

## A NEW ALGORITHM FOR THE EVALUATION OF THE ELASTIC FORCES IN THE ABSOLUTE NODAL COORDINATE FORMULATION

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**Abstract.** *The use of the Absolute Nodal Coordinate Formulation (ANCF) to model flexible multibody systems leads to a system of equations in which the mass matrix is constant. Among other advantages of the ANCF, as the simplicity of the formulation of joint constraint, the constancy of the mass matrix is specially useful since it implies that the mass matrix is calculated once before the analysis and stored as a invariant quantity. On the other hand, the evaluation of the elastic forces has been considered as a difficult task since they result in large nonlinear functions. Both linear and nonlinear formulations based on either Continuum Mechanics or Timoshenko/Euler-Bernoulli approaches have been supposed to lead to the same level of complexity of the elastic force function. The evaluation of the elastic forces when using a nonlinear Continuum Mechanics approach requires, as presented in the literature, the integration over the volume of the element. However, it can be shown that this integration can be avoided for every evaluation since there is a closed form of the elastic force function that can be obtained before the simulation. Finite elements based on ANCF have such a large number of nodal coordinate (24 and 48 for three-dimensional beams and plates, respectively) that the search for a closed form of the elastic force function becomes difficult and costly. This paper presents a set of constant matrices that are integrated in advance (before simulation) and allows the evaluation of the elastic forces without the integration over the volume of the element suggested in the literature. In addition, the amount of data that has to be stored in order to evaluate the function is reduced. These constant matrices together with the mass matrix complete the set of invariants of the ANCF that allows an acceleration of simulations. Numerical simulations comparing CPU time and number of arithmetic operations carried out when these invariant matrices are used with those carried out when they are not used, support the convenience of their use.*

## 1 INTRODUCTION

The absolute nodal coordinate formulation is a non-incremental finite element procedure for the dynamic analysis of flexible bodies that experience rigid body motion and large deformation [1-4]. The use of coordinates that are referred to an inertial frame leads to a constant mass matrix and, as a consequence, neither Coriolis nor centrifugal terms appear in the equations of motion. However, the elastic force term has a complicate and nonlinear expression, even in the case of using the linear theory of elasticity. This fact suggests that the use of a nonlinear strain-displacement would not result in an increment of the complexity of equations of motion. The consistent elastic force term must lead to zero elastic forces for every rigid body motion of the finite element.

If deformation within the element can be considered small, either Euler-Bernoulli or Timoshenko beam theory can be used to formulate the elastic forces [3]. As an assumption of those theories, the section must remain rigid. The corresponding elastic forces have a very complex non-linear form. A Continuum Mechanics approach can avoid the former assumption and allows the incorporation of non linear strain-displacement relations [4]. The expression obtained then seems to be as involved as in the linear case, so that little is gained with the linear model. This paper shows a procedure based on Continuum Mechanics to evaluate the non linear elastic forces in an efficient manner.

With this Continuum Mechanics approach, elements which are straight in the reference configuration are easily dealt with, since the integrals over the volume of the element that appear in the elastic force term is the integral of some polynomials. However, elements that are not straight in their reference configuration lead to very large rational integrals. This paper shows how the elastic forces can be evaluated using some invariant sparse matrices that are calculated only once in advance, regardless of whether the reference configuration is straight or not [5]. At the same time, the information required is stored in a systematic manner and the process of evaluation of the elastic forces becomes very straightforward.

This paper is organized as follows. In Sections 2 the absolute nodal coordinate formulation for beam elements is briefly described. Section 3 deals with the equations of motion. Elastic forces are developed and discussed in Section 4. Section 5 describes the jacobian of the elastic forces, while Section 6 develops the elastic energy. Numerical results are presented in Section 7. Finally, a summary and conclusions drawn from the analysis appear in Section 8.

## 2 FINITE ELEMENT FORMULATION

This section reviews the formulation of beam elements for simplicity, given that other elements, such as plate elements, follow a similar scheme [6]. Assume a flexible body, solid  $i$ , modelled as the assemblage of  $N_i$  beam elements. Nodes  $j$  and  $j+1$  belong to element  $j$  (Figure 1) so that nodal coordinate vector of the element is written as

$$\mathbf{e}_{ij} = \begin{bmatrix} \mathbf{e}^{i,jT} & \mathbf{e}^{i,j+1T} \end{bmatrix}^T, \quad (1)$$

where  $\mathbf{e}_{ij}$  is the coordinate vector of element  $j$  of body  $i$  and  $\mathbf{e}^{i,j}$  and  $\mathbf{e}^{i,j+1}$  are nodal coordinate vectors of nodes  $j$  and  $j+1$  of solid  $i$ , respectively. In this paper, superscripts are used for coordinate vectors of bodies and nodes while subscripts are used for coordinate vectors of elements.

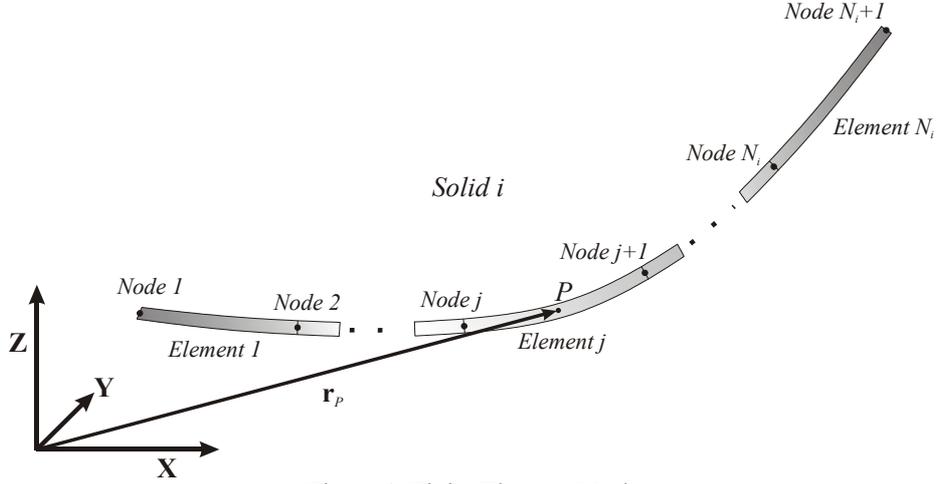


Figure 1. Finite Element Mesh

The position vector,  $\mathbf{r}_p$ , of any point  $P$  belonging to element  $j$  (Figure 1) is interpolated with the shape function of the element as

$$\mathbf{r}_p = \mathbf{S}^{ij}(\mathbf{x}_p)\mathbf{e}_{ij}, \quad (2)$$

where  $\mathbf{x}_p$  is a vector constructed from the three spatial parameters of the shape function. This vector represents a single point  $P$  of a straight element with respect to a reference frame located at one node, as shown in Figure 2. Term  $\mathbf{S}^{ij}(\mathbf{x}_p)$  of Equation (2) is the shape function of element  $j$  of solid  $i$  [3], particularized at point  $P$ .

