



Some Non-Compactness Results for Locally Homogeneous Contact Metric Manifolds

Antonio Lotta  and Verónica Martín-Molina

Abstract. We exhibit some sufficient conditions ensuring the non-compactness of a locally homogeneous, regular, contact metric manifold, under suitable assumptions on the Jacobi operator of the Reeb vector field.

Mathematics Subject Classification. 53C25, 53C30.

Keywords. Locally homogeneous, contact metric manifold, regular contact manifold, Jacobi operator.

1. Introduction

Let (M, η) be a contact manifold of dimension $2n + 1$, $n \geq 1$, i.e. a smooth manifold endowed with a fixed contact form η , that is, a 1-form satisfying:

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M . This condition singles out a distinguished globally defined vector field ξ , called the Reeb vector field, which is transverse to the contact distribution $D = \text{Ker}(\eta)$, and such that $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for every smooth vector field X on M . Many results are known concerning the Riemannian geometry of *associated metrics* on (M, η) (see for instance the standard monograph [1], where the reader can find several examples and an extensive bibliography on this subject).

We recall that an *associated metric* g to η is a Riemannian metric for which there exists a $(1, 1)$ tensor field $\varphi : TM \rightarrow TM$ such that

$$\varphi^2 = -Id + \eta \otimes \xi, \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y),$$

V. Martín-Molina was partially supported by the group FQM-226 of the Junta de Andalucía (Spain) and by the grant VIPPIT-IV.4 of the Universidad de Sevilla (Spain).

for every X, Y vector fields on M . The tensor field φ is uniquely determined by g , and the tensors (φ, ξ, η, g) make up a *contact metric structure* on M . On every contact manifold, associated metrics abound; indeed, by the polarization technique, one can construct them starting from the non-degenerate (symplectic) two-form $d\eta$ on the contact subbundle D . A choice of such a metric endows the contact manifold of an *almost CR*-structure (D, J) , where the complex structure J is the restriction $\varphi|_D$.

The most studied class of associated metrics is that of Sasakian ones, strictly related to Kähler geometry, characterized by the additional requirements that (D, J) should be a (formally integrable) *CR*-structure and moreover ξ should be a Killing vector field. For any associated metric g , this last condition is also equivalent to

$$\mathcal{L}_\xi \varphi = 0,$$

and, when it holds, g is usually called a *K-contact metric*. In the general case, the above Lie derivative is a non-null *symmetric* (1,1) tensor field; in the literature, it is customary to adopt the notation $h := \frac{1}{2}\mathcal{L}_\xi \varphi$. The operator h has a key role in the development of the theory of general contact metric structures. Another important symmetric operator is the Jacobi operator associated to ξ , $l := R(-, \xi)\xi$, where R is the curvature tensor. Observe that h and l both annihilate ξ , so they can be thought of as endomorphisms of the contact subbundle.

The behavior of these two operators, which appear in several key formulas, greatly influence the geometry of the underlying contact metric manifold. The aim of this paper is to discuss an instance of this fact, exhibiting sufficient conditions ensuring the non-compactness of the contact manifold. We consider the class of *locally homogeneous* contact metric manifolds, whose contact form is *regular*, meaning essentially that the orbit space M/ξ determined by the flow of the Reeb vector field is smooth and the canonical projection $\pi : M \rightarrow M/\xi$ is a submersion (for more details, see for instance [1, Chapter 3]). We remark that this class contains all the *homogeneous* contact metric manifolds, due to a general result of Boothy-Wang [8].

We shall restrict to the case where l and h are most nicely related: namely, our basic assumption will be:

$$l|_{D(\lambda)} = c_\lambda Id \tag{*}$$

for every eigenvalue λ of $h : D \rightarrow D$, with eigenbundle $D(\lambda)$, where c_λ is a constant.

Many relevant examples of (locally) homogeneous, non *K*-contact, contact metric manifolds satisfy this condition. We cite here the (κ, μ) -spaces, studied by several authors (see [1, §7.3], [2], [5]), and tangent sphere bundles over symmetric spaces of rank one (see [6]).

Another wide class can be considered in the context of pseudo-Hermitian geometry. Recall that a *pseudo-Hermitian manifold* is a contact metric manifold whose almost CR -structure (D, J) is formally integrable; it is a strongly pseudoconvex CR manifold. For these manifolds, condition (*) holds automatically under the assumption that the Ricci operator Q commutes with φ , i.e., $[Q, \varphi] = 0$. Indeed, in this case, on the contact subbundle, l and h are related by

$$l = Id - h^2 + 2(1 - n)h.$$

In particular, this is true for η -Einstein pseudo-Hermitian manifolds, which are characterized by

$$Q = \alpha Id + \beta\eta \otimes \xi, \quad \alpha, \beta \in \mathbb{R}.$$

Instead of $[Q, \varphi] = 0$, one can also consider the case where Q is of the form:

$$Q = \alpha Id + \beta\eta \otimes \xi + \gamma h, \quad \alpha, \beta, \gamma \in \mathbb{R}.$$

All these relevant cases are examined in the last section (see Theorems 4.5, 4.6 and 4.7).

In the above set up, our main result is the following:

Theorem 1.1. *Let $M(\varphi, \xi, \eta, g)$ be a locally homogeneous, regular contact metric manifold. Assume that the Jacobi operator l satisfies:*

$$l_{|D(\lambda)} = c_\lambda Id,$$

for every eigenvalue of h .

If, for each positive eigenvalue λ of h , we have $c_\lambda \neq (\lambda + 1)^2$ and, for at least one of them, we have:

$$c_\lambda \in [(\lambda + 1)^2 - 4\lambda^2, (\lambda + 1)^2),$$

then M is not compact.

The method of proof consists in constructing a suitable deformation of the original metric g , determining a contact metric structure $(\varphi', \xi, \eta, g')$ that satisfies $\nabla'_\xi h' = 0$ and whose operator h' admits at least one eigenvalue $\lambda \geq 1$. But this turns out to be impossible on a compact, *regular* contact manifold, due to an argument similar to a result about critical metrics in [1, §10.3]. The deformations employed in the proof are given essentially by a rescaling of the original metric by a suitable positive constant factor a_λ on each eigenbundle $D(\lambda)$ (see Definition 3.2).

In Sect. 3, we develop some properties of this kind of deformations for every locally homogeneous contact metric manifold, under our basic assumption (*). In particular, we first discuss when such a deformed structure is a K -contact one (Theorem 3.8). This is also of interest in itself, because it is known that the existence of a K -contact metric is also a strong condition,

imposing topological restrictions in the compact case (for instance, a torus cannot carry a K -contact metric structure).

