Chapter

Dynamic Equations on Time Scales

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Abstract

An extension of differential equations to different underlying time domains are so called dynamic equations on time scales. Time scales calculus unifies the continuous and discrete calculus and extends it to any nonempty closed subset of the real numbers. Choosing the time scale to be the real numbers, a dynamic equation on time scales collapses to a differential equation, while the integer time scale transforms a dynamic equation into a difference equation. Dynamic equations on time scales allow the modeling of processes that are neither fully discrete nor fully continuous. This chapter provides a brief introduction to time scales and its applications by incorporating a selective collection of existing results.

Keywords: time scales, existence, uniqueness, linear, applications

1. Introduction

The modeling of processes using differential equations is a well-established method in multiple branches of sciences. Dependent on the model assumptions, the form of the differential equation can range from a comparably simple ordinary differential equation to more advanced formulations using nonlinear, higher order, and partial differential equations. Reasons to consider difference equations include computational benefits and, even more fundamental, a discrete modeling perspective. For example, when describing a zero-coupon bond where the invested amount at time t, M_t, receives interest r at the end of each year but remains unchanged during each year, the recursive model $M_{t+1} = (1 + r)M_t$ captures the change of the investment from time t to time $t + 1$. Difference equations are also a common tool to describe processes on a macro scale in time, for example, when describing non-overlapping generations. Even though the number of individuals may vary throughout the generation period, one may only be interested in the individuals at the beginning of each generation time, i.e., the size of each cohort. There are however processes that cannot be described accurately using differential or difference equations. For example, when modeling species that are reproducing continuously during certain months of the year before laying eggs right before hibernating. Another example are plant populations that grow continuously during some months of the year and plant their seeds prior to dying out. In [1], Robert May gives examples of insects that exhibit such hybrid continuous–discrete behavior.

Instead of introducing a set of simplifying assumptions and possibly discontinuous model parameters that impact the model analysis, dynamic equations on time scales can provide a simple alternative to describe such processes. Time scales calculus was introduced by Stefan Hilger in 1988 [2]. It unifies the continuous and discrete calculus and extends it to any nonempty closed subset of the real numbers called a time scale, denoted by $\mathbb{T}$. By introducing differentiation and integration on $\mathbb{T}$, the classical theory of differential equations can be extended to time scales, which allows the modeling of processes that are not changing continuously nor solely discretely in time. These so-called dynamic equations are essentially the time scales analogue of differential and difference equations and have gained increasing interest due to their potential in applications. Choosing the time scale to be the real numbers, a dynamic equation transforms into a differential equation and by choosing the time scale to be the integers, a corresponding difference equation is obtained. Thus, instead of studying differential equations and difference equations separately, time scales provides also a tool to investigate both by analyzing the corresponding dynamic equation. This is specifically interesting since certain difference equations exhibit significantly different behavior as their continuous analogues, see for example the "logistic map" and the "logistic differential equation". By analyzing a dynamic equation on time scales, the effect of the underlying time domain onto the behavior of solutions may be revealed.

2. Time scales fundamentals

In this subsection, the basic definitions of time scales calculus are introduced based on the introductory book [3].

Definition 1. A time scale, denoted by $\mathbb{T}$, is a nonempty closed subset of $\mathbb{R}$.

Examples of a time scale are $\mathbb{R}, \mathbb{Z}, h\mathbb{Z}, q^{\mathbb{N}_0} = \{1, q, q^2, q^3, \ldots\} (q > 1), [a, b] \cup \{c, d\}$ where $a < b$ and $a, b, c, d \in \mathbb{R}$, and the Cantor set. It therefore contains the popular cases of the continuous, the discrete, and the quantum calculus.

Operators that aid the description of a time scale are the "forward jump operator", denoted by $\sigma(t)$, the "backward jump operator", denoted by $\rho(t)$, and the "graininess function", denoted by $\mu(t)$. These operators are defined for $t \in \mathbb{T}$ as

$$\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) := \sigma(t) - t. \tag{1}$$

Since $\mathbb{T}$ is closed, $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and $\mu : \mathbb{T} \to [0, \infty)$. **Table 1** provides values of the corresponding operators for different examples of time scales.

$\mathbb{T}$	$\sigma(t)$	$\rho(t)$	$\mu(t)$
$\mathbb{R}$	t	t	0
$\mathbb{Z}$	$t+1$	$t-1$	1
$q^{\mathbb{N}_0}$	qt	$\frac{t}{q}$	$t(q-1)$

Table 1.
The description of the time scales functions σ, ρ, μ for the examples of $\mathbb{R}$, $\mathbb{Z}$, and $q^{\mathbb{N}_0} (q > 1)$.

Using these operators, any $t \in \mathbb{T}$ can be classified as:

- *right-scattered (left-scattered)*, if $\sigma(t) > t$ $(\rho(t) < t)$, and
- *right-dense (left-dense)*, if $\sigma(t) = t$ $(\rho(t) = t)$.

We say that a point $t \in \mathbb{T}$ is isolated, if it is right- and left-scattered. We say that a point $t \in \mathbb{T}$ is dense, if it is right- and left-dense. Note that for $\mathbb{T} = \mathbb{R}$, every point is dense and, for $\mathbb{T} = \mathbb{Z}$, every point is isolated.

Example 2.1. El Niño events can be described using a time scale. El Niño events between 2002 and 2017 have been observed in the time intervals 2002–2003, 2004–2005, 2006–2007, 2009–2010, and 2014–2016 [4], which suggests the corresponding time scale (**Figure 1**, **Table 2**)

$$\mathbb{T} = \cup_{i=0}^{5}[a_i, a_{i+1}]$$

with $(a_0, a_1, a_2, \dots, a_5) = (2002, 2004, 2006, 2009, 2014, 2015)$.

