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A New Approach to Ramanujan's Partition Congruences

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A NEW APPROACH TO RAMANUJAN'S PARTITION CONGRUENCES

A Thesis

by

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ABSTRACT

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MacMahon provided Ramanujan and Hardy a table of values for $p(n)$ with the partitions of the first 200 integers. In order to make the table readable, MacMahon grouped the entries in blocks of five. Ramanujan noticed that the last entry in each block was a multiple of 5. This motivated Ramanujan to make the following conjectures,

$$p(5n+4) \equiv 0 \pmod{5}$$

$$p(7n+5) \equiv 0 \pmod{7}$$

$$p(11n+6) \equiv 0 \pmod{11}$$

which he eventually proved.

The purpose of this thesis is to give new proofs for Ramanujan's partition congruences. This would be done by using theta functions to construct certain vector spaces of modular forms. Computations within these vector spaces result in new proofs for Ramanujan's partition congruences modulo five and seven. Similar techniques will use to derive congruences for a wider class of generating functions.

DEDICATION

I dedicate this thesis to my family, who always stood by me and dealt with all my absences from many family occasions. Especially to my parents, Jose and Maria, for giving me the opportunity to have an education and encouraged and supported me through all my studies.

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CHAPTER I

INTRODUCTION

In this thesis, we will give new proofs for Ramanujan's partition congruences and we will derive new congruences for related modular forms. We will use a set of building blocks for representing generating functions and relevant series dissections. The work employs basic facts from the theory of modular forms and linear algebra. The proofs for the congruences will be given with parallel justifications. We will apply the same technique used to prove Ramanujan's congruence to prove the new congruences. To introduce the ideas involved, we need to consider some definitions and examples. First, we introduce notation and definitions that will be use throughout the entire thesis.

Definition 1.0.1. *A partition of positive integer n is a finite nonincreasing sequence of positive integers* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *such that*

$$\sum_{i=0}^{k} \lambda_i = n.$$

The λ_i *is called the part of the partition. [2, p.1]*

To understand what an integer partition is, we provide the following example. This example illustrates the process of finding the partitions for 4. Hence, we have

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1, \end{aligned}$$

Now recall that two sums that differ only in the order of the summands is consider the same partition; that is, $3+1$ is the same as $1+3$. The number of partitions of n denoted by $p(n)$ which is called the partition function of n. In the previous example, we can see that there are 5 different ways to represent 4 as a sum of positive integers; so we have that $p(4) = 5$. In order to present the generation function, we must first introduce some notation for finite and infinite products.

Definition 1.0.2. *[q-Pochhammer symbol] For $a, q \in \mathbb{C}$, define*

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k).$$

If the limit exists, denote

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n.$$

Now that we have defined the q-Pochhammer symbol notation, we can introduce the generating function for partitions.

Theorem 1.0.1. *The generating function for partitions is given by*

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n, \qquad p(n) \in \mathbb{Z}.$$

Proof. The proof can be found in *Number Theory* by George Andrews [1, p.162]. □

Theorem 1.0.2. *The set of absolutely convergent power series with integer coefficients form a ring.*

Proof. Assume that

$$\sum_{n=0}^{\infty} a_n q^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n q^n$$

are power series with integer coefficients. Then

$$\left(\sum_{k=0}^{\infty} a_k q^k\right)\left(\sum_{i=0}^{\infty} b_i q^i\right) = \sum_{n=0}^{\infty}\left(\sum_{\substack{k+i=n\\k,i\geq 0}}^{\infty} a_k b_r\right) q^n,$$

where the inner sum is an integer. For simplicity, we denote

$$c_n = \sum_{\substack{k+i=n\\k,i\geq 0}}^{\infty} a_k b_r$$

then we can write

$$\sum_{n=0}^{\infty}\left(\sum_{\substack{k+r=n\\k,r\geq 0}}^{\infty} a_k b_r\right) q^n = \sum_{n=0}^{\infty} c_n q^n.$$