In the last section, we apply these properties to prove our main result and some of its applications. In particular, we prove:

Theorem 1.2. *A locally homogeneous, regular, η -Einstein pseudo-Hermitian manifold of dimension $2n + 1$ that satisfies*

$$Ric(\xi, \xi) \leq 2n(1 - n^2) \tag{1.1}$$

is not compact.

We remark that the inequality (1.1) is optimal; this is showed by giving suitable examples of compact homogeneous, η -Einstein pseudo-Hermitian manifolds that satisfy $Ric(\xi, \xi) > 2n(1 - n^2)$ (see Proposition 4.8).

2. Preliminaries

Given a contact manifold (M, η) , we will denote by D the contact distribution, which is the $2n$ -dimensional distribution defined by $\ker(\eta)$. We can write the tangent bundle of M as $TM = D \oplus \mathbb{R}\xi$, where ξ is the Reeb vector field of η .

If g is an associated metric to η , it is known (see e.g. [1], [3]) that:

$$\nabla \xi = -\varphi - \varphi h, \tag{2.1}$$

$$(\nabla_\xi h) = \varphi(Id - h^2 - l), \tag{2.2}$$

$$\varphi l \varphi - l = 2(h^2 + \varphi^2), \tag{2.3}$$

$$Ric(\xi, \xi) = 2n - tr(h^2), \tag{2.4}$$

where $l = R(-, \xi)\xi$ is the Jacobi operator relative to ξ . Moreover, h is a symmetric operator, h anticommutes with φ , and $trh = 0$ (for example, Lemma 6.2. of [1]). An consequence, we recall that, if λ is an eigenvalue of h , then $-\lambda$ is also an eigenvalue. It is also known that $h\xi = 0$. So, whenever convenient, h can also be considered as an endomorphism $h : D \rightarrow D$ of the contact subbundle.

In the case of pseudo-Hermitian manifolds that satisfy $[Q, \varphi] = 0$, on the contact subbundle, l and h are related by

$$l = Id - h^2 + 2(1 - n)h, \tag{2.5}$$

because of the following identity [1, Prop. 10.2]:

$$Q\varphi - \varphi Q = l\varphi - \varphi l + 4(n - 1)h\varphi - \eta \otimes \varphi Q\xi + \eta \circ Q\varphi \otimes \xi.$$

We end this section by recalling the notion of contact metric (κ, μ) -spaces. These are contact metric manifolds satisfying the equation [2]

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

for every vector fields X, Y on M , where κ and μ are constants. The (κ, μ) -spaces obviously include the Sasakian manifolds ($\kappa = 1$ and $h = 0$) and they

are non-Sasakian, locally homogeneous, strongly pseudoconvex CR manifolds when $\kappa < 1$. In this case, the spectrum of h is $\{0, -\sqrt{1 - \kappa}, \sqrt{1 - \kappa}\}$. Applying a D_a -homothetic deformation to a (κ, μ) -space yields another (κ', μ') -space; we recall that such a deformation is given by the following change of the structural tensors of M :

$$\bar{\eta} := a\eta, \quad \bar{\xi} := \frac{1}{a}\xi, \quad \bar{g} = ag + a(a - 1)\eta \otimes \eta,$$

where a is a positive constant. Then one has:

$$\kappa' = \frac{\kappa + a^2 - 1}{a^2}, \quad \mu' = \frac{\mu + 2a - 2}{a}. \tag{2.6}$$

It is known that the Ricci operator of a (κ, μ) -space can be written as

$$Q = (2(n - 1) - n\mu)Id + (2(1 - n) + n(2\kappa + \mu))\eta \otimes \xi + (2(n - 1) + \mu)h. \tag{2.7}$$

Hence, the η -Einstein ones are characterized by $\mu = 2(1 - n)$.

The simply connected, complete, non Sasakian (κ, μ) -spaces are all homogeneous and are completely classified (see [5, 9, 11]) and, considering two such spaces equivalent up to D_a -homothetic deformations, they form a one-parameter family parametrized by \mathbb{R} . This family contains the tangent sphere bundles T_1S where S is a Riemannian space form. The classification relies on a result of Boeckx [5], stating that the number $I_M = \frac{1-\mu/2}{\sqrt{1-\kappa}}$ completely determines a contact metric (κ, μ) -space M locally up to equivalence and up to D_a -homotetic deformations of its contact metric structure. We shall call I_M the Boeckx invariant of the (κ, μ) -space.

3. A Family of Deformations of a Locally Homogeneous Contact Metric Structure

Given a locally homogeneous contact metric manifold $M(\varphi, \xi, \eta, g)$, the symmetric operator h has constant eigenvalues with constant multiplicity, since h is preserved by all the local automorphisms of the geometric structure ([7, Lemma 10]).

We denote by \mathcal{S} the set of eigenvalues of $h|_D$, by λ_i the positive eigenvalues in \mathcal{S} , by $D(\lambda)$ the eigendistribution of h associated with the eigenvalue λ , which is a vector subbundle of TM , and by $[\xi]$ the one-dimensional subbundle spanned by the vector field ξ . Therefore, we can write TM as:

$$TM = \begin{cases} [\xi] \oplus D(0) \oplus D(\lambda_1) \oplus \cdots \oplus D(\lambda_m) \oplus D(-\lambda_1) \oplus \cdots \oplus D(-\lambda_m), & \text{if } 0 \in \mathcal{S}, \\ [\xi] \oplus D(\lambda_1) \oplus \cdots \oplus D(\lambda_m) \oplus D(-\lambda_1) \oplus \cdots \oplus D(-\lambda_m), & \text{if } 0 \notin \mathcal{S}. \end{cases} \tag{3.1}$$

The following simple property of the sections of the subbundles $D(\lambda)$ shall be useful next.

Lemma 3.1. *Let $M(\varphi, \xi, \eta, g)$ be a locally homogeneous contact metric manifold. Then, for every $X \in D(\lambda)$ and $Y \in D(\mu)$, we have:*

$$g((\nabla_\xi h)X, Y) = (\lambda - \mu)g(\nabla_\xi X, Y).$$

Proof. Since h is symmetric and the eigenvalues are constant, we obtain:

$$g((\nabla_\xi h)X, Y) = \lambda g(\nabla_\xi X, Y) - g(\nabla_\xi X, hY) = (\lambda - \mu)g(\nabla_\xi X, Y),$$

thus proving the lemma. □

Using the decomposition (3.1), we shall now define a family of deformed structures $(\varphi', \xi, \eta, g')$, all of which are associated with the given contact form.