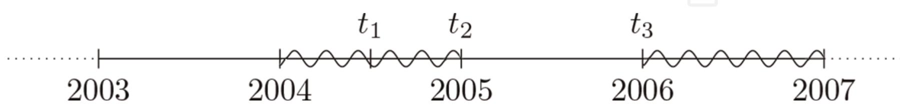

Figure 1.
Part of the time line containing points in the time scale $\mathbb{T}$. Curly lines identify intervals within $\mathbb{T}$. Here, $t_1 \in (2004, 2005)$, t_2 is the last point in the interval $[2004, 2005]$, and $t_3 = 2006$ is the first point in $[2006, 2007]$.

$t \in \mathbb{T}$	$\sigma(t)$	$\mu(t)$	$\rho(t)$
$t_1 \in (2004, 2005)$	$\sigma(t_1) = t_1$	$\mu(t_1) = 0$	$\rho(t_1) = t_1$
$t_2 = 2005$	$\sigma(t_2) = t_3$	$\mu(t_2) = 1$	$\rho(t_2) = t_2$
$t_3 = 2006$	$\sigma(t_3) = t_3$	$\mu(t_3) = 0$	$\rho(t_3) = t_2$

Table 2.
*The functions σ, ρ, μ for the time points $t_1, t_2, t_3 \in \mathbb{T}$ based on **Figure 1**.*

The following notation is commonly used for $t \in \mathbb{T}$,

$$\sigma^n(t) = \underbrace{(\sigma\circ\sigma\circ \dots \circ\sigma)}_{n-\text{times}}(t), \qquad \rho^n(t) = \underbrace{(\rho\circ\rho\circ \dots \circ\rho)}_{n-\text{times}}(t).$$

2.1 Functions on time scales

We can now consider scalar functions on time scales, that is, $f : \mathbb{T} \to \mathbb{R}$, and discuss their properties. We define the subset $\mathbb{T}^\kappa$ as follows: If $\mathbb{T}$ has a left-scattered maximum $m \in \mathbb{T}$, then $\mathbb{T}^\kappa = \mathbb{T}\setminus\{m\}$, else $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2. f: $\mathbb{T} \to \mathbb{R}$ is called regressive, if, for all $t \in \mathbb{T}^\kappa$,

$$1 + \mu(t)f(t) \neq 0$$

and is called positively regressive, if, for all $t \in \mathbb{T}^\kappa$,

$$1 + \mu(t)f(t) > 0.$$

The following are properties of $f : \mathbb{T} \to \mathbb{R}$ that later identify integrability.

Definition 3. $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limit exists (as a finite value) for all right-dense points and its left-sided limit exists (as a finite value) for all left-dense points.

Even though every regulated function on a compact interval is bounded, in general, $\max_{a \le t \le b} f(t)$ and $\min_{a \le t \le b} f(t)$ do not need to exist for regulated $f : \mathbb{T} \to \mathbb{R}$.

Definition 4. $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if f is continuous at all right-dense points and its left-sided limit exists (as a finite value) for all left-dense points. The set of rd-continuous functions is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Note that, if $f : \mathbb{T} \to \mathbb{R}$ is continuous, then f is rd-continuous. If f is rd-continuous, then f is regulated.

The set of rd-continuous and regressive (positively regressive) functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$ $(\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}) = \mathcal{R}^+(\mathbb{T}, \mathbb{R}))$.

Beside the classical addition and subtraction of functions, time scales calculus introduces the so-called "circle plus", denoted by $\oplus$, and "circle minus", denoted by $\ominus$. These operations are defined for $f, g : \mathbb{T} \to \mathbb{R}$ as follows

$$(f \oplus g)(t) = f(t) + g(t) + (\mu f g)(t)$$

$$\text{and, for } g \in \mathcal{R}, \qquad (f \ominus g)(t) = \frac{f(t) - g(t)}{1 + (\mu g)(t)}.$$

A useful property is that if $f, g \in \mathcal{R}$ $(\mathcal{R}^+)$, then $f \oplus g, f \ominus g \in \mathcal{R}$ $(\mathcal{R}^+)$ implying that the (positively) regressive property is being carried over. Furthermore, $(\mathcal{R}, \oplus)$ forms an Abelian group with the inverse elements of $f \in \mathcal{R}$ given by $\ominus f$.

For $\mathbb{T} = \mathbb{R}$, the operators $\oplus$ and $\ominus$ correspond to the classical addition and subtraction.

2.2 Differentiation

Definition 5. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. If there exists $f^\Delta(t) \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \varepsilon |\sigma(t) - s| \quad \text{for all} \quad s \in (t - \delta, t + \delta) \cap \mathbb{T},$$

then we call $f^\Delta(t)$ the delta (or Hilger) derivative of f at $t \in \mathbb{T}^\kappa$.

If $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$, we say that f is delta differentiable (or short: differentiable) and the function $f^\Delta : \mathbb{T} \to \mathbb{R}$ is called delta derivative of f on $\mathbb{T}^\kappa$.

If f is differentiable at $t \in \mathbb{T}^\kappa$, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

The following notations are used equivalently

$$(f^\sigma)(t) = (f \circ \sigma)(t) = f(\sigma(t)).$$

The definition of a delta derivative can be extended to consider higher order derivatives. We say that f is twice delta differentiable with the second (delta) derivative $f^{\Delta\Delta}$, if f^Δ is (delta) differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$.

Note that the definition of delta derivatives focuses on the change forward in time. A corresponding definition that focuses on the change backward in time is referred to as nabla derivative, see for example [5].