Let

$$\alpha = \sum_{n=0}^{\infty} \alpha_n q^n, \quad \beta = \sum_{n=0}^{\infty} \beta_n q^n \quad \text{and} \quad \gamma = \sum_{n=0}^{\infty} \gamma_n q^n$$

then

$$\begin{aligned}
(\alpha+\beta)\gamma &= \left(\sum_{n=0}^{\infty} \alpha_n q^n + \sum_{n=0}^{\infty} \beta_n q^n\right) + \sum_{n=0}^{\infty} \gamma_n q^n \\
&= \left(\sum_{n=0}^{\infty} (\alpha_n + \beta_n)\, q^n\right) + \sum_{n=0}^{\infty} \gamma_n q^n \\
&= \sum_{n=0}^{\infty} (\alpha_n + \beta_n + \gamma_n)\, q^n \\
&= \sum_{n=0}^{\infty} \alpha_n q^n + \left(\sum_{n=0}^{\infty} (\beta_n + \gamma_n)\, q^n\right) \\
&= \sum_{n=0}^{\infty} \alpha_n q^n + \left(\sum_{n=0}^{\infty} \beta_n q^n + \sum_{n=0}^{\infty} \gamma_n q^n\right) \\
&= \alpha + (\beta+\gamma).
\end{aligned}$$

Hence, the ring is associative under addition. Consider

$$0 = \sum_{n=0}^{\infty} 0q^n,$$

then

$$\begin{aligned}
\alpha + 0 &= \sum_{n=0}^{\infty} \alpha_n q^n + \sum_{n=0}^{\infty} 0q^n \\
&= \sum_{n=0}^{\infty} (\alpha_n + 0)q^n \\
&= \sum_{n=0}^{\infty} (\alpha_n)q^n \\
&= \sum_{n=0}^{\infty} (0 + \alpha_n)q^n \\
&= \sum_{n=0}^{\infty} 0q^n + \sum_{n=0}^{\infty} \alpha_n q^n \\
&= 0 + \alpha.
\end{aligned}$$

Thus, 0 is the additive identity. Denote $\alpha^{-1} \in \mathbb{Z}$ as follows:

$$\alpha^{-1} = \sum_{n=0}^{\infty} (-\alpha_n)q^n,$$

then

$$\begin{aligned}
\alpha + \alpha^{-1} &= \sum_{n=0}^{\infty} \alpha_n q^n + \sum_{n=0}^{\infty} (-\alpha_n)q^n \\
&= \sum_{n=0}^{\infty} (\alpha_n - \alpha_n)q^n \\
&= 0.
\end{aligned}$$

Since $\alpha + \alpha^{-1} = 0$ we have that α^{-1} is the additive inverse. Now we have that

$$\begin{aligned}
\alpha + \beta &= \sum_{n=0}^{\infty} \alpha_n q^n + \sum_{n=0}^{\infty} \beta_n q^n \\
&= \sum_{n=0}^{\infty} (\alpha_n + \beta_n) q^n \\
&= \sum_{n=0}^{\infty} (\beta_n + \alpha_n) q^n \\
&= \sum_{n=0}^{\infty} \beta_n q^n + \sum_{n=0}^{\infty} \alpha_n q^n. \\
&= \beta + \alpha
\end{aligned}$$

Hence, addition is commutative and we have an abelian group. Notice that

$$\begin{aligned}
(\alpha \cdot \beta)\gamma &= \left(\left(\sum_{n=0}^{\infty} \alpha_n q^n \right) \left(\sum_{n=0}^{\infty} \beta_n q^n \right) \right) \left(\sum_{n=0}^{\infty} \gamma_n q^n \right) \\
&= \left(\sum_{n=0}^{\infty} \left(\sum_{\substack{i+j=n \\ i,j \geq 0}}^{\infty} \alpha_i \beta_j \right) q^n \right) \left(\sum_{n=0}^{\infty} \gamma_n q^n \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{\substack{i+j+l=n \\ i,j,l \geq 0}}^{\infty} \alpha_i \beta_j \gamma_l \right) q^n \\
&= \left(\sum_{n=0}^{\infty} \alpha_n q^n \right) \left(\sum_{n=0}^{\infty} \left(\sum_{\substack{j+l=n \\ j,l \geq 0}}^{\infty} \beta_j \gamma_l \right) q^n \right) \\
&= \left(\sum_{n=0}^{\infty} \alpha_n q^n \right) \left(\left(\sum_{n=0}^{\infty} \beta_n q^n \right) \left(\sum_{n=0}^{\infty} \gamma_n q^n \right) \right) \\
&= \alpha(\beta \cdot \gamma)
\end{aligned}$$

