Infinite horizon fuzzy optimal control: optimality does not imply asymptotic stability

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ABSTRACT

In this paper, novel stability result for discrete-time infinite horizon optimal control using fuzzy objective functions is presented. For this class of control, the fuzzy goals and the fuzzy constraints introduced in the fuzzy objective function handle the constraints placed on both the state and the control vectors. We analyze the asymptotic stability of the equilibrium for the infinite horizon fuzzy optimal control law using the minimum aggregation operator. We show that the infinite horizon control with the minimum aggregation operator does not guarantee the asymptotic stability of the equilibrium in general. This is done by deriving an analytical solution of the control law for a simple linear system using a fuzzy dynamic programming approach. An example that shows the novel asymptotic stability result of the equilibrium for discrete-time infinite horizon optimal control with fuzzy objective function problem is given.

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1. INTRODUCTION

Receding horizon control (RHC) has become the standard control technique for constrained dynamical systems [1]–[7]. RHC involves an open-loop finite horizon optimal control problem at each sampling instant where an objective function has to be optimized at each sample time. This objective function should well reflect the real performance specifications of the closed loop system. Most RHC algorithms use classical objective functions, namely the 1-norm, the Euclidean norm, or the infinity norm. However, in many cases, those standard objective functions only roughly reflect the real performance requirements. Fuzzy decision-making functions [8] are objective functions that more transparently reflect the performance requirements. A useful overview of decision-making fuzzy objective functions for dynamic systems control is given in [9]. However, only open-loop control applications were considered. It was initially mentioned in [10] how fuzzy decision-making could be applied to RHC. A method for weighing the importance of goals and constraints in RHC with fuzzy objective functions was addressed in [11] and successful applications can be found, e.g., in [12]–[15].

Asymptotic stability of the equilibrium is an overriding performance specification for closed loop control systems. For RHC with standard quadratic objective functions, asymptotic stability of the equilibrium was studied and derived using Lyapunov results [16] with set invariance theory [17]. The appropriate Lyapunov function for establishing the asymptotic stability of RHC with standard objective functions was the infinite horizon objective function associated with an auxiliary local stabilizing controller. In fact, the stabilizing RHC schemes with standard objective functions are based on the result that the infinite horizon optimum controller asymptotically stabilizes the system to equilibrium under specific conditions (stabilizability and detectability). Thus, to derive asymptotically stabilizing RHC with fuzzy objective functions in a manner similar to the standard RHC case, it is important to first investigate the stability of the infinite horizon optimal control with fuzzy objective functions. It is also worth noting that the class of optimal control considered here differs from the fuzzy model optimal control case where only the model under control is fuzzy whereas the objective function is the standard one. For this, latest research on stability has reached a relatively a mature stage, see e.g., [18]–[22].

Asymptotic stability of the equilibrium under an infinite horizon optimal control law with fuzzy objective functions has been studied first in [23]. It has been shown that the product aggregation operator asymptotically stabilizes the infinite horizon optimal control with fuzzy objective functions to the equilibrium. This paper deals with asymptotic stability of equilibrium under an infinite horizon optimal control with fuzzy objective functions that use the minimum operator which is one of the most used aggregation operators. We show that the minimum aggregation operator does not guarantee, in general, the convergence of the infinite horizon optimal control with fuzzy objective functions to the equilibrium for all feasible initial states. We show the non-convergence of this control law by deriving an analytical solution for a simple first-order linear system using fuzzy dynamic programming [24], [25].

The paper's reminder is organized as follows. Infinite horizon optimal control problem with a fuzzy objective function that uses the minimum aggregation operator is introduced for a linear system in section 2. Section 3 presents the analytical solution for a linear system. The non-convergence of the resulting closed-loop linear system is given in section 4 and section 5 presents the conclusion.

2. INFINITE-HORIZON OPTIMAL CONTROL WITH MINIMUM AGGREGATION OPERATOR

Consider the discrete-time linear state space model:

$$x(t+1) = ax(t) + bu(t) \tag{1}$$

where $t \geq 0$ is the current moment, $x(t) \in \mathbb{R}$, and $u(t) \in \mathbb{R}$ are the state and the control input, respectively. The infinite horizon optimal control problem with fuzzy objective function seeks to identify a control sequence $\{u(t)\}_{t\geq 0} = \{u(0), u(1), \cdots, u(\infty)\}$ that maximizes the fuzzy objective function:

$$J_{0,\infty}(x(0), \{u(t)\}_{t\geq 0}) = \overset{t=0}{\underset{\infty}{\wedge}} \{\mu_G(x(t)) \wedge \mu_C(u(t))\} \tag{2}$$

subject to

$$x(t+1) = ax(t) + bu(t) \tag{3}$$

where $\mu_G(x(t))$ and $\mu_C(u(t))$ are the fuzzy goals and the fuzzy constraint imposed on the state variable and the control effort, respectively. Fuzzy goals and fuzzy constraints are characterized by their membership functions which are mappings from the domain of the state variable and control action to [0,1]. Here the operator $\wedge$ denotes the minimum aggregation operator and $J_{0,\infty}(x(0), \{u(t)\}_{t\geq 0})$ given in (2) is the fuzzy objective function. Infinite horizon optimum controller that maximizes (2) subject to (3) is given by $\{u^*(t)\}_{t\geq 0} = arg \max_{\{u(t)\}_{t\geq 0}} J_{0,\infty}(x(0), \{u(t)\}_{t\geq 0})$ which is the point where the confluence between the fuzzy goals and the fuzzy constraints is at its maximum.