Theorem 2.2. [See [3, Theorem 1.16]] Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. Then, the following holds:

i. If t is right-dense, then

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

provided that the limit exists (as a finite number).

ii. If f is continuous at the right-scattered point t, then

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Applying Theorem 2.2 for the case of $\mathbb{T} = \mathbb{R}$, shows that the delta derivative is consistent with the classical derivative, that is, $f^\Delta(t) = f(t)$ for $t \in \mathbb{T} = \mathbb{R}$. For $\mathbb{T} = \mathbb{Z}$, the delta derivative collapses to the forward Euler operator, widely accepted as the discrete analogue of a derivative, that is, $f^\Delta(t) = f(t+1) - f(t)$ if $\mathbb{T} = \mathbb{Z}$ (see **Table 3**).

$\mathbb{T}$	$\mathbb{T} = \mathbb{R}$	$\mathbb{T} = \mathbb{Z}$	$\mathbb{T} = q^{\mathbb{N}_0}$
$f^\Delta(t)$	$f'(t)$	$\Delta f(t)$	$\frac{f(qt)-f(t)}{t(q-1)}$

Table 3.
Derivatives for the examples of $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = q^{N_0}$ $(q > 1)$. Note that $\Delta f(t) = f(t+1) - f(t)$ is the forward Euler operator.

As in the continuous case, the differential operator is linear, that is, for $\alpha, \beta \in \mathbb{R}$, $t \in \mathbb{T}^\kappa$, and for (delta) differentiable functions $f, g : \mathbb{T} \to \mathbb{R}$,

$$(\alpha f + \beta g)^\Delta(t) = \alpha f^\Delta(t) + \beta g^\Delta(t).$$

The analogues of the product and the quotient rule on time scales take on slightly different forms. For (delta) differentiable functions $f, g : \mathbb{T} \to \mathbb{R}$, and $t \in \mathbb{T}^\kappa$,

$$(fg)^\Delta(t) = f^\Delta(t) g^\sigma(t) + f(t) g^\Delta(t) = f^\Delta(t) g(t) + f^\sigma(t) g^\Delta(t)$$

and, for $g(t), g^\sigma(t) \neq 0$,

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t) g(t) - f(t) g^\Delta(t)}{g(t) g^\sigma(t)}.$$

For $\mathbb{T} = \mathbb{R}$, we have $f^\sigma = f$ and $g^\sigma = g$ so that the classical product and quotient rule are retrieved. In the case of $\mathbb{T} = \mathbb{Z}$, we have the correspondent rules consistent with [6], namely

$$\Delta(fg)(t) = (\Delta f(t))g(t+1) + f(t)(\Delta g(t)) = (\Delta f(t))g(t) + f(t+1)(\Delta g(t)).$$

If $g(t), g(t+1) \neq 0$, then

$$\Delta\left(\frac{f(t)}{g(t)}\right) = \left(\frac{f}{g}\right)^{\Delta}(t) = \frac{(\Delta f(t))g(t) - (\Delta g(t))f(t)}{g(t)g(t+1)}.$$

The modifications in the product and quotient rule highlight that some of the well established differentiation rules only carry over to time scales calculus after some adjustments. In fact, the product rule on time scales reveals that the useful property of power functions $f(t) = t^n$ for $n \in \mathbb{N}_0$ is no longer the simple reduction of the power by one, because

$$\left(t^2\right)^{\Delta} = (t \cdot t)^{\Delta} = t + \sigma(t),$$

which may not be delta differentiable. This indicates already that the series representation of functions requires further thought.

Also, considering the chain rule, we note that for $\mathbb{T} = \mathbb{Z}$,

$$\Delta(f \circ f)(t) = f^{\sigma}(t)f^{\Delta}(t) + f(t)f^{\Delta}(t) = f^{\Delta}(t)(f(t) + f^{\sigma}(t)) \neq 2f(t)f^{\Delta}(t),$$

for $f^{\sigma}(t) \neq f(t)$. Thus, the powerful chain rule, often utilized in solving differential equations via a variable transformation, does not apply on time scales. In an attempt to generalize the chain rule for functions on time scales a few identities have been formulated. The next theorem provides such an expression based on works in [7, 8]. Other formulations can be found in [3].

Theorem 2.3. (See [3, Theorem 1.90]). Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is (delta) differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is (delta) differentiable and

$$(f \circ g)^{\Delta}(t) = \left\{\int_0^1 f'\left(g(t) + h\mu(t)g^{\Delta}(t)\right) \mathrm{d}h\right\} g^{\Delta}(t).$$

An interesting observation is that the operators, Δ and σ, do generally not commute, that is, $\left(f^{\Delta}\right)^{\sigma} \neq (f^{\sigma})^{\Delta}$. Take for example $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then

$$\left(f^{\Delta}\right)^{\sigma}(t) = \frac{f(q^2t) - f(qt)}{\mu(qt)} \neq \frac{f(q^2t) - f(qt)}{\mu(t)} = (f^{\sigma})^{\Delta}(t),$$

since $\mu(qt) = qt(q-1) \neq t(q-1) = \mu(t)$.

2.3 Integration

Definition 6. A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with (region of differentiation) D, provided that $D \subset \mathbb{T}^{\kappa}$, $\mathbb{T}^{\kappa} \backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is (delta) differentiable at each $t \in D$.

Theorem 2.4. (See [3, Theorem 1.70]). Let $f : \mathbb{T} \to \mathbb{R}$ be regulated. Then there exists a function $F : \mathbb{T} \to \mathbb{R}$ which is pre-differentiable with region of differentiation D such that

$$F^{\Delta}(t) = f(t) \qquad \text{for all } t \in D.$$

The function F is called an pre-antiderivative of $f(t)$.

