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Residually solvable extensions of pro-nilpotent Leibniz superalgebras ☆

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ABSTRACT

Throughout this paper we show that the method for describing finite-dimensional solvable Leibniz superalgebras with a given nilradical can be extended to infinite-dimensional ones, or so-called residually solvable Leibniz superalgebras. Prior to that, we improve the solvable extension method for the finite-dimensional case obtaining new and important results. Additionally, we fully determine the residually solvable Lie and Leibniz superalgebras with maximal codimension of pro-nilpotent ideals the model filiform Lie and null-filiform Leibniz superalgebras, respectively. Moreover, we prove that the residually solvable superalgebras obtained are complete.

1. Introduction

In general, algebraic tools are very useful in the study of elementary particles in quantum mechanics, in analyzing the properties of solids and crystals and also they can be found in problems of population biology for instance. In particular and since associative algebras defined by a certain identity were started after revealing the property of being closed relative to the usual multiplication of square matrices, their further intensive development led to the creation of the theory of alternative, Lie, Jordan algebras and superalgebras, which are closely intertwined with each other and have numerous connections with various areas of mathematics.

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Remark that the study of solvable Lie algebras with special types of nilradicals is associated with various models of physics. The motivation for studying Lie superalgebras, on the other hand, arose from the properties of supersymmetry in mathematical physics. Moreover, the theory of Lie superalgebras has stablished itself as a universal object in modern algebra. Thus, similarly to the case of Lie algebras, the study of solvable Lie superalgebras with given nilradicals is an urgent problem.

First, we recall that in the 50s it was proved by A.I. Malcev that a solvable Lie algebra is fully determined by its nilradical [19], and after that was stablished the method for describing solvable Lie algebra in terms of its nilradical and its nil-independent derivations [22]. Since then, the aforementioned method has been used in numerous papers to obtain the classification of solvable Lie algebras with different types of nilradical [5,6,23,26–28]. More recently, the solvable extension method was extended to Leibniz algebras and a great deal of papers have been devoted to it [9,12,13,18]. Special mention deserves the solvable Lie and Leibniz algebras with maximal codimension of a given nilradical, due to its properties, they are in some cases complete and cohomologically rigid [4]. The next natural step was to extend the method for superalgebras but let us note that the structures of solvable Lie and Leibniz superalgebras are more complex than structures of the corresponding solvable algebras [25]. In fact, Lie's theorem is not true for solvable Leibniz superalgebras and even the square of a solvable superalgebra can be non-nilpotent [24]. Despite all the difficulties, recently in [10,11] the solvable extension method for Leibniz superalgebras was stablished. Of special relevance are those superalgebras of maximal codimension of nilradical in the same way as occurs for algebras.

Thus, having analyzed the structure of solvable finite-dimensional Leibniz superalgebras with a given nilradical and the maximal complementary subspace to it, we wonder if this structure can be extended to the infinite-dimensional case, or so-called residually solvable Leibniz superalgebras. With respect to the nilradical we consider the infinite-dimensional analog of nilpotent superalgebras, that is, pro-nilpotent superalgebras. Recall that pro-nilpotent superalgebras are defined by two properties: the intersection of all members of the lower central series is zero (so-called residually nilpotent property), and the quotient algebra by any member of the central series is finite-dimensional.

In this paper firstly, we obtain some important results regarding the solvable extension method for the finite-dimensional case. After, we give an explicit structure of residually solvable Lie superalgebras whose maximal pro-nilpotent ideal is infinite-dimensional model filiform Lie superalgebra. Recall that, on one hand, the filiform Lie superalgebra is one of most relevant nilpotent Lie superalgebra [8] and on the other hand, Lie superalgebras are particular cases of Leibniz superalgebras. Next, we consider residually solvable Leibniz non-Lie superalgebras obtaining all the residually solvable Leibniz superalgebras whose maximal pro-nilpotent ideal is infinite-dimensional null-filiform Leibniz superalgebra. Additionally, we prove that the residually solvable superalgebras obtained, Lie and non-Lie Leibniz, are complete.

2. Preliminary results

In this section both notions and results which are the same for Lie and Leibniz superalgebras will be given only for Leibniz superalgebras.

A vector space V is said to be $\mathbb{Z}_2$-*graded* if it admits a decomposition into a direct sum, $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $\bar{0}, \bar{1} \in \mathbb{Z}_2$. An element $x \in V$ is called *homogeneous of degree* $\bar{i}$ if it is an element of $V_{\bar{i}}$, $\bar{i} \in \mathbb{Z}_2$. In particular, the elements of $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are also called *even* (resp. *odd*). For a homogeneous element $x \in V$ we denote $|x|$ the degree of x (either $\bar{0}$ or $\bar{1}$).

Definition 2.1. A *Lie superalgebra* (see [15]) is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, with an even bilinear commutation operation (or "supercommutation") $[\cdot,\cdot]$, which for an arbitrary homogeneous elements x, y, z satisfies the conditions

1. $[x, y] = -(-1)^{|x||y|}[y, x]$,
2. $(-1)^{|z||x|}[x, [y, z]] + (-1)^{|x||y|}[y, [z, x]] + (-1)^{|y||z|}[z, [x, y]] = 0$ *(super Jacobi identity).*

Thus, $\mathfrak{g}_{\bar{0}}$ is an ordinary Lie algebra, and $\mathfrak{g}_{\bar{1}}$ is a module over $\mathfrak{g}_{\bar{0}}$; the Lie superalgebra structure also contains the symmetric pairing $S^2\mathfrak{g}_{\bar{1}} \longrightarrow \mathfrak{g}_{\bar{0}}$.

Definition 2.2. [3]. A $\mathbb{Z}_2$-graded vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is called a *Leibniz superalgebra* if it is equipped with a product $[\cdot,\cdot]$ which for an arbitrary element x and homogeneous elements y, z satisfies the condition

$$[x, [y, z]] = [[x, y], z] - (-1)^{|y||z|}[[x, z], y] \quad \textit{(super Leibniz identity)}.$$

Note that if a Leibniz superalgebra L satisfies the identity $[x, y] = -(-1)^{|x||y|}[y, x]$ for any homogeneous elements $x, y \in L$, then the super Leibniz identity becomes the super Jacobi identity. Consequently, Leibniz superalgebras are a generalization of Lie superalgebras. Also and in the same way as for Lie superalgebras, isomorphisms are assumed to be consistent with the $\mathbb{Z}_2$-graduation.

Definition 2.3. For a Leibniz superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ we define the *right annihilator of* L as the set $Ann_r(L) := \{x \in L : [L, x] = 0\}$.

It is easy to see that $Ann_r(L)$ is a two-sided ideal of L and $[x,x] \in Ann_r(L)$ for any $x \in L_{\bar{0}}$. This notion is compatible with the right annihilator in Leibniz algebras. If we consider the ideal $I := ideal < [x,y] + (-1)^{|x||y|}[y,x] >$, then $I \subset Ann_r(L)$ and we have a Lie superalgebra L/I.