3. ANANALYTICAL SOLUTION

Let us first consider the optimal controller $u^*(t)$ with finite horizon and fuzzy objective function:

$$u^*(t) = arg \max_{\{u(t)\}_{0\leq t\leq N-1}} \bar{J}_{0,N}(x(0), \{u(t)\}_{0\leq t\leq N-1}), \tag{4}$$

subject to

$$x(t+1) = ax(t) + bu(t) \tag{5}$$

where $\{u(t)\}_{0\le t\le N-1}=\{u(0),u(1),\cdots,u(N-1)\}$ and $N\ge 1$ is the finite horizon. The fuzzy objective function $\bar{J}_{0,N}(x(0),\{u(t)\}_{0\le t\le N-1})$ with a finite horizon is given by (6).

$$\bar{J}_{0,N}(x(0),\{u(t)\}_{0\le t\le N-1})=min\Big(\mu_C\big(u(0)\big),\mu_G\big(x(1)\big),\cdots,\mu_C\big(u(N-1)\big),\mu_G\big(x(N)\big)\Big) \quad (6)$$

The membership functions $\mu_G\big(x(t)\big)$, $1\le t\le N$ and $\mu_C\big(u(t)\big)$, $0\le t\le N-1$ are given in Figure 1(a) and Figure 1(b), respectively.

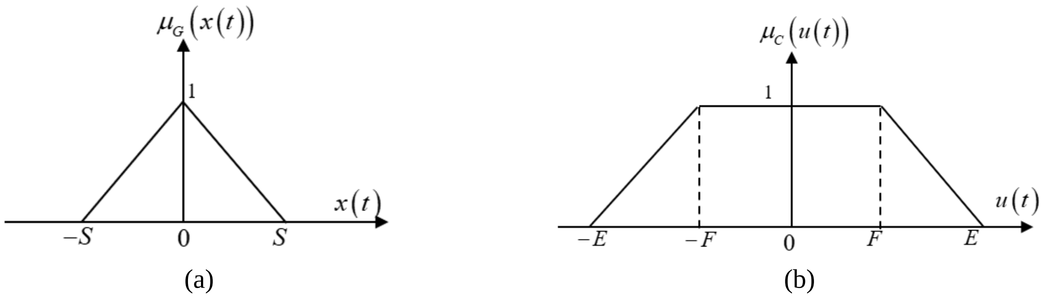

Figure 1. Membership functions (a) fuzzy goal and (b) fuzzy constraint

In this scenario, if the state x equals zero, i.e., the equilibrium is reached, then the goal is fully achieved. Nevertheless, if $-F\le u(t)\le F$, the constraint is fully satisfied. Moving away from zero for the state and away from the interval $[-F,F]$ for the control action, the level of the fuzzy goal's and constraint's fulfillment would steadily decline. Note that these membership function choices satisfy the hard constraints $-E\le u(t)\le E$ and $-S\le x(t)\le S$.

In the sequel, the optimal partial objective function of (6) from stage $N-r$ to stage N, where $r=1,2,\cdots,N$, is given by (7),

$$\bar{J}^*_{N-r,N}\big(x(N-r)\big)=\max_{\{u(t)\}_{N-r\le t\le N-1}}\bar{J}_{N-r,N}(x(N-r),\{u(t)\}_{N-r\le t\le N-1}) \quad (7)$$

where

$$\{u(t)\}_{N-r\le t\le N-1}=\{u(N-r),u(N-r+1),\cdots,u(N-1)\} \quad (8)$$

and $\bar{J}_{N-r,N}(x(N-r),\{u(t)\}_{N-r\le t\le N-1})$ is the partial objective function from stage $N-r$ to stage N. It would be given for a minimum aggregation operator by (9):

$$\bar{J}_{N-r,N}(x(N-r),\{u(t)\}_{N-r\le t\le N-1})=min\Big(\mu_C\big(u(N-r)\big),\mu_G\big(x(N-r+1)\big),\cdots,\mu_C\big(u(N-1)\big),\mu_G\big(x(N)\big)\Big) \quad (9)$$