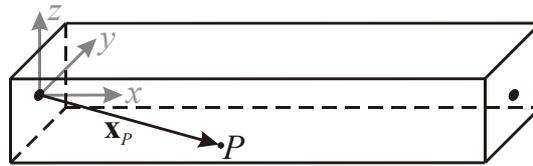


Figure 2. Local element frame

The interpolation polynomials of the shape function are cubic in the longitudinal direction, local coordinate  $x$ , and linear in the transverse direction, local coordinates  $y$  and  $z$  (Figure 2), in such a way that the cross section remains planar after deformation [3]. Nodal coordinate vectors include the position vector of the node and global slopes, that is, the derivatives of the position vector with respect to the three local coordinates of the element,  $x$ ,  $y$  and  $z$ . Nodal coordinates are written as

$$\mathbf{e}^{i,j} = \left[ \mathbf{r}^{ijT} \quad \left. \frac{\partial \mathbf{r}}{\partial x} \right|_{ij}^T \quad \left. \frac{\partial \mathbf{r}}{\partial y} \right|_{ij}^T \quad \left. \frac{\partial \mathbf{r}}{\partial z} \right|_{ij}^T \right]^T \quad (3)$$

In what follows, the nomenclature  $\mathbf{r}_{,\alpha}^{ij} = \left. \frac{\partial \mathbf{r}}{\partial \alpha} \right|_{ij}$  is used, where  $\alpha = x, y$  or  $z$ . At any point in the centreline of the beam,  $\mathbf{r}(x,0,0)$ , the vector  $\mathbf{r}_{,x}$  is tangent to the centreline of the beam element, and  $\mathbf{r}_{,y}$  and  $\mathbf{r}_{,z}$  are contained in the cross section (Figure 3). These three vectors are mutually perpendicular in the straight configuration [3].

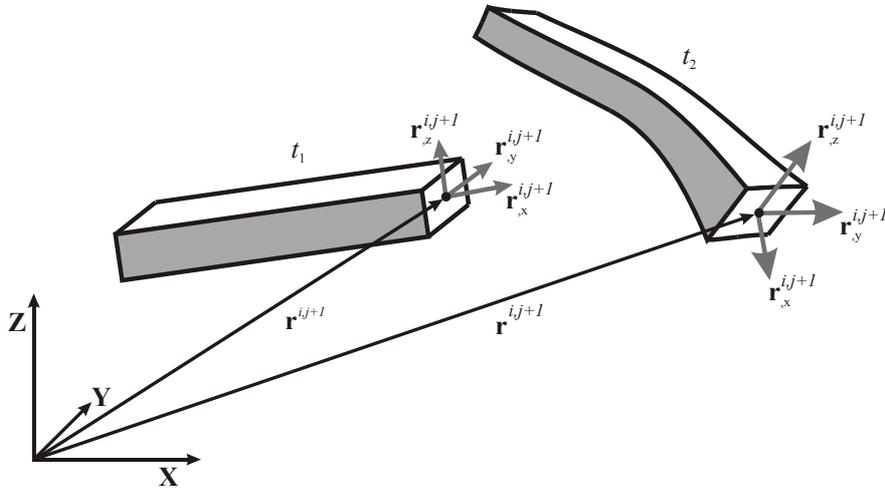


Figure 3. Finite element before and after a general motion

The previous kinematic description leads to a constant mass matrix of the element, which is simply written as

$$\mathbf{M}_e^{ij} = \int_{V^{ij}} \rho^{ij} \mathbf{S}^{ijT} \mathbf{S}^{ij} dV^{ij} \quad (4)$$

where  $V^{ij}$  is the element volume and  $\rho^{ij}$  is the mass density. As a consequence of the constancy of the mass matrix, neither Coriolis nor centrifugal force terms appear in the equations of motion. These equations are obtained by assembling the element equations as in any other finite element procedure, with the exception of flexible bodies that connect beams at non-zero angles. In this particular case, a different element parametrization is needed in order to use the standard assembling process [7]. The equations of motion of the flexible body  $i$  are written as

$$\mathbf{M}^i \ddot{\mathbf{e}}^i = \mathbf{Q}^i \quad (5)$$

where  $\mathbf{M}^i$  is the constant mass matrix of body  $i$ ,  $\mathbf{Q}^i$  is the vector of generalized forces, which includes external and elastic forces, and  $\mathbf{e}^i$  is the vector of nodal coordinates of flexible body  $i$ . This vector is written as

$$\mathbf{e}^i = \begin{bmatrix} \mathbf{e}^{i,1T} & \mathbf{e}^{i,2T} & \dots & \mathbf{e}^{i,N_i+1T} \end{bmatrix}^T \quad (6)$$

The generalized elastic force term is a non-linear function of the nodal coordinates. It can be obtained through partial differentiation of the elastic strain energy with respect to the nodal coordinates. The deformation energy can be obtained by using the non-linear Green-Lagrange strain-displacement relationship [4]. In spite of accounting for non linear effects, as will be shown later, the elastic forces can be calculated in a simple manner as a function of some invariant matrices that are obtained in advance.

### 3 EQUATIONS OF MOTION OF THE SYSTEM

Many different approaches, such as Lagrange equations, can be used to obtain the equations of motion of the system. When constraint equations are accounted for through the inclusion of Lagrange multipliers, the mechanical problem of various flexible bodies being interconnected leads to a system of differential algebraic equations, DAE, of Index 3 [8]. This system is usually written as

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{e}} + \Phi_e^T \boldsymbol{\lambda} &= \mathbf{Q} \\ \Phi(\mathbf{e}, t) &= 0 \end{aligned} \quad (7)$$

where  $\Phi_e$  is the jacobian matrix of the constraints,  $\Phi$ ,  $\mathbf{Q}$  includes external and elastic forces, and  $\boldsymbol{\lambda}$  is the vector of Lagrange multipliers.

This system of equations can be solved by the differentiation of the constraints, leading to an index 1 DAE system as

$$\begin{bmatrix} \mathbf{M} & \Phi_e^T \\ \Phi_e & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{e}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{Q}_d \end{bmatrix} \quad (8)$$

where  $\mathbf{Q}_d$  is built from partial derivatives of the constraints. However, there exist other methods for solving the system of differential algebraic equations of Equation (7) that do not need for differentiating the constraints [8]. In this paper, the integration procedure is illustrated by using Equation (8).

Solving Equations (8) requires the use of a stable enough numerical method and very often an implicit method is used. An implicit method requires solving a non linear system of equations in each time step [9] and the jacobian of this system has to be calculated.

Calculus of the jacobian matrix is the most cumbersome task when solving Equations (8). When the exact jacobian of Equation (8) is not available, numerical differentiation techniques can be used, although the computational cost is hugely increased. Numerical differentiation involves a large number of function evaluations. Nevertheless, there are some approaches that

give a good estimation of the jacobian, avoiding the numerical evaluation in each time step, such as the one proposed by Broyden [10]. On the other hand, the use of these approaches implies the undesirable consequence of missing the quadratic convergency of the Newton-Raphson algorithm. However, if it is possible to efficiently evaluate the exact jacobian of the system of Equation (8), the quadratic convergency can be preserved.

