Journal of Logical and Algebraic Methods in Programming 120 (2021) 100643

Contents lists available at ScienceDirect

Journal of Logical and Algebraic Methods in Programming

www.elsevier.com/locate/jlamp

Schreier split extensions of preordered monoids ☆

Nelson Martins-Ferreira [a], Manuela Sobral [b,*,1]

[a] Instituto Politécnico de Leiria, Leiria, Portugal
[b] Universidade de Coimbra, CMUC e Departamento de Matemática, Coimbra, Portugal

ARTICLE INFO

Article history:
Received 14 March 2020
Received in revised form 16 January 2021
Accepted 25 January 2021
Available online 26 January 2021

Dedicated to J.M. Esgalhado Valença on the occasion of his 70th birthday

ABSTRACT

Properties of preordered monoids are investigated and important subclasses of such structures are studied. The corresponding full subcategories are related between them by appropriate functors as well as with the categories of preordered sets and of monoids. Schreier split extensions are described in the full subcategory of preordered monoids whose preorder is determined by the corresponding positive cone.

1. Introduction

Preordered monoids are monoids equipped with a preorder compatible with the monoid operation. They are relevant tools in many areas, for instance, in computer science where they are used in the theory of language recognition (see [24]), as well as in non-classical logics, namely in fuzzy logics (see [8] and [11]).

Many fundamental results have been obtained by switching from categories of monoids to categories of preordered or ordered monoids, and the same for semigroups. Examples of this fact are new proofs of two remarkable results that we mention next.

A celebrated result of I. Simon [25] on the classification of recognizable languages in terms of $\mathcal{J}$-triviality of the corresponding syntactic monoids has a radically new proof in [26], where it is proved that every finite $\mathcal{J}$-trivial monoid (for the Green's $\mathcal{J}$-equivalence relation [6]) is a quotient of an ordered monoid satisfying the identity $x \leq 1$. In [9], the authors give another proof of this result and explain its relevance in the theory of finite semigroups. A systematic use of ordered monoids in language theory, was initiated by J.-E. Pin [20] and developed in [21], [22] and other subsequent papers.

The second example is a new proof of a well-known and important result of A. Tarski that gives a criterion for the existence of a monoid homomorphism from a given commutative monoid A to the extended positive real line $\overline{\mathbb{R}^+}$ that sends a fixed element $a \in A$ to 1. In [27], F. Wehrung proves that this is a Hahn-Banach type property, stating the injectivity of $\overline{\mathbb{R}^+}$, not in the category of commutative monoids, where there are no nontrivial injectives, but in the category of commutative monoids equipped with a preorder that makes every element positive, called there "positively ordered monoids" or P.O.M., for short.

Preordered monoids have a much richer diversity of features than preordered groups. In contrast with the case of preordered groups, in preordered monoids the submonoid of positive elements, called the positive cone, neither determines

☆ This work was partially supported by the Fundação para a Ciência e a Tecnologia (FCT) and Centro2020 through the following Projects: UIDB/04044/2020, UIDP/04044/2020, PAMI - ROTEIRO/0328/2013 (Nº 022158), Next.parts (17963); and MATIS (CENTRO-01-0145-FEDER-000014 - 3362). Also by CDRSP and ESTG from the Polytechnic Institute of Leiria.

* Corresponding author.
E-mail addresses: martins.ferreira@ipleiria.pt (N. Martins-Ferreira), sobral@mat.uc.pt (M. Sobral).

1 The second author was partially supported by the Centre for Mathematics of the University of Coimbra – UIDB/MAT/00324/2020, funded by the Portuguese Government through FCT/MCTES.

https://doi.org/10.1016/j.jlamp.2021.100643
2352-2208/

the preorder nor is a cancellative monoid, in general. These features of preordered groups are rescued in the new context by considering convenient subcategories of the category of preordered monoids, **OrdMon**, satisfying these properties or appropriate generalizations, covering a wide range of structures.

In particular, the failure of the first property gives rise to a classification of preordered monoids according to the relation between its preorder and the preorder induced by the corresponding positive cone considered here that is, for P.O.M., the opposite of Green's preorder $\mathcal{L}$ as explained in Section 2. Furthermore, this last preorder may or may not be compatible with the monoid operation. The characterization of the positive cones inducing compatible preorders provides a reason as to why the commutativity of the underlying monoid is often assumed in the literature.

This classification gives rise to several categories and functors between them, some of them being part of adjoint situations.

The cancellation property is often replaced by weaker conditions like the "pseudo-cancellation" introduced in [27] that plays an important role in the characterization of the injective objects presented there.

Let **Ord** be the category of preordered sets and monotone maps, and **Mon** the category of monoids and monoid homomorphisms. We recall that the forgetful functor **Ord** $\to$ **Set** is a topological functor [10] (like the one from the category of topological spaces to the category of sets it has initial and final structures) and that the forgetful functor **Mon** $\to$ **Set** (such as the underlying functor of any variety of algebras) is monadic [17, p. 156]. We prove that the forgetful functors from **OrdMon** to **Mon** and to **Ord** are topological and monadic functors, respectively, and derive some consequences of these facts.

Due to the fact that **OrdMon** is the category **Mon**(**Ord**) of internal monoids in **Ord** (which fails to be so in **OrdGrp**), we show that the construction of the left adjoint to U_1: **OrdMon** = **Mon**(**Ord**) $\to$ **Ord** as well as its monadicity can be derived from general results for the forgetful functor **Mon**(**C**) $\to$ **C**, when **C** is a symmetric monoidal category satisfying some additional conditions [12,13,23].

In [11] coextensions of commutative pomonoids (monoids equipped with a compatible partial order) are introduced, generalizing similar constructions due to P. A. Grillet [7] and J. Leech [14,15], in the unordered case.

It is well known that in the category of groups there is an equivalence between group actions and split extensions (which are, in this case, nothing but split epimorphisms), obtained via the semidirect product construction.

Schreier split extensions of monoids, that first appeared in [18], correspond to an important class of split epimorphisms of monoids, the Schreier split epimorphisms (whose name was inspired by the Schreier internal categories in monoids introduced by Patchkoria in [19]). Indeed, they are exactly those split epimorphisms that correspond to monoid actions: an action of a monoid B on a monoid X being a monoid homomorphism $\varphi\colon B \to \mathrm{End}(X)$ from B to the monoid of endomorphisms of X. Also this class of split epimorphisms has essentially all homological and algebraic properties of the split homomorphisms in groups (see [2] and [3]).

Schreier split extensions have already been defined in categories of monoids with operations [18] and in the categories of cancellative conjugation monoids [5].

In this paper we describe Schreier split extensions in the full subcategory **OrdMon*** of **OrdMon** with objects all preordered monoids whose preorder is induced by the corresponding positive cone.

In [4] the structure of the split extensions in the category of preordered groups is studied and the case where the restriction to the positive cones gives a Schreier split epimorphism in **Mon** is analysed. Also the behaviour of the category **Mon**(**Ord**) and, more generally, the one **Mon**(**C**) where **C** satisfies suitable conditions, is considered in the last section.