Lemma 1: If $b>0$ and $(a-1)S>bE$, then the optimal control sequence solution of (7)-(9) subject to system (5) is given by (10a) to (10f):

$$u^*(N-r)=\frac{(F-E)a^r x(N-r)+FS}{b(E-F)(a^r-1)/(a-1)+S}\ if\ \frac{-S-bE(a^r-1)/(a-1)}{a^r}<x(N-r)\le\frac{bFS}{(1-a)S+b(E-F)} \quad (10a)$$

$$\frac{FS-a(E-F)x(N-r)}{S+b(E-F)}\ if\ \frac{bFS}{(1-a)S+b(E-F)}\le x(N-r)\le\frac{-bF}{a} \quad (10b)$$

$$-\frac{a}{b}x(N-r)\ if\ \frac{-bF}{a}\le x(N-r)\le\frac{bF}{a} \quad (10c)$$

$$\frac{-FS-a(E-F)x(N-r)}{S+b(E-F)}\ if\ \frac{bF}{a}\le x(N-r)\le\frac{-bFS}{(1-a)S+b(E-F)} \quad (10d)$$

$$\frac{(F-E)a^r x(N-r)-FS}{b(E-F)(a^r-1)/(a-1)+S}\ if\ \frac{-bFS}{(1-a)S+b(E-F)} \leq x(N-r) < \frac{S+bE(a^r-1)/(a-1)}{a^r} \tag{10e}$$

$$infeasible\ if\ |x(N-r)| \geq \frac{S+bE(a^r-1)/(a-1)}{a^r} \tag{10f}$$

for all $r = 1, 2, \cdots, N$, and the optimal partial fuzzy objective function is given by (11a) to (11f):

$$\bar{J}^*_{N-r,N}\big(x(N-r)\big) = \frac{a^r x(N-r)+bE(a^r-1)/(a-1)+S}{b(E-F)(a^r-1)/(a-1)+S}$$
$$if\ \frac{-S-bE(a^r-1)/(a-1)}{a^r} < x(N-r) \leq \frac{bFS}{(1-a)S+b(E-F)} \tag{11a}$$

$$\frac{S+ax(N-r)+bE}{S+b(E-F)}\ if\ \frac{bFS}{(1-a)S+b(E-F)} \leq x(N-r) \leq \frac{-bF}{a} \tag{11b}$$

$$1\ if\ \frac{-bF}{a} \leq x(N-r) \leq \frac{bF}{a} \tag{11c}$$

$$\frac{S-ax(N-r)+bE}{S+b(E-F)}\ if\ \frac{bF}{a} \leq x(N-r) \leq \frac{-bFS}{(1-a)S+b(E-F)} \tag{11d}$$

$$\frac{-a^r x(N-r)+bE(a^r-1)/(a-1)+S}{b(E-F)(a^r-1)/(a-1)+S}\ if\ \frac{-bFS}{(1-a)S+b(E-F)} \leq x(N-r) < \frac{S+bE(a^r-1)/(a-1)}{a^r} \tag{11e}$$

$$0\ if\ |x(N-r)| \geq \frac{S+bE(a^r-1)/(a-1)}{a^r} \tag{11f}$$

for all $r = 1, 2, \cdots, N$.

Proof: the proof is done by induction. First, note that applying the principle of optimality to problem (7)-(9) subject to system (5) we can write:

$$\bar{J}^*_{N,N}\big(x(N)\big) = 0 \tag{12}$$

and

$$\bar{J}^*_{N-r,N}\big(x(N-r)\big) = \max_{u(N-r)}$$
$$\min\big\{\mu_C\big(u(N-r)\big), \mu_G\big(x(N-r+1)\big), \bar{J}^*_{N-r+1,N}\big(x(N-r+1)\big)\big\} \tag{13}$$

where

$$x(N-r+1) = ax(N-r) + bu(N-r), r = 1, \cdots, N \tag{14}$$

The optimal control sequence and the optimal partial fuzzy objective functions will be derived using (12)-(14). From Figure 1(a) and Figure 1(b), it follows that

$$0\ if\ |u(N-r)| \geq E \tag{15a}$$

$$\frac{u(N-r)+E}{E-F}\ if\ -E \leq u(N-r) \leq -F \tag{15b}$$

$$1\ if\ |u(N-r)| \leq F \tag{15c}$$

$$\frac{-u(N-r)+E}{E-F}\ if\ F \leq u(N-r) \leq E \tag{15d}$$

and

$$\mu_G\big(x(N-r+1)\big) = 0\ if\ |x(N-r+1)| \geq S \tag{16a}$$

$$\frac{x(N-r+1)+S}{S}\ if\ -S \leq x(N-r+1) \leq 0 \tag{16b}$$

$$\frac{-x(N-r+1)+S}{S} \quad if\ 0 \le x(N-r+1) \le S \tag{16c}$$

for all $r = 1, \cdots, N$. Substituting (14) into (16a)-(16c) we obtain:

$$\mu_G\big(x(N-r+1)\big) = \mu_G\big(ax(N-r) + bu(N-r)\big) = 0$$
$$if\ u(N-r) \le \frac{-ax(N-r)-S}{b}\ or\ (N-r) \ge \frac{-ax(N-r)+S}{b} \tag{17a}$$

$$\frac{ax(N-r)+bu(N-r)+S}{S} \quad if\ \frac{-ax(N-r)-S}{b} \le u(N-r) \le \frac{-ax(N-r)}{b} \tag{17b}$$

$$\frac{-ax(N-r)-bu(N-r)+S}{S} \quad if\ \frac{-ax(N-r)}{b} \le u(N-r) \le \frac{-ax(N-r)+S}{b} \tag{17c}$$

for all $r = 1, \cdots, N$.