#### 4 NONLINEAR ELASTIC FORCE DEFINITION

This section is concerned with the elastic forces of an arbitrary element. Subscripts and superscripts are omitted for simplicity. Nonlinear strain-displacement relations are used to develop an expression for the elastic forces in the element. Figure 4, shows a sketch of the general motion of a deformable body. The reference configuration is represented by the region  $\mathcal{B}_0$ , in which vector  $\mathbf{r}_0$  points to the location of an arbitrary material point  $P_0$  in  $\mathcal{B}_0$ . After a general motion, the body occupies region  $\mathcal{B}$ , and  $\mathbf{r}$  is now the vector pointing to point  $P$ , which is the position of material point  $P_0$  after the motion.

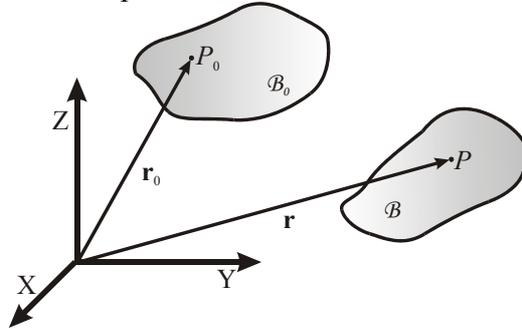


Figure 4. General motion of a deformable body

The nonlinear Green-Lagrange strain-displacement relationship is calculated from the deformation gradient, which is written as follows

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_0} \quad (9)$$

With this expression of the deformation gradient the strain tensor can be written as [4]

$$\boldsymbol{\varepsilon} = \frac{1}{2} (\mathbf{J}^T \mathbf{J} - \mathbf{I}), \quad (10)$$

where  $\mathbf{I}$  is the identity matrix.

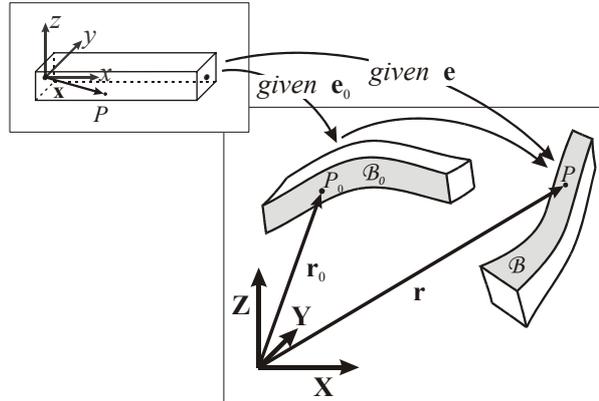


Figure 5. General motion of a curved element

It is possible to verify that if the body experiences a rigid motion, matrix  $\mathbf{J}$  is orthonormal [11], and thus it is clear from Equation (10) that there is no strain. The situation depicted in Figure 4 can be reproduced if the bodies in the figure are elements similar to the ones shown in Figure 5.

Assume that the position vector,  $\mathbf{r}_0$ , of any arbitrary material point,  $P_0$ , in  $\mathcal{B}_0$  (Figure 5) can be interpolated using the shape function as follows

$$\mathbf{r}_0 = \mathbf{S}(\mathbf{x})\mathbf{e}_0, \quad (11)$$

where  $\mathbf{e}_0$  is the element coordinate vector that defines the reference configuration, and  $\mathbf{x}$  is the position vector in the local element frame of a straight element (Figure 5). This vector is written as follows:

$$\mathbf{x} = [x \quad y \quad z]^T \quad (12)$$

Equation (11) assumes that for a given coordinate vector of the reference configuration,  $\mathbf{e}_0$ , there is a one-to-one correspondence between the points in the straight element of Figure 5 and the points in the reference configuration, non straight in the general case (Figure 5). In fact, each configuration of the element accepts a one-to-one correspondence with the straight element if the nodal coordinate vector,  $\mathbf{e}$ , of the element is given. Thus, the deformation gradient can be expressed as a function of  $\mathbf{x}$  using the chain rule as follows:

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{r}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{r}}{\partial \mathbf{x}} \mathbf{J}_0^{-1} \quad (13)$$

where

$$\mathbf{J}_0 = \frac{\partial \mathbf{r}_0}{\partial \mathbf{x}} \quad (14)$$

However, using Equation (13) to obtain the strain tensor, as it is proposed in Reference [4], leads to an expression of the strain tensor that is not manageable, so that a different

procedure is presented here. It is possible to express the deformation gradient in a more suitable form with the help of the chain rule and the local coordinate system of the straight element. The gradient of one component of the position vector with respect to the local coordinates of the straight element is first introduced

$$\left[ \frac{\partial r_i}{\partial \mathbf{x}} \right]^T = \left[ \frac{\partial r_i}{\partial \mathbf{r}_0} \frac{\partial \mathbf{r}_0}{\partial \mathbf{x}} \right]^T = \mathbf{J}_0^T \left[ \frac{\partial r_i}{\partial \mathbf{r}_0} \right]^T \quad i=1,2,3, \quad (15)$$

The partial derivative of a component of the position vector  $\mathbf{r}$  with respect to an arbitrary component of  $\mathbf{r}_0$  can be obtained from Equation (15) as follows:

$$\frac{\partial r_i}{\partial r_{0,j}} = \left( \mathbf{J}_0^{-T} \right)_j \frac{\partial r_i}{\partial \mathbf{x}}^T = \left( \mathbf{J}_0^{-T} \right)_j \begin{bmatrix} \mathbf{S}_{i,x} \\ \mathbf{S}_{i,y} \\ \mathbf{S}_{i,z} \end{bmatrix} \mathbf{e} \quad (16)$$

where  $\mathbf{e}$  is the element coordinate vector,  $\mathbf{J}_0^{-T}$  represents the transpose of the inverse of  $\mathbf{J}_0$ ,  $\mathbf{S}_{i,x}$ ,  $\mathbf{S}_{i,y}$ , and  $\mathbf{S}_{i,z}$  are derivatives of row  $i$  of the shape function with respect to  $x$ ,  $y$ , and  $z$ , respectively, and  $\left( \mathbf{J}_0^{-T} \right)_j$  is the  $j^{\text{th}}$  row of the matrix  $\mathbf{J}_0^{-T}$ . By grouping together the elements defined in Equation (16), the deformation gradient is built. Thus, the following equation represents the rows of three important matrices of the procedure

$$\left( \mathbf{S}_{\cdot,j} \right)_i = \left( \mathbf{J}_0^{-T} \right)_j \left[ \mathbf{S}_{i,x}^T \quad \mathbf{S}_{i,y}^T \quad \mathbf{S}_{i,z}^T \right]^T \quad i,j=1,2,3 \quad (17)$$

where  $\left( \mathbf{S}_{\cdot,j} \right)_i$  is row  $i$  of matrix  $\mathbf{S}_{\cdot,j}$  and  $\mathbf{S}_{\cdot,j}$  is the derivative of matrix  $\mathbf{S}$  with respect to  $r_{0,j}$ . Note that the inverse of  $\mathbf{J}_0$  in Equation (17) converts  $\mathbf{S}_{\cdot,j}$  in rational functions of  $x$ ,  $y$  and  $z$ . In the case of an initially straight element,  $\mathbf{J}_0$  is a constant matrix. These matrices allow for writing the deformation gradient in a suitable form, so as to calculate the elastic energy, as follows:

$$\mathbf{J} = \left[ \mathbf{S}_{\cdot,1} \mathbf{e} \quad \mathbf{S}_{\cdot,2} \mathbf{e} \quad \mathbf{S}_{\cdot,3} \mathbf{e} \right] \quad (18)$$