This paper is organized as follows. In Section 2 we give several examples of preordered monoids and characterize the submonoids of a monoid A that induce a compatible preorder in A. Some full subcategories of the category **OrdMon** are defined and an isomorphism is established between the full subcategory **OrdMon*** and the category **RNMono**(**Mon**) of right normal monomorphisms in monoids, in a sense introduced there, which plays a central role to obtain the main result of the last section. In Section 3 we study in detail functorial relations between the main categories involved in this paper that, being quite simple, give much information about these categories. We also include a brief but complete account of the general categorical results from which they can be derived. In Section 4 we introduce the notion of Schreier split extensions in the category **OrdMon*** and show how they are related with what we call preordered actions via an appropriate concept of semidirect product. Finally, we point out some special cases and present an example that helps to show the real character of the notions introduced.

Throughout we will denote preordered monoids additively, say by $(A, +, 0, \leq)$ where the monoid $(A, +, 0)$ is not necessarily commutative and $\leq$ is a preorder compatible with $+$, that is, where $+\colon A \times A \to A$ is a monotone map.

For concepts of category theory that are not defined here we suggest MacLane's book [17].

2. The category of preordered monoids

We start by recalling that if $(A, +, 0, \leq)$ is a preordered group, i.e. $(A, +, 0)$ is a (not necessarily abelian) group and the preorder $\leq$ is compatible with the group operation

$$\forall a, b, c, d \in A \quad a \leq b \quad \text{and} \quad c \leq d \quad \Longrightarrow \quad a + c \leq b + d,$$

then $P = \{a \in A \mid 0 \leq a\}$ is a submonoid of A closed under conjugation. Furthermore, this monoid P, called the *positive cone* of the preordered group, determines the preorder, i.e.,

$$a \leq b \Longleftrightarrow b - a \in P.$$

Indeed, if $a \leq b$, since $-a \leq -a$, then

$$0 = a - a \leq b - a.$$

Conversely, if $b - a \geq 0$, since $a \geq a$ then

$$b = b - a + a \geq a.$$

In this case, defining

$$a \leq_P b \Longleftrightarrow b \in P + a,$$

we have that $\leq$ coincides with $\leq_P$ and $P + a = a + P$, because P is closed under conjugation: $x + a = a + y \Longleftrightarrow y = -a + x + a \Longleftrightarrow x = a + y - a$.

If P is the submonoid of positive elements in a preorder monoid, we define the relation $\leq_P$ by

$$a \leq_P b \quad \text{if} \quad b \in P + a,$$

and get a preorder $\leq_P$ which is contained in the original preorder.

Proposition 1. *If* $(A, +, 0, \leq) \in \mathbf{OrdMon}$ *then* $P = \{a \in A \mid 0 \leq a\}$ *is a submonoid of* A *and*

$$a \leq_P b \Longrightarrow a \leq b.$$

Proof. We have that $0 \in P$ and if $a, b \in P$ then $a \geq 0$ and $b \geq 0$ implies that $a + b \geq 0$ and so P is a submonoid of A.

If $b = x + a$ with $x \in P$, since $x \geq 0$ and $a \geq a$, then $b = x + a \geq a$. □

The converse of this result is false, in general, as the following example shows.

Example 1. Let $(A, +, 0)$ be the monoid with the following addition table

+	0	1	2	3	4
0	0	1	2	3	4
1	1	1	4	4	4
2	2	2	4	4	4
3	3	3	4	4	4
4	4	4	4	4	4

equipped with the preorder $\leq$ with $P = A$ and generated by the following diagram (where the arrows from zero have been omitted)

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & 2 \arrow[d] \arrow[dl] \\ 3 \arrow[r] & 4 \end{tikzcd}.$$

Then $(A, +, 0, \leq) \in \mathbf{OrdMon}$ and $\leq_P$ is the preorder

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & 2 \arrow[d] \\ 3 \arrow[r] & 4 \end{tikzcd},$$

that is strictly contained in $\leq$.

In the previous example one can easily check that $\leq_P$ is compatible with $+$ and so $(A, +, 0, \leq_P)$ is also a preordered monoid. The following example shows that this is not always the case.

Example 2. We consider the monoid $(A, +, 0)$ with addition table

+	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	2	4
2	2	1	2	1	4
3	3	1	2	0	4
4	4	4	4	4	4

and the preorder generated by

$$\begin{tikzcd} 1 \arrow[r] \arrow[dr] & 2 \arrow[d] \\ 3 \arrow[u] \arrow[ur] \arrow[r] & 4, \end{tikzcd}$$

$3 \leq 0$, and $P = A$, that is, $0 \leq x$, for all $x \in A$. It is easy to check that $(A, +, 0, \leq)$ is a preordered monoid. However, $\leq_P$ being the following preorder, plus $3 \leq_P 0$,

$$\begin{tikzcd} 1 \arrow[dr] & 2 \arrow[d] \\ 3 \arrow[u] \arrow[ur] \arrow[r] & 4 \end{tikzcd}$$

is not compatible with the monoid operation. Indeed, $2 \geq_P 0$ and $1 \geq_P 1$ but $1 + 2 = 2 \not\geq_P 1$ since $2 \notin A + 1 = \{1, 4\}$.

The following is an example of a preordered monoid where the two preorders coincide.

Example 3. Let $(A, +, 0)$ be the monoid of Example 1 now with a different positive cone, $P = \{0, 1\}$, and the preorder sketched below

$$\begin{tikzcd} 0 \arrow[r] & 1 & 2 \arrow[d] \\ & 3 \arrow[r] & 4 \end{tikzcd}$$

which is exactly $\leq_P$, i.e. $\leq$ is the same as $\leq_P$.

Now we characterize the submonoids of a preordered monoid which induce a compatible preorder.

Definition 1. Given a monoid A and a submonoid M of A we say that M is

- *right normal* if $a + M \subseteq M + a$, for every $a \in A$;
- *left normal* if $M + a \subseteq a + M$, for every $a \in A$;
- *normal* if it is both right and left normal.

Proposition 2. *Let P be the positive cone of a preordered monoid $(A, +, 0, \leq)$. Then the monoid operation is monotone with respect to $\leq_P$ if and only if P is right normal.*

Proof. If $\leq_P$ is compatible with $+$ and $b = a + x$ with $x \in P$ then

$$x \geq_P 0 \quad \text{and} \quad a \geq_P a \quad \Longrightarrow \quad b = a + x \geq_P a$$

and so there exists an $y \in P$ such that $a + x = y + a$, i.e. $a + P \subseteq P + a$.

Conversely, if $a \leq_P b$ and $c \leq_P d$ then $b = x + a$ and $d = y + c$, for some $x, y \in P$ and so, because P is right normal, we can find $z \in P$ for which $a + y = z + a$, hence

$$b + d = x + a + y + c = x + z + a + c$$

and so $a + c \leq_P b + d$. $\square$

In Example 1 we have $P = A$, the so-called positively preordered monoid, and the left and right cosets are the following

a	$a+A$	$A+a$
0	A	A
1	$\{1,4\}$	{1,2,3,4}
2	{2,4}	{2,4}
3	{3,4}	{3,4}
4	{4}	{4}

Since P is right normal (for all $a \in A$, $a + A \subseteq A + a$) then $\leq_P$ is compatible with $+$.