If $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$, then F is called antiderivative of f.

We define the indefinite integral of a regulated function f by $\int f(t)\,\Delta t = F(t) + C$, where $C \in \mathbb{R}$ is an arbitrary integration constant and F is a pre-antiderivative of f. The Cauchy integral is defined by $\int_a^b f(t)\,\Delta t = F(b) - F(a)$ for all $a, b \in \mathbb{T}$.

Theorem 2.5. (See [3, Theorem 1.74]). Every rd-continuous function f has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by

$$F(t) := \int_{t_0}^{t} f(s)\,\Delta s \qquad \text{for all } t \in \mathbb{T}$$

is an antiderivative of f.

For $\mathbb{T} = \mathbb{R}$, the integral is consistent with the Rieman integral (see **Table 4**).

$\mathbb{T}$	$\mathbb{T} = \mathbb{R}$	$\mathbb{T} = \mathbb{Z}$	$\mathbb{T} = q^{\mathbb{N}_0}$	$\mathbb{T}_{\mathcal{I}}$
$\int_s^t f(\tau)\,\Delta\tau$	$\int_s^t f(\tau)\,\mathrm{d}\tau$	$\sum_{\tau=s}^{t-1} f(\tau)$	$\sum_{n=0}^{k} sq^k(q-1)f(q^k s)$	$\sum_{\tau \in [a,b) \cap \mathbb{T}} \mu(\tau) f(\tau)$

Table 4.
Integrals for the examples of $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = q^{N_0}$ $(q > 1)$, and isolated time scales $\mathbb{T}_{\mathcal{I}}$, for which all points in $\mathbb{T}_{\mathcal{I}}$ are assumed to be isolated. In all cases, $s, t \in \mathbb{T}$ and $s < t$. In the case of $\mathbb{T} = q^{\mathbb{N}_0}$, we assume $t = q^k s$.

The integral operator is linear so that for $f, g \in C_{rd}$ and $a < b$, $a, b \in \mathbb{T}$, and $\alpha, \beta \in \mathbb{R}$,

$$\int_a^b (\alpha f + \beta g)(s)\,\Delta s = \alpha \int_a^b f(s)\,\Delta s + \beta \int_a^b g(s)\,\Delta s.$$

With the definition of integration on time scales, we have the machinery to introduce a series representation for time scales functions. In [9], see also [3], a time scales analogue of polynomials that allows a corresponding Taylor series expression was introduced using the recursive formulation

$$g_0(t,s) = h_0(t,s) \equiv 1 \qquad \text{for all } t, s \in \mathbb{T},$$

and, for every $k \in \mathbb{N}_0$,

$$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau), s)\,\Delta\tau \qquad \text{for all } s, t \in \mathbb{T},$$

and

$$h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\,\Delta\tau \qquad \text{for all } s, t \in \mathbb{T}.$$

Now, $h_k^{\Delta}(t,s) = h_k(t,s)$ and $g_k^{\Delta}(t,s) = g_k(\sigma(t),s)$ for $k \in \mathbb{N}$ and t,s T^{κ}. Two Taylor series representations can be formulated for a time scales function f, one that uses the time scales polynomials g_k and one that uses the polynomials h_k, see Section 1.6 in [3] for more details.

3. Linear dynamic equations

This chapter provides a brief introduction to first order dynamic equations and provides a selected summary of [3], extended by applications. A first order dynamic equation is of the form

$$y^{\Delta}(t) = f(t, y, y^{\sigma}), \tag{2}$$

for $y : \mathbb{T} \to \mathbb{R}^n$ and $f : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $n \in \mathbb{N}_1 = \{1, 2, 3, \ldots\}$. A first order initial value problem (short: IVP) is then given by (2) with an initial condition $y(t_0) = y_0 \in \mathbb{R}^n$ for $t_0 \in \mathbb{T}$. A function $y : \mathbb{T} \to \mathbb{R}^n$ is called a *solution* of (2) if y satisfies the equation for all $t \in \mathbb{T}^{\kappa}$.

We call (2) linear if

$$f(t, y, y^{\sigma}) = f_1(t)y + f_2(t), \quad \text{or } f(t, y, y^{\sigma}) = f_1(t)y^{\sigma} + f_2(t),$$

where $f_1, f_2 : \mathbb{T} \to \mathbb{R}^n$. We say the linear dynamic equation is homogeneous, if $f_2 \equiv 0$.

3.1 Scalar case

We first focus on the scalar case of (2), that is, $f : \mathbb{T} \to \mathbb{R}$. Based on the above definition of linearity, there are two forms a linear, homogeneous, first order dynamic equation can have:

$$y^{\Delta} = p(t)y, \tag{3}$$

$$y^{\Delta} = p(t)y^{\sigma}, \quad \text{for } p : \mathbb{T} \to \mathbb{R} \tag{4}$$

Note that for $\mathbb{T} = \mathbb{R}$, $y^{\sigma} = y$ and therefore $y' = p(t)y^{\sigma} = p(t)y$ so that both, (3) and (4), are the time scales analogues of $y' = p(t)y$.

If $p \in \mathcal{R}$, then (3) is called *regressive* and if $-p \in \mathcal{R}$, then (4) is called *regressive*.

The unique solution to (3) with initial condition $y(t_0) = 1$ for some $t_0 \in \mathbb{T}$ is denoted by $y(t) = e_p(t, t_0)$ and is called the time scales exponential function. The unique solution to (4) with initial condition $y(t_0) = 1$ is $y(t) = e_{\ominus(-p)}(t, t_0)$.