Hence, the six different components of the symmetric strain tensor are

$$\varepsilon_{11} = \frac{1}{2} \left( \mathbf{e}^T \mathbf{S}_{\cdot,1}^T \mathbf{S}_{\cdot,1} \mathbf{e} - 1 \right), \quad \varepsilon_{22} = \frac{1}{2} \left( \mathbf{e}^T \mathbf{S}_{\cdot,2}^T \mathbf{S}_{\cdot,2} \mathbf{e} - 1 \right), \quad \varepsilon_{33} = \frac{1}{2} \left( \mathbf{e}^T \mathbf{S}_{\cdot,3}^T \mathbf{S}_{\cdot,3} \mathbf{e} - 1 \right) \quad (19)$$

$$\varepsilon_{12} = \frac{1}{2} \left( \mathbf{e}^T \mathbf{S}_{\cdot,1}^T \mathbf{S}_{\cdot,2} \mathbf{e} \right), \quad \varepsilon_{13} = \frac{1}{2} \left( \mathbf{e}^T \mathbf{S}_{\cdot,1}^T \mathbf{S}_{\cdot,3} \mathbf{e} \right), \quad \varepsilon_{23} = \frac{1}{2} \left( \mathbf{e}^T \mathbf{S}_{\cdot,2}^T \mathbf{S}_{\cdot,3} \mathbf{e} \right) \quad (20)$$

Once the strain tensor is written in this form, the elastic strain energy is directly obtained using Lamé's constitutive equations as

$$U_e = \int_{V_e} \frac{\lambda + 2G}{2} (\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2) + \lambda (\varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{22}\varepsilon_{33}) + 2G (\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2) dV_e \quad (21)$$

where  $V_e$  is the volume of the element,  $\lambda$  is the Lamé's constant and  $G$  is the shear modulus of the material. Thus, the elastic forces are written as the gradient of the elastic potential as

$$\mathbf{F}_e^T = -\frac{\partial U_e}{\partial \mathbf{e}} \quad (22)$$

The elastic force vector is obtained by differentiating the elastic energy with respect to the nodal coordinate vector:

$$\begin{aligned} \mathbf{F}_e^T = & -\int_{V_e} \frac{\lambda + 2G}{2} \left( 2\varepsilon_{11} \frac{\partial \varepsilon_{11}}{\partial \mathbf{e}} + 2\varepsilon_{22} \frac{\partial \varepsilon_{22}}{\partial \mathbf{e}} + 2\varepsilon_{33} \frac{\partial \varepsilon_{33}}{\partial \mathbf{e}} \right) + \\ & \lambda \left( \varepsilon_{11} \frac{\partial \varepsilon_{22}}{\partial \mathbf{e}} + \frac{\partial \varepsilon_{11}}{\partial \mathbf{e}} \varepsilon_{22} + \varepsilon_{11} \frac{\partial \varepsilon_{33}}{\partial \mathbf{e}} + \frac{\partial \varepsilon_{11}}{\partial \mathbf{e}} \varepsilon_{33} + \varepsilon_{22} \frac{\partial \varepsilon_{33}}{\partial \mathbf{e}} + \frac{\partial \varepsilon_{22}}{\partial \mathbf{e}} \varepsilon_{33} \right) + \\ & 2G \left( 2\varepsilon_{12} \frac{\partial \varepsilon_{12}}{\partial \mathbf{e}} + 2\varepsilon_{13} \frac{\partial \varepsilon_{13}}{\partial \mathbf{e}} + 2\varepsilon_{23} \frac{\partial \varepsilon_{23}}{\partial \mathbf{e}} \right) dV_e \end{aligned} \quad (23)$$

where the derivatives of the components of the strain tensor can be written as

$$\frac{\partial \varepsilon_{ij}}{\partial \mathbf{e}} = \frac{1}{2} \mathbf{e}^T (\mathbf{S}_{,i}^T \mathbf{S}_{,j} + \mathbf{S}_{,j}^T \mathbf{S}_{,i}) \quad i, j = 1, 2, 3 \quad j \geq i \quad (24)$$

Substituting the derivatives, the strains of Equation (24) into Equation (23), it can be seen that the elastic forces can be written as the product of a coordinate-dependent stiffness matrix and the nodal coordinate vector. Thus, the elastic forces are calculated as

$$\mathbf{F}_e = -\mathbf{K}(\mathbf{e})\mathbf{e} \quad (25)$$

This matrix is written as

$$\begin{aligned} \mathbf{K}(\mathbf{e}) = & \sum_{\alpha=1}^3 \frac{\lambda + 2G}{2} \int_{V_e} (\mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} \mathbf{e} \mathbf{e}^T \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} - \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha}) dV_e + \\ & \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^3 \frac{\lambda}{2} \int_{V_e} (\mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} \mathbf{e} \mathbf{e}^T \mathbf{S}_{,\beta}^T \mathbf{S}_{,\beta} - \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha}) dV_e + \\ & \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^3 G \int_{V_e} (\mathbf{S}_{,\alpha}^T \mathbf{S}_{,\beta} \mathbf{e} \mathbf{e}^T \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\beta}) dV_e \end{aligned} \quad (26)$$

From Equation (26), if one expands the integrals of each coefficient of  $\mathbf{K}$ , it can be seen that it would be possible to evaluate the integrals once before the analysis since the nodal coordinates can be factorized out of the integrals for each coefficient of the matrix. In fact, the

coefficients of the stiffness matrix result in polynomial functions of the nodal coordinates. Such integration allows for expressing the stiffness matrix as an explicit function of the nodal coordinates. However, the integration of the coefficients of the stiffness matrix requires managing very large expressions since the number of nodal coordinates when using the absolute nodal coordinate formulation is very large (24 and 48 for three-dimensional beams and plates, respectively).