In Example 2, again $P = A$ but P is not right normal and so $\leq_P$ is not compatible with $+$.

a	$a+A$	$A+a$
0	A	A
1	{1,2,4}	{1,4}
2	{1,2,4}	{2,4}
3	{0,1,2,3,4}	{0,1,2,3,4}
4	{4}	{4}

We remark that, in this case, A is not right normal in itself but it is left normal ($A + a \subseteq a + A$, for every $a \in A$) and so if we consider the preorder

$$a \leq'_P b \Longleftrightarrow b \in a + P$$

then, using a result similar to the one of Proposition 2, we conclude that $(A, +, 0, \leq'_P) \in$ **OrdMon**.

Remark 1. For a submonoid M of a monoid A, we can define two preorders on A

$$a \leq_M b \Leftrightarrow b \in M + a$$

and

$$a {\leq'}_M b \Leftrightarrow b \in a + M,$$

whose positive cones are precisely M. When $M = A$ we have that $\leq_M = \leq_{\mathcal{L}}^{op}$ and ${\leq'}_M = \leq_{\mathcal{R}}^{op}$, where $\mathcal{L}$ and $\mathcal{R}$ are the Green's relations defined, in additive notation, by

$$a \leq_{\mathcal{L}} b \Leftrightarrow M + a \subseteq M + b,$$

$$a \leq_{\mathcal{R}} b \Leftrightarrow a + M \subseteq b + M.$$

Indeed,

$$a \leq_M b \Leftrightarrow b = x + a, \text{ for some } x \in M \Leftrightarrow M + b \subseteq M + a \Leftrightarrow b \leq_{\mathcal{L}} a,$$

and the same for $\leq_{\mathcal{R}}$.

Corollary 1. *For every submonoid M of a commutative preordered monoid $(A, +, 0, \leq)$, the preorders $\leq_M$ and $\leq'_M$ coincide and, moreover, $(A, +, 0, \leq_M)$ is a preordered monoid.*

The positive cone of a commutative preordered monoid need not determine the preorder: for

+	0	1	2
0	0	1	2
1	1	1	1
2	2	1	1

with $P = A$ and $\leq$ as sketched below

$$1 \rightleftarrows 2$$

the right (= left) cosets are

a	$P+a$
0	P
1	$\{1\}$
2	$\{1,2\}$

and so $\leq_P$ is

$$1 \longleftarrow 2\,,$$

but $1 \not\leq_P 2$ because $2 \notin P+1$.

Let us denote by $\mathbf{OrdMon}^*$ the full subcategory of **OrdMon** with objects the preordered monoids such that $\leq = \leq_P$. And the same for the commutative case, $\mathbf{OrdCMon}^*$.

Proposition 3. *The subcategory* $\mathbf{OrdCMon}^*$ *is coreflective in the category* **OrdCMon**.

Proof. To each preorder commutative monoid we can associate a special one with the preorder induced by its positive cone. This is expressed by saying that the subcategory is coreflective. Indeed, if $(A,+,0,\leq)$ is a preordered commutative monoid and P is its positive cone then, by Corollary 1, $(A,+,0,\leq_P) \in \mathbf{OrdCMon}^*$. Furthermore, the identity morphism on A defines a monotone one $c_{(A,\leq)}\colon (A,\leq_P) \to (A,\leq)$, by Proposition 1.

We prove that it is the coreflection of $(A,+,0,\leq)$ in $\mathbf{OrdMon}^*$. Indeed, given a morphism $f\colon (A',\leq_{P'}) \to (A,\leq)$ in **OrdCMon** if $a' \in P'$ then $f(a') \in P$ $(a' \geq 0 \Rightarrow f(a') \geq 0)$ and so $f(P') \subseteq P$. Consequently f factors through $c_{(A,\leq)}$

$$\begin{tikzcd}
(A,\leq_P) \arrow[r, "c_{(A,\leq)}"] & (A,\leq) \\
(A',\leq_{P'}) \arrow[u, dashed, "\bar{f}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

by a unique homomorphism $\bar{f} \in \mathbf{OrdCMon}^*$ because if $a'_1 \leq_{P'} a'_2$ then $a'_2 \in P' + a'_1$ and so

$$f(a'_2) \in f(P') + f(a'_1) \subseteq P + f(a'_1).$$

Hence, $f(a'_1) \leq_P f(a'_2)$ and so $\bar{f}(a'_1) \leq_P \bar{f}(a'_2)$ for all $a'_1 \leq_{P'} a'_2$ in A'. Thus $\mathbf{OrdCMon}^*$ being a full coreflective subcategory is closed under colimits in **OrdCMon**. $\square$

Definition 2. We say that a monomorphism $m\colon S \to A$ of monoids is *right normal* if its image $m(S)$ is a right normal submonoid of A and denote by **RNMono**(**Mon**) the corresponding full subcategory of the category of monomorphisms of monoids, **Mono**(**Mon**).

Example 2 shows that the identity morphism may not be a right normal monomorphism.

Theorem 1. *The category* $\mathbf{OrdMon}^*$ *is isomorphic to the one of right normal monomorphisms in* **Mon**, **RNMono**(**Mon**).

Proof. The functor $G\colon \mathbf{OrdMon}^* \to \mathbf{RNMono}(\mathbf{Mon})$ defined by

$$\begin{tikzcd}
(A,\leq_P) \arrow[d, "f"] & \mapsto & P \arrow[r] \arrow[d, "f|_P"'] & A \arrow[d, "f"] \\
(A',\leq_{P'}) & & P' \arrow[r] & A'
\end{tikzcd}$$

has an inverse $F\colon \mathbf{RNMono}(\mathbf{Mon}) \to \mathbf{OrdMon}^*$ assigning

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "f'"'] & A \arrow[d, "f"] & \mapsto & (A,\leq_S) \arrow[d, "f"'] \\
S' \arrow[r] & A' & & (A',\leq_{S'})
\end{tikzcd}$$

where $f(S) \subseteq S'$ implies that $f \in \mathbf{OrdMon}^*$. Then $GF(S \to A) = G(A,\leq_S) = (S \to A)$ and $FG(A,\leq_P) = F(P \to A) = (A,\leq_P)$. $\square$

The following are examples, inspired by [27], of objects in $\mathbf{OrdMon}^*$.

(1) The set of all R-submodules of a module A over a ring R, equipped with the “Minkowski sum”

$$U + V = \{u + v : u \in U \text{ and } v \in V\}$$

and the order defined by the inclusion. Indeed, in this case every element is positive (i.e. the positive cone is the set of all R-submodules) and $U \subseteq V$ if and only if $V = V + U$.