Let us now denote by R_x the right multiplication operator, i.e., $R_x : L \to L$ given as $R_x(y) := [y,x]$ for $y \in L$, then the super Leibniz identity can be expressed as $R_{[x,y]} = R_y R_x - (-1)^{|x||y|} R_x R_y$.

It should be noted that in the case of Lie (super)algebras, instead of the operator of right multiplication R_x it is considered the operator of left multiplication denoted by ad_x.

If we denote by $R(L)$ the set of all right multiplication operators, then $R(L)$ with respect to the following multiplication

$$< R_a, R_b > := R_a R_b - (-1)^{|a||b|} R_b R_a \tag{2.1}$$

forms a Lie superalgebra.

Let us recall the definition of superderivations for Leibniz superalgebras [15,17]. A superderivation of degree s of a Leibniz superalgebra L, $s \in \mathbb{Z}_2$, is an endomorphism $d \in End_s(L)$ with the property

$$d([x,y]) = (-1)^{s|y|}[d(x), y] + [x, d(y)]$$

If we denote $Der_s(L) \subset End_s(L)$ the space of all superderivations of degree s, then $Der(L) = Der_{\bar{0}}(L) \oplus Der_{\bar{1}}(L)$ is the Lie superalgebra of superderivations of L, with $Der_{\bar{0}}(L)$ composed by even superderivations and $Der_{\bar{1}}(L)$ by odd ones.

Note that for any $z \in L$ the operator R_z is a superderivation (such kind of superderivations are called inner). It follows from the following equality

$$R_z([x,y]) = (-1)^{|z||y|}[R_z(x), y] + [x, R_z(y)],$$

which can be rewritten as the super (graded) Leibniz identity

$$[[x,y],z] = (-1)^{|z||y|}[[x,z],y] + [x,[y,z]].$$

Recall, the *descending central sequence* of a Leibniz superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is defined in the same way as for Leibniz algebras: $\mathcal{C}^0(L) := L$, $\mathcal{C}^{k+1}(L) := [\mathcal{C}^k(L), L]$ for all $k \geq 0$, [7]. Consequently, if $\mathcal{C}^k(L) = \{0\}$ for some k, then the Leibniz superalgebra is called *nilpotent*. Then, the smallest integer k such that $\mathcal{C}^k(L) = \{0\}$ is called the *nilindex* of the Leibniz superalgebra L.

In the same way, the *derived sequence* of L is defined by $\mathcal{D}^0(L) := L$, $\mathcal{D}^{k+1}(L) := [\mathcal{D}^k(L), \mathcal{D}^k(L)]$ for all $k \geq 0$. If this sequence is stabilized in zero, then the Leibniz superalgebra is said to be *solvable*. All nilpotent Leibniz superalgebras are solvable ones.

Engel's theorem and its direct consequences remain valid for Leibniz superalgebras. In particular, a Leibniz superalgebra L is nilpotent if and only if R_x is nilpotent for every homogeneous element x of L. Moreover, for solvable Leibniz superalgebras we have that a Leibniz superalgebra L is solvable if and only if its Leibniz algebra $L_{\bar{0}}$ is solvable. Nevertheless, we do not have the analog of Lie's Theorem and neither its corollaries even for solvable Lie superalgebras.

Definition 2.4. A nilpotent Lie superalgebra is called characteristically nilpotent if all its superderivations are nilpotent.

Recall, a Leibniz superalgebra is said to be *complete*, if all its superderivations are inner and center of the superalgebra is trivial.

3. Solvable extensions of finite-dimensional nilpotent Lie and Leibniz superalgebras

In this section we establish some additional results on solvable Leibniz superalgebras obtained in [10] and we extend them to the infinite-dimensional analogous of solvable Leibniz superalgebras.

Firstly, we consider Lie superalgebras which are a particular case of Leibniz superalgebras. Note that structure of solvable Lie superalgebras is much more complicated than structure of solvable Lie algebras. Indeed, for solvable Lie superalgebras the analog of Lie's theorem and its corollaries are not true. Moreover, for a solvable Lie superalgebra its derived superalgebra is not nilpotent, in general. It should be noted that imposing the constraint that derived superalgebra of solvable Lie superalgebra is nilpotent, we have the following results showed in [10].

Let L be a solvable Lie superalgebra such that L^2 is nilpotent, then $L^2 \subset \mathfrak{n}$, where $\mathfrak{n}$ is the maximal nilpotent ideal of L, so-called nilradical. Under this condition any solvable Lie superalgebra over the real or complex field admits a decomposition

$$L = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$$

where $\overrightarrow{\oplus}$ denotes the semidirect sum and satisfying the relations

$$[\mathfrak{t}, \mathfrak{n}] \subset \mathfrak{n}, \quad [\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}, \quad [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{n}.$$

By means of the super Jacobi identity it can be seen that ad_z for any $z \in \mathfrak{t}$, acts as a derivation of the nilpotent superalgebra $\mathfrak{n}$.

Since for the elements $z \notin \mathfrak{n}$ derivations ad_z are not nilpotent (otherwise we get a contradiction with maximality of $\mathfrak{n}$), for a given basis $\{z_1, \dots, z_n\}$ of $\mathfrak{t}$ and arbitrary non-null scalars $\alpha_1, \dots, \alpha_n$ it is satisfied that

$$(\alpha_1 ad_{z_1} + \cdots + \alpha_n ad_{z_n})^k \neq 0, \;\; k \geq 1,$$

i.e., the derivation $\alpha_1 ad_{z_1} + \cdots + \alpha_n ad_{z_n}$ is not nilpotent. In the same way as for Lie algebras we say that the elements $ad_{z_1}, \dots, ad_{z_n}$ are nil-independent [22].

Thus, the solvable Lie superalgebra L can be always described by means of nil-independent derivations of the nilradical $\mathfrak{n}$. Hence the dimension of a solvable Lie superalgebra having a given nilradical is bounded by the maximal number of nil-independent derivations of the nilradical.

On the other hand and with respect to Leibniz superalgebras, we extend in a natural way the above results in the same way that it has been already done for Leibniz algebras [7,12].

We explore now the differences between even and odd superderivations with respect to solvable extensions of nilpotent Lie superalgebras.

Proposition 3.1. *Let $N = N_{\bar{0}} \oplus N_{\bar{1}}$ be a nilpotent (non-characteristically nilpotent) Lie superalgebra with $\{x_1, \dots, x_n\}$ a basis of $N_{\bar{0}}$ and $\{y_1, \dots, y_m\}$ a basis of $N_{\bar{1}}$. If D is a non-nilpotent even superderivation of N, then $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with $L_{\bar{1}} := N_{\bar{1}}$ and $L_{\bar{0}} := N_{\bar{0}} \oplus (\mathbb{K}z)$ being $ad_z := D$, is a solvable Lie superalgebra with nilradical N.*

Proof. Let D be an even superderivation of N, then replacing D by ad_z in the even superderivation condition we get

$$[z,[a,b]] = [[z,a],b] + [a,[z,b]]$$

for any $a, b \in N$. Taking into account that z is an even element we shall check that this equality is nothing but the super Jacobi identity for the triples $\{z, a, b\}$. Indeed, the general super Jacobi identity

$$(-1)^{|z||x|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0$$

for the triple $\{z, a, b\}$ remains

$$[a,[b,z]] + (-1)^{|a||b|}[b,[z,a]] + [z,[a,b]] = 0.$$

Then $[z,[a,b]] = -(-1)^{|a||b|}[b,[z,a]] - [a,[b,z]] = -(-1)^{|a||b|}[b,[z,a]] + [a,[z,b]]$.