Now, let us show the validity of (10a)-(10f) and (11a)-(11f) for $r = 1$. Letting $r = 1$ in (15a)-(15d) and (17a)-(17c), we get the membership functions $\mu_G\big(ax(N-1) + bu(N-1)\big)$ and $\mu_C\big(u(N-1)\big)$. They are shown in Figure 2.

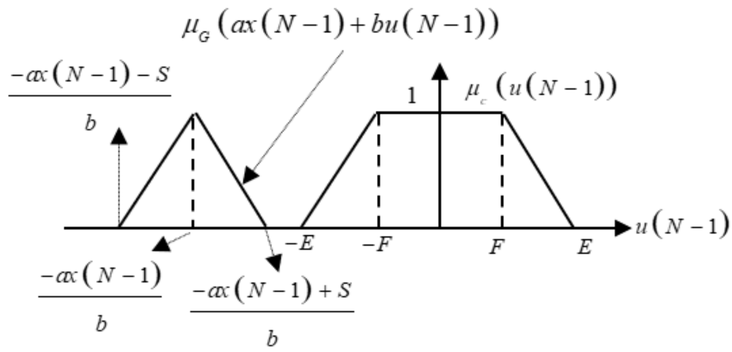

Figure 2. Variation of the membership functions $\mu_C\big(u(N-1)\big)$ and $\mu_G\big(x(N)\big)$

From Figure 2, we conclude that:

- Case 1: If $\frac{-ax(N-1)+S}{b} \le -E$ or $\frac{-ax(N-1)-S}{b} \ge E$, or equivalently if $|x(N-1)| \ge \frac{S+bE}{a}$, then there is no intersection between the two membership functions. This shows (10f) and (11f) for $r = 1$.
- Case 2: If $\frac{-ax(N-1)+S}{b} > -E$ and $\frac{-ax(N-1)}{b} \le -F$ or equivalently if $\frac{bF}{a} \le x(N-1) < \frac{S+bE}{a}$, then the control action that maximizes the fuzzy objective function $\bar{J}_{N-1,N}\big(x(N-1), u(N-1)\big) = min\Big(\mu_C\big(u(N-1)\big), \mu_G\,\big(ax(N-1) + bu(N-1)\big)\Big)$ is the intersection between $\mu_C\big(u(N-1)\big)$ and $\mu_G\big(ax(N-1) + bu(N-1)\big)$ given in (15b) and (17c) for $r = 1$. It satisfies

$$\frac{u^*(N-1)+E}{E-F} = \frac{-ax(N-1)-bu^*(N-1)+S}{S} \tag{18}$$

  Hence, (10d), (10e), (11d) and (11e) follow for $r = 1$.
- Case 3: If $-F \le \frac{-ax(N-1)}{b} \le F$ or equivalently if $\frac{-bF}{a} \le x(N-1) \le \frac{bF}{a}$, then $u^*(N-1) = \frac{-ax(N-1)}{b}$ and $\bar{J}^*_{N-1,N}\big(x(N-1)\big) = 1$. Thus, proving (10c) and (11c) for $r = 1$.

Equations (10a), (10b), (11a) and (11b), for $r = 1$, can be shown in a similar fashion. Now assume, as induction hypothesis, that (10a)-(10f) and (11a)-(11f) hold for $r-1$ and we will show their validity for r. The membership functions $\mu_C\big(u(N-r)\big)$, $\mu_G\big(x(N-r+1)\big)$ and $\bar{J}^*_{N-r+1,N}\big(x(N-r+1)\big)$ are given in Figure 3, Figure 4, and Figure 5, respectively with $z = \frac{(a-1)S-bE}{(a-1)S-b(E-F)}$. Figure 3 is obtained from (15a)-(15d), Figure 4 from (16a)-(16c) and Figure 5 from (11a)-(11f) substituting r with $r-1$.

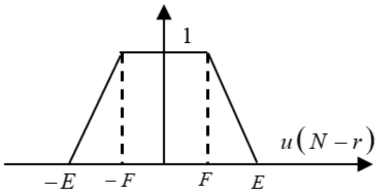

Figure 3. $\mu_C(u(N-r))$

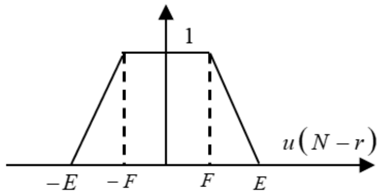

Figure 4. $\mu_G(x(N-r+1))$

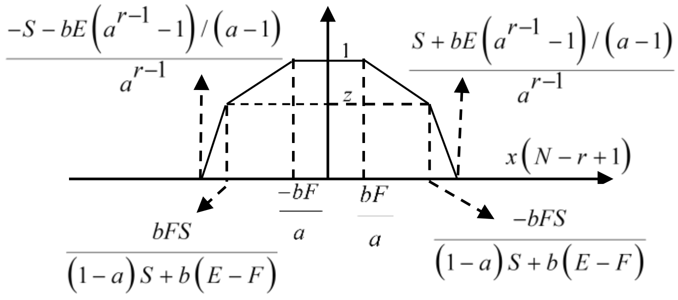

Figure 5. $\bar{J}^*_{N-r+1,N}(x(N-r+1))$

If we pose $\mu(x(N-r+1)) = min\{\mu_G(x(N-r+1)), \bar{J}^*_{N-r+1,N}(x(N-r+1))\}$, then (13) might be rewritten as (19).