Definition 3.2. We take φ' as

$$\varphi'\xi = 0, \quad \varphi' = a_\lambda\varphi \text{ on } D(\lambda),$$

and the metric g' as

$$g'(X, Y) = \begin{cases} 1, & \text{if } X = Y = \xi, \\ a_\lambda g(X, Y), & \text{if } X, Y \in D(\lambda), \\ 0, & \text{otherwise,} \end{cases}$$

with a_λ a constant attached to each λ in the spectrum of h ; these constants are required to satisfy the following constrains:

$$a_\lambda \begin{cases} = 1 & \text{if } \lambda = 0, \\ > 0 & \text{if } \lambda \neq 0, \\ = \frac{1}{a_{-\lambda}}. \end{cases}$$

We will now give sufficient conditions for this family of structures to be contact metric.

Proposition 3.3. *Given a locally homogeneous contact metric manifold $(M, \varphi, \xi, \eta, g)$, then each structure $(\varphi', \xi, \eta, g')$ of the family defined in Definition 3.2 is a contact metric structure.*

Proof. If the manifold is K -contact, then the deformation leaves invariant the contact metric structure.

If the manifold is not K -contact, we start by verifying $\varphi'^2 = -Id + \eta \otimes \xi$. Indeed, taking into account the splitting (3.1), when computing $\varphi'^2(X)$, we just have to consider two possible cases:

- Case $X = \xi$. Then $\varphi'\xi = \varphi\xi = 0$, thus $\varphi'^2\xi = 0$. On the other hand, $-\xi + \eta(\xi)\xi = -\xi + \xi = 0$.
- Case $X \in D(\lambda)$. Then $\varphi X \in D(-\lambda)$, so that

$$\varphi'^2 X = a_\lambda \varphi'(\varphi X) = a_\lambda \left(\frac{1}{a_{-\lambda}} \right) \varphi^2 X = \varphi^2 X = -X + \eta(X)\xi.$$

Thus the structure is almost contact.

The new metric, g' , is compatible with the almost contact structure, i.e. $g'(\varphi'X, \varphi'Y) = g'(X, Y) - \eta(X)\eta(Y)$. This follows by considering the following cases:

- Case $X, Y = \xi$. Then $g'(\varphi'\xi, \varphi'\xi) = 0$ and $g'(\xi, \xi) - \eta(\xi)\eta(\xi) = 1 - 1 = 0$.
- Case $X = \xi, Y \in D(\lambda)$. Then $g'(\varphi'\xi, \varphi'Y) = 0$ and $g'(\xi, Y) - \eta(\xi)\eta(Y) = 0 - 1 \cdot 0 = 0$.
- Case $X, Y \in D(\lambda)$. Then

$$g'(\varphi'X, \varphi'Y) = a_\lambda^2 g'(\varphi X, \varphi Y) = a_\lambda^2 (a_{-\lambda}) g(\varphi X, \varphi Y) = a_\lambda^2 \left(\frac{1}{a_\lambda}\right) g(\varphi X, \varphi Y) \\ = a_\lambda g(\varphi X, \varphi Y) = a_\lambda (g(X, Y) - \eta(X)\eta(Y)) = a_\lambda g(X, Y)$$

and

$$g'(X, Y) - \eta(X)\eta(Y) = a_\lambda g(X, Y).$$

- Case $X \in D(\lambda), Y \in D(\mu)$ with $\lambda \neq \mu$. Then

$$g'(\varphi'X, \varphi'Y) = a_\lambda a_\mu g'(\varphi X, \varphi Y) = 0$$

and

$$g'(X, Y) - \eta(X)\eta(Y) = 0 + 0.$$

Thus our structure is almost contact metric. Finally, we can check that g' is associated to the contact form η , i.e. that $d\eta(X, Y) = g'(X, \varphi'Y)$. This also can be proved case by case. For instance, if $X \in D(\lambda)$ and $Y \in D(-\lambda)$, then

$$g'(X, \varphi'Y) = a_{-\lambda} g'(X, \varphi Y) = a_{-\lambda} a_\lambda g(X, \varphi Y) \\ = \left(\frac{1}{a_\lambda}\right) a_\lambda g(X, \varphi Y) = g(X, \varphi Y).$$

In all the remaining cases, the verification is similar and straightforward and we omit it for the sake of brevity. □

From now on, we shall restrict our study to locally homogeneous contact metric manifolds whose Jacobi operator l has the simplest behavior with respect to the splitting (3.1). Namely, we shall require that l should preserve the splitting and that, for every eigenvalue λ of h :

$$l|_{D(\lambda)} = c_\lambda Id,$$

where c_λ is a constant.

Under this assumption, we shall determine the operator $h' = \frac{1}{2} \mathcal{L}_\xi \varphi'$ of a deformed contact metric structure and its covariant derivative $\nabla'_\xi h'$ with respect to the new metric. This information will be used in the next section to prove our main results.

Let us first see a couple of necessary lemmas.

Lemma 3.4. *Let $M(\varphi, \xi, \eta, g)$ be a locally homogeneous contact metric manifold. Assume that $l_{|D(\lambda)} = c_\lambda Id$ for every eigenvalue λ of $h_{|D}$. Then,*

$$c_{-\lambda} = -c_\lambda - 2\lambda^2 + 2, \tag{3.2}$$

In particular, if $\lambda = 0$, then $c_\lambda = 1$.