Only rest to note that $-(-1)^{|a||b|}[b,[z,a]] = [[z,a],b]$. If a is an even basis vector, then $-(-1)^{|a||b|}[b,[z,a]] = -[b,[z,a]] = [[z,a],b]$. If a and b are odd basis vectors then $-(-1)^{|a||b|}[b,[z,a]] = +[b,[z,a]] = [[z,a],b]$.

On the other hand, the super Jacobi identity for the triples $\{z, z, b\}$, $\{z, z, z\}$ trivially vanishes for being z an even basis vector. Thus, we have a Lie superalgebra L and since ad_z is non-nilpotent we conclude the proof of the statement. □

A different situation happens if we involve an odd superderivation.

Remark 3.1. The above extension method described in Proposition 3.1 does not hold in the case of an odd superderivation D. Replacing D by ad_z in the odd superderivation condition we get

$$[z,[a,b]] = [[z,a],b] + (-1)^{|a|}[a,[z,b]]$$

for any $a, b \in N$. Similarly as in the proof of Proposition 3.1, it can be seen that this condition is super Jacobi identity for the triples $\{z, a, b\}$. However, the super Jacobi identity for the triples $\{z, z, b\}$, $\{z, z, z\}$ do not vanish on account of z is an odd basis vector. Thus, the super Jacobi identity does not hold, in general. As an example we consider the abelian (and then nilpotent) Lie superalgebra $N = (\mathbb{K}x)_{\bar{0}} \oplus (\mathbb{K}y)_{\bar{1}}$ and its odd superderivation D such that $D(x) = y$ and $D(y) = x$. If we consider $L_{\bar{0}} \oplus L_{\bar{1}}$ with $L_{\bar{0}} := (\mathbb{K}x)$ and $L_{\bar{1}} := (\mathbb{K}y) \oplus (\mathbb{K}z)$ being $ad_z := D$, then we have the bracket products

$$[z,x] = -[x,z] = y, \qquad [z,y] = [y,z] = x, \qquad [z,z] = ax \qquad \text{for some } a \in \mathbb{K}.$$

Then the super Jacobi identity for the triple $\{z, z, y\}$ does not hold. Indeed, we have

$$[y,[z,z]] + [z,[z,y]] + [z,[y,z]] = 2[z,[y,z]] = 2[z,x] = 2y \neq 0.$$

Therefore, L is not a Lie superalgebra.

Moreover, in the case when by adding only odd nil-independent superderivations the super Jacobi identity holds true we get a nilpotent Lie superalgebra, that is, the resultant Lie superalgebra is always nilpotent. Thus, we have the following result.

Lemma 3.1. *Let N be a nilpotent Lie superalgebra $N = N_{\bar{0}} \oplus N_{\bar{1}}$. If L is a solvable extension Lie superalgebra $\mathfrak{t} \overrightarrow{\oplus} N$ (with $[\mathfrak{t}, N] \subset N$, $[N, N] \subset N$, $[\mathfrak{t}, \mathfrak{t}] \subset N$) by means of odd superderivations (that is $\mathfrak{t} = \mathfrak{t}_{\bar{1}}$), then L is also nilpotent.*

Proof. Suppose $DerN = Der_{\bar{0}}N \oplus Der_{\bar{1}}N$ the Lie superalgebra of superderivations of N, with $Der_{\bar{0}}N$ composed by even superderivations and $Der_{\bar{1}}N$ by odd ones. The product that provides $DerN$ with Lie superalgebra structure is

$$< d_1, d_2 >:= d_1 d_2 - (-1)^{|d_1||d_2|} d_2 d_1. \tag{3.1}$$

It is known that a Lie superalgebra is nilpotent if and only if the adjoint operator is nilpotent for any homogeneous element of the superalgebra. Therefore, in order to prove the lemma we have show that for any $z \in \mathfrak{t}_{\bar{1}}$ the operator ad_z is a nilpotent odd superderivation of N. Let us assume that there exists $z \in \mathfrak{t}_{\bar{1}}$ such that ad_z is a non-nilpotent odd superderivation of N. From the structure of Lie superalgebra over $DerL$ we have in particular that $ad_{[z,z]} = [ad_z, ad_z] = 2ad_z ad_z$ and then $ad_z^2 = \frac{1}{2} ad_{[z,z]}$ is a non-nilpotent even superderivation of N. At the same time the condition $L^2 \subseteq N$ implies that $[z, z] \in N_{\bar{0}}$. So, $ad_{[z,z]}$ is nilpotent. Thus we get a contradiction which completes the proof of the lemma. □

Theorem 3.1. *Let N be a nilpotent Lie superalgebra $N = N_{\bar{0}} \oplus N_{\bar{1}}$. If L is a solvable non-nilpotent Lie superalgebra $\mathfrak{t} \overrightarrow{\oplus} N$ with nilradical N, then $\mathfrak{t}_{\bar{1}} = \{0\}$.*

Proof. If $\mathfrak{t}_{\bar{1}} \neq \{0\}$ then we have at least a non-null basis vector $z \in \mathfrak{t}_{\bar{1}}$. From the proof of Lemma 3.1 ad_z is always nilpotent. Therefore, $N' = N'_{\bar{0}} \oplus N'_{\bar{1}}$ with $N'_{\bar{0}} := N_{\bar{0}}$ and $N'_{\bar{1}} := N_{\bar{1}} \oplus (\mathbb{K}z)$ is a nilpotent Lie superalgebra verifying $N \subset N'$ and $dim(N') = dim(N) + 1$ which contradicts the maximality of N. □

On account of Theorem 3.1 from now on, we consider only solvable extensions by means of even superderivations. Next we explore under which conditions the solvable extension method developed by Mubarakzjanov [22] remains true for Lie and Leibniz superalgebras.

Suppose given a nilpotent and non-characteristically nilpotent Lie superalgebra $N = N_{\bar{0}} \oplus N_{\bar{1}}$ verifying $[N_{\bar{1}}, N_{\bar{1}}] = \{0\}$. Then from all the possible solvable extensions $\mathfrak{t} \overrightarrow{\oplus} N$ with nilradical N ($\mathfrak{t} = \mathfrak{t}_{\bar{0}}$), the maximal-dimensional one which satisfies the condition $dim(\mathfrak{t}) = dim(N/N^2)$ is unique and will be given by considering $\mathfrak{t}_{\bar{0}}$ the maximal torus T of even derivations (which is in particular, abelian and simultaneously diagonalizable).

