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FJ-Invex control problem

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ABSTRACT

This paper introduces a new condition on the functionals of a control problem and extends a recent characterization result of KT-invexity. We prove that the new condition, the FJ-invexity, is both necessary and sufficient in order to characterize the optimal solution set using Fritz John points.

1. Introduction

Invexity plays an important role in optimization. Mond and Smart [1] proved that invexity of a functional is necessary and sufficient for its critical points to be global minimum, which coincides with the original concept of an invex function [2] for mathematical programming problems being one for which critical points are global minima [3]. From this characterization result for invex functionals, Arana et al. [4] extend this result to the control problems, that is, they provide a class of functionals, called KT-invex, that is characterized by a Kuhn–Tucker point being an optimal solution for the control problem. The aim of this paper is to extend the recent work by Arana et al. [4] to characterize the optimal solution set of a control problem with a new weaker condition on the involved functionals.

These types of problems are applied, for example, to engineering problems, like the control design for autonomous vehicles or impulsive control problems (see [5,6]), electrical power production (see [7]), economy (see [8]), medicine (see [9]) and ecology (see [10]), among others.

Let us consider the following mathematical formulation of a control problem, commonly used in the literature (see [11,12,1]):

$$\text{(CP)} \quad \text{Minimize } F(x,u) = \int_a^b f(t,x,u)\mathrm{d}t$$

subject to :

$$x(a) = \alpha, \qquad x(b) = \beta \tag{1}$$

$$g(t,x,u) \le 0 \tag{2}$$

$$h(t,x,u) = \dot{x}. \tag{3}$$

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Let $I = [a, b]$ be a real interval. Each $f : I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $g : I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, $h : I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuously differentiable function.

Denote the partial derivatives of f by f_t, f_x and f_u, where

$$f_t = \frac{\partial f}{\partial t}, \qquad f_x = \left[\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \ldots, \frac{\partial f}{\partial x^n}\right], \qquad f_u = \left[\frac{\partial f}{\partial u^1}, \frac{\partial f}{\partial u^2}, \ldots, \frac{\partial f}{\partial u^m}\right]$$

where the superscripts denote the vector components. Similarly, we have g_t, g_x, g_u and h_t, h_x, h_u.

X is the space of piecewise smooth state functions $x : I \to \mathbb{R}^n$ such that $x(a) = \alpha$ and $x(b) = \beta$ and that is equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$; similarly, Y is the space of piecewise continuous control functions $u : I \to \mathbb{R}^m$, and has the uniform norm $\|\cdot\|_\infty$. $F : X \times Y \to \mathbb{R}$. We denote by K the set of feasible solutions of (CP).

The objective of the present work is to find the optimal solutions of this control problem, which means to obtain the minimum for the objective functional. In order to achieve it we consider the following papers: Bhatia and Kumar [13], Mond and Hanson [12] and Arana et al. [4]. We are going to extend the study given in [4] for the case of Fritz John points, giving a new generalized convexity notion, that is weaker than the Kuhn–Tucker one presented in the mentioned paper [4]. These Fritz John optimality conditions were originally developed by Chandra et al. [11], and they require the assumption of a surjective condition, but not of a normal condition (necessary for Kuhn–Tucker conditions). Let us remind, briefly, this condition. Following Craven and Mond [14], the differential equation (3) for $x(t)$, with initial conditions, expressed as $x(t) = x(a) + \int_a^t h(s, x(s), u(s))\mathrm{d}s$, $t \in I$, may be written as $Dx = H(x, u)$, where $H : X \times U \to C(I, \mathbb{R}^n)$ is defined by $H(x, u)(t) = h(t, x(t), u(t))$. In order to provide the following theorem giving Fritz John type optimality conditions, Chandra et al. [11] needed for the equality constraint to be locally solvable (see [15]). For this purpose, they needed

$$Q'(\bar{x}, \bar{u}) = [D - H_x(\bar{x}, \bar{u}), -H_u(\bar{x}, \bar{u})]$$

to be surjective; that is, it is necessary to assume that the differential equation

$$Dp(t) - h_x(t, \bar{x}(t), \bar{u}(t))p(t) - h_u(t, \bar{x}(t), \bar{u}(t))q(t) = z(t)$$

can be solved for piecewise smooth $p(\cdot)$ and piecewise continuous $q(\cdot)$, with boundary conditions $p(a) = 0 = p(b)$, whatever $z(\cdot)$ is chosen. Here, we write $(\bar{x}, \bar{u})$ as the optimal solution.