Table 5 contains the time scales analogues of the exponential function for the dense time scale $\mathbb{T} = \mathbb{R}$, the discrete time scale $\mathbb{T} = \mathbb{Z}$, and the quantum time scale $\mathbb{T} = q^{\mathbb{N}_0}$.

$\mathbb{T}$	**Dynamic Eq. (3)**	$e_p(t, t_0)$
$\mathbb{R}$	$y' = p(t)y$	$\exp\left\{\int_{t_0}^{t} p(s)\,ds\right\}$
$\mathbb{Z}$	$\Delta y = p(t)y$	$\prod_{i=t_0}^{t-1}(1 + p(i))$
$q^{\mathbb{N}_0}$	$y^{\Delta} = p(t)y$	$\prod_{s \in [t_0, t) \cap \mathbb{T}}(1 + s(q-1)p(s))$

Table 5.
The exponential function for the continuous, discrete, and quantum time scale ($q > 1$), assuming $p \in \mathcal{R}$.

The **Table 5** reveals a crucial aspect of the time scales exponential function, namely that the positivity property, known for the traditional exponential function, does not uphold on time scales. Take for example, $\mathbb{T} = \mathbb{Z}$ and $p = -3$, then $p \in \mathcal{R}$ as $1 + p = -2 \neq 0$, but $e_p(t, 0) = (-2)^t$ which is negative for odd values of t. If however $p \in \mathcal{R}^+$, then $e_p(t, t_0) > 0$, restoring the positivity property. Note that if $\mathbb{T} = \mathbb{R}$, then any function $p \in \mathcal{R}^+$ since $1 + \mu(t)p(t) = 1 > 0$.

Some of the properties of the time scales exponential function are consistent with the convenient properties in the continuous case. If $p, q \in \mathcal{R}$ and $t, s \in \mathbb{T}$, then

i. $e_0(t,s) = 1$, $e_p(t,t) = 1$,

ii. $e_{p \oplus q}(t,s) = e_p(t,s)e_q(t,s)$,

iii. $e_{\ominus p}(t,s) = e_p(s,t) = \frac{1}{e_p(t,s)}$,

iv. $e_p(t,r)e_p(r,s) = e_p(t,s)$,

v. $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$.

Theorem 3.1. [See [3, Theorem 2.39]] If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then

$$\int_a^b p(t)e_p(t,c)\,\Delta t = e_p(b,c) - e_p(a,c)$$
$$\int_a^b p(t)e_p(c,\sigma(t))\,\Delta t = e_p(c,a) - e_p(c,b).$$

As an application of linear, homogeneous, first order dynamic equations, one may consider the Malthusian growth model. In "An essay on the principle of population" from 1798, Thomas Robert Malthus proposed an exponential law of population growth with the corresponding differential equation

$$P' = rP, \quad P(t_0) = P_0,$$

where P is the population at time t, r is the inherent growth rate, and P_0 is the initial population level at time $t_0 \in \mathbb{R}$. This linear, homogeneous, first order differential equation has the solution $P(t) = e^{r(t-t_0)}P_0$. Assuming a positive initial population level $P_0 > 0$, it follows that for a positive growth rate $r > 0$, the population increases exponentially. If instead $r < 0$ and $P_0 > 0$, then the population goes extinct as $\lim_{t\to\infty} e^{r(t-t_0)}P_0 = 0$. Despite its simplicity and the unrealistic behavior of unbounded population levels for r > 0, the Malthusian model can sometimes serve short-term predictions.

Let us now consider the corresponding time scales model (3) with initial condition $P(t_0) = P_0 > 0$ and inherent growth rate $r > 0$, that is, $P^\Delta = rP$ with $P(t_0) = P_0$ for $t_0 \in \mathbb{T}$. The respective solution is then $P(t) = e_r(t, t_0)P_0$, which is unbounded for $r, P_0 > 0$, see **Figure 2**. Thus, for $r, P_0 > 0$, the behavior of the solution is consistent with the solution in the continuous case. However, for $r < 0$, the population does not have to go extinct but can result in biologically unmeaningful behavior as solutions can become negative.

Using the time scales exponential function that solves a linear, homogeneous, first order dynamic equation, we can use the variation of constants formula to obtain the solution to a linear, nonhomogeneous, first order dynamic equation.

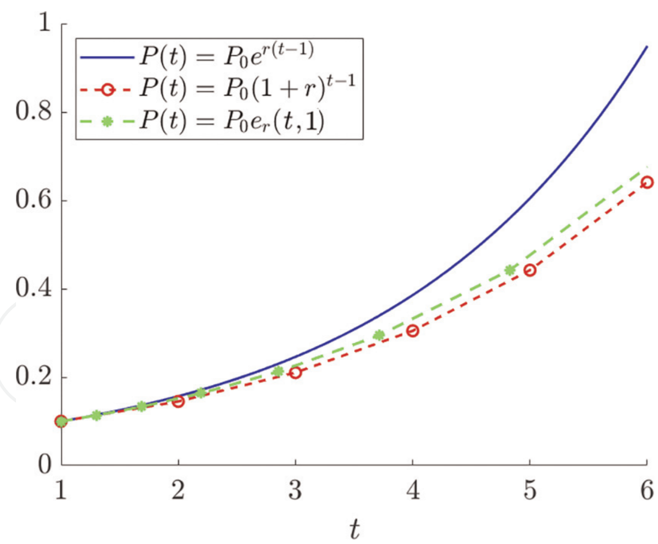

Figure 2.
The behavior of the solution to $P^\Delta = rP$ with $P(t_o) = P_o$ where $r = 0.45$, $t_o = 1$, and $P_o = 0.1$, for $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = 1.3^{\mathbb{N}_0}$. The solid line represents the solution in the continuous case, the open circle represents the solution in the discrete case, and the stars represent the solution in the quantum calculus case with q=1.3.