$$\bar{J}^*_{N-r,N}(x(N-r)) = \max_{u(N-r)} min\{\mu_C(u(N-r)), \mu(x(N-r+1))\} = \max_{u(N-r)} min\{\mu_C(u(N-r)), \mu(ax(N-r)+bu(N-r))\} \quad (19)$$

In view of Figure 4 and Figure 5, and since $\mu_G\left(x(N-r+1) = \pm\frac{bFS}{(1-a)S+b(E-F)}\right) = \frac{(a-1)S-bE}{(a-1)S-b(E-F)}$ (denoted z in Figure 5), which is obtained by direct substitution into (16b) and (16c), $\mu(x(N-r+1))$ would be as shown with thick line in Figure 6, where $m_1 = \frac{-bFS}{(1-a)S+b(E-F)}$ and $m_2 = \frac{S+bE(a^{r-1}-1)/(a-1)}{a^{r-1}}$. Thus,

$$\mu(x(N-r+1)) = 0 \; if \; |x(N-r+1)| \geq \frac{S+bE(a^{r-1}-1)/(a-1)}{a^{r-1}} \quad (20a)$$

$$\frac{a^{r-1}x(N-r+1)+bE(a^{r-1}-1)/(a-1)+S}{b(E-F)(a^{r-1}-1)/(a-1)+S} \; if \; \frac{-S-bE(a^{r-1}-1)/(a-1)}{a^{r-1}} < x(N-r+1) \leq \frac{bFS}{(1-a)S+b(E-F)} \quad (20b)$$

$$\frac{x(N-r+1)+S}{S} \; if \; \frac{bFS}{(1-a)S+b(E-F)} \leq x(N-r+1) \leq 0 \quad (20c)$$

$$\frac{-x(N-r+1)+S}{S} \; if \; 0 \leq x(N-r+1) \leq \frac{-bFS}{(1-a)S+b(E-F)} \quad (20d)$$

$$\frac{-a^{r-1}x(N-r+1)+bE(a^{r-1}-1)/(a-1)+S}{b(E-F)(a^{r-1}-1)/(a-1)+S} \; if \; \frac{-bFS}{(1-a)S+b(E-F)} \leq x(N-r+1) < \frac{S+bE(a^{r-1}-1)/(a-1)}{a^{r-1}} \quad (20e)$$

Substituting (14) into (20a)-(20e), we obtain:

$$\mu(ax(N-r)+bu(N-r)) = 0 \; if \; u(N-r) \geq \frac{S+bE(a^{r-1}-1)/(a-1)-a^r x(N-r)}{ba^{r-1}} \quad (21a)$$

$$\frac{a^r x(N-r)+a^{r-1}bu(N-r)+\frac{bE(a^{r-1}-1)}{(a-1)}+S}{\frac{b(E-F)(a^{r-1}-1)}{(a-1)}+S} \; if \; \frac{-S-bE(a^{r-1}-1)/(a-1)-a^r x(N-r)}{ba^{r-1}} u(N-r) \leq \frac{FS}{(1-a)S+b(E-F)} - \frac{a}{b}x(N-r) \quad (21b)$$

$$\frac{ax(N-r)+bu(N-r)+S}{S} \ if \ \frac{FS}{(1-a)S+b(E-F)}-\frac{a}{b}x(N-r)\le u(N-r)\le -\frac{a}{b}x(N-r) \tag{21c}$$

$$\frac{-ax(N-r)-bu(N-r)+S}{S} \ if \ -\frac{a}{b}x(N-r)\le u(N-r)\le \frac{-FS}{(1-a)S+b(E-F)}-\frac{a}{b}x(N-r) \tag{21d}$$

$$\frac{-a^r x(N-r)-a^{r-1}bu(N-r)+\frac{bE\left(a^{r-1}-1\right)}{(a-1)}+S}{\frac{b(E-F)\left(a^{r-1}-1\right)}{(a-1)}+S} \text{ if}$$
$$\frac{-FS}{(1-a)S+b(E-F)}-\frac{a}{b}x(N-r)\le u(N-r) < \frac{S+bE\left(a^{r-1}-1\right)/(a-1)-a^r x(N-r)}{ba^{r-1}} \tag{21e}$$

Figure 7 shows the variation of the membership functions $\mu_C\big(u(N-r)\big)$ given in (15a)-(15d), and $\mu\big(ax(N-r)+bu(N-r)\big)$ given in (21a)-(21e), with respect to $u(N-r)$. Where: $g_0=\frac{-ax(N-r)}{b}$, $g_1=-\frac{S+bE\left(a^{r-1}-1\right)/(a-1)+a^r x(N-r)}{ba^{r-1}}$, $g_2=\frac{FS}{(1-a)S+b(E-F)}-\frac{a}{b}x(N-r)$, $g_3=\frac{-FS}{(1-a)S+b(E-F)}-\frac{a}{b}x(N-r)$ and $g_4=\frac{S+bE\left(a^{r-1}-1\right)/(a-1)-a^r x(N-r)}{ba^{r-1}}$. e is obtained by solving the equation $\frac{-u(N-r)+E}{E-F}=z$ ($z=\frac{(a-1)S-bE}{(a-1)S-b(E-F)}$) for $u(N-r)$. It is given by $e=\frac{(1-a)FS}{(1-a)S+b(E-F)}$. Thus, the position of $\mu\big(ax(N-r)+bu(N-r)\big)$ with respect to $\mu_C\big(u(N-r)\big)$ depends on $x(N-r)$.