In the following theorem, we have a Fritz John type necessary optimality condition.

Theorem 1 (*Fritz John Necessary Optimality Conditions*). *If $(\bar{x}, \bar{u}) \in K$ is a local solution for* (CP) *and Q' is surjective, then $(\bar{x}, \bar{u})$ is a Fritz John point, that is, there exist $\tau \in \mathbb{R}$ and piecewise smooth functions $\lambda : I \to \mathbb{R}^k$ and $\mu : I \to \mathbb{R}^n$ satisfying the following:*

$$\tau f_x(t, \bar{x}, \bar{u}) + \lambda(t)^{\mathrm{T}} g_x(t, \bar{x}, \bar{u}) + \mu(t)^{\mathrm{T}} h_x(t, \bar{x}, \bar{u}) + \dot{\mu}(t) = 0 \tag{4}$$

$$\tau f_u(t, \bar{x}, \bar{u}) + \lambda(t)^{\mathrm{T}} g_u(t, \bar{x}, \bar{u}) + \mu(t)^{\mathrm{T}} h_u(t, \bar{x}, \bar{u}) = 0 \tag{5}$$

$$\lambda(t)^{\mathrm{T}} g(t, \bar{x}, \bar{u}) = 0 \tag{6}$$

$$(\tau, \lambda(t)) \geq 0, \qquad (\tau, \lambda(t)) \neq 0 \tag{7}$$

for all $t \in I$, except at discontinuities.

Remark 1. Note that if constraints (2) and (3) are removed from (CP) then a Fritz John point reduces to a critical point of F. And if $\tau \neq 0$, then $(\bar{x}, \bar{u})$ is said to be a Kuhn–Tucker point, and $(\bar{x}, \bar{u})$ is called normal (see [11]). But to ensure that $(\bar{x}, \bar{u})$ is normal it will be necessary to assume and add to the previous theorem some constraint qualification condition; for example, Zowe's form of the Slater condition (see [16]). That is, it will be necessary to assume local solvability for the whole constraint system. At this point, we can say that no normality conditions are required.

Fritz John type optimality conditions are weaker than Kuhn–Tucker type; that is, for some control problems, it is possible to obtain a Fritz John point which is not a Kuhn–Tucker point. In order to show it, we present the following example.

Example 1. Let us consider the following control problem:

$$(\mathrm{CP}_{ex}) \quad \text{Minimize } F(x, u) = \int_0^{10} \left(\frac{x^2(t)}{2} + \frac{(u(t)+1)^2}{2}\right) \mathrm{d}t$$

subject to :

$$x(0) = 0, \qquad x(10) = 0$$

$$\dot{x}(t) = x(t) \cdot u(t)$$

$$x(t) \cdot u(t) \leq 0$$

where $t \in I = [0, 10]$, $f(t, x, u) = \frac{x^2(t)}{2} + \frac{(u(t)+1)^2}{2}$, $g(t, x, u) = x(t) \cdot u(t)$ and $h(t, x, u) = x(t) \cdot u(t)$ are continuously differentiable. Conditions (4)–(7) reduce to

$$\tau x(t) + \lambda(t)u(t) + \mu(t)u(t) + \dot{\mu}(t) = 0$$
$$\tau(u(t) + 1) + \lambda(t)x(t) + \mu(t)x(t) = 0$$
$$\lambda(t)x(t)u(t) = 0$$
$$(\tau, \lambda(t)) \geq 0, \qquad (\tau, \lambda(t)) \neq 0$$