(2) All injective objects in **OrdMon** with respect to embeddings (not to monomorphisms) are objects in **OrdMon***. In fact, let M be the submonoid of the monoid $\mathbb{N} \times \mathbb{N}$, generated by $(1,0)$ and $(1,1)$ with the order induced by the product order and $i\colon M \to \mathbb{N} \times \mathbb{N}$ the embedding. If $a \le b$ in an injective object A then there exists a (unique) morphism in **OrdMon**, $u\colon M \to A$ such that $u(1,0) = a$ and $u(1,1) = b$, defined by $u(n+m, m) = na + mb$, for every $n, m \in \mathbb{N}$. By injectivity of A, there exists a morphism $v\colon \mathbb{N} \times \mathbb{N} \to A$

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "u"'] & \mathbb{N} \times \mathbb{N} \arrow[dl, "v"] \\ A & \end{tikzcd}$$

extending u, that is such that $v \cdot i = u$. Then taking $c = v(0,1)$ we have that $b = c + a \in P + a$ and so the preorder in A coincides with the one induced by its positive cone. Indeed, since $(0,0) \le (0,1)$ and v preserves the order then $0 \le c$.

Let **OrdMon**$^\square$ be the full subcategory of **OrdMon** with objects all preordered monoids whose positive cone is a right normal monoid. Note that Example 1 describes an object of **OrdMon**$^\square$ that does not belong to **OrdMon***

Proposition 4. *The category* **OrdMon*** *is coreflective in* **OrdMon**$^\square$.

Proof. Essentially the same as the one of Proposition 3. □

Summing up, we have the following commutative diagram of categories and functors

$$\begin{tikzcd} \mathbf{OrdMon} \arrow[rrr] & & & \mathbf{Mono}(\mathbf{Mon}) \\ & \mathbf{OrdMon}^\square \arrow[ul] \arrow[dl, shift left] & & \\ \mathbf{OrdMon}^* \arrow[uu] \arrow[ur, shift left] \arrow[rrr, shift left, "\cong"] & & & \mathbf{RNMono}(\mathbf{Mon}) \arrow[uu] \arrow[lll, shift left] \end{tikzcd}$$

where **OrdMon*** is coreflective in **OrdMon**$^\square$ but **OrdMon**$^\square$ is not coreflective in **OrdMon** as we prove in the following section.

3. The forgetful functors

Let us consider the following commutative diagram of forgetful functors

$$\begin{tikzcd} \mathbf{OrdMon} \arrow[r, "U_2"] \arrow[d, "U_1"'] & \mathbf{Mon} \arrow[d, "V_1"] \\ \mathbf{Ord} \arrow[r, "V_2"'] & \mathbf{Set} \end{tikzcd}$$

where V_2 is topological and V_1 is a monadic functor. We are going to prove that also U_2 is a topological functor and U_1 is a monadic one.

We recall that $U\colon \mathbf{C} \to \mathbf{D}$ is a topological functor if every family $(f_i\colon D \to U(C_i))_{i \in I}$, where I may be a proper class, has a unique U-initial lift:

(i) there exists a family $(\overline{f}_i\colon C \to C_i)_{i \in I}$ such that $U(\overline{f}_i) = f_i$ for $i \in I$;

(ii) if $h\colon U(C') \to U(C)$ is a morphism in $\mathbf{D}$ such that $U(\overline{f}_i) \cdot h = U(g_i)$, for each i in I, there exists a unique morphism $\overline{h}\colon C' \to C$ in $\mathbf{C}$ such that $U(\overline{h}) = h$ and $\overline{f}_i \cdot \overline{h} = g_i$.

The uniqueness, up to isomorphism, of the U-lift comes from the uniqueness of $\overline{h}$.

If $U\colon \mathbf{C} \to \mathbf{D}$ is a topological functor then the same holds for its dual, $U^{op}\colon \mathbf{C}^{op} \to \mathbf{D}^{op}$, which means that every family $(f_i\colon U(C_i) \to D)_{i\in I}$ has a unique U-final lift, a generalization of the well-known fact that each meet-complete partially ordered set is also join-complete. This implies that a topological functor has a left adjoint (the discrete object functor) and a right adjoint (the indiscrete object functor).

In this case we say that the category $\mathbf{C}$ is topological over $\mathbf{D}$ which is a powerful condition with nice consequences.

It is easy to prove that the functor V_2 is topological. However, if we replace **Ord** by the category **Pos** of partially ordered sets then it is no longer the case. This is a reason why the category of preordered sets is better behaved than the one of partially ordered sets, for our purposes.

Proposition 5. *The functor* $U_2\colon \mathbf{OrdMon} \to \mathbf{Mon}$ *is a topological functor.*

Proof. Given a family of monoid homomorphisms

$$f_i\colon (X,+,0) \to U_2(A_i,+,0,\leq_i),$$

for $i \in I$, defining for $x, x' \in X$

$$x \leq x' \Leftrightarrow f_i(x) \leq_i f_i(x'), \forall i \in I,$$

we obtain a preorder which, in addition, is compatible with the monoid operation:

$$\begin{aligned} x \leq x' \text{ and } y \leq y' \quad &\Leftrightarrow \quad \forall i \in I,\, f_i(x) \leq f_i(x') \text{ and } f_i(y) \leq f_i(y') \\ &\Leftrightarrow \quad \forall i \in I,\, f_i(x) + f_i(y) \leq f_i(x') + f_i(y') \\ &\Leftrightarrow \quad \forall i \in I,\, f_i(x+y) \leq f_i(x'+y') \\ &\Leftrightarrow \quad x+y \leq x'+y'. \end{aligned}$$

It is easy to check that condition (ii) above holds and so that the family has a unique U_2-initial lift. $\square$

From this we conclude that:

(1) U_2 has a left and a right adjoint defined by equipping each monoid with the discrete and the total preorder, respectively;
(2) **OrdMon** is complete and cocomplete, since **Mon** is complete and cocomplete, and U_2 preserves limits and colimits.

Proposition 6. *The functor* $U_1\colon \mathbf{OrdMon} \to \mathbf{Ord}$ *has a left adjoint.*

Proof. For $(X,\leq)$, let $F_1(X,\leq) = (X^*,\cdot,\epsilon,\leq)$, where X^* is the set of all words in the alphabet X with the operation of concatenation, having the empty word ϵ as identity (i.e., X^* is the free monoid on the set X), equipped with the preorder

$$w = [w_1 \cdots w_n] \leq w' = [w'_1 \cdots w'_m]$$

if and only if $n = m$ and $w_i \leq w'_i$ for $i = 1, 2, \ldots, n$. In this way we define a preorder compatible with concatenation.