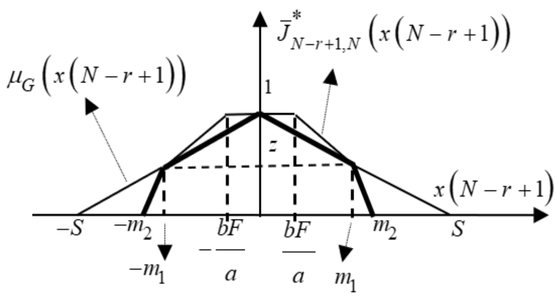

Figure 6. $\mu\big(x(N-r+1)\big)$

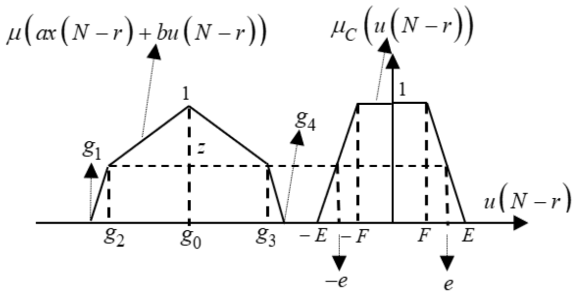

Figure 7. $\mu_C\big(u(N-r)\big)$ and $\mu\big(ax(N-r)+bu(N-r)\big)$

From Figure 7, we conclude that:

- Case 1: If $g_4\le -E$ or $g_1\ge E$, or equivalently if $|x(N-r)|\ge \frac{S+bE(a^r-1)/(a-1)}{a^r}$, then there is no intersection between the two membership functions $\mu_C\big(u(N-r)\big)$ and $\mu\big(ax(N-r)+bu(N-r)\big)$ and hence $min\Big(\mu_C\big(u(N-r)\big),\mu\big(ax(N-r)+bu(N-r)\big)\Big)=0$. From (19), we conclude that $\bar{J}^*_{N-r,N}\big(x(N-r)\big)=0$. Thus (10f) and (11f) follow.
- Case 2: If $g_4>-E$ and $g_3\le -e$, or equivalently if $\frac{-bFS}{(1-a)S+b(E-F)}\le x(N-r)<\frac{S+bE(a^r-1)/(a-1)}{a^r}$, then the maximizer $u^*(N-r)$ is the intersection of $\mu_c\big(u(N-r)\big)$ and $\mu\big(ax(N-r)+bu(N-r)\big)$ given in (15b) and (21e), respectively. It satisfies (22).

$$\frac{-a^r x(N-r)-a^{r-1}bu^*(N-r)+bE\left(a^{r-1}-1\right)/(a-1)+S}{b(E-F)\left(a^{r-1}-1\right)/(a-1)+S}=\frac{u^*(N-r)+E}{E-F} \tag{22}$$

  Then, (10e) follows. (11e) follows easily substituting the maximizer $u^*(N-r)$ into one side of (22).
- Case 3: If $g_3\ge -e$ and $g_0\le -F$ or equivalently if $\frac{bF}{a}\le x(N-r)\le \frac{-bFS}{(1-a)S+b(E-F)}$, then from (15d) and (21d), we get:

$$\frac{u^*(N-r)+E}{E-F}=\frac{-ax(N-r)-bu^*(N-r)+S}{S} \tag{23}$$

  which shows (10d). (11d) is obtained by substituting $u^*(N-r)$ into one side of (23).

– Case 4: If $-F \leq g_0 \leq F$ or equivalently $\frac{-bF}{a} \leq x(N-r) \leq \frac{bF}{a}$, then $u^*(N-r) = \frac{-ax(N-r)}{b}$ and $\bar{J}^*_{N-r,N}\big(x(N-r)\big) = 1$. Thus, (10c) and (11c) follow.

The proof of (10a), (10b), (11a) and (11b) can be done in a similar fashion. This concludes the proof of the lemma. Lemma 1 provides an analytical solution of the finite horizon optimal control problem (7)-(9) subject to system (5) under assumptions $b > 0$, $(a-1)S > bE$ and the fuzzy goal and the fuzzy constraint given in Figure 1(a) and Figure 1(b), respectively.

4. OPTIMALITY DOES NOT IMPLY ASYMPTOTIC STABILITY OF THE EQUILIBRIUM

The following theorem shows that the optimality of the infinite horizon optimal control with the minimum aggregation operator does not imply asymptotic stability of the equilibrium for the closed loop linear system (1).