with $\tau \in \mathbb{R}$, λ and μ piecewise smooth functions. Let us consider $(\bar{x}, \bar{u})$ defined as $\bar{x}(t) = 0 = \bar{u}(t)$, $t \in [0, 10]$. Obviously, $(\bar{x}, \bar{u})$ is feasible for our control problem. Let us see that $(\bar{x}, \bar{u})$ is a Fritz John point, that is, there exist τ, λ and μ satisfying the previous conditions. For this purpose, we take $\tau = 0$, $\lambda(t) = 2$ and $\mu(t) = -2$, and it is easy to see that Fritz John conditions are fulfilled. Otherwise, let us prove that $(\bar{x}, \bar{u})$ is not a Kuhn–Tucker point. For this, and since (5) has to be satisfied for some τ, λ and μ, we have that $\tau(0 + 1) + \lambda(t) \cdot 0 + \mu(t) \cdot 0 = 0$, and it follows that $\tau = 0$, and therefore it is not possible for $(\bar{x}, \bar{u})$ to be a Kuhn–Tucker point. Therefore, $(\bar{x}, \bar{u})$ is a Fritz John point, but it is not a Kuhn–Tucker point.

Mond and Smart [1] proved that invexity of F is necessary and sufficient for its critical points to be global minimum, which coincides with the original concept of an invex function [2] for mathematical programming problems being one for which critical points are global minimum [3]. Recently, Arana et al. [4] have extended this result to the control problem (CP) by introducing the concept KT-invexity of (CP), which characterizes the optimal solutions of (CP) by Kuhn–Tucker points. Now, the aim of this paper is to extend this optimality result to consider Fritz John conditions, for which we need new conditions on the involved functionals of the control problem.

2. FJ-Invexity: Main result

Before presenting the main result of this paper we need the following definition:

Definition 1. The control problem (CP) is said to be FJ-invex at $(\bar{x}, \bar{u}) \in K$ if for all $(x, u) \in K$, and for all $\tau \in \mathbb{R}$, $\lambda : I \to \mathbb{R}^k$, which verify (6) and (7), and $\mu : I \to \mathbb{R}^n$ piecewise smooth functions, there exist differentiable vector functions $\eta(t, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)$ and $\xi(t, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)$ with $\eta(a, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu) = 0 = \eta(b, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)$ and $\xi(a, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu) = 0 = \xi(b, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)$ such that

$$\begin{aligned} F(x, u) - F(\bar{x}, \bar{u}) < 0 \Rightarrow \int_a^b \Big(& (\tau f_x(t, \bar{x}, \bar{u}) + \lambda(t)^{\mathrm{T}} g_x(t, \bar{x}, \bar{u}) + \mu(t)^{\mathrm{T}} h_x(t, \bar{x}, \bar{u}))\eta(t, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu) \\ & - \mu(t)^{\mathrm{T}} \dot{\eta}(t, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu) + (\tau f_u(t, \bar{x}, \bar{u}) + \lambda(t)^{\mathrm{T}} g_u(t, \bar{x}, \bar{u}) \\ & + \mu(t)^{\mathrm{T}} h_u(t, \bar{x}, \bar{u}))\xi(t, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)\Big)\, \mathrm{d}t < 0. \end{aligned}$$

The control problem (CP) is said to be FJ-invex if it is FJ-invex for all $(\bar{x}, \bar{u}) \in K$. If τ is replaced by 1 in Definition 1, then the problem (CP) is said to be KT-invex, such as Arana et al. [4] defined. Thus, if (CP) is a KT-invex control problem, then (CP) is FJ-invex, and therefore, FJ-invexity extends, in fact, KT-invexity. In what follows, we present the aimed optimality result for FJ-invex control problems, which extends those given by Arana et al. [4] for KT-invex control problems. We prove that FJ-invexity of (CP) is both a sufficient and necessary condition for a Fritz John point to be an optimal solution.