Proof. Given $X \in D(\lambda)$, we have that $hX = \lambda X$ and $lX = c_\lambda X$. Moreover, since $\varphi X \in D(-\lambda)$, then $h\varphi X = -\lambda\varphi X$ and $l\varphi X = c_{-\lambda}\varphi X$. Substituting these formulas in Eq. (2.3) and using standard contact metric properties give us

$$-(c_{-\lambda} + c_\lambda)X = 2(\lambda^2 - 1)X,$$

from which Eq. (3.2) follows. When $\lambda = 0$, we have that $-(c_0 + c_0) = -2$, so $c_0 = 1$. □

Lemma 3.5. *Let $M(\varphi, \xi, \eta, g)$ be a locally homogeneous contact metric manifold. Assume that*

$$l_{|D(\lambda)} = c_\lambda Id$$

for every eigenvalue λ of $h_{|D}$. Then, for every $X \in D(\lambda)$ and $Y \in D(\mu)$ satisfying $\lambda + \mu \neq 0$, we have:

$$g([\xi, X], \varphi Y) = \left(\frac{1 - \lambda^2 - c_\lambda}{\lambda + \mu} + \lambda + 1 \right) g(X, Y).$$

Proof. Using standard properties of the Levi-Civita connection ∇ , Lemma 3.1 and Eq. (2.1), it follows that

$$\begin{aligned} g([\xi, X], \varphi Y) &= g(\nabla_\xi X, \varphi Y) - g(\nabla_X \xi, \varphi Y) \\ &= \frac{1}{\lambda + \mu} g((\nabla_\xi h)X, \varphi Y) + (\lambda + 1)g(X, Y). \end{aligned}$$

Applying now Eq. (2.2) and the hypothesis on l , we obtain that

$$\begin{aligned} g([\xi, X], \varphi Y) &= \frac{1}{\lambda + \mu} g(X - h^2 X - lX, Y) + (\lambda + 1)g(X, Y) \\ &= \frac{1}{\lambda + \mu} g(X - \lambda^2 X - c_\lambda X, Y) + (\lambda + 1)g(X, Y) \\ &= \left(\frac{1 - \lambda^2 - c_\lambda}{\lambda + \mu} + \lambda + 1 \right) g(X, Y), \end{aligned}$$

concluding the proof. □

We can now determine the operator h' of the deformed contact metric structures.

Proposition 3.6. *Let $M(\varphi, \xi, \eta, g)$ be a locally homogeneous contact metric manifold. Assume that*

$$l_{|D(\lambda)} = c_\lambda Id,$$

for every eigenvalue λ of $h|_D$. Consider a deformation $(\varphi', \xi, \eta, g')$ of the contact metric structure according to Definition 3.2. Then the corresponding operator h' is determined as follows:

$$h'|_{D(\lambda)} = \rho_\lambda Id,$$

where

$$\rho_\lambda = \begin{cases} \frac{(-(\lambda+1)^2 + 4\lambda^2 + c_\lambda)a_\lambda^2 + (\lambda+1)^2 - c_\lambda}{4\lambda a_\lambda} & \text{for } \lambda \neq 0, \\ 0 & \text{for } \lambda = 0. \end{cases} \tag{3.3}$$

Proof. Since the deformed structure is contact metric, $h'\xi = 0$. Therefore, in order to determine h' completely, we only have to check $h'|_D$.

Given $X \in D(\lambda)$ and $Y \in D(\mu)$, by the definition of h' and φ' , we have that:

$$2g(h'X, Y) = g([\xi, \varphi'X], Y) - g(\varphi'[\xi, X], Y) = a_\lambda g([\xi, \varphi X], Y) - g(\varphi'[\xi, X], Y). \tag{3.4}$$

We will now consider two cases: $\mu = -\lambda$ and $\mu \neq -\lambda$.

If $\mu = -\lambda$, we obtain the next equation from the definition of h :

$$2g(h'X, Y) = a_\lambda g([\xi, \varphi X], Y) - a_\lambda g(\varphi[\xi, X], Y) = 2a_\lambda g(hX, Y) = 2a_\lambda \lambda g(X, Y).$$

Now, in the subcase $\lambda = 0$ this gives directly $g(h'X, Y) = 0$. Assuming $\lambda \neq 0$, then $g(X, Y) = 0$ since eigenvectors belonging to different eigendistributions are orthogonal, and hence we have $g(h'X, Y) = 0$ again.

If $\mu \neq -\lambda$, then substituting $[\xi, \varphi X] = 2hX + \varphi[\xi, X] = 2\lambda X + \varphi[\xi, X]$ in Eq. (3.4) gives us:

$$2g(h'X, Y) = 2a_\lambda \lambda g(X, Y) + a_\lambda g(\varphi[\xi, X], Y) - g(\varphi'[\xi, X], Y).$$

Using the definition of φ' and that φ is anti-symmetric, we obtain:

$$\begin{aligned} 2g(h'X, Y) &= 2a_\lambda \lambda g(X, Y) + (a_\lambda - a_{-\mu})g(\varphi[\xi, X], Y) \\ &= 2a_\lambda \lambda g(X, Y) - (a_\lambda - a_{-\mu})g([\xi, X], \varphi Y). \end{aligned}$$

It now follows from Lemma 3.5 that

$$2g(h'X, Y) = \left[2a_\lambda \lambda - (a_\lambda - a_{-\mu}) \left(\frac{1 - \lambda^2 - c_\lambda}{\lambda + \mu} + \lambda + 1 \right) \right] g(X, Y).$$

In the subcase $\mu \neq \lambda$, we know that $g(X, Y) = 0$, so $g(h'X, Y) = 0$.

In the subcase $\mu = \lambda \neq 0$, we have that

$$\begin{aligned} g(h'X, Y) &= \frac{1}{2} \left[2a_\lambda \lambda - (a_\lambda - a_{-\lambda}) \left(\frac{1 - \lambda^2 - c_\lambda}{2\lambda} + \lambda + 1 \right) \right] g(X, Y) \\ &= \frac{(4\lambda^2 - (\lambda + 1)^2 + c_\lambda)a_\lambda^2 + (\lambda + 1)^2 - c_\lambda}{4\lambda a_\lambda} g(X, Y), \end{aligned}$$

thus ending the proof. □

Lemma 3.7. *Let $M(\varphi, \xi, \eta, g)$ be a locally homogeneous contact metric manifold. Assume that $l_{|D(\lambda)} = c_\lambda Id$ for every eigenvalue λ of $h_{|D}$. Then,*

$$\rho_{-\lambda} = -\rho_\lambda, \tag{3.5}$$

for every eigenvalue $\lambda \in \mathcal{S}$.

Proof. Equation (3.5) can be obtained by substituting (3.2) and $a_{-\lambda} = \frac{1}{a_\lambda}$ in (3.3). □

We can give sufficient and necessary conditions for the deformed structures to be K -contact.