Note that $[N_{\bar{1}}, N_{\bar{1}}] = \{0\}$ implies that N is not only a Lie superalgebra but also a $\mathbb{Z}_2$-graded Lie algebra. Now if $\mathfrak{t}$ is composed only by even elements then for any $z \in \mathfrak{t}$ the adjoint operator ad_z is an even superderivation of the Lie superalgebra N and, in particular, a Lie derivation of N regarded as $\mathbb{Z}_2$-graded Lie algebra. Therefore, the maximal solvable extensions under the condition $dim(\mathfrak{t}) = dim(N/N^2)$ can be described in explicit form in terms of the multiplication table obtained in [16].

Remark 3.2. In fact, under the condition $dim(\mathfrak{t}) = dim(N/N^2)$, that is the dimension of $\mathfrak{t} = k$ is exactly the number of generators of the nilradical N, $R = \mathfrak{t} \overrightarrow{\oplus} N$ admits a basis

$$\{z_1, z_2, \dots z_{k_1}, z'_1, \dots, z'_{k_2}, x_1, \dots, x_{k_1}, \dots, x_n, y_1, \dots, y_{k_2}, \dots, y_m\}$$

where $\{x_1, \dots, x_{k_1}, y_1, \dots, y_{k_2}\}$ are generators of N being $k = k_1 + k_2$ and such that the table of multiplications of R has the following form:

$$\begin{cases} [x_i, x_j] = -[x_j, x_i] = \sum\limits_{t=k_1+1}^{n} \gamma_{i,j}^t x_t, & 1 \le i < j \le n, \\ [x_i, y_j] = -[y_j, x_i] = \sum\limits_{t=k_2+1}^{m} \delta_{i,j}^t y_t, & 1 \le i \le n, 1 \le j \le m, \\ [x_i, z_i] = -[z_i, x_i] = x_i, & 1 \le i \le k_1, \\ [y_j, z'_j] = -[z'_j, y_j] = y_j, & 1 \le j \le k_2, \\ [x_i, z_j] = -[z_j, x_i] = \alpha_{i,j} x_i, & k_1 + 1 \le i \le n, \ 1 \le j \le k_1, \\ [x_i, z'_j] = -[z'_j, x_i] = \alpha'_{i,j} x_i, & k_1 + 1 \le i \le n, \ 1 \le j \le k_2, \\ [y_i, z_j] = -[z_j, y_i] = \beta_{i,j} y_i, & k_2 + 1 \le i \le m, \ 1 \le j \le k_1, \\ [y_i, z'_j] = -[z'_j, y_i] = \beta'_{i,j} y_i, & k_2 + 1 \le i \le m, \ 1 \le j \le k_2, \end{cases}$$

where the omitted products are zero and

- $\alpha_{i,j}$ is the number of entries of a generator basis element x_j involved in forming non generator basis element x_i,

- $\alpha'_{i,j}$ is the number of entries of a generator basis element y_j involved in forming non generator basis element x_i,
- $\beta_{i,j}$ is the number of entries of a generator basis element x_j involved in forming non generator basis element y_i,
- $\beta'_{i,j}$ is the number of entries of a generator basis element y_j involved in forming non generator basis element y_i.

Remark 3.3. It should be noted that the above remark is also extendable for Leibniz superalgebras having been described the corresponding multiplication table in [1].

Note that the maximal solvable Lie (resp. Leibniz) superalgebras with model filiform nilradical, $SL^{n,m} = \mathfrak{t}\overrightarrow{\oplus} L^{n,m}$, obtained in [11] verifies the multiplication table of Remark 3.2. Moreover it was proved that this superalgebra is complete, that is, it is centerless and all the superderivations are inner.

Note, on the other hand, that maximal nilindex for Leibniz superalgebras is exactly $(n+m)$ - a unit greater than the maximal nilindex for Lie superalgebras - and it is obtained by the null-filiform superalgebra. Note that the only one, up to isomorphism, null-filiform Leibniz superalgebra (non Leibniz algebra) is $NF^{n,m}$, that can be expressed by the law:

$$NF^{n,m}: \begin{cases} [y_i, y_1] = x_i, & 1 \le i \le n, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \le i \le m-1, \\ [y_j, x_1] = y_{j+1}, & 1 \le j \le m-1, \\ [x_i, x_1] = x_{i+1}, & 1 \le i \le n-1, \end{cases}$$

where $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_m\}$ are bases of the even and odd parts respectively. Moreover, in order to have a non-trivial odd part we have only two possibilities for m ($m = n$ or $m = n+1$). For more details it can be consulted [3]. In [10] the authors studied the maximal complex solvable Leibniz superalgebras with nilradical $NF^{n,m}$, showing that there is only one, up to isomorphism, which corresponds exactly with $\mathfrak{t}\overrightarrow{\oplus} NF^{n,m}$:

$$SNF^{n,m}: \begin{cases} [y_i, y_1] = x_i, & 1 \le i \le n, & [y_j, x_1] = y_{j+1}, & 1 \le j \le m-1, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \le i \le m-1, & [x_i, x_1] = x_{i+1}, & 1 \le i \le n-1, \\ [x_i, z] = 2i x_i, & 1 \le i \le n, & [z, x_1] = -2x_1, & \\ [y_j, z] = (2j-1) y_j, & 1 \le j \le m, & [z, y_1] = -y_1, & \end{cases}$$

where the omitted brackets are equal to zero.

Moreover, a straightforward computation leads to the following result.

Proposition 3.2. *$SNF^{n,m}$ is a complete Leibniz superalgebra.*

Proof. By the multiplication table it is easy to check that $SNF^{n,m}$ is centerless. A straightforward computation leads to

$$Der_{\overline{0}}(SNF^{n,m}) = \langle R_z \rangle \text{ and } Der_{\overline{1}} SNF^{n,m} = \langle R_{y_1} \rangle$$

which means that all superderivations are inner. $\square$

4. Residually solvable extensions of infinite-dimensional Lie and Leibniz superalgebras

Throughout this section we explore the method to obtain an equivalent of the previous section for the case of infinite-dimensional Lie and Leibniz superalgebras. In particular, we will obtain an infinite-dimensional version of the Lie and non-Lie Leibniz superalgebras $\mathfrak{t}\overrightarrow{\oplus} L^{n,m}$ and $\mathfrak{t}\overrightarrow{\oplus} NF^{n,m}$. Additionally we are going to study whether obtained superalgebras are complete.

The following definition was used in the paper [20] for Lie algebras and after was generalized for Leibniz algebras [2]. Now we introduce these definitions for superalgebras.

Definition 4.1. A superalgebra L is called residually nilpotent (respectively, solvable) if $\bigcap\limits_{i=0}^{\infty} \mathcal{C}^i(L) = \{0\}$ (respectively, $\bigcap\limits_{i=0}^{\infty} \mathcal{D}^i(L) = \{0\}$).