Theorem 1: The infinite horizon optimal control with fuzzy objective function (2), the minimum aggregation operator and the fuzzy goal and constraint given in Figures 1(a) and 1(b) do not converge to the equilibrium for all initial feasible states.

Proof: We also assume here that $b > 0$ and$(a-1)S > bE$. Equation (2) can be written as (24).

$$J_{0,\infty}(x(0), \{u(t)\}_{t\geq 0}) = min\{\mu_G\big(x(0)\big), min\big(\mu_C\big(u(0)\big), \mu_G\big(x(1)\big), \mu_C\big(u(1)\big), \cdots\big)\} \tag{24}$$

Letting $N \to \infty$ into (6) and substituting it into (24), we can write as (25).

$$J_{0,\infty}(x(0), \{u(t)\}_{t\geq 0}) = min\{\mu_G\big(x(0)\big), \bar{J}_{0,\infty}(x(0), \{u(t)\}_{t\geq 0})\} \tag{25}$$

Substituting r with N in (10a)-(10f), (11a)-(11f) and then letting $N \to \infty$ one obtains the first optimal control action $u^*(0)$ of the optimal sequence $\{u^*(t)\}_{t\geq 0}$solution of (2)-(3) and the corresponding optimal fuzzy objective function $\bar{J}^*{}_{0,\infty}\big(x(0)\big)$. Note that since $bE > 0$ and $(a-1)S > bE$ then $a > 1$ and hence $\lim_{N\to\infty} a^N = \infty$. Thus, the first optimal control action $u^*(0)$ is given by (26a)-(26b).

$$u^*(0) = \frac{1-a}{b}x(0)\ if\ \frac{-bE}{(a-1)} < x(0) \leq \frac{bFS}{(1-a)S+b(E-F)} \tag{26a}$$

$$\frac{S+ax(N-r)+bE}{S+b(E-F)}\ if\ \frac{bFS}{(1-a)S+b(E-F)} \leq x(N-r) \leq \frac{-bF}{a} \tag{26b}$$

$$-\frac{a}{b}x(0)\ if\ \frac{-bF}{a} \leq x(0) \leq \frac{bF}{a} \tag{26c}$$

$$\frac{-FS-a(E-F)x(0)}{S+b(E-F)}\ if\ \frac{bF}{a} \leq x(0) \leq \frac{-bFS}{(1-a)S+b(E-F)} \tag{26d}$$

$$\frac{1-a}{b}x(0)\ if\ \frac{-bFS}{(1-a)S+b(E-F)} \leq x(0) < \frac{bE}{a-1} \tag{26e}$$

$$infeasible\ if\ |x(0)| \geq \frac{bE}{a-1} \tag{26f}$$

The associated optimal fuzzy objective function is given by (27a)-(27f).

$$\bar{J}^*_{0,\infty}\big(x(0)\big) = \frac{(a-1)x(0)+bE}{b(E-F)}\ if\ -\frac{bE}{a-1} < x(0) \leq \frac{bFS}{(1-a)S+b(E-F)} \tag{27a}$$

$$\frac{ax(0)+bE+S}{S+b(E-F)}\ if\ \frac{bFS}{(1-a)S+b(E-F)} \leq x(0) \leq \frac{-bF}{a} \tag{27b}$$

$$1\ if\ \frac{-bF}{a} \leq x(0) \leq \frac{bF}{a} \tag{27c}$$

$$\frac{-ax(t)+bE+S}{S+b(E-F)}\ if\ \frac{bF}{a} \leq x(0) \leq \frac{-bFS}{(1-a)S+b(E-F)} \tag{27d}$$

$\frac{(1-a)x(0)+bE}{b(E-F)}\ if\ \frac{-bFS}{(1-a)S+b(E-F)} \leq x(0) < \frac{bE}{a-1}$ (27e)

$0\ if\ |x(0)| \geq \frac{bE}{a-1}.$ (27f)