so multiplication is associative. Then

$$\alpha(\beta + \gamma) = \left(\sum_{n=0}^{\infty} \alpha_n q^n \right) \left(\sum_{n=0}^{\infty} \beta_n q^n + \sum_{n=0}^{\infty} \gamma_n q^n \right)$$

$$
\begin{aligned}
&= \left(\sum_{n=0}^{\infty} \alpha_n q^n\right)\left(\sum_{n=0}^{\infty} (\beta_n + \gamma_n) q^n\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{\substack{i+j=n \\ i,j\geq 0}}^{\infty} \alpha_i(\beta_j + \gamma_j)\right) q^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{\substack{i+j=n \\ i,j\geq 0}}^{\infty} \alpha_i\beta_j + \sum_{\substack{i+j=n \\ i,j\geq 0}}^{\infty} \alpha_i\gamma_j\right) q^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{\substack{i+j=n \\ i,j\geq 0}}^{\infty} \alpha_i\beta_j\right) q^n + \sum_{n=0}^{\infty} \left(\sum_{\substack{i+j=n \\ i,j\geq 0}}^{\infty} \alpha_i\gamma_j\right) q^n \\
&= \left(\sum_{n=0}^{\infty} \alpha_n q^n\right)\left(\sum_{n=0}^{\infty} \beta_n q^n\right) + \left(\sum_{n=0}^{\infty} \alpha_n q^n\right)\left(\sum_{n=0}^{\infty} \gamma_n q^n\right) \\
&= \alpha\beta + \alpha\gamma.
\end{aligned}
$$

Thus, the left distributive property holds. Similarly, we can check that the right distributive property holds. Therefore, we have shown a power series with integer coefficients forms a ring. □

With the preceding notation and terminology, we are ready to state the main results of this thesis. The purpose of this thesis is to provide new proofs the following congruences for the partition function:

$$
\begin{aligned}
p(5n+4) &\equiv 0 \pmod{5} \\
p(7n+5) &\equiv 0 \pmod{7}.
\end{aligned}
$$

We also provide proofs for the new congruences using the same process.

1.1 History

During their research on $p(n)$, Hardy and Ramanujan needed a table of values of $p(n)$ to check their work. Which was supplied by P.A. MacMahon. He made the table of the values of $p(n)$ for $1 \leq n \leq 200$. MacMahon did this by grouping the entries in blocks of five, like in Table

1.1, to make the table readable.

n	**p(n)**	**n**	**p(n)**	**n**	**p(n)**
0	1	10	42	20	627
1	1	11	56	21	792
2	2	12	77	22	1002
3	3	13	101	23	1255
4	5	14	135	24	1575
5	7	15	176	25	1958
6	11	16	231	26	2436
7	15	17	297	27	3010
8	22	18	385	28	3718
9	30	19	490	29	4565

Table 1.1: Values for $p(n)$ where $1 \leq n \leq 29$

Ramanujan noticed that in each block, the last $p(n)$ entry was a multiple of 5. This motivated Ramanujan to make conjectures which he eventually proved [3, p.51]. Ramanujan announced, in 1919, that he had found three congruences satisfied by $p(n)$,

$$p(5n+4) \equiv 0 \pmod{5} \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod{7} \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

In [18], he proved (1.1) and (1.2) and later in [19], Ramanujan announced, [22], that he had found a proof for (1.3). In [19], he remarked that there does not exist similar properties for other primes besides the ones above. [6, p.27]

In his book, Bruce C. Berndt gave two elementary proofs of Theorem 2.3.1 [4, p.31] where the first proof was in Ramanujan's [18], [22, pp.210-213] and reproduced in Hardy's book [12, pp.87-88]. He also gave a proof of the Theorem 2.4.1 [4, p.31] which was also taken from Ramanujan's paper [18] and it was then sketched by Hardy [12, p.88]. John Drost, [11] gives another proof of Theorem 2.3.1. Also, another elementary proof of the theorem is given in the paper