Theorem 3.8. *Let $M(\varphi, \xi, \eta, g)$ be a locally homogeneous contact metric manifold. Assume that $l_{|D(\lambda)} = c_\lambda Id$ for every eigenvalue λ of h . Then M admits a deformation $(\varphi', \xi, \eta, g')$ of its contact metric structure according to Definition 3.2 which is K -contact if and only if, for every positive eigenvalue λ of h , we have:*

$$c_\lambda \notin [(\lambda + 1)^2 - 4\lambda^2, (\lambda + 1)^2]. \tag{3.6}$$

In this case, such a deformation is unique and it is determined by setting:

$$a_\lambda = \sqrt{\frac{c_\lambda - (\lambda + 1)^2}{c_\lambda - (\lambda + 1)^2 + 4\lambda^2}}, \quad \text{for every } \lambda \in \mathcal{S}. \tag{3.7}$$

Proof. First we remark that, if (3.6) holds for all positive eigenvalues, then it actually holds for all non-null eigenvalues. This is a simple consequence of (3.2).

Since the structure $(\varphi', \xi, \eta, g')$ is contact metric, then it is K -contact if and only if $h' = 0$. By Proposition 3.6, this is equivalent to requiring that $\rho_\lambda = 0$ for all $\lambda \neq 0$, that is, that

$$(c_\lambda - (\lambda + 1)^2 + 4\lambda^2)a_\lambda^2 = c_\lambda - (\lambda + 1)^2,$$

for all $\lambda \neq 0$.

Assume first that $h' = 0$. Then, observe that, for each non-null λ , we must have $c_\lambda \neq (\lambda + 1)^2 - 4\lambda^2$, because otherwise we should also have $c_\lambda = (\lambda + 1)^2$; but that would mean that $(\lambda + 1)^2 - 4\lambda^2 = (\lambda + 1)^2$ and therefore $\lambda = 0$, which is a contradiction.

Hence we obtain:

$$a_\lambda^2 = \frac{c_\lambda - (\lambda + 1)^2}{c_\lambda - (\lambda + 1)^2 + 4\lambda^2},$$

for all $\lambda \neq 0$.

This is possible provided the right-hand side of this equality is positive for all $\lambda \neq 0$, which is equivalent to the condition (3.6). Vice versa, if condition (3.6) holds true, a deformation of the contact metric structure can be defined by the formula in (3.7), since the argument of each square root is positive. Moreover, if $\lambda = 0$ is in the spectrum of h , we have $a_0 = 1$ and,

thanks to Eq. (3.2), we obtain that $a_{-\lambda} = \frac{1}{a_\lambda}$ for all λ , thus satisfying all the conditions of Definition 3.2. Clearly, this deformed structure is K -contact by construction. \square

We can also give sufficient conditions for the locally homogeneous, pseudo-Hermitian manifolds with $[Q, \varphi] = 0$ to carry a K -contact structure.

Theorem 3.9. *If M is a locally homogeneous, pseudo-Hermitian manifold with $[Q, \varphi] = 0$, then M admits a K -contact structure, compatible with the original contact form, provided*

$$Ric(\xi, \xi) > 2n(1 - n). \tag{3.8}$$

Proof. From (3.8) we see that, for each positive $\lambda \in \mathcal{S}$, we have $\lambda < n$. Indeed, assume by contradiction $\lambda \geq n$ for some positive λ . From (2.4), we have $Ric(\xi, \xi) = 2n - tr(h^2)$, which implies $tr(h^2) \geq 2n^2$ and so $Ric(\xi, \xi) \leq 2n - 2n^2$, contrary to the hypothesis. On the other hand, since $[Q, \varphi] = 0$, the Jacobi operator l is determined according to (2.5) and we have at once that

$$c_\lambda < (\lambda + 1)^2 - 4\lambda^2,$$

for every positive $\lambda \in \mathcal{S}$. Hence, according to Theorem 3.8, we can perform a deformation of the contact metric structure in order to get a K -contact metric structure on M . \square

Since a torus does not admit a K -contact structure, we obtain the following result.

Corollary 3.10. *A torus of $\dim > 3$ cannot carry a Ricci flat, locally homogeneous, pseudo-Hermitian structure.*

Next we determine the covariant derivative $\nabla'_\xi h'$ of the deformed contact metric structures.

Proposition 3.11. *Under the same assumptions of Proposition 3.6, for every λ, μ eigenvalues of h and for every $X \in D(\lambda)$ and $Y \in D(\mu)$, we have:*

$$g((\nabla'_\xi h')X, Y) = \begin{cases} -\frac{\rho_\lambda}{2\lambda}((c_\lambda - (\lambda + 1)^2 + 4\lambda^2)a_\lambda^2 + 4\lambda a_\lambda \\ \quad + c_\lambda - (\lambda + 1)^2)g(\varphi X, Y), & \text{if } \lambda = -\mu \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

Proof. Observe first that h' is symmetric also with respect to the original metric g , owing to Proposition 3.6. Thus, given $X \in D(\lambda)$ and $Y \in D(\mu)$, by the definition of h' we have that:

$$g((\nabla'_\xi h')X, Y) = \rho_\lambda g(\nabla'_\xi X, Y) - g((\nabla'_\xi X), h'Y) = (\rho_\lambda - \rho_\mu)g(\nabla'_\xi X, Y).$$

If $\lambda = \mu$, then $\rho_\lambda = \rho_\mu$ and thus $g((\nabla'_\xi h')X, Y) = 0$.

If $\lambda \neq \mu$, then using standard properties of the Levi-Civita connection ∇' and Eq. (2.1) applied to the deformed structure gives:

$$g((\nabla'_\xi h')X, Y) = (\rho_\lambda - \rho_\mu)[g([\xi, X], Y) - g(\varphi'X, Y) - g(\varphi'h'X, Y)].$$

It follows from the definition of φ' and from Proposition 3.6 that

$$g((\nabla'_\xi h')X, Y) = (\rho_\lambda - \rho_\mu)[g([\xi, X], Y) - (a_\lambda + a_\lambda\rho_\lambda)g(\varphi X, Y)].$$

Since $\lambda \neq \mu$, we have by Lemma 3.5 that

$$\begin{aligned} g([\xi, X], Y) &= -g([\xi, X], \varphi(\varphi Y)) = -\left(\frac{1 - \lambda^2 - c_\lambda}{\lambda - \mu} + \lambda + 1\right)g(X, \varphi Y) \\ &= \left(\frac{1 - \lambda^2 - c_\lambda}{\lambda - \mu} + \lambda + 1\right)g(\varphi X, Y). \end{aligned}$$

If we substitute this last equation in the previous one, we obtain

$$g((\nabla'_\xi h')X, Y) = (\rho_\lambda - \rho_\mu)\left(\frac{1 - \lambda^2 - c_\lambda}{\lambda - \mu} + \lambda + 1 - a_\lambda - a_\lambda\rho_\lambda\right)g(\varphi X, Y).$$

Now, in the subcase $\lambda \neq -\mu$, we know that $\varphi X \in D(-\lambda)$ and $Y \in D(\mu)$ are orthogonal, and therefore $g(\varphi X, Y) = 0$, so that $g((\nabla'_\xi h')X, Y) = 0$.