The integrals involved in Equation (26) are rational in the general case of initially curved elements due to the presence of the inverse of the jacobian of the mapping in Equation (17). Thus, given the nodal coordinate vector that defines the initial configuration,  $\mathbf{e}_0$ , the symbolic integration of Equation (26) is possible. Computationally, it is more systematic to perform the integrals using a numerical quadrature [12] because the dependence of the integrals of Equation (26) on the initial configuration would require a different factorization of the rational integrals for each initial configuration. It is clear that one would like to avoid carrying out a numerical integration each time the elastic forces are evaluated and it would be desirable to solve the integrals without operating with the nodal coordinates. In order to accomplish these two goals, a procedure based on the definition of some constant matrices, which are referred as invariant matrices or simply invariants, is presented. Grouping all terms of the matrix  $\mathbf{K}(\mathbf{e})$  that depend on  $\mathbf{e}$ , the non linear stiffness matrix is written as the sum of two terms

$$\mathbf{K}(\mathbf{e}) = \mathbf{K}_2(\mathbf{e}) + \mathbf{K}_1 \quad (27)$$

where

$$\begin{aligned} \mathbf{K}_2(\mathbf{e}) = & \frac{\lambda + 2G}{2} \sum_{\alpha=1}^3 \int_{V_e} (\mathbf{s}_{,\alpha}^T \mathbf{S}_{,\alpha} \mathbf{e} \mathbf{e}^T \mathbf{s}_{,\alpha}^T \mathbf{S}_{,\alpha}) dV_e + \\ & \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^3 \int_{V_e} \left( \frac{\lambda}{2} (\mathbf{s}_{,\alpha}^T \mathbf{S}_{,\alpha} \mathbf{e} \mathbf{e}^T \mathbf{s}_{,\beta}^T \mathbf{S}_{,\beta}) + G (\mathbf{s}_{,\alpha}^T \mathbf{S}_{,\beta} \mathbf{e} \mathbf{e}^T \mathbf{s}_{,\alpha}^T \mathbf{S}_{,\beta}) \right) dV_e \end{aligned} \quad (28)$$

$$\mathbf{K}_1 = -\frac{3\lambda + 2G}{2} \sum_{\alpha=1}^3 \int_{V_e} \mathbf{s}_{,\alpha}^T \mathbf{S}_{,\alpha} dV_e \quad (29)$$

Two-dimensional beam elements with absolute nodal coordinates [2] also accept the same formulation for the elastic forces. In that case, Equations (28) and (29) must be slightly modified as follows

$$\begin{aligned} \mathbf{K}_2^{2D}(\mathbf{e}) = & \frac{\lambda + 2G}{2} \sum_{\alpha=1}^2 \int_{V_e} (\mathbf{s}_{,\alpha}^T \mathbf{S}_{,\alpha} \mathbf{e} \mathbf{e}^T \mathbf{s}_{,\alpha}^T \mathbf{S}_{,\alpha}) dV_e + \\ & \sum_{\alpha=1}^2 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^2 \int_{V_e} \left( \frac{\lambda}{2} (\mathbf{s}_{,\alpha}^T \mathbf{S}_{,\alpha} \mathbf{e} \mathbf{e}^T \mathbf{s}_{,\beta}^T \mathbf{S}_{,\beta}) + G (\mathbf{s}_{,\alpha}^T \mathbf{S}_{,\beta} \mathbf{e} \mathbf{e}^T \mathbf{s}_{,\alpha}^T \mathbf{S}_{,\beta}) \right) dV_e \end{aligned} \quad (30)$$

$$\mathbf{K}_1^{2D} = -(\lambda + G) \sum_{\alpha=1}^2 \int_{V_e} \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} dV_e \quad (31)$$

where the shape function and the nodal coordinate vector are those of the two-dimensional elements. The complete procedure previously presented is the same for two-dimensional elements.

In order to simplify Equation (28) some algebraic relationships are now introduced. Products of the form  $\mathbf{A} \mathbf{e} \mathbf{e}^T \mathbf{B}$ , where  $\mathbf{A}$  and  $\mathbf{B}$  are two arbitrary matrices, are matrices whose components can be written in the form  $\mathbf{e}^T \mathbf{C}^{ij} \mathbf{e}$ , being  $\mathbf{C}^{ij}$  a matrix built as the product of the transposition of row  $i$  of matrix  $\mathbf{A}$  and row  $j$  of matrix  $\mathbf{B}^T$ . To ensure this, one component of the product  $\mathbf{A} \mathbf{e} \mathbf{e}^T \mathbf{B}$  is developed

$$\left( \mathbf{A} \mathbf{e} \mathbf{e}^T \mathbf{B} \right)_{ij} = \sum_k \sum_l A_{ik} e_k e_l B_{lj} = \sum_k \sum_l e_k (\mathbf{A})_{ik} (\mathbf{B}^T)_{jl} e_l = \mathbf{e}^T \mathbf{C}^{ij} \mathbf{e} \quad (32)$$

where matrix  $\mathbf{C}^{ij}$  is built as follows

$$\mathbf{C}^{ij} = \mathbf{A}_i^T (\mathbf{B}^T)_j \quad (33)$$

Components of the first term of Equation (27) are quadratic forms of the nodal coordinate vector while the second matrix is constant. Using Equation (33) the components of matrix  $\mathbf{K}_2$  can be written as

$$\left( \mathbf{K}_2(\mathbf{e}) \right)_{ij} = \mathbf{e}^T \left( \mathbf{C}_{\mathbf{K}_2}^{ij} \right) \mathbf{e}, \quad (34)$$

where  $\mathbf{C}_{\mathbf{K}_2}^{ij}$  is a very sparse matrix that is the addition of the corresponding  $\mathbf{C}^{ij}$ -type matrix after application of Equation (33) to each  $\mathbf{A} \mathbf{e} \mathbf{e}^T \mathbf{B}$ -type term in Equation (28). The final expression of matrix  $\mathbf{C}_{\mathbf{K}_2}^{ij}$  is written as

$$\begin{aligned} \mathbf{C}_{\mathbf{K}_2}^{ij} = & \sum_{\alpha=1}^3 \frac{\lambda + 2G}{2} \int_V \left( \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} \right)_i^T \left( \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} \right)_j dV + \\ & \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^3 \frac{\lambda}{2} \int_V \left( \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} \right)_i^T \left( \mathbf{S}_{,\beta}^T \mathbf{S}_{,\beta} \right)_j dV + \\ & \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^3 G \int_V \left( \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\beta} \right)_i^T \left( \mathbf{S}_{,\beta}^T \mathbf{S}_{,\alpha} \right)_j dV \end{aligned} \quad (35)$$

where  $\left( \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} \right)_i$  is row  $i$  of matrix  $\mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha}$ . Matrix  $\mathbf{C}_{\mathbf{K}_2}^{ij}$  of Equation (35) is referred to in this paper as an invariant matrix, and there exists one invariant matrix associated to each one of the components of matrix  $\mathbf{K}_2(\mathbf{e})$ . A careful look at Equation (28) shows that matrix  $\mathbf{K}_2(\mathbf{e})$  is symmetric. Therefore, invariant matrices only have to be calculated for the upper (lower) triangle and the diagonal of matrix  $\mathbf{K}_2(\mathbf{e})$ .

The sparsity of  $\mathbf{C}_{\mathbf{K}_2}^{ij}$  matrices allows for the storage of a small quantity of data leading to a fast evaluation of the elastic forces by matrix multiplications. The procedure proposed in this section to evaluate the elastic forces leads to a considerable reduction of computational time because it avoids the need for integrating large matrices in each time step, even in the case in which the element is initially curved. Similar invariant matrices can be obtained in the two-dimensional case.