Theorem 3.2. [See [3, Theorems 2.74 & 2.77]] Suppose $p \in \mathcal{R}, f \in C_{rd},\ t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$ then the unique solution to

$$y^\Delta = p(t)y + f(t), \qquad y(t_0) = y_0$$

is given by

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(s))f(s)\,\Delta s.$$

Furthermore, the unique solution to

$$y^\Delta = -p(t)y^\sigma + f(t), \qquad y(t_0) = y_0$$

is given by

$$y(t) = e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, s)f(s)\,\Delta s.$$

Example 3.3. Suppose that the life span of a certain species is one time unit. Suppose that just before the species dies out, eggs are laid that are hatch after one time unit. The species is therefore only alive on $\mathbb{T} = \cup_{k=0}^{\infty}[2k, 2k+1]$, see also [3, Example 1.39] and [10]. Suppose further that during the specie's, life cycle, the species reduces due to external factors with rate $d \in (0, 1)$ and at the end of the life cycle $t = 2k + 1$, the individuals alive in $(2k, 2k + 1)$ lay eggs that result in the reproduction rate $r > 0$. The corresponding dynamic equation for the species $N(t)$ at time t, is then

$$N^\Delta(t) = p(t)N(t), \qquad \textit{with} \quad p(t) = \begin{cases} -d & t \in [2k, 2k+1) \\ r & t = 2k+1 \end{cases}$$

and initial condition $N(0) = N_0$. We note that even though $p(t)$ is discontinuous at $t = 2k + 1, p(t) \in \mathcal{R}$. Theorem 3.2 gives the population at time $t \in [2m, 2m + 1]$ as

$$N(t) = N_0 e_p(t, t_0) = N_0 e_p(t, 2m) \prod_{k=0}^{m-1} \left(e_p(2k+1, 2k) e_p(2k+2, 2k+1)\right)$$
$$= N_0 \exp\left\{\int_{2m}^{t} -d\, \mathrm{d}s\right\} \left\{\prod_{k=0}^{m-1} \exp\left\{\int_{2k}^{2k+1} -d\, \mathrm{d}s\right\}(1+r)\right\} = N_0 e^{-d(t-m)}(1+r)^m.$$

Example 3.4. Newton's law of cooling suggests that the temperature of an object at time t, $T(t)$, changes dependent on the temperature of its surrounding, T_m. Then, $T'(t) = -\kappa(T - T_m)$, where κ is the heat transfer coefficient. Suppose that an object with initial temperature T_0 is cooled in a lab environment. Due to safety regulations, once the lab assistant leaves the work space, the object can only be exposed to an environment that preserves the current temperature of the object. The cooling of the object can be modeled using time scales with the underlying time domain to be the working hours of the lab assistant. Assume that the lab assistant's working hours, and therefore the time scale, is of the form $\mathbb{T} = \cup_{i=0}^{\infty}[a_i, b_i] \cup [c_i, d_i]$, where the interval $[a_i, b_i]$ are the working hours prior to lunch, and $[c_i, d_i]$ are the working hours of the lab assistant after lunch of day i. One way of modeling this scenario on time scales is

$$T^\Delta = -p(t)(T - T_m), \qquad p(t) = \begin{cases} \kappa & t \in [a_i, b_i) \cup [c_i, d_i) \\ 0 & t \in \{b_i, d_i\} \end{cases}$$

with initial temperature $T(t_0) = T_0$ for $t_0 \in \mathbb{T}$. Since $p(t)$ is rd-continuous and regressive, the theorems above can be applied despite the discontinuity of $p(t)$.

Example 3.5. The following example is from [11], where a Keynesian-Cross model with lagged income is considered. Here, the aggregated income y changes according to

$$y^\Delta = \delta[d^\sigma(t) - y], \qquad t \geq t_0 \in \mathbb{T},$$

where $d(t)$ is the aggregated demand at time t and $\delta \in (0,1)$ is the "adjustment speed". Since $d(t)$ can be expressed as the addition of aggregated consumption (c), aggregated investment (I), and governmental spending (G), we have $d(t) = c(t) + I + G$ for $I, G \in (0, \infty)$. Under the assumption that aggregated consumption is itself linear in the aggregated income, we have $c(t) = a + by(t)$ with $a, b > 0$ so that the model reads as

$$y^\Delta = \delta[a + by^\sigma + I + G - y].$$

Under the assumption that $p(t) := 1 - \delta b\mu(t) \neq 0$, we can apply $y^\sigma = y + \mu y^\Delta$, and express the dynamic equation as

$$y^\Delta = \frac{\delta(a + I + G)}{p(t)} + \frac{\delta(b-1)}{p(t)} y.$$

which is a linear, non-homogeneous, first order dynamic equation. It is left as an exercise to apply the techniques of this subsection to derive an explicit solution to this dynamic equation.

Example 3.6. Let us consider a time scales analogue of the popular logistic growth model $y' = ry\left(1 - \frac{y}{K}\right)$, namely,

$$y^{\Delta} = ry^{\sigma}\left(1 - \frac{y}{K}\right), \qquad y(t_0) = y_0, \tag{5}$$

with growth rate $r > 0$, and carrying capacity $K > 0$, and initial population size $y(t_0) > 0$ at time $t_0 \in \mathbb{T}$. Even though this is an example of a nonlinear dynamic equation of first order, we can apply the substitution $z = \frac{1}{y}$ for $y \neq 0$, to obtain the linear dynamic equation

$$z^{\Delta} = \frac{-y^{\Delta}}{yy^{\sigma}} = -rz + \frac{r}{K}, \qquad z(t_0) = \frac{1}{y_0}.$$

For $-r \in \mathcal{R}$, the solution is then given by Theorem 3.2. Using also Theorem 3.1 and resubstituting yields

$$y(t) = \frac{y_0 K}{e_{-r}(t, t_0)(K - y_0) + y_0}. \tag{6}$$

It can be easily checked that $y(t_0) = y_0$ and that y solves (5), see also [12].