Theorem 2. *All Fritz John points are optimal solutions for* (CP) *if and only if* (CP) *is FJ-invex.*

Proof. (i) (Necessary condition) Let (x, u) and $(\bar{x}, \bar{u})$ be feasible points for (CP), $\tau \in \mathbb{R}$, and $\lambda : I \to \mathbb{R}^k$ and $\mu : I \to \mathbb{R}^n$ piecewise smooth functions such that (6) and (7) are verified and $F(x, u) - F(\bar{x}, \bar{u}) < 0$, since otherwise, the result would already be proved. We look for differentiable vector functions $\eta(t, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)$ and $\xi(t, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)$ with $\eta(a, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu) = 0 = \eta(b, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)$ and $\xi(a, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu) = 0 = \xi(b, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)$ such that $W(\eta(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu), \xi(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)) < 0$, where W is defined as follows:

$$\begin{aligned} & W(\eta(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu), \xi(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)) \\ & \quad = \int_a^b \Big(\big(\tau f_x(t, \bar{x}, \bar{u}) + \lambda(t)^{\mathrm{T}} g_x(t, \bar{x}, \bar{u}) + \mu(t)^{\mathrm{T}} h_x(t, \bar{x}, \bar{u})\big)\, \eta(t, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu) \\ & \qquad - \mu(t)^{\mathrm{T}} \dot{\eta}(t, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu) + (\tau f_u(t, \bar{x}, \bar{u}) + \lambda(t)^{\mathrm{T}} g_u(t, \bar{x}, \bar{u}) + \mu(t)^{\mathrm{T}} h_u(t, \bar{x}, \bar{u}))\xi(t, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)\Big)\, \mathrm{d}t. \end{aligned}$$

Suppose that $W(\eta(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu), \xi(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)) < 0$, is not verified for any $\eta(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)$, $\xi(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)$ and in consequence

$$W(\eta(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu), \xi(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)) > 0$$

is not verified either (take $-\eta(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)$, $-\xi(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)$). Therefore,

$$W(\eta(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu), \xi(\cdot, \bar{x}, x, \bar{u}, u, \tau, \lambda, \mu)) = 0$$

for all differentiable vector functions $\eta(\cdot,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$, $\xi(\cdot,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$ such that $\eta(a,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)=0=\eta(b,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$ and $\xi(a,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)=0=\xi(b,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$. If we set $\xi(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)=0$ for all $t\in I$, then we have that

$$W(\eta(\cdot,\bar{x},x,\bar{u},u,\tau,\lambda,\mu),0)=0$$

for all differentiable vector function $\eta(\cdot,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$ with $\eta(a,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)=0=\eta(b,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$, i.e.,

$$\int_a^b \big((\tau f_x(t,\bar{x},\bar{u})+\lambda(t)^{\mathrm{T}}g_x(t,\bar{x},\bar{u})+\mu(t)^{\mathrm{T}}h_x(t,\bar{x},\bar{u}))\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)-\mu(t)^{\mathrm{T}}\dot{\eta}(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\big)\,\mathrm{d}t=0.$$

From the generalized Dubois–Raymond's lemma (see [17]), we have

$$\tau f_x(t,\bar{x},\bar{u})+\lambda(t)^{\mathrm{T}}g_x(t,\bar{x},\bar{u})+\mu(t)^{\mathrm{T}}h_x(t,\bar{x},\bar{u})+\dot{\mu}(t)=0. \tag{8}$$

On the other hand, we set $\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)=0$ for all $t\in I$; then it is verified that

$$W(0,\xi(\cdot,\bar{x},x,\bar{u},u,\tau,\lambda,\mu))=0$$

for all differentiable vector function $\xi(\cdot,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$ such that $\xi(a,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)=0=\xi(b,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$, i.e.,

$$\int_a^b (\tau f_u(t,\bar{x},\bar{u})+\lambda(t)^{\mathrm{T}}g_u(t,\bar{x},\bar{u})+\mu(t)^{\mathrm{T}}h_u(t,\bar{x},\bar{u}))\xi(\cdot,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\mathrm{d}t=0$$

and then

$$\tau f_u(t,\bar{x},\bar{u})+\lambda(t)^{\mathrm{T}}g_u(t,\bar{x},\bar{u})+\mu(t)^{\mathrm{T}}h_u(t,\bar{x},\bar{u})=0. \tag{9}$$