## 5 JACOBIAN OF THE ELASTIC FORCE FUNCTION

In order to obtain an explicit expression of the jacobian of the system of Equation (8), the inverse of the matrix of Equation (8) is obtained by carrying out some algebraic manipulations. The second time derivative of the nodal coordinate is written as

$$\ddot{\mathbf{e}} = \mathbf{M}^{-1}\mathbf{Q} + \mathbf{M}^{-1}\Phi_e^T(\Phi_e\mathbf{M}^{-1}\Phi_e^T)^{-1}(\mathbf{Q}_d - \Phi_e\mathbf{M}^{-1}\mathbf{Q}) = \mathbf{g}(\mathbf{e}, \dot{\mathbf{e}}, t) \quad (36)$$

The above-mentioned jacobian is obtained from equation (36) by simple partial differentiation. Equation (36) offers great advantages, in view of the constancy of the mass matrix when the absolute nodal coordinate formulation is used to model a multibody system. Moreover, some of the usual kinematic joints lead to constraint equations that are linear or quadratic with the ANCF [13], so that the jacobian of the constraints has a very simple structure.

It is also possible to evaluate the jacobian of Equation (36) using a mixed procedure in which some of the derivatives are numerically evaluated. Even in these cases, some benefit can be achieved from the exact jacobian of the elastic forces, owing to its simple structure, as it is shown in this section. This procedure can not be followed if the equations include non generalized coordinates. Because non generalized coordinates have neither mass nor inertia associated [13], block  $\mathbf{M}$  of Equation (8) is singular, so Equation (36) can not be applied. Nevertheless, this does not mean that the matrix in Equation (8) is singular. An alternative to Equation (36) can be the numerical differentiation of the inverse of the matrix of Equation (8) and exact evaluation of the derivative of the vector on the right hand side of the same equation.

By substituting Equation (26) into Equation (25) and differentiating, the following expression of the jacobian is obtained

$$\begin{aligned} \frac{\partial \mathbf{F}_e}{\partial \mathbf{e}} = & - \sum_{\alpha=1}^3 \frac{\lambda + 2G}{2} \int_{V_e} \left( (\mathbf{e}^T \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} \mathbf{e}) \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} + 2 \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} \mathbf{e} \mathbf{e}^T \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} - \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} \right) dV_e \\ & - \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^3 \frac{\lambda}{2} \int_{V_e} \left( (\mathbf{e}^T \mathbf{S}_{,\beta}^T \mathbf{S}_{,\beta} \mathbf{e}) \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} + 2 \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} \mathbf{e} \mathbf{e}^T \mathbf{S}_{,\beta}^T \mathbf{S}_{,\beta} - \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\alpha} \right) dV_e \\ & - \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^3 G \int_{V_e} \left( (\mathbf{e}^T \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\beta} \mathbf{e}) \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\beta} + \mathbf{S}_{,\alpha}^T \mathbf{S}_{,\beta} \mathbf{e} \mathbf{e}^T (\mathbf{S}_{,\alpha}^T \mathbf{S}_{,\beta} + \mathbf{S}_{,\beta}^T \mathbf{S}_{,\alpha}) \right) dV_e \end{aligned} \quad (37)$$

As it is shown in Equation (37), integration in the volume of the element would also be needed in order to calculate the jacobian of the elastic forces in a matrix form. However, the expression obtained in the previous section for the elastic forces based on the invariant matrices leads to a jacobian of the elastic forces that requires no integration in the volume of the element.

Using the invariant matrices, a simple partial derivation of the elastic forces of Equation (25) with respect to the nodal coordinate vector leads to the following expression for the components of the jacobian matrix

$$\left(\frac{\partial \mathbf{F}_e}{\partial \mathbf{e}}\right)_{ik} = \frac{\partial F_{ei}}{\partial e_k} = -\mathbf{K}_{ik}(\mathbf{e}) - \sum_j \sum_s e_s \left( \mathbf{C}_{\mathbf{K}_2}^{ij} + \mathbf{C}_{\mathbf{K}_2}^{ijT} \right)_{sk} e_j, \quad (38)$$

where the two subscript indices refer to the component of a matrix and the two superscript indices indicate which component of the non-linear stiffness matrix is calculated with that invariant matrix. As shown in this section, obtaining the invariant matrices not only allows the reduction of arithmetic operations when evaluating the elastic forces, but also when evaluating their jacobian. If the sparsity of the  $\mathbf{C}^{ij}$ -type matrices is efficiently used, the evaluation of the jacobian matrix of the elastic forces is carried out with few operations. Calculus of the jacobian of the elastic forces is no longer cumbersome and this fact makes it possible to efficiently evaluate the jacobian of the equations of motion avoiding a costly numerical differentiation.

## 6 STRAIN ENERGY

Calculus of the elastic energy also involves the invariant matrices previously presented. By substituting Equations (19) and (20) into Equation (21), it can be shown that the elastic energy is written as

$$U_e = \frac{1}{4} \mathbf{e}^T \mathbf{K}_2(\mathbf{e}) \mathbf{e} + \frac{1}{2} \mathbf{e}^T \mathbf{K}_1 \mathbf{e} + U_0 \quad (39)$$

where

$$U_0 = \frac{(9\lambda + 6G)}{8} V \quad (40)$$

where  $V$  is the volume of the element. For two-dimensional elements, the same expression is valid except for the value of the constant, which is

$$U_0^{2D} = \frac{\lambda + G}{2} V \quad (41)$$

## 7 NUMERICAL RESULTS

A linear example is first solved with the aim of comparing the numerical results with the analytical solution of a well-known problem, so that the procedure proposed can be verified. Dynamic analysis of a cantilever beam is carried out in order to make a comparison between

the proposed procedure for evaluating the elastic forces and that of the references [4, 6]. The beam of the problem is assumed to have a square cross section of  $0.1 \times 0.1 \text{ m}^2$  and 5 m of length. A concentrated time-dependent load is applied at the free end of the beam, as shown in Figure 6. The beam is assumed to have a mass density of  $8245.2 \text{ Kg/m}^3$  and a Young modulus of 132 GPa. The variation law of the external load is written as

$$\mathbf{F}(t) = \begin{cases} \frac{F}{2} \left( 1 - \cos\left(\frac{\pi t}{t_c}\right) \right) & \text{if } t \leq t_c \\ F & \text{if } t > t_c \end{cases} \quad (42)$$

where  $F$  has a value of 300 N.