[13] by Michael D. Hirschhorn. The latest edition of Ramanujan's *Collected Papers* [22, pp.372-375] has been used to further prove Theorem 3.2.1. In his book, Berndt references other proofs of Theorem 2.4.1. In the summer of 1918 from the nursing home Fitzrog House, Ramanujan, in a letter to Hardy, summarized the congruences he had proved and the method used to prove them. He remarked in [4, pp.192-193] that the divisibility by $5^a 7^b 11^c$ where $a = 0,1,2,3$, $b = 0,1,2,3$, and $c = 0,1,2$ amounting to $4 \times 4 \times 3 - 1$ or 47 cases of the conjecture theorem were proved. Ramanujan's statement is interesting for the following reasons. First, he proved special cases of his general conjecture without leaving any proofs of these special cases. Second, in [21, pp.133-177] he began a proof, which he was not able to complete, for an arbitrary b and $a = c = 0$, but the conjecture he made was false in this case. He formulated his conjecture using a table of values of $p(n)$ where $0 \leq n \leq 200$, and it was not until after his death that Cholwa [8] found that $p(243)$ is not divisible by 7^3. Lastly, the proofs that Ramanujan gave for his conjecture were for arbitrary powers of 5 and it was established after the letter was written. [6, pp.49-50]

1.2 Theorems and Definitions

In this section, we introduce definitions and theorems that will be needed to prove Ramanujan's congruences. We begin by introducing Ramanujan's Theta function.

Definition 1.2.1. *[Ramanujan's Theta function] For $|ab| < 1$, let*

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

When working with Ramanujan Theta functions, we often require the following product formulation.

Theorem 1.2.1. *[The Jacobi Triple Product Identity]*

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Proof. The proof for this theorem can be found in *Ramanujan's Notebooks, Part III* by B.C.

Berndt [5, p.35]. □

Lemma 1.2.2. *For $a, b \in \mathbb{N}$*

$$\frac{1}{f(-q^a, -q^b)}$$

is a power series in q with integer coefficients.

Proof. By the Jacobi Triple Product,

$$\frac{1}{f(-q^a, -q^b)} = \frac{1}{(q^a; q^{a+b})_\infty (q^b; q^{a+b})_\infty (q^{a+b}; q^{a+b})_\infty}. \tag{1.4}$$

The product on right side of (1.4) is the generating function for the number of partitions of n into parts congruent to a, b and $a+b$ congruent to modulo $a+b$. This comes from the fact that

$$\prod_{i \in K} \frac{1}{(1-q^i)} = \sum_{n=0}^{\infty} p_K(n) q^n$$

where $p_K(n)$ is the number of parts of n into parts form K. Thus,

$$\frac{1}{f(-q^a, -q^b)} \qquad a, b \in \mathbb{N}$$

is a power series in q with integer coefficients. □

Throughout the thesis, we will be dissecting power series. Note that the dissection is analogous to the dissection of functions into even and odd parts.

Definition 1.2.2. *For any $h(q)$ having an absolutely convergent series expansion about $q = 0$, say*

$$h(q) = \sum_{n=0}^{\infty} r_n q^n$$

we define the dissection components of $h(q)$ modulo k as follows:

$$(h(q))_j = \sum_{n=0}^{\infty} r_{kn+j} q^n, \qquad j = 0, 1, \ldots, k-1.$$

Note that

$$\begin{aligned} h(q) &= \sum_{n=0}^{\infty} r_{kn} q^{kn} + \sum_{n=0}^{\infty} r_{kn+1} q^{kn+1} + \cdots + \sum_{n=0}^{\infty} r_{kn+(k-1)} q^{kn+(k-1)} \\ &= (h(q^k))_0 + (qh(q^k))_1 + \cdots + (q^{k-1}h(q^k))_{k-1} \\ &= \sum_{m=0}^{k-1} (q^m h(q^k))_m. \end{aligned}$$

Most of the dissections will be in the variable $q^{1/k}$ instead of the variable q. In this case, we will write

$$\begin{aligned} h(q^{1/k}) &= \sum_{m=0}^{k-1} (q^{m/k} h(q))_m \\ &= (h(q))_0 + (q^{1/k} h(q))_1 + \cdots + (q^{(k-1)/k} h(q))_{k-1}. \end{aligned}$$

Throughout the thesis, we will need to dissect the product of a power series in q and a power series in $q^{1/p}$. The following theorem shows that j dissection of the product equals the j dissection of the power series in $q^{1/p}$ times the power series in q.