Note that for $\mathbb{T} = \mathbb{R}$, (5) collapses to the Verhulst model $y' = ry\left(1 - \frac{y}{K}\right)$ and the solution (6) reads in this case as

$$y(t) = \frac{y_0 K}{e^{-r(t-t_0)}(K - y_0) + y_0},$$

which coincides with the classical solution.

3.2 Linear systems

Let us now consider (2) with $f\colon \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. In order to extend the solution methods for linear first order dynamic equations that were introduced in the previous section for scalar functions, the definitions of rd-continuity and delta differentiability have to be first extended to matrix valued functions $A : \mathbb{T} \to \mathbb{R}^{m \times n}$. This adjustment is mostly proposed element-wise. More precisely, A is rd-continuous on $\mathbb{T}$ if a_{ij} is rd-continuous on $\mathbb{T}$ for all $1 \le i \le n$, $1 \le j \le m$. The class of all such rd-continuous $m \times n$-matrix-valued functions on $\mathbb{T}$ is then denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^{m \times n})$. Similarly, we say that A is delta differentiable (or short: differentiable), if a_{ij} is delta differentiable for all $1 \le i \le n$, $1 \le j \le m$. Similar to the scalar case, the following identity holds for any matrix-valued (delta) differentiable function A,

$$A^{\sigma}(t) = A(t) + \mu(t)A^{\Delta}(t).$$

The property of regressive is however not defined elementwise. Instead, we say that $A \in \mathbb{R}^{n \times n}$ is regressive if $I_n + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^{\kappa}$, where $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix. The class of rd-continuous and regressive functions is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$ (or short $\mathcal{R}$).

Note that even if all entries of A are regressive, A does not have to be regressive. Take for example $\mathbb{T} = \mathbb{Z}$ with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & 3 \end{bmatrix}.$$

Then all entries are regressive as $1 + a_{ij} \neq 0$ for all $1 \le i, j \le 2$ but $\det(I + A) = 0$.

As for the scalar case, differentiation is linear, that is,

$$(\alpha A + \beta B)^{\Delta}(t) = \alpha A^{\Delta}(t) + \beta B^{\Delta}(t)$$

for differentiable $m \times n$-matrix-valued functions A, B, and $\alpha, \beta \in \mathbb{R}$.

We consider

$$y^{\Delta} = A(t)y \tag{7}$$

to be the system analogue of (3). If A is $n \times n$ matrix valued function, then, the unique solution to (7) with $y(t_0) = I_n$, where I_n is the $n \times n$ identity matrix, is denoted by $y(t) = e_A(t, t_0)$. If $A \in \mathbb{R}^{n \times n}$ and $\mathbb{T} = \mathbb{R}$ then $e_A(t, t_0) = e^{A(t-t_0)}$, and if $\mathbb{T} = \mathbb{Z}$, then $e_A(t, t_0) = (I + A)^{t-t_0}$. The analogue of (4) in higher dimensions is

$$y^{\Delta} = -A^*(t)y^{\sigma},$$

where $A^*(t)$ is the conjugate transpose of $A(t)$.

Theorem 3.7. (See [3, Theorems 5.24 & 5.27]). Let $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n}, \mathbb{R}^{n \times n})$ and suppose that $f: \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then, the initial value problem

$$y^{\Delta} = A(t)y + f(t), \qquad y(t_0) = y_0$$

is given by

$$y(t) = e_A(t, t_0)y_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\,\Delta\tau.$$

The unique solution to

$$y^{\Delta} = -A^*(t)y^{\sigma} + f(t), \qquad y(t_0) = y_0$$

is given by

$$y(t) = e_{\ominus A^*}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus A^*}(t, \tau)f(\tau)\,\Delta\tau.$$

Example 3.8. In [13], the authors consider the Cucker-Smale type model on an isolated $\mathbb{T}$ (i.e., every $t \in \mathbb{T}$ is isolated) with $\sup\mathbb{T} = \infty$ and $\sup\{\mu(t) : t \in \mathbb{T}\} < \infty$,

$$\begin{aligned} x_i^{\Delta} &= v_i \\ v_i^{\Delta} &= \frac{1}{N}\sum_{j=1}^{N} a_{ij}(v_j - v_i), \end{aligned} \tag{8}$$

where $a_{ij} \in \mathbb{R}_0^+ = [0, \infty)$ and $i \in \{1, 2, \dots, N\}$ represents the impact of agent's j opinion onto the agent's i opinion. The variable x_i represents the state of agent i, and v_i is the consensus parameter of agent i. The original Cucker-Smale model, see [14], is a discrete time system discussing the flock behavior of birds, where v_i represents the velocity of bird i and x_i is its position. The weights a_{ij} quantify the way the birds influence each other.

Note that since $\mathbb{T}$ is isolated, we can equivalently write (8) as

$$x_i(\sigma(t)) = x_i(t) + \mu(t)v_i(t), \qquad v_i(\sigma(t)) = v_i(t) + \frac{\mu(t)}{N}\sum_{j=1}^{N} a_{ij}(v_j(t) - v_i(t)),$$

or in form of a system in $y = (x_1, x_2, \dots, x_N, v_1, v_2, \dots, v_N)^T$,

$$y^\Delta = By, \qquad B = \frac{1}{N}\begin{bmatrix} 0_N & NI_N \\ 0_N & A - D, \end{bmatrix}, \tag{9}$$

where $(A)_{ij} = a_{ij}$ for $i, j \in \{1, 2, \dots, N\}$, $D = diag(d_1, d_2, \dots, d_N)$ with $d_k = \sum_{j=1}^{N} a_{kj}$, 0_N is a matrix of dimension $N \times N$ with all entries being zero, and I_N is the identity matrix of dimension $N \times N$.