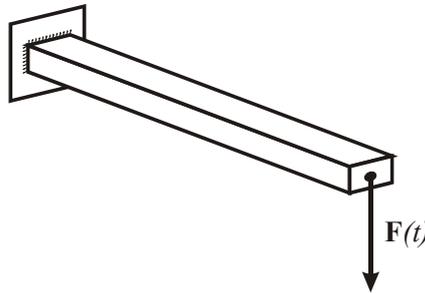


Figure 6. Cantilever beam

The three-dimensional beam element used in this paper can lead to an incorrect solution due to Poisson effect as pointed out by Sopenan and Mikkola [14]. These authors explain that, since the cross-section of the element is not kinematically able to deform in a trapezoid when the applied load causes bending, some residual transverse normal stresses appear. These residual stresses affect the longitudinal normal stress due to Poisson effect making the element to converge to an incorrect solution [14]. In order to avoid incorrect results, Poisson's modulus has been considered zero.

Integration over the volume of the element is carried out with five integration points in the longitudinal dimension and three integration points in each transverse dimension. The number of elements and the integration time is varied to show how the saving in arithmetic operations and CPU time increases with the number of elements and simulation time.

The same expression of the jacobian matrix is used regardless of the procedure to evaluate the elastic forces (with or without invariants). That expression of the jacobian is built from the knowledge of the invariant matrices since an important save in time is achieved with it. Therefore, the differences observed solving the problem with and without invariant matrices can only be attributed to the evaluation of the elastic force term. This jacobian matrix leads to a quadratic rate of convergence of the Newton-Raphson process required by the implicit integrator. The equations of motion were transformed in an index 1 DAE system. The solution of this DAE system may not fulfil the constraint equations due to the accumulation of numerical errors. This error accumulation can be avoided using Baumgarte stabilization of the constraints or projection methods [15, 16]. In this example, the differential equations are

integrated with a second order Adams Moulton integrator, with projection in position and velocity [16].

The transverse displacement of the tip of the cantilever beam and the fundamental frequency are used in this example to validate the models. The displacement of the tip of a cantilever beam due to a concentrated force in the free end under the assumption of small deformations is

$$\delta_L = \frac{PL^3}{3EI} \quad (43)$$

The fundamental frequency of vibration of a cantilever beam is

$$\omega_n = (1.8751)^2 \sqrt{\frac{EI}{mL^3}} \quad (44)$$

Using Equations (43) and (44), the values of the transverse displacement of the tip and the first natural frequency of the beam of this problem are 0.0114 m and 16.24 rad/s, respectively. As a first verification, the simulation was carried out for four seconds with a value of time  $t_c$  of 3.5 seconds. The same time period was analyzed with three different models that differ in the number of elements used. Two, five and ten elements have been used. The results have been obtained using a Pentium 4, 3.06 GHz, and 2 GB of RAM. Table I shows that usage of the invariant matrices leads to a considerable saving of CPU time and arithmetic operations, and that the saving increases as the number of elements increases.

Table I. Cantilever beam simulation. Effect of the number of elements

Number of elements	$\delta$ (cm)	$\omega_n$ (rad/s)	Evaluations of $\mathbf{F}_e$	Invariant matrices	Part of the analysis	Arithmetic operations	CPU time (s)
2	1.07	16.89	301	With	Pre-processing	$1.48 \cdot 10^9$	40.2
					Simulation	$2.37 \cdot 10^8$	68.7
					Total analysis	$1.71 \cdot 10^9$	108.9
				Without	Total analysis	$7.78 \cdot 10^9$	312.4
5	1.13	16.41	301	With	Pre-processing	$3.72 \cdot 10^9$	100.6
					Simulation	$1.31 \cdot 10^9$	136.4
					Total analysis	$4.03 \cdot 10^9$	237.0
				Without	Total analysis	$2.21 \cdot 10^{10}$	910.4
10	1.14	16.28	301	With	Pre-processing	$7.45 \cdot 10^9$	201.3
					Simulation	$6.82 \cdot 10^9$	296.2
					Total analysis	$1.42 \cdot 10^{10}$	497.5
				Without	Total analysis	$5.45 \cdot 10^{10}$	2930.8

In this table,  $\delta$  represents the transverse displacement at the tip of the beam when the applied load is constant and  $\omega_n$  represents the frequency of vibration of the beam in the steady state. The transverse displacement of the tip and the frequency of vibration measured in the finite element model have a good agreement with the values obtained with Equations (43) and (44). The number of evaluations of the elastic force functions is shown in the fourth column since it is representative of the cost of the simulation. The higher the number of function evaluations, the larger the difference between the use or not of the invariants. The last two columns show the number of arithmetic operations and the CPU time necessary to solve the problem with and without the use of invariants. When the invariants are used the computational cost represented by the last two columns has been split in two parts corresponding to the calculus of the invariant matrices and the simulation itself, respectively. It can be observed in Table I that the difference in CPU time becomes very important when the number of elements increases.

Table II. Cantilever beam simulation. Simulation times

<i>Simulation time (s)</i>	<i>Number of evaluations of <math>F_e</math></i>	<i>Use of invariant matrices</i>	<i>Part of the analysis</i>	<i>Arithmetic operations</i>	<i>CPU time (s)</i>
10	754	With	<i>Pre-processing</i>	$3.72 \cdot 10^9$	100.6
			<i>Simulation</i>	$3.29 \cdot 10^9$	372.4
			<i>Total analysis</i>	$7.01 \cdot 10^9$	473.0
		Without	<i>Total analysis</i>	$5.53 \cdot 10^{10}$	2466.2
20	1450	With	<i>Pre-processing</i>	$3.72 \cdot 10^9$	100.6
			<i>Simulation</i>	$6.42 \cdot 10^9$	1226.0
			<i>Total analysis</i>	$1.01 \cdot 10^{10}$	1326.6
		Without	<i>Total analysis</i>	$1.07 \cdot 10^{11}$	8446.0
30	2039	With	<i>Pre-processing</i>	$3.72 \cdot 10^9$	100.6
			<i>Simulation</i>	$9.27 \cdot 10^9$	1570.5
			<i>Total analysis</i>	$1.30 \cdot 10^{10}$	1671.1
		Without	<i>Total analysis</i>	$1.54 \cdot 10^{11}$	10424.1

The number of arithmetic operations required for one evaluation of the elastic forces of one element without the use of the invariants is  $9.16 \cdot 10^6$ , while only  $5.52 \cdot 10^4$  arithmetic operations are needed using the invariants. However, the calculus of the invariant matrices implies a computational cost that is constant and independent from the simulation time. The number of arithmetic operations carried out in the calculus of the invariant matrices of one element is  $7.45 \cdot 10^8$  and the time elapsed in the calculus is 20.13 seconds. Given the former numbers, it is clear that for a number of evaluations of the elastic forces higher than eighty it is advisable to use these invariants, above all in problems with large simulation times and lots of elements. The benefit of the use of the invariant matrices undoubtedly appears in long time simulations, as shown in Table II. The results of Table II have been obtained for a five-element model by varying the simulation time from 10 to 30 seconds as it is shown in the first

column of Table II. In each case, the time of application of the load ( $t_c$ ) is the 8.75 % of the simulation time. The second column shows how the number of function evaluations increases as the simulation time increases. Again, the computational cost is represented by the number of arithmetic operations and the time of CPU required in the analysis that are shown in the last two columns. As can be seen in Table II, the computational cost of the calculus of the invariant matrices becomes less important when the simulation time increases. In fact, the pre-processing consumes the 21.3 % of CPU time when the simulation time is 10 seconds while the 6 % for a 30 seconds simulation. Note that if the jacobian of the elastic forces had not been evaluated using the invariants, the differences in number of arithmetic operations and CPU time would have been much larger since another numeric quadrature should have been carried out in virtue of Equation (37).