Figure 8 shows the membership functions $\mu_G(x(0))$ and $\bar{J}^*_{0,\infty}(x(0)) = \max_{\{u(t)\}_{t\geq 0}} \bar{J}_{0,\infty}(x(0),\{u(t)\}_{t\geq 0})$. This latest is given in (27a)-(27f). It is clear that $(a-1)S > bE > 0$ and $S > 0$ then $S > \frac{bE}{a-1}$ and hence (see Figure 8) there exists a right hand neighborhood $\mathcal{N}_1$ of $\frac{bE}{a-1}$, and a left hand neighborhood $\mathcal{N}_2$ of $-\frac{bE}{a-1}$, such that $\mu_G(x(0)) > \bar{J}^*_{0,\infty}(x(0)) > 0$, for all $x(0) \in \{\mathcal{N}_1 \cup \mathcal{N}_2\}$. Since by optimality $\bar{J}^*_{0,\infty}(x(0)) > \bar{J}_{0,\infty}(x(0),\{u(t)\}_{t\geq 0})$, it follows that $\mu_G(x(0)) > \bar{J}_{0,\infty}(x(0),\{u(t)\}_{t\geq 0})$, for all $x(0) \in \{\mathcal{N}_1 \cup \mathcal{N}_2\}$. Thus, from (25): $J_{0,\infty}(x(0),\{u(t)\}_{t\geq 0}) = \bar{J}_{0,\infty}(x(0),\{u(t)\}_{t\geq 0})$, for all $x(0) \in \{\mathcal{N}_1 \cup \mathcal{N}_2\}$. Therefore, if $x(0) \in \{\mathcal{N}_1 \cup \mathcal{N}_2\}$, then the first control action $u^*(0)$ of the sequence $\{u^*(t)\}_{t\geq 0}$ solution of the infinite horizon optimal control problem that maximizes the fuzzy objective function (2) with the minimum aggregation operator and subject to linear system (1) could be obtained from (26a) and (26e). It is given by $u^*(0) = \frac{1-a}{b}x(0)$, for all $x(0) \in \{\mathcal{N}_1 \cup \mathcal{N}_2\}$. Therefore, $x(1) = ax(0) + b\left(\frac{1-a}{b}x(0)\right) = x(0)$. In this case, we can show using the principle of optimality that $u^*(t) = \frac{1-a}{b}x(t)$, $\forall t \geq 1$. It follows that $x(t+1) = ax(t) + b\left(\frac{1-a}{b}x(t)\right) = x(t)$, $\forall t \geq 1$ and thus the state of the closed-loop system will not reach the origin equilibrium point.

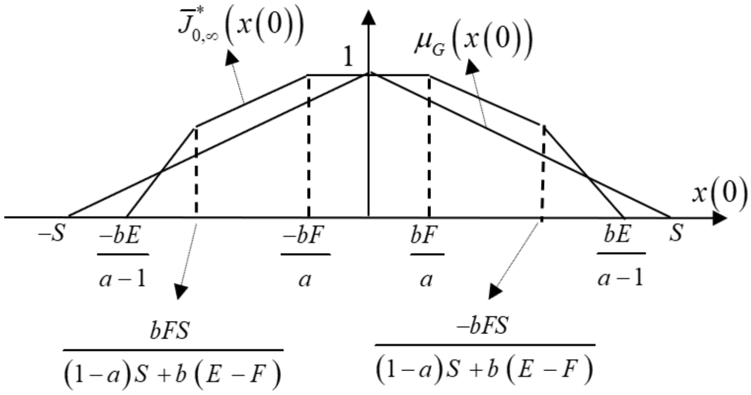

Figure 8. The membership functions $\mu_G(x(0))$ and $\bar{J}^*_{0,\infty}(x(0))$

4.1. Example

Consider the first order linear system: $x(t+1) = ax(t) + bu(t)$ where $a = 2$, $b = 1$ and $x(0) = 4.45$. The aim is to stabilize asymptotically the system to the equilibrium point $(x_e, u_e) = (0,0)$. The fuzzy goal and fuzzy constraint are given in Figures 1(a) and 1(b) with $S = 10$, $E = 5$, and $F = 4$. Using (26a)-(26f), one can calculate the first sample of the infinite horizon fuzzy optimal control sequence as:

$$u^*(0) = \begin{cases} -x(0) & if \quad -5 < x(0) \leq \frac{-40}{9} \\ \frac{-2x(0)+40}{11} & if \quad \frac{-40}{9} \leq x(0) \leq -2 \\ -2x(0) & if \quad -2 \leq x(0) \leq 2 \\ \frac{-2x(0)-40}{11} & if \quad 2 \leq x(0) \leq \frac{40}{9} \\ -x(0) & if \quad \frac{40}{9} \leq x(0) < 5 \\ infeasible & if \quad x(0) \geq 5 \text{ or } x(0) \leq -5 \end{cases}$$

We have $\frac{40}{9} \leq x(0) = 4.45 < 5$ then $u^*(0) = -x(0)$. In this case, $x(1) = 2x(0) - x(0) = x(0) = 4.45$. For an infinite horizon optimal controller and using the principle of optimality, if $x(1) = x(0)$ then $u^*(1) = u^*(0)$. Hence, the infinite horizon fuzzy optimal control sequence would be given by $[u^*(0), u^*(1), u^*(2), \dots, u^*(\infty)] = [-4.45, -4.45, -4.45, \dots, -4.45]$ and the resulting state trajectory is given by $[x^*(0), x^*(1), x^*(2), \dots, x^*(\infty)] = [4.45, 4.45, 4.45, \dots, 4.45]$. We conclude that the state of the system would never reach the equilibrium point $(x_e, u_e) = (0,0)$.

5. CONCLUSION

In this paper, novel stability result for the infinite horizon optimal control with the fuzzy objective function was given. It has been shown that the infinite horizon optimal control law does not make the resulting closed-loop system converge to the equilibrium in general if the minimum operator is used to aggregate the fuzzy goals and the fuzzy constraints in the fuzzy objective function. This has been achieved by solving analytically the infinite horizon optimal control problem for a linear system. Another interesting area for research is the investigation of asymptotic stability with various aggregation operators and deriving practical implementable RHC algorithms with asymptotic stability guarantees.

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