Theorem 1.2.3.

$$\left(\sum_{n=0}^{\infty} k_n q^{n/p} \times \sum_{n=0}^{\infty} r_n q^n \right)_j = \left(\sum_{n=0}^{\infty} k_n q^{n/p} \right)_j \times \sum_{n=0}^{\infty} r_n q^n.$$

Proof. Assume ℓ_n is a sequence such that

$$\ell_n = \begin{cases} r_{n/p} & \text{if } p \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=0}^{\infty} r_n q^n = \sum_{n=0}^{\infty} \ell_n q^{n/p}.$$

Let the subscript j denote j decomposition of the power series with respect to the variable $q^{1/p}$.

We have

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} k_n q^{n/p} \times \sum_{n=0}^{\infty} r_n q^n\right)_j &= \left(\sum_{n=0}^{\infty} k_n q^{n/p} \times \sum_{n=0}^{\infty} r_n q^{np/p}\right)_j \\
&= \left(\sum_{r=0}^{\infty} k_n q^{n/p} \times \sum_{s=0}^{\infty} \ell_n q^{sp/p}\right)_j \\
&= \left(\sum_{n=0}^{\infty} \left(\sum_{\substack{r+s=n \\ 0\le r,s\le n}}^{\infty} k_r \ell_s\right) q^{n/p}\right)_j \\
&= \sum_{n=0}^{\infty} \left(\sum_{\substack{r+s=nj+p \\ 0\le r,s\le n}}^{\infty} k_r \ell_s\right) q^{(np+j)/p} \\
&= q^{j/p} \sum_{n=0}^{\infty} \left(\sum_{\substack{r+s=pn+j \\ 0\le r,s\le n}}^{\infty} k_r \ell_s\right) q^n \\
&= q^{j/p} \sum_{n=0}^{\infty} \left(\sum_{\substack{r+sp=pn+j \\ 0\le r,s\le n}}^{\infty} k_r r_s\right) q^n \\
&= q^{j/p} \sum_{n=0}^{\infty} \left(\sum_{\substack{r=p(n-s)+j \\ 0\le s\le n}}^{\infty} k_r r_s\right) q^n \\
&= q^{j/p} \sum_{n=0}^{\infty} \left(\sum_{0\le s\le n}^{\infty} k_{p(n-s)+j} r_s\right) q^n.
\end{aligned}$$

On the other hand,

$$\left(\sum_{n=0}^{\infty} k_n q^{n/p}\right)_j \times \sum_{n=0}^{\infty} r_n q^n = \sum_{n=0}^{\infty} k_{pn+j} q^{(pn+j)/p} \times \sum_{n=0}^{\infty} r_n q^n$$

$$
\begin{aligned}
&= q^{j/p} \sum_{d=0}^{\infty} k_{pd+j} q^n \times \sum_{s=0}^{\infty} r_s q^i \\
&= q^{j/p} \sum_{n=0}^{\infty} \left(\sum_{\substack{d+s=n \\ 0 \le d,s \le n}} k_{pd+j} r_s \right) q^n \\
&= q^{j/p} \sum_{n=0}^{\infty} \left(\sum_{0 \le s \le n} k_{p(n-s)+j} r_s \right) q^n.
\end{aligned}
$$

Hence,

$$
\left(\sum_{n=0}^{\infty} k_n q^{n/p} \times \sum_{n=0}^{\infty} r_n q^n \right)_j = \left(\sum_{n=0}^{\infty} k_n q^{n/p} \right)_j \times \sum_{n=0}^{\infty} r_n q^n.
$$

□

To prove Ramanujan's congruence, we will use basic facts from the theory of modular forms. The remaining part of the chapter is devoted to theorems that will be used throughout the rest of the thesis.

Definition 1.2.3. *A modular form of weight k for a subgroup Γ of the modular group*

$$
SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.
$$

A function f is a modular form of weight k for Γ and multiplier $v(\gamma)$ if

1. *For any matrix in $SL_2 \in \Gamma$ we have that*

$$
f\left(\frac{a\tau + b}{c\tau + d}\right) = v(\gamma)(c\tau + d)^k f(\tau);
$$

2. *f is a holomorphic function;*

3. *f is holomorphic at $z = \infty$.*

Definition 1.2.4. *Let $M_k(\Gamma)$ be the set of modular form of weight k for Γ.*

Now let us define the congruence subgroups that will be relevant for later discussions.