The morphism

$$\eta_{(X,\leq)}\colon (X,\leq) \to U_1(X^*,\cdot,\epsilon,\leq),$$

which assigns to each $x \in X$ the singleton word $[x]$, is universal from $(X,\leq)$ to U_1:

$$\begin{tikzcd} (X,\leq) \arrow[r, "\eta_{(X,\leq)}"] \arrow[dr, "f"'] & U_1(X^*,\cdot,\epsilon,\leq) \arrow[d, dashed, "U_1\bar{f}"] \\ & U_1(A,+,0,\leq) \end{tikzcd} \qquad \begin{tikzcd} (X^*,\cdot,\epsilon,\leq) \arrow[d, dashed, "\bar{f}"'] \\ (A,+,0,\leq) \end{tikzcd}$$

for each f in **Ord** there exists a unique $\bar{f} \in \mathbf{OrdMon}$ such that $\bar{f}([x]) = f(x)$ and so $\bar{f}([x_1 x_2 \cdots x_n]) = f(x_1) + f(x_2) + \cdots + f(x_n)$, because $\bar{f} \in \mathbf{Mon}$. And $\bar{f}$ is monotone: if $x = [x_1 x_2 \cdots x_n] \leq y = [y_1 y_2 \cdots y_n]$, since $x_i \leq y_i$, $i = 1, \cdots, n$, we have $f(x_1) + f(x_2) + \cdots + f(x_n) \leq f(y_1) + f(y_2) + \cdots + f(y_n)$, i.e. $f(x) \leq f(y)$.

Consequently, this defines a functor

$$F_1\colon \mathbf{Ord} \to \mathbf{OrdMon}$$

that is left adjoint of U_1 with unit η. $\square$

Proposition 7. *The functor* $U_1 \colon \mathbf{OrdMon} \to \mathbf{Ord}$ *is monadic.*

Proof. We recall that, by Beck's monadicity criterion (see e.g. [16, Thm. 2.4]), a right adjoint functor U_1 is monadic if and only if

- U_1 reflects isomorphisms;
- **OrdMon** has and U_1 preserves coequalizers of all parallel pairs (f, g) such that its image under U_1, $(U_1(f), U_1(g))$, has a contractible coequalizer in **Ord**.

Given a morphism $f \colon (A, +, 0, \leq) \to (B, +, 0, \leq)$ in **OrdMon** such that $U_1(f)$ is an isomorphism in **Ord** then, being also a bijective homomorphism of monoids, it is an isomorphism of monoids and so it is also an isomorphism in **OrdMon**. Hence U_1 reflects isomorphisms.

For a parallel pair of morphisms $f, g \colon (A, +, 0, \leq) \to (B, +, 0, \leq)$ in **OrdMon** let $q \colon (B, +, 0) \to (C, +, 0)$ be a coequalizer of $(U_2(f), U_2(g))$ in the category of monoids. Considering in C the preorder that is the transitive closure of the image by q of the preorder in B, it is easy to prove that this preorder is compatible with the monoid operation, so that $(C, +, 0, \leq) \in$ **OrdMon**, and also that

$$q \colon (B, +, 0, \leq) \to (C, +, 0, \leq)$$

is the coequalizer of (f, g) in this category.

Let us assume that $(U_1(f), U_1(g))$ has a contractible coequalizer $(U_1(f), U_1(g), h; i, j)$ in **Ord**. We have to prove that the unique morphism $t \in \mathbf{Ord}$ such that $t \cdot h = U_1(q)$ is an isomorphism.

Since $V_2 U_1 = V_1 U_2$ and V_1 is monadic, we know that $V_2(t)$ is a bijection. Furthermore, if $c = t(x) \leq t(y) = d$ then $x \leq y$. Indeed, by definition of the preorder in C, there exists a zig-zag in B

$$b_1 \leq b_2 \sim {b'}_2 \leq b_3 \sim \cdots \leq b_{n-1} \sim {b'}_{n-1} \leq b_n,$$

such that $q(b_1) = c, q(b_n) = d$ and $q(b_i) = q({b'}_i)$ for $i = 2, \ldots, n-1$. Thus $x = h(b_1) \leq h(b_n) = y$. □

Proposition 8. *The subcategory* $\mathbf{OrdMon}^{\square}$ *is not coreflective in the category* **OrdMon**.

Proof. For every preordered set $(X, \leq)$, $F_1(X, \leq) = (X^*, \cdot, \epsilon, \leq)$ has positive cone $P = \{\epsilon\}$ that is a right normal (indeed a normal) submonoid. Hence the preordered monoid $F_1(X, \leq)$ lies in $\mathbf{OrdMon}^{\square}$ and we have the following situation

$$\begin{tikzcd}
\mathbf{OrdMon}^{\square} \arrow[rr] \arrow[dr, shift left, "U_1^{\square}"] & & \mathbf{OrdMon} \simeq \mathbf{Ord}^T \arrow[dl, shift right, "U_1"'] \\
& \mathbf{Ord} \arrow[ul, shift left, "F_1^{\square}"] \arrow[ur, shift right, "F_1"'] &
\end{tikzcd}$$

where $U_1^{\square}$ is the restriction of U_1 to $\mathbf{OrdMon}^{\square}$, $F_1^{\square}$ is the corestriction of F_1 giving a left adjoint to $U_1^{\square}$, and T is the monad that both adjunctions induce in **Ord**.

From the above we conclude that $\mathbf{OrdMon}^{\square}$ cannot be coreflective in **OrdMon** otherwise, being closed under coequalizers, $U_1^{\square}$ would be monadic and so $\mathbf{OrdMon}^{\square} \cong \mathbf{Ord}^T \cong \mathbf{OrdMon}$ which is false as Example 2 shows. □

Direct proofs presented in this section are simple and informative about the categories involved.

However, since **OrdMon** is the category **Mon**(**Ord**) of internal monoids in the category of preordered sets (which is not true for preordered groups) these results can be derived from more general ones relative to categories of models of the theory of monoids in monoidal categories. In our case, since **Ord** is a cartesian closed category which, furthermore, is locally finitely presentable (see [1]), the construction of the left adjoint of $U_1 \colon \mathbf{OrdMon} = \mathbf{Mon}(\mathbf{Ord}) \to \mathbf{Ord}$ is a special case of the construction of the left adjoint of the forgetful functor of $\mathbf{Mon}(\mathbf{C}) \to \mathbf{C}$, when $\mathbf{C}$ is a symmetric monoidal category, satisfying some additional conditions, presented by G. M. Kelly in [12], see also [13]. Also the monadicity of U_1 was proved by H. Porst [23, Cor. 2.6].

In more detail, S. Lack [13] proves that the forgetful functor of $\mathbf{Mon}(\mathbf{C}) \to \mathbf{C}$ has a left adjoint when $\mathbf{C}$ is a symmetric monoidal category with countable coproducts that are preserved by tensoring on either side, with the free monoid over an object $X \in \mathbf{C}$ given by

$$1 + X + X^2 + \cdots$$

where X^n means the n-fold tensor product of X.

H. Porst [23] deals with "admissible monoidal categories" which are locally presentable categories that, in addition, are symmetric monoidal with the property that tensoring by a fixed object defines a finitary functor (i.e., a functor preserving directed colimits).

In the cartesian case, that is when the tensor is given by the direct product and the identity is the terminal object in the monoidal category, if **C** is locally presentable and cartesian closed it is clearly admissible, in the above sense, and so, by Corollary 2.6 in [23] we conclude the monadicity of **Mon**(**C**) over **C**.