Since $\mathcal{D}^i(L) \subset \mathcal{C}^i(L)$ for any i we conclude that residually nilpotency implies residually solvability. The converse does not always hold, as an example let us consider the Lie superalgebra with even and odd basis $\{e_0, e_1\} \oplus \{e_2, \dots\}$ such that it has the following multiplication table:

$$[e_0, e_1] = -[e_1, e_0] = e_0, \qquad [e_0, e_i] = -[e_i, e_0] = e_{i-1}, \ i \ge 3,$$

where the omitted products are zero. Clearly, this superalgebra is residually solvable but not residually nilpotent.

Other example which will be important in our study is the infinite-dimensional model filiform Lie superalgebra defined by the non-null bracket products that follows:

$$L:\begin{cases}[x_1,x_i]=-[x_i,x_1]=x_{i+1}, & i\geq 2,\\ [x_1,y_j]=-[y_j,x_1]=y_{j+1}, & j\geq 1,\end{cases}$$

where $\{x_1,x_2,\dots\}$ are even basis vectors and $\{y_1,y_2,\dots\}$ odd ones. Then one can check that L is residually nilpotent and then residually solvable. Remark also, that the infinite-dimensional Lie algebra even part of L was already considered in [14].

Let us introduce now the concepts of pro-nilpotent and pro-solvable for superalgebras, which was introduced in [21] for the Lie algebra case and after extended for Leibniz algebras [2].

Definition 4.2. A superalgebra L is said to be pro-nilpotent if

$$\bigcap_{i=0}^{\infty}\mathcal{C}^i(L)=\{0\} \text{ and } dim(\mathcal{C}^i(L)/\mathcal{C}^{i+1}(L))<\infty \quad \text{for any } i\geq 0$$

(respectively, pro-solvable if $\bigcap_{i=0}^{\infty}\mathcal{D}^i(L)=\{0\}$ and $dim(\mathcal{D}^i(L)/\mathcal{D}^{i+1}(L))<\infty$).

It can be easily checked that the infinite-dimensional model filiform Lie superalgebra L is pro-nilpotent but not pro-solvable. Note that $dim(\mathcal{C}^i(L)/\mathcal{C}^{i+1}(L))<\infty$ is equivalent to $dim(L/\mathcal{C}^i(L))<\infty$ (respectively, $dim(\mathcal{D}^i(L)/\mathcal{D}^{i+1}(L))<\infty$ is equivalent to $dim(L/\mathcal{D}^i(L))<\infty$).

Let us note that every quotient $L/\mathcal{C}^i(L)$ of a pro-nilpotent Lie or Leibniz superalgebra is a finite-dimensional nilpotent Lie or Leibniz superalgebra. Moreover, any pro-nilpotent superalgebra is a finitely generated superalgebra.

Definition 4.3. A linear map $\rho: L\to L$ is called residually nilpotent, if $\bigcap_{i=1}^{\infty} Im\ \rho^i=\{0\}$ holds.

Example: If we consider the infinite-dimensional model filiform Lie superalgebra L, then the adjoint operator ad_{x_2}, for instance, is clearly residually nilpotent.

Below we introduce the analog of the concept of nil-independency.

Definition 4.4. Derivations $d_1,d_2,\dots,d_n$ of a Lie (respectively, Leibniz) superalgebra over a field $\mathbb{K}$ are said to be residually nil-independent, if the map $f=\alpha_1 d_1+\alpha_2 d_2+\dots+\alpha_n d_n$ is not residually nilpotent for any non-null scalars $\alpha_1,\alpha_2,\dots,\alpha_n\in\mathbb{K}$. In other words, $\bigcap_{i=1}^{\infty} Im\ f^i=\{0\}$ if and only if $\alpha_1=\alpha_2=\dots=\alpha_n=0$.

4.1. Infinite-dimensional Lie superalgebras

Along this subsection we study residually solvable extension of maximal pro-nilpotent ideal the infinite-dimensional model filiform Lie superalgebra. Among all of them there will be of special interest those of maximal codimension of the infinite-dimensional model filiform Lie superalgebra, that is $R=\mathfrak{t}\overrightarrow{\oplus}L$ with $dim(\mathfrak{t})$ maximal. This case is important because for the finite-dimensional case provides a complete Lie superalgebra. In particular, this problem was solved in [11] for the finite-dimensional case, obtaining the unique Lie superalgebra $SL^{n,m}=\mathfrak{t}\overrightarrow{\oplus}L^{n,m}$ which is defined in a basis $\{x_1,\dots,x_n,t_1,t_2,t_3\}\oplus\{y_1,\dots,y_m\}$ by the only non-zero bracket products

$$SL^{n,m}:\begin{cases}[x_1,x_i]=-[x_i,x_1]=x_{i+1}, & 2\leq i\leq n-1, & [t_1,y_j]=-[y_j,t_1]=jy_j, & 1\leq j\leq m,\\ [x_1,y_j]=-[y_j,x_1]=y_{j+1}, & 1\leq j\leq m-1, & [t_2,x_i]=-[x_i,t_2]=x_i, & 2\leq i\leq n,\\ [t_1,x_i]=-[x_i,t_1]=ix_i, & 1\leq i\leq n, & [t_3,y_j]=-[y_j,t_3]=y_j, & 1\leq j\leq m.\end{cases}$$

Thanks to Proposition 3.1, it is enough to study even derivations of L, the infinite-dimensional model filiform Lie superalgebra. In particular we have the following result.

Proposition 4.1. *The space of even derivations of the superalgebra L is the following:*

$$Der_{\overline{0}}(L):\begin{cases} d(x_1)=\sum\limits_{i=1}^{t}\alpha_i x_i, \\ d(x_k)=((k-2)\alpha_1+\beta_2)x_k+\sum\limits_{i=3}^{t}\beta_i x_{i+k-2}, & \text{where } k\geq 2, \\ d(y_j)=((j-1)\alpha_1+\gamma_1)y_j+\sum\limits_{i=2}^{t}\gamma_i y_{i+j-1}, & \text{where } j\geq 1. \end{cases}$$

Proof. Since $\{x_1, x_2, y_1\}$ are the generators of L, we set

$$d(x_1)=\sum_{i=1}^{l}\alpha_i x_i, \quad d(x_2)=\sum_{i=1}^{m}\beta_i x_i, \quad d(y_1)=\sum_{i=1}^{n}\gamma_i y_i.$$

If $t := max\{l, m, n\}$, then we can suppose

$$d(x_1)=\sum_{i=1}^{t}\alpha_i x_i, \quad d(x_2)=\sum_{i=1}^{t}\beta_i x_i, \quad d(y_1)=\sum_{i=1}^{t}\gamma_i y_i.$$

Finally and by using the derivation condition, the expression desired is obtained. Note that in particular from the derivation condition applied over the pair $\{x_2, x_3\}$ we get $\beta_1 = 0$. □

Now, we search for a maximal set of residually nil-independent derivations. Then we have