Definition 1.2.5. *Let $N \in \mathbb{N}$,*

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| a \equiv d \equiv 1 (mod\, N), b \equiv c \equiv 0\, (mod\, N) \right\},$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| a \equiv d \equiv 1 (mod\, N), c \equiv 0\, (mod\, N) \right\}.$$

Theorem 1.2.4. *Let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$. Then $M_k(\Gamma)$, is a vector space of finite dimension over $\mathbb{C}$.*

Proof. The proof is given by P. Deligne and M. Rapoport in *Les schémas de modules de courbes elliptiques* [9]. □

The following theorem shows that if we start with a modular form of weight one on $(\Gamma_1(N))$ and we replace τ with τ/N, we get a modular form of weight one on $(\Gamma(N))$

Theorem 1.2.5. *If $f(\tau) \in M_1(\Gamma_1(N))$, then $g(\tau) = f(\tau/N) \in M_1(\Gamma(N))$.*

Proof. Assume that $f(\tau) \in M_1(\Gamma_1(N))$ and let $g(\tau) = f(\tau/N)$. Then, for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$ we have $\begin{pmatrix} a & \frac{b}{N} \\ Nc & d \end{pmatrix} \in \Gamma_1(N)$, so that

$$\begin{aligned} g\left(\frac{a\tau+b}{c\tau+d}\right) &= f\left(\frac{1}{N} \cdot \frac{a\tau+b}{c\tau+d}\right) \\ &= f\left(\frac{a\frac{\tau}{N} + \frac{b}{N}}{Nc\frac{\tau}{N}+d}\right) \\ &= \left(Nc\frac{\tau}{N}+d\right) f(\tau/N) \\ &= (c\tau+d)g(\tau). \end{aligned}$$

Thus, if $f(\tau) \in M_1(\Gamma_1(N))$, then $f(\tau/N) \in M_1(\Gamma(N))$. □

Lemma 1.2.6. *The product of a weight k_1 modular form and a weight k_2 modular form form the same congruence group of $SL(2,\mathbb{Z})$ is a weight $k_1 + k_2$ modular form.*

Proof. Suppose that f is a modular form of weight k_1 and g be a modular form of weight k_2. Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

then

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k_1} f(\tau)$$

and

$$g\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k_2} g(\tau).$$

Hence,

$$\begin{aligned} fg\left(\frac{a\tau+b}{c\tau+d}\right) &= f\left(\frac{a\tau+b}{c\tau+d}\right) g\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= (c\tau+d)^{k_1+k_2} fg(\tau). \end{aligned}$$

Since f and g are modular forms we have that f and g are holomorphic functions. Then

$$\begin{aligned} (fg)'(\tau) &= \lim_{h\to 0} \frac{f(\tau+h)g(\tau+h) - f(\tau)g(\tau)}{h} \\ &= \lim_{h\to 0} \frac{f(\tau+h)g(\tau+h) - f(\tau)g(\tau+h) + f(\tau)g(\tau+h) - f(\tau)g(\tau)}{h} \\ &= \lim_{h\to 0} \frac{f(\tau+h)g(\tau+h) - f(\tau)g(\tau+h)}{h} + \lim_{h\to 0} \frac{f(\tau)g(\tau+h) - f(\tau)g(\tau)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(\tau+h) - f(\tau)}{h} g(\tau+h)\right) + \lim_{h\to 0} \left(f(\tau)\frac{g(\tau+h) - g(\tau)}{h}\right) \\ &= f'(\tau)g(\tau) + f(\tau)g'(\tau). \end{aligned}$$