4. Schreier split extensions

We recall that, in the category of monoids, a Schreier split epimorphism is a split epimorphism (A, B, p, s), $ps = 1_B$, such that for each $a \in A$ there exists a unique x in the kernel of p such that $a = x + sp(a)$ [2]. This can be seen as a diagram

$$\begin{tikzcd} X \arrow[r, shift left, "k"'] & A \arrow[l, shift left, dashed, "q"'] \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd} \tag{1}$$

where k, p and s are monoid homomorphisms, $ps = 1_B$, k is the kernel of p and q is a map (called the Schreier retraction map), such that,

(S1) $kq + sp = 1_A$, and
(S2) $q(k(x) + s(b)) = x$, for every $x \in X$ and $b \in B$.

To the Schreier split epimorphism above corresponds an action

$$\varphi\colon B \to \mathrm{End}(X)$$

defined by $\varphi(b)(x) = q(s(b) + k(x))$ that we will denote by $b \cdot x$ [18, Thm. 2.9].

As consequence, the monoid A is isomorphic to the semidirect product $X \rtimes_\varphi B$, that is the set $X \times B$ with the operation defined by

$$(x_1, b_1) + (x_2, b_2) = (x_1 + \varphi(b_1)(x_2), b_1 + b_2).$$

Further consequences, that will be used in the sequel, are the following:

(C1) $k(b \cdot x) + s(b) = s(b) + k(x)$, for all $b \in B$ and $x \in X$;
(C2) $q(a_1 + a_2) = q(a_1) + q(sp(a_1) + kq(a_2)) = q(a_1) + p(a_1) \cdot q(a_2)$, for all $a_1, a_2 \in A$;
(C3) A is isomorphic to the semi-direct product $X \rtimes_\varphi B$ with isomorphisms defined by $\alpha(a) = (q(a), p(a))$ and $\beta(x, b) = k(x) + s(b)$;
(C4) p is the cokernel of k and so, since the sequence is exact, we speak of Schreier split extensions.

Note that condition $(C1)$ follows from $(S1)$ by taking $a = s(b) + k(x)$. A detailed proof of these conditions can be found in [2].

This definition can be extended to the category of preordered monoids by keeping q a set-theoretical map and assuming that k, p and s are monotone homomorphisms.

In this section we are going to introduce the notion of Schreier split extensions in **OrdMon***. For that we use the isomorphism defined in Theorem 1 and work in the category **RNMono**(**Mon**). For simplicity, we assume that the objects in this category are inclusions and we denote the right normal submonoids of a monoid M by P_M, since they are the positive cones of a compatible preorder in M.

Definition 3. A Schreier split epimorphism in **RNMono**(**Mon**) is a diagram

$$\begin{tikzcd} P_X \arrow[r, "\bar{k}"] \arrow[d] & P_A \arrow[r, shift left, "\bar{p}"] \arrow[d] & P_B \arrow[l, shift left, "\bar{s}"] \arrow[d] \\ X \arrow[r, shift left, "k"'] & A \arrow[l, shift left, dashed, "q"'] \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd} \tag{2}$$

in which the lower row is a Schreier split epimorphism in **Mon**, and the upper row consists of right normal submonoids, the positive cones P_X, P_A, and P_B, that make X, A, and B, objects in **OrdMon***. The morphisms $\bar{k}$, $\bar{p}$, and $\bar{s}$, are the corresponding restrictions.

We point out that we do not assume the monotonicity of q.

We will show that for every two objects (X, P_X) and (B, P_B) in **RNMono**(**Mon**), there is an equivalence between Schreier split extensions of (X, P_X) by (B, P_B) and a certain kind of actions that we will call preordered actions for the purpose of this paper.

Definition 4. Let (X, P_X) and (B, P_B) be two objects in the category **RNMono**(**Mon**). A *preordered action* of (B, P_B) on (X, P_X), that will be denoted by $(X, B, P_X, P_B, \varphi, \xi)$, consists of a monoid action of the underlying monoid B on X, i.e. a monoid homomorphism

$$\varphi\colon B \to \mathrm{End}(X),$$

together with a set-theoretical mapping

$$\xi\colon X \times P_B \to X,$$

satisfying the following conditions:

(A1) $\xi(0, b) = 0$, for all $b \in P_B$
(A2) if $x \in P_X$ then $\xi(x, 0) = x$
(A3) if $\xi(x, b) = x$ and $\xi(x', b') = x'$ then

$$\xi(x + b \cdot x', b + b') = x + b \cdot x'$$

(A4) for all $x, u \in X$, $v \in P_B$, $b \in B$, if $\xi(u, v) = u$, then there exists $u' \in X$ such that

$$x + b \cdot u = u' + v' \cdot x$$

and

$$\xi(u', v') = u'$$

where $v' \in P_B$ is such that $b + v = v' + b$, which exists because P_B is right normal.

A morphism (f_0, f_1, f_2) between two Schreier split extensions in the category **RNMono**(**Mon**) is a commutative diagram of the form

$$\begin{tikzcd}
& P_X \arrow[rr, "\bar{k}"] \arrow[dl, "\bar{f}_0"'] \arrow[dd] & & P_A \arrow[rr, shift left, "\bar{p}"] \arrow[dl, "\bar{f}_1"'] \arrow[dd] & & P_B \arrow[ll, shift left, "\bar{s}"] \arrow[dl, "\bar{f}_2"] \arrow[dd] \\
P_{X'} \arrow[rr, "\bar{k'}"] \arrow[dd] & & P_{A'} \arrow[rr, shift left, "\bar{p}'"] \arrow[dd] & & P_{B'} \arrow[ll, shift left, "\bar{s}'"] \arrow[dd] & \\
& X \arrow[rr, shift right, "k"'] \arrow[dl, "f_0"'] & & A \arrow[ll, dashed, shift right, "q"'] \arrow[rr, shift left, "p"] \arrow[dl, "f_1"'] & & B \arrow[ll, shift left, "s"] \arrow[dl, "f_2"] \\
X' \arrow[rr, shift right, "k'"'] & & A' \arrow[ll, dashed, shift right, "q'"'] \arrow[rr, shift left, "p'"] & & B'. \arrow[ll, shift left, "s'"] &
\end{tikzcd}$$