Theorem 4.1. *Let $R = \mathfrak{t}\overrightarrow{\oplus} L$ be a residually solvable Lie superalgebra with maximal pro-nilpotent ideal L and maximal dimension of $\mathfrak{t}$. Then R can be expressed in a basis $\{z_1, z_2, z_3, x_1, x_2, \dots, y_1, y_2, \dots\}$ by the only non-zero bracket products*

$$R(\beta,\gamma):\begin{cases} [x_1,x_i]=-[x_i,x_1]=x_{i+1}, & i\geq 2, \\ [x_1,y_j]=-[y_j,x_1]=y_{j+1}, & j\geq 1, \\ [z_1,x_1]=-[x_1,z_1]=x_1, & \\ [z_1,x_k]=-[x_k,z_1]=(k-2)x_k+\sum\limits_{i=3}^{t}\beta_i x_{i+k-2}, & k\geq 2, \\ [z_1,y_j]=-[y_j,z_1]=(j-1)y_j+\sum\limits_{i=2}^{t}\gamma_i y_{i+j-1}, & j\geq 1, \\ [z_2,x_k]=-[x_k,z_2]=x_k, & k\geq 2, \\ [z_3,y_j]=-[y_j,z_3]=y_j, & j\geq 1, \end{cases}$$

with $\beta = (\beta_3, \dots, \beta_t) \in \mathbb{C}^{t-2}$ and $\gamma = (\gamma_2, \dots, \gamma_t) \in \mathbb{C}^{t-1}$, for some $t \in \mathbb{N}$.

Proof. In order to obtain a non-residually nilpotent derivation we get the condition $(\alpha_1, \beta_2, \gamma_1) \neq (0, 0, 0)$ (otherwise we have always a residually nilpotent derivation). Thus, the maximal number of derivations nil-independent and linearly independent is equal to three. We denote them by d_1 if $(\alpha_1, \beta_2, \gamma_1) = (1, 0, 0)$, d_2 for $(\alpha_1, \beta_2, \gamma_1) = (0, 1, 0)$ and finally d_3 with $(\alpha_1, \beta_2, \gamma_1) = (0, 0, 1)$. After setting $ad_{z_i} := d_i$ for $1 \leq i \leq 3$ we have the following products:

$$[z_1,x_1]=x_1+\sum_{i=2}^{t}\alpha_i x_i,$$

$$[z_1,x_k]=(k-2)x_k+\sum_{i=3}^{t}\beta_i x_{i+k-2}, \quad k\geq 2,$$

$$[z_1,y_j]=(j-1)y_j+\sum_{i=2}^{t}\gamma_i y_{i+j-1}, \quad j\geq 1,$$

$$[z_2,x_1]=\sum_{i=2}^{t}\alpha'_i x_i,$$

$$[z_2, x_k] = x_k + \sum_{i=3}^{t} \beta_i' x_{i+k-2}, \quad k \geq 2,$$

$$[z_2, y_j] = \sum_{i=2}^{t} \gamma_i' y_{i+j-1}, \quad j \geq 1,$$

$$[z_3, x_1] = \sum_{i=2}^{t} \alpha_i'' x_i,$$

$$[z_3, x_k] = \sum_{i=3}^{t} \beta_i'' x_{i+k-2}, \quad k \geq 2,$$

$$[z_3, y_j] = y_j + \sum_{i=2}^{t} \gamma_i'' y_{i+j-1}, \quad j \geq 1,$$

$$[z_1, z_2] = \sum_{i=1}^{t} a_i x_i, \quad [z_1, z_3] = \sum_{i=1}^{t} b_i x_i, \quad [z_2, z_3] = \sum_{i=1}^{t} c_i x_i.$$

Taking $x_1' = x_1 + \alpha_2 x_2$ we can suppose $\alpha_2 = 0$. After the following change:

$$z_1' = z_1 + \sum_{i=2}^{t-1} \alpha_{i+1} x_i, \qquad z_2' = z_2 - a_1 x_1 + \sum_{i=2}^{t-1} \alpha_{i+1}' x_i, \qquad z_3' = z_3 - b_1 x_1 + \sum_{i=2}^{t-1} \alpha_{i+1}'' x_i,$$

one can assume $[z_1, x_1] = x_1$, $[z_2, x_1] = \alpha_2' x_2$ and $a_1 = 0$, and $[z_3, x_1] = \alpha_2'' x_2$ and $b_1 = 0$.

Now, by application of the super Jacobi identity in the following cases we get the constraints given in the table:

Super Jacobi identity	Constraint
$\{x_1, z_1, z_2\}$	$\alpha_2' = 0$, $a_i = 0$, $2 \leq i \leq t$
$\{x_2, z_1, z_2\}$	$\beta_i' = 0$, $3 \leq i \leq t$
$\{x_1, z_1, z_3\}$	$\alpha_2'' = 0$, $b_i = 0$, $2 \leq i \leq t$
$\{x_2, z_1, z_3\}$	$\beta_i'' = 0$, $3 \leq i \leq t$
$\{x_i, z_2, z_3\}$, $1 \leq i \leq 2$	$c_i = 0$, $1 \leq i \leq t$
$\{y_1, z_1, z_i\}$, $2 \leq i \leq 3$	$\alpha_i' = \alpha_i'' = 0$, $2 \leq i \leq t$

Thus, we obtain the multiplication table of the statement. □

Theorem 4.2. *Any superalgebra of the family $R(\beta, \gamma)$ is complete.*

Proof. By the multiplication table it is easy to check that $R(\beta, \gamma)$ is centerless. We re-write now the basis of $R(\beta, \gamma)$ by $\{z_1, z_2, z_3, x_1, y_1, x_2, y_2, \dots\}$. Recall that all the basis vectors are even except for y_i which are odd elements for all $i \geq 1$. Let us note that $\{z_1, z_2, z_3, x_1, y_1, x_2\}$ are generators and since a derivation is completely determined by its values over the generators then it will be sufficient to show the existence of $a \in R(\beta, \gamma)$ such that $d(z) = ad_a(z)$ for $z \in \{z_1, z_2, z_3, x_1, y_1, x_2\}$.

Let us define now for each $k \in \mathbb{N}$, $L_k := span\{y_k, x_{k+1}, y_{k+1}, x_{k+2}, \dots\}$ and the quotient algebra

$$R_k(\beta, \gamma) := R(\beta, \gamma)/L_k = \mathfrak{t} \overrightarrow{\oplus} (L/L_k) = \mathfrak{t} \overrightarrow{\oplus} \overline{L}$$

is finite-dimensional solvable Lie superalgebra (maximal extension) with nilradical the model filiform Lie superalgebra $SL^{k,k-1}$. By changing of basis it can be seen that the family $R_k(\beta, \gamma)$ is isomorphic to $R_k(0, 0)$ which is itself isomorphic to $SL^{k,k-1}$. In [11] the authors showed that $SL^{n,m}$ has all the derivations inner.