Hence, fg is holomorphic. Now we need to show that fg is holomorphic at τ_0. Then,

$$
\begin{aligned}
(fg)'(\tau_0) &= \lim_{\tau\to\tau_0} \frac{f(\tau)g(\tau)-f(\tau_0)g(\tau_0)}{\tau-\tau_0} \\
&= \lim_{\tau\to\tau_0} \frac{f(\tau)g(\tau)-f(\tau_0)g(\tau)+f(\tau_0)g(\tau)-f(\tau_0)g(\tau_0)}{\tau-\tau_0} \\
&= \lim_{\tau\to\tau_0} \frac{f(\tau)g(\tau)-f(\tau_0)g(\tau)}{\tau-\tau_0} + \lim_{\tau\to\tau_0} \frac{f(\tau_0)g(\tau)-f(\tau_0)g(\tau_0)}{\tau-\tau_0} \\
&= \lim_{\tau\to\tau_0} \left(\frac{f(\tau)-f(\tau_0)}{\tau-\tau_0} g(\tau)\right) + \lim_{\tau\to\tau_0} \left(f(\tau_0)\frac{g(\tau)-g(\tau_0)}{\tau-\tau_0}\right) \\
&= f'(\tau_0)g(\tau_0)+f(\tau_0)g'(\tau_0).
\end{aligned}
$$

Thus, fg is holomorphic at τ_0. Therefore, the product of two modular forms of weight k_1 and k_2 is a modular form of weight k_1+k_2. □

In the thesis, we work with functions denoted with T which are connected to Ramanujan's Theta functions. To understand what T functions are, we introduce the following definition.

Definition 1.2.6. *For $N>3$ and for odd integers r with $1\le r\le N-1$,*

$$
T_r(\tau) = \exp\left(-\frac{\pi i r}{N(N-3)}\right)\eta^{-\frac{6}{N(N-3)}}(\tau)\cdot\theta^{\frac{2}{N-3}}\begin{bmatrix} r/N \\ 1 \end{bmatrix}(0,N\tau),
$$

where

$$
\eta(\tau) = q^{1/24}(q;q)_\infty,
$$

and

$$
\theta\begin{bmatrix} r/N \\ 1 \end{bmatrix}(N\tau) = \exp\left(\frac{\pi i r}{2N}\right) q^{r^2/(8N)}(q^{(N-r)/2};q^N)_\infty(q^{(N+r)/2};q^N)_\infty(q^N;q^N)_\infty.
$$

The next theorem classifies the T functions as fractional weight modular forms for the principal congruence subgroup $\Gamma(N)$.

Theorem 1.2.7. *For a fixed N an appropriate fixed branch of the logarithm, $T_r(\tau)$ are weight $\frac{1}{N}$ modular forms on $\Gamma(N)$ with the same multiplier satisfying $v(\gamma)^N = 1$.*

Proof. A proof for this theorem can be given based on results in Ibukiyama's paper [16]. $\square$

As previously mentioned, we use parallel techniques to derive Ramanujan's congruences for the the partition function modulo 5 and 7. This is accomplished by through the following 3-step process:

1. Determine respective bases of weight one forms for $\Gamma_1(p)$ and for $\Gamma(p)$. This is done through the following recipe:

   (a) Determine the respective dimensions of $M_1(\Gamma_1(p))$ and $M_1(\Gamma_1(p))$.

   (b) Use Theorem 1.2.7 to find respective sets of linearly independent elements of $M_1(\Gamma_1(p))$ and $M_1(\Gamma_1(p))$ with cardinality equal to the dimension of the vector space.

2. Apply Theorem 1.2.5 to express each basis element for $M_1(\Gamma_1(p))$ at argument $q^{1/p}$ as a linear combination of the elements in the basis for $M_1(\Gamma_1(p))$.

3. Find the dissection of the product of the elements forming a basis for $M_1(\Gamma_1(p)$, and use the resulting representation to prove Ramanujan's congruences.

CHAPTER II

LEVEL FIVE RESULTS

The purpose of this chapter is to show the following congruence:

$$p(5n+4) \equiv 0 \pmod{5}. \tag{2.1}$$

We will prove the above congruence by using properties of modular forms and linear algebra. In order to do this, we first introduce some theorems and definitions. We start by stating the relations connecting the T_i functions defined in Definition 1.2.6 with Ramanujan Theta functions Definition 1.2.1.

