\beta)$, hence $P(\phi) = 0$.

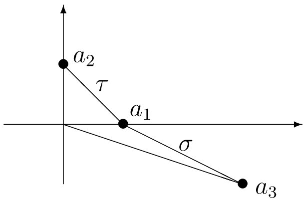

Fig. 1. Regular triangulation of A.

By Lemma 2.7 and using that $\partial_j[p_u] = 0$ for all $j \notin \tau$, we have that

$$\partial_j^{m_j}[x^{v+u}p_u(\log(x_\tau))] = x^{v+u-m_je_j}\lambda_{m_j}p_u(\log(x_\tau)) \neq 0$$

where $\lambda_{m_j} = [v_j + u_j]_{m_j} \neq 0$ because $v_j + u_j < 0$ by assumption. Since $P(\phi) = 0$ and $v + u - m_je_j = v + u + u(m)_-$ belongs to the support of the series $\partial^{u(m)_-}(\phi) = \partial^{u(m)_+}(\phi)$, we have that $v + u + u(m)$ must be in the support of ϕ, but then $u + u(m) \in C$ and $v_j + u_j + u(m)_j < 0$. On the other hand, $\text{in}_w(P) = -\partial_j^{m_j}$ because $\partial_\sigma^{m_\sigma} \notin \text{in}_w(I_A)$ for $\sigma \in T(\tau) = T_w$, see [18, Corollary 8.4]. Thus, $\langle w, u(m)\rangle < 0$ and we obtain that $\langle w, u + u(m)\rangle < \langle w, u\rangle$ which is in contradiction with our assumption.

Finally, let us see that each f_α is convergent at $p \in \mathcal{U}_{w(\tau)}$. Since ϕ is basic, there is an open and strongly convex cone $\mathcal{C} \subseteq \mathcal{C}_w$ such that $w \in \mathcal{C}$ and $\langle w', u\rangle > 0$ for all $w' \in \mathcal{C}$ and all $u \in C \setminus \{0\}$. In particular, $\text{in}_{w'}(\phi) = \text{in}_w(\phi)$ for all $w' \in \mathcal{C}$. This implies that for fixed α and all $w' \in \mathcal{C}$ with $w'_{\overline{\tau}} = w_{\overline{\tau}}$ there exists the initial form $\text{in}_{w'_\tau}(f_\alpha)$. Notice that the set of all w'_τ for w' as above is a neighborhood of $w_\tau \in \mathbb{R}^\tau$ and we can find a smaller neighborhood U of w_τ such that $\text{in}_{w'_\tau}(f_\alpha) = \text{in}_{w(\tau)}(f_\alpha)$ for all $w'_\tau \in U$. Then, by [3, Remark 2.7], $f_\alpha \in \mathcal{N}_{w(\tau)}(H_{A_\tau}(\beta - A_{\overline{\tau}}\alpha))$ and by Theorem 1.12, f_α is convergent at any point $p \in \mathcal{U}_{w(\tau)}$.

Thus, ϕ is a formal solution of $M_A(\beta)$ along Y_τ, which implies it is Gevrey of some order because $M_A(\beta)$ is holonomic.

For $\beta \in \mathbb{C}^d \setminus \varepsilon(A)$, $\dim(\mathcal{N}_w(H_A(\beta))) = \text{vol}(\tau)$ by Corollary 2.2, where $\cup_{\sigma\in T_w}\Delta_\sigma = \Delta_\tau$ in our case, and Theorem 1.6. □

Remark 3.5. The additional condition $\text{pos}(A) = \text{pos}(A_\tau)$ in Theorem 3.4 is necessary, even if $\beta \in \mathbb{C}^d$ is generic, as shown by the following example.

Example 3.6. Let us consider the system $H_A(\beta)$ for

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix},$$

and a generic parameter $\beta \in \mathbb{C}^2$. For $\tau = \{1, 2\}$, we have $Y_\tau = \{x_3 = 0\}$. Let $w = (0, 0, 1) + \epsilon(w(\tau), 0)$ where $\epsilon > 0$ is small enough and $w(\tau) \in \mathbb{R}^2$ is a perturbation of $(1, 1)$. We have that $T_w = \{\tau, \sigma\}$ for $\sigma = \{1, 3\}$, see Fig. 1. Moreover, $T(\tau) = \{\tau\} \subseteq T_w$.

Notice that both simplices τ, σ have normalized volume one. According to Proposition 2.3, $\{\phi_1 := \phi_{v^0_\tau}, \phi_2 := \phi_{v^0_\sigma}\}$ is a basis of $\mathcal{N}_w(H_A(\beta))$, where $v^0_\tau = (\beta_1, \beta_2, 0)$ and $v^0_\sigma = (\beta_1 + 3\beta_2, 0, -\beta_2)$. Moreover $\ker_{\mathbb{Z}}(A) = \mathbb{Z}(-3, 1, 1)$. Thus, $\text{supp}(\phi_1) = \{(\beta_1 - 3m, \beta_2 + m, m)|\ m \in \mathbb{N}\}$ and $\text{supp}(\phi_2) = \{(\beta_1 + 3\beta_2 - 3m, m, -\beta_2 + m)|\ m \in \mathbb{N}\}$. It follows from Theorem 1.5 that ϕ_1 generates the space of Gevrey solutions of $H_A(\beta)$ along $Y_\tau = \{x_3 = 0\}$ at any point $p \in \{x_3 = 0; x_1x_2 \neq 0\}$.

In particular, the space $\mathcal{N}_w(H_A(\beta))$ is not contained in the space of Gevrey solutions of $M_A(\beta)$ along Y_τ, although all the assumptions in Theorem 3.4 except the condition $\text{pos}(A) = \text{pos}(A_\tau)$ are satisfied.

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