A morphism of preordered actions,

$$(f_0, f_2)\colon (X, B, P_X, P_B, \varphi, \xi) \to (X', B', P'_X, P'_B, \varphi', \xi')$$

consists of two monoid homomorphisms $f_0\colon X \to X'$ and $f_2\colon B \to B'$ which restrict to the respective positive cones giving $\bar{f}_0\colon P_X \to P_{X'}$ and $\bar{f}_2\colon P_B \to P_{B'}$, such that

$$f_0(b \cdot x) = f_2(b) \cdot f_0(x)$$

and

$$\xi'(f_0(u), \bar{f}_2(v)) = f_0(u),$$

whenever $\xi(u, v) = u$. In other words, the diagram where the horizontal arrows are defined by the monoid actions, $(b, x) \mapsto b \cdot x$,

$$\begin{tikzcd}
B \times X \arrow[r] \arrow[d, "f_0 \times f_2"'] & X \arrow[d, "f_0"] \\
B' \times X' \arrow[r] & X'
\end{tikzcd}$$

is commutative and the diagram

$$\begin{tikzcd} X \times P_B \arrow[r, "\xi"] \arrow[d, "f_0 \times \bar{f}_2"'] & X \arrow[d, "f_0"] \\ X' \times P_{B'} \arrow[r, "\xi'"] & X' \end{tikzcd}$$

commutes only when restricted to those pairs $(u, v) \in X \times P_B$ for which $\xi(u, v) = u$. That is, there exists $g\colon P_\xi \to P_{\xi'}$, such that the left square and the outer rectangle commute

$$\begin{tikzcd} P_\xi \arrow[r] \arrow[d, dashed, "g"'] & X \times P_B \arrow[r, "\xi"] \arrow[d, "f_0 \times \bar{f}_2"'] & X \arrow[d, "f_0"] \\ P_{\xi'} \arrow[r] & X' \times P_{B'} \arrow[r, "\xi'"] & X' \end{tikzcd} \tag{3}$$

where $P_\xi = \{(u, v) \in X \times P_B \mid \xi(u, v) = u\}$ and similarly for $P_{\xi'}$.

In this way we define a category $\mathcal{S}$ of Schreier split extensions in **RNMono**(**Mon**) and a category $\mathcal{A}$ of preordered actions.

Theorem 2. *There is an equivalence of categories between the category $\mathcal{A}$ of preordered actions and the category $\mathcal{S}$ of Schreier split extensions in* **RNMono**(**Mon**).

Proof. We define a functor $G\colon \mathcal{S} \to \mathcal{A}$ assigning to a Schreier split epimorphism in **RNMono**(**Mon**) as displayed in (2), a preordered action as follows:

(1) $\varphi_b(x) = q(s(b) + k(x))$, for all $x \in X$ and $b \in B$;
(2) $\xi(u, v) = u$ if $k(u) + s(v) \in P_A$ and $\xi(u, v) = 0$ otherwise.

These maps φ and ξ satisfy the conditions of Definition 4:

- The first condition above defines an action of B on X [18, Thm. 2.9].
- $\xi(0, b) = 0$ for $b \in P_B$.
- $\xi(x, 0) = x$ if $x \in P_X$, since $k(x) + s(0) = k(x) \in P_A$.
- If $\xi(x, b) = x$ and $\xi(x', b') = x'$ then $k(x) + s(b), k(x') + s(b') \in P_A$. Since P_A is a monoid then

  $$k(x) + s(b) + k(x') + s(b') \in P_A,$$

  but $s(b) + k(x') = k(b \cdot x') + s(b)$ and so we have that

  $$k(x + b \cdot x') + s(b + b') \in P_A.$$

  Consequently, $\xi(x + b \cdot x', b + b') = x + b \cdot x'$.
- $P_A \to A \cong X \rtimes_\varphi B$ right normal means that for all $(x, b) \in X \rtimes_\varphi B$, $(u, v) \in P_A$, there exists $(u', v') \in P_A$ such that

  $$(x, b) + (u, v) = (u', v') + (x, b)$$

  that is

  $$(x + b \cdot u, b + v) = (u' + v' \cdot x, v' + b)$$

  which implies $x + b \cdot u = u' + v' \cdot x$ and $b + v = v' + b$.

Defining $G(f_0, f_1, f_2) = (f_0, f_2)$ we obtain a functor $G\colon \mathcal{S} \to \mathcal{A}$.

Conversely, given a preordered action $(X, B, P_X, P_B, \varphi, \xi)$ we construct a Schreier split extension in **RNMono**(**Mon**) as follows (using the same notation as in (2)):

(1) $A = X \rtimes_\varphi B$ is the semi-direct product of the underlying monoids induced by the monoid action φ. This means that A is the set $X \times B$ with the monoid operation

$$(x, b) + (x', b') = (x + b \cdot x', b + b')$$

and neutral element $(0, 0) \in X \times B$.

(2) the right normal submonoid of A, $P_A = P_\xi$, is defined by

$$(x, b) \in P_A \Leftrightarrow b \in P_B \text{ and } \xi(x, b) = x.$$

This gives a Schreier split extension in **RNMono**(**Mon**). Indeed:
(a) P_ξ is a submonoid of $X \rtimes_\varphi B$ by (A3) and the fact that P_B is a monoid.
(b) The right normality of P_A comes from (A4).
(c) The morphism $\langle 1, 0\rangle\colon X \to A$ restricts to $P_X \to P_A$ by (A2).
(d) The morphism $\langle 0, 1\rangle\colon B \to A$ restricts to $P_B \to P_A$ by (A1).

Moreover, we define a functor $H\colon \mathcal{A} \to \mathcal{S}$ assigning to each morphism

$$(f_0, f_2)\colon (X, B, P, P_B, \varphi, \xi) \to (X', B', P_{X'}, P_{B'}, \varphi', \xi'),$$

the object $H(f_0, f_2) = (f_0, f_1, f_2)$ where $f_1 = g\colon P_\xi \to P_{\xi'}$ as in diagram (3).

Then $HG \cong 1_{\mathcal{S}}$: in the diagram

$$\begin{tikzcd}
P_X \arrow[rr, "\bar{k}"] \arrow[dr] \arrow[dd] & & P_A \arrow[rr, shift left, "\bar{p}"] \arrow[dd] & & P_B \arrow[ll, shift left, "\bar{s}"] \arrow[dd] \\
& P_\xi \arrow[ur, "\bar{\beta}"] \arrow[rr, shift left] \arrow[dd] & & P_B \arrow[ll, shift left] \arrow[ur, equal] \arrow[dd] & \\
X \arrow[rr, shift right, "k"'] \arrow[dr] & & A \arrow[ll, dashed, shift right, "q"'] \arrow[rr, shift left, "p"] & & B \arrow[ll, shift left, "s"] \\
& X \rtimes_\varphi B \arrow[ur, "\beta"] \arrow[rr, shift left] & & B \arrow[ll, shift left] \arrow[ur, equal] &
\end{tikzcd}$$

since $\beta(x, b) = k(x) + s(b)$, by definition of P_ξ, we conclude that $\bar{\beta}\colon P_\xi \to P_A$ is an isomorphism.

It is easy to check that also $GH = 1_{\mathcal{A}}$, thus giving the desired equivalence of categories. $\square$

Finally, we point out two interesting special cases:

- When q is a monotone map then it restricts to $\bar{q}\colon P_A \to P_X$ and ξ is trivial, in the sense that $\xi(x, b) = x$ when $x \in P_X$ and $b \in P_B$ and it is zero otherwise. In this case, the upper row of the diagram (2) is a Schreier split epimorphism of monoids and hence P_A is isomorphic to the semidirect product $P_X \times_{\bar{\varphi}} P_B$.
- When q is an homomorphism then the monoid action φ is trivial, i.e. $\varphi_b(x) = x$, for all $b \in B$. However, we may still have a non trivial ξ in this case, as the following example shows.