If $d \in Der(R(\beta, \gamma))$ we consider an induced derivation $\overline{d} \in Der(R_k(\beta, \gamma))$ verifying $\overline{d}(\overline{v}) = \overline{d(v)}$ with $\overline{v} = v + L_k$. Since $d(L_k) \subseteq L_k$, that is, L_k is invariant under d we have that $\overline{d}$ is well-defined. In fact, as $[\mathfrak{t}, L] = L$, we get for any even or odd derivation

$$d(L) = d([\mathfrak{t}, L]) = [d(\mathfrak{t}), L] + [\mathfrak{t}, d(L)] \subseteq L.$$

On account of the derivation property we get $d(L_k) \subseteq L_k$ for any $k \in \mathbb{N}$ and then $\overline{d}$ is well-defined. Let us set for any even derivation d

$$d(x_1) = \sum_{i=1}^{s} a_i x_i, \quad d(x_2) = \sum_{i=1}^{s} b_i x_i, \quad d(y_1) = \sum_{i=1}^{s} c_i y_i, \quad d(z_1) = \sum_{i=1}^{s} \alpha_i x_i + \alpha_{1,1} z_1 + \alpha_{1,2} z_2 + \alpha_{1,3} z_3,$$

$$d(z_2)=\sum_{i=1}^{s}\varphi_i x_i+\varphi_{1,1}z_1+\varphi_{1,2}z_2+\varphi_{1,3}z_3,\quad d(z_3)=\sum_{i=1}^{s}\rho_i x_i+\rho_{1,1}z_1+\rho_{1,2}z_2+\rho_{1,3}z_3.$$

We consider $k \geq max\{s,t\}$, then we have $\overline{d(v)}=\overline{d}(\overline{v})=ad_{\overline{e}_k}$ for some even element $\overline{e}_k=e_k+L_k$ and any $\overline{v}=v+L_k$. We put then

$$\overline{e}_k=\sum_{i=1}^{k}\lambda_i^k x_i+\lambda_{k,1}z_1+\lambda_{k,2}z_2+\lambda_{k,3}z_3+L_k.$$

From the expressions

$$\overline{d(x_1)}=[\overline{e}_k,\overline{x}_1],\quad \overline{d(x_2)}=[\overline{e}_k,\overline{x}_2],\quad \overline{d(y_1)}=[\overline{e}_k,\overline{y}_1].$$

We derive

$$\begin{aligned}&\sum_{i=1}^{s}a_i x_i-\sum_{i=2}^{k}\lambda_i^k x_{i+1}+\lambda_{k,1}x_1\in L_k,\\&\sum_{i=1}^{s}b_i x_i+\lambda_1^k x_3+\lambda_{k,2}x_2\in L_k,\\&\sum_{i=1}^{s}c_i y_i+\lambda_1^k y_2+\lambda_{k,3}y_1\in L_k.\end{aligned}$$

Considering the coefficients over the basis vectors we have in particular that

$$\lambda_{k,1}=-a_1,\ \lambda_{k,2}=-b_2,\ \lambda_{k,3}=-c_1,\ \lambda_1^k=-b_3=-c_2,\ \lambda_i^k=a_{i+1}\text{ for }2\leq i\leq s-1$$

and finally $\lambda_i^k=0$ for $s\leq i\leq k$, which leads to $e_k=e_{k+1}$ for any $k\geq max\{s,t\}$. Therefore, let us set $e:=e_k$ and $W_k=span\{z_1,z_2,z_3,x_1,y_1,x_2,y_2,\dots,y_{k-1},x_k\}$ then we get $d(z)_{|W_k}=ad_e(z)_{|W_k}$ for any $z\in\{z_1,z_2,z_3,x_1,y_1,x_2\}$ any $k\geq max\{s,t\}$.

On account of $\cup_{k=1}^{\infty}W_k=R(\beta,\gamma)$ we get $d=ad_e$. Analogously it can obtain the result for any odd derivation. □

4.2. Infinite-dimensional Leibniz superalgebras

Similarly as in the previous subsection, we study residually solvable extension with maximal pro-nilpotent ideal the infinite-dimensional null-filiform Leibniz superalgebra. In the same way as the case of infinite-dimensional Lie superalgebras we provide the completeness of the considered Leibniz superalgebra. In particular, this problem was solved in [10] for the finite-dimensional case, it obtains the unique Leibniz superalgebra $SNF^{n,m}=T\overrightarrow{\oplus}NF^{n,m}$ which is defined in a basis $\{x_1,\dots,x_n,z\}\oplus\{y_1,\dots,y_m\}$ by the only non-zero bracket products

$$SNF^{n,m}:\begin{cases}[y_i,y_1]=x_i, & 1\leq i\leq n, & [x_i,y_1]=\frac{1}{2}y_{i+1}, & 1\leq i\leq m-1,\\ [y_j,x_1]=y_{j+1}, & 1\leq j\leq m-1, & [x_i,x_1]=x_{i+1}, & 1\leq i\leq n-1,\\ [x_i,z]=2ix_i, & 1\leq i\leq n, & [y_j,z]=(2j-1)y_j, & 1\leq j\leq m,\\ [z,x_1]=-2x_1, & & [z,y_1]=-y_1.\end{cases}$$

Now, we construct $R=\mathfrak{t}\overrightarrow{\oplus}NF$ with $dim(\mathfrak{t})$ maximal and NF the infinite-dimensional null-filiform Leibniz superalgebra defined by the non-null bracket products that follows:

$$NF:\begin{cases}[y_i,y_1]=x_i, & i\geq 1, & [x_i,y_1]=\frac{1}{2}y_{i+1}, & i\geq 1,\\ [y_j,x_1]=y_{j+1}, & j\geq 1, & [x_i,x_1]=x_{i+1}, & i\geq 1,\end{cases}$$

where $\{y_1,x_1,y_2,x_2,\dots\}$ are basis vectors. Then one can check that NF is residually nilpotent and then residually solvable.

Thanks to Proposition 3.1, it is enough to study even derivations. In particular we have the following result.

Proposition 4.2. *The space of even derivations of the superalgebra NF is the following:*

$$Der_{\overline{0}}(NF):\begin{cases}d(y_1)=\sum_{i=1}^{t}\alpha_i y_i,\\ d(y_k)=(2k-1)\alpha_1 y_k+\sum_{i=2}^{t}\alpha_i y_{i+k-1},\quad where\ k\geq 2,\\ d(x_k)=2k\alpha_1 x_k+\sum_{i=2}^{t}\alpha_i x_{i+k-1},\quad where\ k\geq 1,\end{cases}$$

for some $t\in\mathbb{N}$.

Proof. Since $\{y_1\}$ is the unique generator of NF, we set $d(y_1)=\sum_{i=1}^{t}\alpha_i y_i$ for some $t\in\mathbb{N}$.

Finally and by using the even superderivation condition, the expression desired is obtained. □

Note that, the maximal set of residually nil-independent derivations has dimension 1. Then we have the following theorem.