In the subcase $\lambda = -\mu$, we have that $\varphi X, Y \in D(-\lambda)$, and using (3.5) gives us

$$g((\nabla'_\xi h')X, Y) = \frac{\rho_\lambda}{\lambda}((\lambda + 1)^2 - c_\lambda - 2\lambda(\rho_\lambda + 1)a_\lambda)g(\varphi X, Y),$$

where $\lambda = -\mu \neq 0$ because $\lambda \neq \mu$. Then, Eq. (3.9) follows from (3.3) and the definition of ρ_λ . □

4. Main Results

Theorem 4.1. *Let $M(\varphi, \xi, \eta, g)$ be a locally homogeneous, regular contact metric manifold. Assume that the Jacobi operator l satisfies:*

$$l_{|D(\lambda)} = c_\lambda Id,$$

for every eigenvalue of h .

If, for each positive eigenvalue λ of h , we have $c_\lambda \neq (\lambda + 1)^2$ and, for at least one of them, we have:

$$c_\lambda \in [(\lambda + 1)^2 - 4\lambda^2, (\lambda + 1)^2), \tag{4.1}$$

then M is not compact.

Proof. Consider a locally homogeneous, regular contact metric manifold whose operator h behaves as in the statement.

We consider first the case where $\nabla_\xi h = 0$. In this case, according to (2.2), we have $l = Id - h^2$, so that

$$c_\lambda = 1 - \lambda^2$$

and, by assumption, h admits at least a positive eigenvalue $\lambda_0 \geq 1$. We shall adopt a reasoning similar to the proof of [1, Theorem 10.12].

Assume by contradiction that M is compact. Hence, by the regularity assumption, M is a principal fibre bundle over a symplectic manifold B with group and fibre \mathcal{S}^1 ([8]). More precisely, the structure group \mathcal{S}^1 is generated by a new Reeb vector field $\bar{\xi}$, which is associated to a suitable contact form $\bar{\eta} = f\eta$, with $f \neq 0$ a scalar function.

Fix $p \in M$, and let $\gamma : \mathbb{R} \rightarrow M$ be the maximal integral curve of ξ passing through p . Then γ is closed ([8, p. 722]) and therefore a periodic geodesic. Fix a unit eigenvector $v \in T_pM$ of h with eigenvalue λ_0 . Then v can be extended to a parallel vector field $X = X(t)$ along γ , so that $X(0) = v$. Since $\nabla_\xi h = 0$, for each $t \in \mathbb{R}$, $X(t)$ is also an eigenvector of h with the same eigenvalue λ_0 . Since $\nabla_\xi \varphi = 0$, $\varphi X(t)$ is also parallel and an eigenvector with eigenvalue $-\lambda_0$, for all t .

On the other hand, v can also be extended to a vector field $Y \in \mathfrak{X}(M)$, which is a section of the contact distribution D such that $[Y, \xi] = 0$. Indeed, consider the bundle projection $\pi : (M, \bar{\eta}) \rightarrow B$ and the vector $u = \pi_*(v)$. Then u can be extended to a vector field Z on B , and Z admits a unique horizontal lift $Y = Z^*$ ([10, p. 64]), with respect to the connection form $\bar{\eta}$. By construction, $Y_p = v$ and moreover, Y is invariant under the flow of $\bar{\xi}$, by a general property of horizontal lifts ([10, Prop. 1.2]). Hence $[Y, \bar{\xi}] = 0$. Now, $\xi = f\bar{\xi}$, and thus

$$[Y, \xi] = (Yf)\bar{\xi} + f[Y, \bar{\xi}] = (Yf)\bar{\xi}.$$

But $[Y, \xi]$ is in the contact distribution D , since Y is, so that $[Y, \xi] = 0$.

Now consider the functions α and β defined as:

$$\alpha(t) = g(Y_{\gamma(t)}, X(t)), \quad \beta(t) = g(Y_{\gamma(t)}, \varphi X(t)).$$

Then $\alpha(0) = g(v, v) = 1$ and $\beta(0) = g(v, \varphi v) = 0$. Since M is compact, the norm of Y must be bounded on M , and thus α and β are also bounded:

$$\alpha(t) \leq \|Y_{\gamma(t)}\| \cdot \|X(t)\| \leq \|Y_{\gamma(t)}\|, \quad \beta(t) \leq \|Y_{\gamma(t)}\| \cdot \|\varphi X(t)\| \leq \|Y_{\gamma(t)}\|.$$

We compute

$$\begin{aligned} \dot{\alpha} &= g(\nabla_\xi Y, X) = g(\nabla_Y \xi, X) = -g(\varphi Y, X) - g(\varphi h Y, X) \\ &= g(Y, \varphi X) - \lambda_0 g(Y, \varphi X), \end{aligned}$$

which means that

$$\dot{\alpha} = (1 - \lambda_0)\beta.$$

In a similar fashion, we also get:

$$\dot{\beta} = -(\lambda_0 + 1)\alpha.$$

If $\lambda_0 = 1$, then $\dot{\alpha} = 0$ and $\dot{\beta} = -2\alpha$, so $\alpha(t) = 1$ and $\beta(t) = -2t$. But this would mean that β is unbounded above, which is impossible.

If $\lambda_0 > 1$, then $1 - \lambda_0^2 < 0$ and we have that

$$\ddot{\alpha} + (1 - \lambda_0^2)\alpha = 0, \text{ with } \alpha(0) = 1 \text{ and } 1 - \lambda_0^2 < 0.$$

Therefore, $\alpha(t) = (1 - c)e^{\sqrt{\lambda_0^2 - 1} \cdot t} + ce^{-\sqrt{\lambda_0^2 - 1} \cdot t}$, for some constant c . But this would mean that α is unbounded above, which is again impossible. Therefore, M cannot be compact.

Now we come to the general case. We shall construct a deformation $(\varphi', \xi, \eta, g')$ as in Definition 3.2, for which $\nabla'_\xi h' = 0$. Observe that it suffices to impose that the restriction of $\nabla'_\xi h'$ to the contact subbundle vanishes.