If $B \in \mathcal{R}$, then the solution to (9) with initial condition $y(t_0) = y_0$ is $y(t) = e_B(t, t_0)\, y_0$. In order for $B \in \mathcal{R}$, $NI_N + \mu(t)(A\text{-}D)$ must be invertible because

$$\tilde{B}(t) = I_{2N} + \mu(t)B = \begin{bmatrix} I_N & \mu(t)I_n \\ 0_N & C(t) \end{bmatrix}, \qquad C(t) = I_N + \mu(t)\frac{1}{N}(A - D),$$

and

$$\det\left(\tilde{B}(t)\right) = \det(I_{2N} + \mu(t)B) = \det(I_N)\det(C(t)).$$

We conclude this section by examples of nonlinear dynamic equations that can be transformed into a system of linear dynamic equations of first order, so that Theorem 3.7 provides its solution.

Example 3.9. Let $\mathbb{T}$ be again an isolated time scale, that is, every point in $\mathbb{T}$ is isolated and $inf\{\mu(t) : t \in \mathbb{T}\} > 0$. Consider

$$x^{\sigma^k} = \frac{Kx}{(1 - \mu(t)\alpha)K + \mu(t)\alpha x}, \tag{10}$$

with initial values $\vec{x}_0 = (x_0, x_1, \dots, x_{k-1}) \in (0, \infty)^k$, $K > 0$, and $-\alpha \in \mathbb{R}^+$. Eq. (10) is a delayed Beverton-Holt model and can be used to model mature individuals of a population, assuming that it takes k reproductive cycles for an individual to become mature, where the length of a reproductive cycle starting at t is $\mu(t)$. An application may be populations where the lengths between breeding cycles is temperature dependent. Model (10) has been considered in [15] (and, for $\mathbb{T} = \mathbb{Z}$, in [16]), where the authors applied the transformation $y := \frac{K}{x}$ for $x \neq 0$ to obtain

$$Y^\Delta = A(t)Y + \mathbf{b}(t) \quad \text{with} \quad A(t) = \frac{1}{\mu(t)}\begin{bmatrix} \mathbf{0}_{k-1} & I_{k-1} \\ -\mu\alpha & -\mathbf{s} \end{bmatrix}, \ \mathbf{b}(t) = \begin{pmatrix} \mathbf{0}_{k-1} \\ \alpha \end{pmatrix}, \tag{11}$$

where $\mathbf{s} = \left(\binom{k}{1}, \binom{k}{2}, \binom{k}{3}, \dots, \binom{k}{k-1}\right)$ and $\mathbf{0}_{k-1} \in \mathbb{R}^{k-1\times 1}$ is vector of zeros. Applying Theorem 3.7, to (11) yields the solution.

Example 3.10. In [17], the authors proposed the following nonlinear system of dynamic equations to model the spread of a contagious disease,

$$S^\Delta = -\beta(t)S^\sigma I - \nu(t)S + \gamma(t)I + \nu(t)\kappa,$$
$$I^\Delta = \beta(t)S^\sigma I - \gamma(t)I - \nu(t)I.$$

In line with well-established epidemic models, the population was compartmentalized into susceptible S and infected I individuals. The model assumes that the disease is spread by contact with an infected individual with a transmission rate of $\beta > 0$. The recovery rate is assumed to be $\gamma > 0$ and recovered individuals rejoin the group of susceptible individuals. The death rate is $\nu(t)$ across the population and $\nu(t)\kappa$ newborns join the group of susceptibles.

By introducing a new variable $w := S + I$, $w^\Delta = -\nu(t)w + \nu(t)\kappa$. This first order, linear, nonhomogeneous dynamic equation can be solved using Theorem 3.2, assuming $-\nu(t) \in \mathcal{R}$. The solution is then $w(t) = e_{-\nu}(t, t_0)(I_0 + S_0 - \kappa) + \kappa$, so that, after recalling that $S = w(t) - I$, the dynamic equation in I can be expressed as

$$I^\Delta = \beta(t)(w^\sigma - I^\sigma)I - \gamma(t)I - \nu(t)I.$$

Although the dimension has been reduced to one, the dynamic equation is still nonlinear. Defining however $y = \frac{1}{I}$ for $I \neq 0$ yields again a linear dynamic equation, namely

$$y^\Delta = (-\beta(t)w^\sigma(t) + \gamma(t) + \nu(t))y^\sigma + \beta(t).$$

Applying Theorem 3.2 gives the solution

$$y(t) = e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, s)\beta(s)\,\Delta s,$$

where $p(t) = \beta(t)w(\sigma(t)) - (\gamma(t) + \nu(t))$ is assumed to be an element of $\mathcal{R}$. Resubstituting yields then the solution I and using $S = w - I$ yields S.

For more epidemic models on time scales that are systems of first order nonlinear dynamic equations, see [18–21]. While the dynamic Susceptible-Infected-Recovered epidemic model introduced in [18] can be solved explicitly via variable transformations, in most cases, including [19], explicit solutions to nonlinear dynamic equations are not available. In these cases, properties of solutions such as existence and uniqueness are of fundamental interest. The interested reader is referred to [22, Section 2] and [3, Section 8.2].

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