In the diagram (2) if q is a monoid homomorphism then $A \cong X \times B$ but the upper row need not be a Schreier split epimorphism.

Example 4. Let us consider the following diagram

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \mathbb{N} \times \mathbb{N} \arrow[r, shift left, "+"] \arrow[d] & \mathbb{N} \arrow[l, shift left, "\langle 0,1\rangle"] \arrow[d] \\
\mathbb{Z} \arrow[r, shift right, "\langle 1,-1\rangle"'] & \mathbb{Z} \times \mathbb{Z} \arrow[l, dashed, shift right, "\pi_1"'] \arrow[r, shift left, "+"] & \mathbb{Z}, \arrow[l, shift left, "\langle 0,1\rangle"]
\end{tikzcd} \tag{4}$$

which is an example of a Schreier split epimorphism in the category **RNMono**(**Mon**). The left $\mathbb{Z}$ has the discrete order because its positive cone is $\{0\}$, while the one on the right has the usual order since its positive cone is $\mathbb{N}$. The positive cone $\mathbb{N} \times \mathbb{N}$ and the corresponding order on $\mathbb{Z} \times \mathbb{Z}$ will be described below.

In this case we have a non trivial $\xi\colon \mathbb{Z} \times \mathbb{N} \to \mathbb{Z}$, defined by

$$\xi(u, v) = \begin{cases} u & \text{if } u \in \mathbb{N} \text{ and } u \leq v \\ 0 & \text{otherwise} \end{cases}$$

giving a preordered action $(\mathbb{Z}, \mathbb{Z}, \{0\}, \mathbb{N}, \varphi, \xi)$ where φ is trivial, which induces a Schreier split extension in **RNMono**(**Mon**)

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & P_\xi \arrow[r, shift left] \arrow[d] & \mathbb{N} \arrow[l, shift left] \arrow[d] \\
\mathbb{Z} \arrow[r, shift right, "\langle 1,0\rangle"'] & \mathbb{Z}\times\mathbb{Z} \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & \mathbb{Z} \arrow[l, shift left, "\langle 0,1\rangle"]
\end{tikzcd} \tag{5}$$

where $P_\xi = \{(u, v) \in \mathbb{Z}\times\mathbb{Z} \mid 0 \le u \le v\}$, with $0 \le u \le v$ in the usual order of $\mathbb{N}$. This defines the positive cone $P = P_\xi$ and the order of $\mathbb{Z}\times\mathbb{Z}$ in (4).

Declaration of competing interest

The authors declare that there is no conflict of interest.

Acknowledgements

The authors are grateful to the referees for their valuable comments and suggestions that greatly contributed to the improvement of a first version of the present paper.

References

[1] J. Adámek, J. Rosický, Locally Presentable and Accessible Categories, Cambridge Univ. Press, 1994.
[2] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, Schreier Split Epimorphisms in Monoids and in Semirings, Textos de Matemática Série B, vol. 45, Departamento de Matemática, Universidade de Coimbra, ISBN 978-972-8564-49-0, 2013, vi+116 pp.
[3] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, Schreier split epimorphisms between monoids, Semigroup Forum 88 (2014) 739–752.
[4] M.M. Clementino, N. Martins-Ferreira, A. Montoli, On the categorical behaviour of preordered groups, J. Pure Appl. Algebra 223 (2019) 4226–4245.
[5] A.P. Garrão, N. Martins-Ferreira, M. Raposo, M. Sobral, Cancellative conjugation semigroups and monoids, Semigroup Forum 100 (2020) 806–836, https://doi.org/10.1007/s00233-019-10070-9.
[6] J.A. Green, On the structure of semigroups, Ann. Math. (2) 54 (1) (1951) 163–172.
[7] P.A. Grillet, Left coset extensions, Semigroup Forum 7 (1974) 200–263.
[8] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1988.
[9] K. Henckell, J.-E. Pin, Ordered monoids and J-trivial monoids, in: J.C. Birget, S. Margolis, J. Meakin, M. Sapir (Eds.), Algorithmic Problems in Groups and Semigroups, in: Trends in Mathematics, Birkhäuser, Boston, 2000, pp. 121–137.
[10] H. Herrlich, Topological functors, Gen. Topol. Appl. 4 (1974) 125–142.
[11] J. Janda, T. Vetterlein, The coextension of commutative pomonoids and its application to triangular norms, Quaest. Math. 42 (3) (2019) 319–345.
[12] G.M. Kelly, A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on, Bull. Aust. Math. Soc. 22 (1) (1980) 1–83.
[13] S. Lack, Note on the construction of free monoids, Appl. Categ. Struct. 18 (2010) 17–29.
[14] J. Leech, Extending groups by monoids, J. Algebra 74 (1982) 1–19.
[15] J. Leech, H-coextensions of monoids, Mem. Am. Math. Soc. 157 (1975) 1–66.
[16] J. MacDonald, M. Sobral, Aspects of monads, in: M. Pedicchio, W. Tholen (Eds.), Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory, in: Encyclopedia of Mathematics and Its Applications, Cambridge University Press, 2004, pp. 213–268.
[17] S. Mac Lane, Categories for the Working Mathematician, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer, 1998.
[18] N. Martins-Ferreira, A. Montoli, M. Sobral, Semidirect products and crossed modules in monoids with operations, J. Pure Appl. Algebra 217 (2) (2013) 334–347.
[19] A. Patchkoria, Crossed semimodules and Schreier internal categories in the category of monoids, Georgian Math. J. 5 (6) (1998) 575–581.
[20] J.-E. Pin, A variety theorem without complementation, Izv. VUZ Math. 39 (1995) 80–90; English version, Russ. Math. (Izv. VUZ) 39 (1995) 74–83.
[21] J.-E. Pin, P. Weil, Polynomial closure and unambiguous product, Theory Comput. Syst. 30 (1997) 1–39.
[22] J.-E. Pin, P. Weil, Semidirect product of ordered monoids, Commun. Algebra 30 (1) (2002) 149–169.
[23] H.-E. Porst, On categories of monoids, comonoids and bimodules, Quaest. Math. 31 (2) (2008) 127–139.
[24] G. Rozenberg, A. Salomaa (Eds.), Handbook of Formal Languages, vol. 1, Springer-Verlag, 1997.
[25] I. Simon, Piecewise testable events, in: Proc. 2nd GI Conf., in: Lect. Notes in Comp. Sci., vol. 33, Springer Verlag, Berlin, Heidelberg, New York, 1975, pp. 214–222.
[26] H. Straubing, D. Thérien, Partially ordered finite monoids and a theorem of I. Simon, J. Algebra 119 (1985) 393–399.
[27] F. Wehrung, Injective positively ordered monoids I, J. Pure Appl. Algebra 83 (1) (1992) 43–82.