Given $X \in D(\lambda)$ and $Y \in D(\mu)$, we will now examine when $g((\nabla'_\xi h')X, Y) = 0$ holds. Firstly, we know from Eq. (3.9) that the left-hand side can be written as

$$g((\nabla'_\xi h')X, Y) = \begin{cases} -\frac{\rho_\lambda}{2\lambda}((c_\lambda - (\lambda + 1)^2 + 4\lambda^2)a_\lambda^2 + 4\lambda a_\lambda + c_\lambda - (\lambda + 1)^2)g(\varphi X, Y), & \text{if } \lambda = -\mu \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\nabla'_\xi h' = 0$ if and only if

$$\rho_\lambda((c_\lambda - (\lambda + 1)^2 + 4\lambda^2)a_\lambda^2 + 4\lambda a_\lambda + c_\lambda - (\lambda + 1)^2) = 0, \text{ for every } \lambda \neq 0. \tag{4.2}$$

Now, we know from Eq. (3.3) that $\rho_\lambda = 0$ is equivalent to

$$a_\lambda^2 = \frac{c_\lambda - (\lambda + 1)^2}{c_\lambda - (\lambda + 1)^2 + 4\lambda^2}, \tag{4.3}$$

which is possible if $c_\lambda \notin ((\lambda + 1)^2 - 4\lambda^2, (\lambda + 1)^2)$. We remark that, by (3.2), this condition is invariant under the change $\lambda \rightarrow -\lambda$, and, under this change, the positive solution to the Eq. (4.3) transforms to its inverse.

On the other hand, $(c_\lambda - (\lambda + 1)^2 + 4\lambda^2)a_\lambda^2 + 4\lambda a_\lambda + c_\lambda - (\lambda + 1)^2 = 0$ is a second-order equation of the type

$$\nu_\lambda a_\lambda^2 + 4\lambda a_\lambda - \mu_\lambda = 0, \tag{4.4}$$

with coefficients

$$\nu_\lambda := c_\lambda - (\lambda + 1)^2 + 4\lambda^2, \quad \mu_\lambda := (\lambda + 1)^2 - c_\lambda.$$

If $c_\lambda \in ((\lambda + 1)^2 - 4\lambda^2, (\lambda + 1)^2)$, this equation admits a unique positive solution a_λ . This condition is again invariant under the change $\lambda \rightarrow -\lambda$. Moreover,

$$\nu_{-\lambda} = \mu_\lambda, \quad \mu_{-\lambda} = \nu_\lambda$$

and this guarantees again that the positive solution to the Eq. (4.4) transforms to its inverse.

Next, we consider the case where λ is a positive eigenvalue for which $c_\lambda = (\lambda + 1)^2 - 4\lambda^2$. Then $\nu_\lambda = 0$ and $\mu_\lambda = 4\lambda^2$, hence the Eq. (4.4) admits the unique solution

$$a_\lambda = \lambda.$$

Notice that, for $-\lambda$, we have $c_{-\lambda} = (1-\lambda)^2$, so that the corresponding Eq. (4.4) admits the unique solution $\frac{1}{\lambda}$.

In conclusion, we can now choose, for each eigenvalue λ of h , an appropriate $a_\lambda > 0$ so that the deformed structure $(\varphi', \xi, \eta, g')$ satisfies $\nabla'_\xi h' = 0$. Firstly, we define $a_\lambda = 1$ if $\lambda = 0$. If $\lambda \neq 0$, and $c_\lambda \notin ((\lambda + 1)^2 - 4\lambda^2, (\lambda + 1)^2)$, we consider the positive solution to (4.3); if $c_\lambda \in ((\lambda + 1)^2 - 4\lambda^2, (\lambda + 1)^2)$, we choose the positive solution to (4.4). If $\lambda > 0$ is an eigenvalue for which $c_\lambda = (\lambda + 1)^2 - 4\lambda^2$, we choose $a_\lambda = \lambda$ and $a_{-\lambda} = \frac{1}{\lambda}$. Then, we have that $a_{-\lambda} = \frac{1}{a_\lambda}$ in all cases, so that the deformed structure is well defined.

Lastly, we have by hypothesis (4.1) that there is at least one $\lambda > 0$ such that c_λ belongs to the interval $[(\lambda + 1)^2 - 4\lambda^2, (\lambda + 1)^2)$. Then a_λ satisfies Eq. (4.4) with $\nu_\lambda \geq 0$ and thus h' admits the eigenvalue

$$\rho_\lambda = \frac{\nu_\lambda a_\lambda^2 + \mu_\lambda}{4\lambda a_\lambda} = 1 + \frac{\nu_\lambda a_\lambda}{2\lambda} \geq 1.$$

So we are reduced to the particular case already discussed, yielding that M cannot be compact. □

Corollary 4.2. *A locally homogeneous, regular contact metric manifold with vanishing Jacobi operator is not compact.*

Proof. Indeed, the previous result includes the case $l = 0$, because, in that case, Eq. (2.2) yields $h^2 = Id$, so that $\lambda = 1$ is the unique positive eigenvalue of h . Therefore, $c_1 = 0$ and (4.1) is satisfied. □

Another consequence of our main result is the following.

Corollary 4.3. *A locally homogeneous, regular contact metric manifold whose ξ -sectional curvatures satisfy:*

$$K(\xi, X) = K(\xi, \varphi X) < 0, \tag{4.5}$$

for every tangent vector X orthogonal to ξ , is not compact.

Proof. It follows from (4.5) that

$$g((l + h^2 - Id)X, X) = 0,$$

for every tangent vector X orthogonal to ξ . Therefore, for tangent vector fields X, Y orthogonal to ξ ,

$$0 = g((l + h^2 - Id)(X + Y), X + Y) = 2g((l + h^2 - Id)X, Y),$$

which implies that $l = Id - h^2$ in the contact subbundle.

So $l|_{D(\lambda)} = c_\lambda Id$, with $c_\lambda = 1 - \lambda^2$, for every eigenvalue of h . Moreover, $c_\lambda < 0$ since $K(\xi, X) = g(lX, X) = c_\lambda g(X, X) < 0$ by assumption. Hence $\lambda > 1$ and (4.1) holds true for all positive eigenvalues of h , and thus Theorem 4.1 is applicable. □

Corollary 4.4. *Let M be a regular non-Sasakian (κ, μ) -space. If $-1 < I_M \leq 1$, then M is not compact.*

Proof. After performing a D_a -homothetic deformation of the contact metric structure, which preserves the Boeckx invariant, we may suppose that $\kappa = 0$. So $l = \mu h$ and $I_M = 1 - \frac{\mu}{2}$ and it follows that $\mu \in [0, 4)$. Moreover, $\kappa = 0$ means that h has a unique positive eigenvalue $\lambda = 1$, so all the hypotheses of Theorem 4.1 hold, yielding the conclusion. \square

Next we discuss some applications of Theorem 4.1 to pseudo-Hermitian geometry.