As a second example, a cantilever rotating beam problem was solved. The geometric stiffening effect that appears in this problem requires a non-linear formulation to be taken into account [17, 18]. Rotating beams are extensively studied in the references as it is considered a benchmark for geometrically nonlinear formulations.

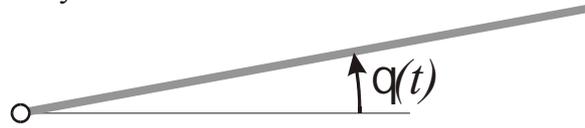


Figure 7. Rotating beam

The beam in Figure 7 is assumed to have a cross sectional area of  $7.299 \cdot 10^{-5} \text{ m}^2$  and a second moment of area of the section of  $8.215 \cdot 10^{-9} \text{ m}^4$ . The length of the beam is assumed to be 8 m, and the Young modulus, 68.95 GPa. The beam is assumed to have a mass density of  $2766.67 \text{ kg/m}^3$  and it rotates around its end section with the following law of motion:

$$\theta(t) = \begin{cases} \left( \frac{\omega_s}{T_s} \right) \left[ \left( \frac{t^2}{2} \right) + \left( \frac{T_s}{2\pi} \right)^2 \left( \cos \left( \frac{2\pi t}{T_s} \right) - 1 \right) \right] & 0 \leq t \leq T_s \\ \omega_s \left( t - \frac{T_s}{2} \right) & T_s \leq t \end{cases} \quad (45)$$

where  $T_s$  is 15 seconds and  $\omega_s$  is 4 rad/s.

The beam was modeled with ten two-dimensional beam elements using three integration points in the transverse direction and five in the longitudinal direction. An implicit second order Adams Moulton method was used in the integration of the equations of motion.

In Figure 8 the transverse displacements of the free end of the beam are shown. The results obtained show a good agreement with those of the reference [17, 18]. The importance of this example is the fact that 20072 evaluations of the elastic forces were needed to solve the problem. With this number of function evaluations, there is no doubt about the advantage of using the invariants in this problem.

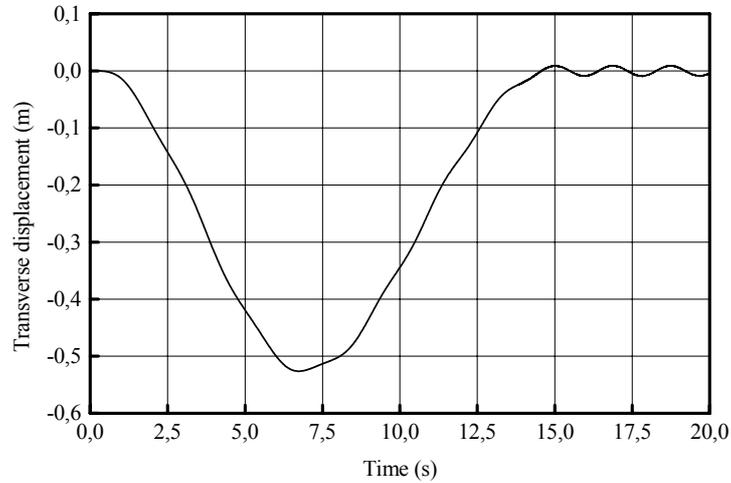


Figure 8. Transverse displacements of the free end

The elastic energy of the rotating beam is shown in Figure 9. As is expected, the elastic deformation energy increases and decreases in the same way as the deflection of the free end. After the spin angular velocity reaches a constant value, a small quantity of axial deformation remains in the beam due to the centrifugal force.

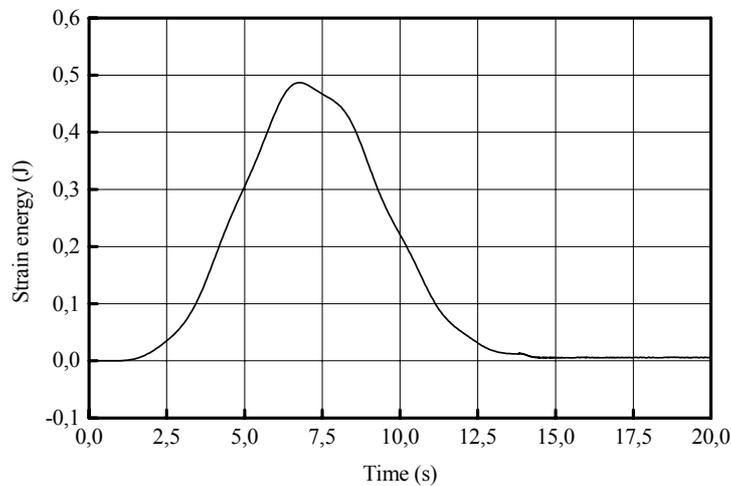


Figure 9. Elastic energy of the rotating beam

## 8 CONCLUSIONS

When the absolute nodal coordinate formulation is used to model flexible multibody systems, the mass matrix is constant. This implies that this matrix and its inverse are calculated just

once before the simulation at the expense of storing certain quantity of data. On the other hand, the elastic force function results in a nonlinear expression, regardless of the assumptions made in its formulation (linear or nonlinear elasticity). Two-dimensional elements have been shown in the literature [1] to have a closed expression for the elastic forces. However, the elastic force term for shear deformable beam (2D or 3D) and plates have been shown in the literature [2-4, 6] to require the solution of some integrals over the volume of the element for every evaluation. As it has been shown later in the literature [5] and resumed in this paper, there is a set of constant matrices that avoids the integration over the volume of the element, transforming the evaluation of the forces in a simple sequence of matrix multiplications. Therefore, shear deformable element also have a closed expression for the elastic forces. The information required to evaluate the elastic forces together with the mass matrix complete the amount of data that has to be obtained in the pre-processing stage. The procedure proposed in this paper has shown to considerably reduce the cost of the simulation stage. However, it has to be noticed that a model with a large number of element may require excessive provision of memory to store data.

The same invariant matrices are involved in the evaluation of the strain energy and the jacobian of the elastic forces. Calculating the jacobian using numeric differentiation, results in a very expensive since it requires a large number of evaluations of the force term. Therefore, an efficient evaluation of the jacobian matrix allows preserving the quadratic convergence of the Newton-Raphson algorithm at a small expense. This is the case when the jacobian of the elastic forces can be calculated using the invariant matrices.

Regarding the calculation of these invariant matrices some properties can be used to reduce the cost of this process. The symmetry of the stiffness matrix implies that only the invariant matrices corresponding to the coefficients at the diagonal and upper/lower triangle must be obtained. At the same time, the invariant matrices are very sparse matrices and so only a small quantity of data has to be stored and matrix product can also be optimized.

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