Since $(\bar{x},\bar{u},\tau,\lambda,\mu)$ verifies conditions (6)–(9), then $(\bar{x},\bar{u})$ is a Fritz John point, and therefore $(\bar{x},\bar{u})$ is an optimal solution for (CP), which stands in contradiction to $F(x,u)-F(\bar{x},\bar{u})<0$. Therefore, (CP) is FJ-invex.

(ii) (Sufficient condition) Let $(\bar{x},\bar{u})$ be a Fritz John point, i.e., there exist $\tau\in\mathbb{R}$ and piecewise smooth functions $\lambda: I\to\mathbb{R}^m$ and $\mu: I\to\mathbb{R}^n$ satisfying (4)–(7). On the other hand, for all $(x,u)\in K$, and for λ (which verifies (6) and (7)) and μ there exist vector functions $\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$ and $\xi(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$ with $\eta(a,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)=0=\eta(b,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$ and $\xi(a,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)=0=\xi(b,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$ which verify the FJ-invex definition. We have:

$$\begin{aligned}
&\int_a^b \big((\tau f_x(t,\bar{x},\bar{u})+\lambda(t)^{\mathrm{T}}g_x(t,\bar{x},\bar{u})+\mu(t)^{\mathrm{T}}h_x(t,\bar{x},\bar{u}))\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\\
&\quad-\mu(t)^{\mathrm{T}}\dot{\eta}(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)+(\tau f_u(t,\bar{x},\bar{u})+\lambda(t)^{\mathrm{T}}g_u(t,\bar{x},\bar{u})+\mu(t)^{\mathrm{T}}h_u(t,\bar{x},\bar{u}))\xi(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\big)\,\mathrm{d}t\\
&=\int_a^b \big((\tau f_x(t,\bar{x},\bar{u})+\lambda(t)^{\mathrm{T}}g_x(t,\bar{x},\bar{u})+\mu(t)^{\mathrm{T}}h_x(t,\bar{x},\bar{u})+\dot{\mu}(t))\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\\
&\quad+(\tau f_u(t,\bar{x},\bar{u})+\lambda(t)^{\mathrm{T}}g_u(t,\bar{x},\bar{u})+\mu(t)^{\mathrm{T}}h_u(t,\bar{x},\bar{u}))\xi(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\big)\,\mathrm{d}t\\
&\quad-\mu(t)^{\mathrm{T}}\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\big|_a^b \quad \text{(by integration by parts)}\\
&=\int_a^b \big((\tau f_x(t,\bar{x},\bar{u})+\lambda(t)^{\mathrm{T}}g_x(t,\bar{x},\bar{u})+\mu(t)^{\mathrm{T}}h_x(t,\bar{x},\bar{u})+\dot{\mu}(t))\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\\
&\quad+(\tau f_u(t,\bar{x},\bar{u})+\lambda(t)^{\mathrm{T}}g_u(t,\bar{x},\bar{u})+\mu(t)^{\mathrm{T}}h_u(t,\bar{x},\bar{u}))\xi(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\big)\,\mathrm{d}t\\
&=0 \quad \text{(by (4) and (5))}.
\end{aligned}$$

Since (CP) is FJ-invex, we obtain $F(x,u)-F(\bar{x},\bar{u})\geq 0$ for all $(x,u)\in K$. Therefore, $(\bar{x},\bar{u})$ is an optimal solution for (CP). □

Therefore, FJ-invexity of (CP) is the weakest condition in order to characterize the optimal solution set of (CP) by Fritz John points.

3. Conclusion

We have provided the condition of FJ-invexity for a control problem and we have proved that FJ-invexity is a necessary and sufficient condition for a Fritz John point to be an optimal solution of a control problem. This result extends the characterization result recently given by Arana et al. [4].

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