Theorem 4.5. *Let M be locally homogeneous, regular pseudo-Hermitian manifold of dimension $2n + 1$, whose Ricci operator Q satisfies:*

$$[Q, \varphi] = 0.$$

Then M is not compact, provided that h admits at least one eigenvalue $\geq n$.

Proof. Under the above assumptions, (2.5) holds and we see at once that all the hypotheses of Theorem 4.1 are satisfied, yielding that M is not compact. \square

A similar argument applies also to the following situation:

Theorem 4.6. *Let M be locally homogeneous, regular pseudo-Hermitian manifold of dimension $2n + 1$, whose Ricci operator Q satisfies:*

$$Q = \alpha Id + \beta \eta \otimes \xi + \gamma h, \quad \alpha, \beta, \gamma \in \mathbb{R}.$$

Then M is not compact, provided $\frac{\gamma-2n}{2}$ is not in the spectrum of h and h admits at least one positive eigenvalue in the interval $[-\frac{\gamma-2n}{2}, \frac{\gamma-2n}{2})$.

Proof. Assuming the above expression for Q , one gets the following identity on the contact subbundle:

$$l = Id - h^2 + (\gamma + 2(1 - n))h.$$

The rest of the proof is similar to that of Theorem 4.5. \square

Theorem 4.7. *A locally homogeneous, regular, η -Einstein, pseudo-Hermitian manifold of dimension $2n + 1$ that satisfies*

$$Ric(\xi, \xi) \leq 2n(1 - n^2) \tag{4.6}$$

is not compact.

Proof. This follows from Theorem 4.5. Indeed, the hypothesis $[Q, \varphi] = 0$ holds true because the manifold is η -Einstein. There must be an eigenvalue of h greater than or equal to n because (2.4) and the hypothesis mean that

$$Ric(\xi, \xi) = 2n - tr(h^2) \leq 2n(1 - n^2)$$

and thus $tr(h^2) \geq 2n^3$. \square

Lastly, we prove that the inequality (4.6) is optimal.

Proposition 4.8. *For any $n \geq 1$ and r with $2n(1 - n^2) < r \leq 2n$, there exists a homogeneous, η -Einstein, pseudo-Hermitian manifold of dimension $2n + 1$ that satisfies $\text{Ric}(\xi, \xi) = r$ and is compact.*

Proof. Let us distinguish two cases: $r = 2n$ and $2n(1 - n^2) < r < 2n$.

Case $r = 2n$. It suffices to consider a sphere S^{2n+1} endowed with the standard Sasakian structure.

Case $2n(1 - n^2) < r < 2n$. If we define $c := \frac{(\sqrt{2n^3} - \sqrt{2n-r})^2}{2n^3 - (2n-r)}$, then $0 < c < 1$ and we can take a sphere $S = S^{n+1}$ with constant curvature c . Then its tangent sphere bundle $M = T_1S$, with the standard contact metric structure, is a compact homogeneous (κ, μ) -space M^{2n+1} with $\kappa = c(2 - c) < 1$ and $\mu = -2c < 0$ ([2]). If we now perform a D_a -homothetic deformation with constant $a = \frac{c+1}{n} > 0$, then it follows from (2.6) that the deformed manifold is a non-Sasakian (κ', μ') -space with $\kappa' = \frac{r}{2n} < 1$ and $\mu' = 2(1 - n)$ and thus it is η -Einstein. Since $Q'\xi' = 2n\kappa'\xi' = r\xi'$, it also satisfies that $\text{Ric}'(\xi', \xi') = r$. \square

Funding Open access funding provided by Università degli Studi di Bari Aldo Moro within the CRUI-CARE Agreement.

Data availability statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- [1] Blair, D.E.: Riemannian geometry of contact and symplectic manifolds, 2nd edn. Progress in Mathematics 203. Birkhäuser, Boston (2010)
- [2] Blair, D.E., Koufogiorgos, T., Papantoniou, B.J.: Contact metric manifolds satisfying a nullity condition. Israel J. Math. **91**, 189–214 (1995)

- [3] Blair, D.E., Perrone, D.: Conformally Anosov flows in contact metric geometry. *Balkan J. Geom. Appl.* **3**, 33–46 (1998)
- [4] Boeckx, E.: A class of locally φ -symmetric contact metric spaces. *Arch. Math.* **72**, 466–472 (1999)
- [5] Boeckx, E.: A full classification of contact metric (κ, μ) -spaces. *Illinois J. Math.* **44**, 212–219 (2000)
- [6] Boeckx, E., Vanhecke, L.: Geometry of Riemannian manifolds and their unit sphere bundles. *Publ. Math. Debrecen* **57**(3–4), 509–533 (2000)
- [7] Boeckx, E., Perrone, D., Vanhecke, L.: Unit tangent sphere bundles and two-point homogeneous spaces. *Period. Math. Hungar.* **36**, 79–95 (1998)
- [8] Boothby, W.M., Wang, H.C.: On contact manifolds. *Ann. of Math.* **2**(68), 721–734 (1958)
- [9] Cho, J.T.: Contact hypersurfaces and CR-symmetry. *Ann. Mat. Pura Appl.* (4) **199**(5), 1873–1884 (2020)
- [10] Kobayashi, S., Nomizu, K.: *Foundations of differential geometry*, vol. I. John Wiley & Sons, New York (1963)
- [11] Loiudice, E., Lotta, A.: Canonical fibrations of contact metric (κ, μ) -spaces. *Pacific J. Math.* **300**(1), 39–63 (2019)

Antonio Lotta
Dipartimento di Matematica
Università degli Studi di Bari Aldo Moro
Via E. Orabona 4
70125 Bari
Italy
e-mail: antonio.lotta@uniba.it

Verónica Martín-Molina
Departamento de Didáctica de las Matemáticas,
Facultad de Ciencias de la Educación, c/ Pirotecnia s/n
Universidad de Sevilla
Sevilla 41013
Spain
e-mail: veronicamartin@us.es

Received: June 25, 2021.

Accepted: May 25, 2022.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.