Theorem 4.3. *Let $R=\mathfrak{t}\overrightarrow{\oplus}NF$ be a residually solvable Leibniz superalgebra with maximal pro-nilpotent ideal NF and maximal dimension of $\mathfrak{t}$. Then $\mathfrak{t}\overrightarrow{\oplus}NF$ can be expressed in a basis $\{z,x_1,x_2,\dots,y_1,y_2,\dots\}$ by the only non-zero bracket products*

$$R(\alpha):\begin{cases}[y_i,y_1]=x_i, & i\geq 1, \quad [x_i,y_1]=\frac{1}{2}y_{i+1}, & i\geq 1,\\ [y_j,x_1]=y_{j+1}, & j\geq 1,\quad [x_i,x_1]=x_{i+1}, & i\geq 1,\\ [x_i,z]=2ix_i+\displaystyle\sum_{k=3}^{t}\alpha_k x_{k+i-1}, & i\geq 1,\quad [y_j,z]=(2j-1)y_j+\displaystyle\sum_{k=3}^{t}\alpha_k y_{k+i-1}, & j\geq 1,\\ [z,x_1]=-2x_1, & \quad [z,y_1]=-y_1, \\ [z,z]=-2\displaystyle\sum_{k=2}^{t-1}\alpha_{k+1}x_k, & \end{cases}$$

with $\beta=(\alpha_3,\dots,\alpha_t)\in\mathbb{C}^{t-2}$ and for some $t\in\mathbb{N}$.

Proof. In order to have a non-residually nilpotent derivation we get the condition $\alpha_1\neq 0$ (otherwise we have always a residually nilpotent derivation). Thus, the maximal number of derivations nil-independent is equal to one. We denote it by d if $\alpha_1=1$. After setting $R_z=d$ we have the following bracket products:

$$\begin{cases}[y_i,y_1]=x_i, & i\geq 1, \quad [x_i,y_1]=\frac{1}{2}y_{i+1}, & i\geq 1,\\ [y_j,x_1]=y_{j+1}, & j\geq 1,\quad [x_i,x_1]=x_{i+1}, & i\geq 1,\\ [x_1,z]=2x_1+\sum_{i=2}^{t}\alpha_i x_i, & \quad [x_k,z]=2kx_k+\sum_{i=2}^{t}\alpha_i x_{i+k-1}, & k\geq 2,\\ [y_1,z]=y_1+\sum_{i=2}^{t}\alpha_i y_i, & \quad [y_k,z]=(2k-1)y_k+\sum_{i=2}^{t}\alpha_i y_{i+k-1}, & k\geq 2.\end{cases}$$

Let $[z,x_1]=\sum_{i=1}^{t}\beta_i x_i+\delta z$ be. Application of the super Leibniz identity for the triple $\{x_1,z,x_1\}$ we get $\delta=0$ and $\beta_1=-2$. Thus, $[z,x_1]=-2x_1+\sum_{i=2}^{t}\beta_i x_i$.

Now the super Leibniz identity for the triple $\{z,x_1,x_1\}$ leads to $[z,x_2]=0$ and by induction method it is easy to prove that $[z,x_i]=0$ with $i\geq 3$.

Let $z'=z-\sum_{i=1}^{t-1}\beta_{i+1}x_i$ be, then it is easy to prove that $[z,x_1]=-2x_1$.

Consider now $[z,y_1]=\sum_{i=1}^{t}\gamma_i y_i$. Using the super Leibniz identity for the triple $\{z,y_1,y_1\}$ leads to $\gamma_1=-1$ and $\gamma_i=0$, $2\leq i\leq t$, that is, $[z,y_1]=-y_1$.

Applying the super Leibniz identity for the triple $\{z,y_1,x_1\}$ we have $[z,y_2]=0$ and using the induction method $\Rightarrow$ $[z,y_i]=0$, $i\geq 3$.

Let $[z,z]=\sum_{i=1}^{s}c_i x_i+\tau z$ be. Now, we impose the super identity and derive a set of constraints for the structure constants as follows

Super Leibniz identity	Constraint
$\{z,z,z\}$	$c_1=\tau=0$,
$\{z,z,x_1\}$	$\alpha_2=0$, $c_i=-2\alpha_{i+1}$, $2\leq i\leq t-1$, and $c_i=0$, $i\geq t$.

Thus, we have the multiplication table of the statement. □

Similar to Lie superalgebra, we prove that any superalgebra of the above family is complete. This result is tested below.

Theorem 4.4. *Any superalgebra of the family $R(\alpha)$ is complete.*

Proof. This proof is similar to corresponding proof of Theorem 4.2 for Lie superalgebra. Thus, we only give some notes.

It is easy to check that $R(\alpha)$ is centerless. Note that $\{z, y_1\}$ are the generators of $R(\alpha)$ and since a derivation is completely determined by its values over the generators then it will be sufficient to show the existence of $a \in R(\alpha)$ such that $d(v) = R_a(v)$ for $v \in \{z, y_1\}$.

Let us define now for each $k \in \mathbb{N}$, $L_k := span\{y_k, x_k, y_{k+1}, x_{k+1}, \dots\}$ and the quotient algebra

$$R_k(\alpha) := R(\alpha)/L_k = \mathfrak{t} \overrightarrow{\oplus} (NF/L_k) = \mathfrak{t} \overrightarrow{\oplus} \overline{NF}$$

is finite-dimensional solvable Leibniz superalgebra (maximal extension) with nilradical the null-filiform Leibniz superalgebra $NF^{k-1,k-1}$. By changing of basis it can be seen that the family $R_k(\alpha)$ is isomorphic to $R_k(0)$ which is itself isomorphic to $SNF^{k-1,k-1}$. In Proposition 3.2 showed that $SNF^{n,m}$ has all the derivations inner.

Let us set for any even derivation d:

$$d(y_1) = \sum_{i=1}^{s} a_i y_i, \quad d(z) = \sum_{i=1}^{s} b_i x_i + bz.$$

We consider $k \geq max\{s, t\}$, then we have $\overline{d(v)} = \overline{d}(\overline{v}) = R_{\overline{e}_k}$ for some even element $\overline{e}_k = e_k + L_k$ and any $\overline{v} = v + L_k$. We put then

$$\overline{e}_k = \sum_{i=1}^{k-1} \lambda_i^k x_i + \lambda_k z + L_k$$

From the expressions $\overline{d(z)} = [\overline{z}, \overline{e}_k]$, $\quad \overline{d(y_1)} = [\overline{y}_1, \overline{e}_k]$ we derive

$$\sum_{i=1}^{s} a_i y_i - \lambda_1^k y_2 - \lambda_k y_1 - \lambda_k \sum_{i=3}^{t-1} \alpha_i y_i \in L_k,$$
$$\sum_{i=1}^{s} b_i x_i + bz + 2\lambda_1^k x_1 + 2\lambda_k \sum_{i=2}^{t-1} \alpha_{i+1} x_i \in L_k.$$

Considering the coefficients over the basis vectors we have in particular that $\lambda_1^k = a_2$, $\lambda_k = a_1$. Thus, $e_k = a_2 x_1 + a_1 z$. Analogously to Lie superalgebra, we conclude that $e_k = e_{k+1}$ for any $k \geq max\{s, t\}$. Let l be the smallest k satisfying this condition. Then setting $e := e_k$ we get $d = R_e$.

Analogously it can obtain the result for any odd derivation. □

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