Theorem 2.0.1. *Let $q = e^{2\pi i\tau}$. Then*

$$T_3 = q^{1/5}\frac{f(-q,-q^4)}{(q;q)_\infty^{3/5}}(q^5;q^5)_\infty,$$
$$T_1 = \frac{f(-q^2,-q^3)}{(q;q)_\infty^{3/5}}(q^5;q^5)_\infty.$$

The quotients above will be denoted by

$$A := T_3 \qquad \text{and} \qquad B := T_1.$$

2.1 Modular Forms and Linearly Independence

In this section, we use the theory of modular forms and linear algebra to find representations for $A^5(q^{1/5})$ and $B^5(q^{1/5})$ in terms of a basis for weight one forms on $\Gamma(5)$. We begin by proving the following theorem.

Theorem 2.1.1. *A^5 and B^5 are linearly independent over $\mathbb{C}$.*

Proof. The functions

$$A^5(q) = q\frac{f^5(-q,-q^4)}{(q;q)_\infty^3}(q^5;q^5)_\infty^5 \quad \text{and} \quad B^5(q) = \frac{f^5(-q^2,-q^3)}{(q;q)_\infty^3}(q^5;q^5)_\infty^5$$

are linearly independent if the only solution to

$$c_1 A^5(q) + c_2 B^5(q) = 0, \tag{2.2}$$

is the trivial solution. Let

$$G(q) = c_1 q\frac{f^5(-q,-q^4)}{(q;q)_\infty^3} + c_2 \frac{f^5(-q^2,-q^3)}{(q;q)_\infty^3}.$$

The coefficients of equation (2.2) are linear equations in c_1 and c_2. Where the restrictions on c_1 and c_2 are obtained by equating coefficients on both sides of $G(q)$. From equating coefficients of q, we get the equation $c_1 + 3c_2 = 0$, and $-2c_1 + 4c_2 = 0$. Therefore, we have the following system of equations

$$-2c_1 + 4c_2 = 0,$$
$$-2c_1 + 4c_2 = 0,$$

and by solving the system of equations we obtain $c_1 = c_2 = 0$. Thus, we have that A^5+ and B^5 are linearly independent over $\mathbb{C}$. □

The subsequent theorems provide the necessary evidence we need to conclude that $A^5(q)$ and $B^5(q)$ form a basis for $M_1(\Gamma_1(5))$.

Theorem 2.1.2. *A^5 and B^5 are modular forms of weight one for $\Gamma_1(5)$.*

Proof. From [14, Theorem 2.2], $A^5(\tau)$ and $B^5(\tau)$ are linear combinations of Eisenstein series of weight one twisted by Dirichlet characters modulo five. Since the twisted Eisenstein series are modular forms on $\Gamma_1(5)$, we conclude that $A^5(\tau)$ and $B^5(\tau)$ are elements of $M_1(\Gamma_1(5))$. □

Theorem 2.1.3. $\dim M_1(\Gamma_1(5)) = 2$.

Proof. The dimension formula can be derived from [7, 10, p.91]. □

Since $A^5(q)$ and $B^5(q)$ are linearly independent over $\mathbb{C}$ and $A^5(q), B^5(q) \in M_1(\Gamma_1(5)$ and since $M_1(\Gamma_1(5)$ has dimension 3, we obtain the following corollary.

Corollary 2.1.3.1. *A basis for* $M_1(\Gamma_1(5))$ *is* $\{A^5, B^5\}$.

We need to find representations for $A^5(q^{1/5})$ and $B^5(q^{1/5})$ in terms of the basis for another vector space spanned by monomials of degree five in A and B. We need to show that this monomials in the other vector space are linearly independent.

Theorem 2.1.4. *$A^5(q)$, $A^4(q)B(q)$, $A^3(q)B^2(q)$, $A^2(q)B^3(q)$, $A(q)B^4(q)$ and $B^5(q)$ are linearly independent over $\mathbb{C}$.*

Proof. Let $G(q)$ be a linear combination of the 6 monomials; that is

$$G(q) = c_1 A^5(q) + c_2 B^5(q) + c_3 A^4(q)B(q) + c_4 A^3(q)B^2(q) + c_5 A^2(q)B^3(q) + c_6 A(q)B^4(q).$$

We need to show that the only possible solution for $G(q) = 0$ is $c_k = 0$ for $k = 1, 2, \cdots, 6$. To obtain the restrictions on c_k, we equate coefficients of successive powers of q on both sides of $G(q)$. We do this for $n = 1, 2, \cdots, 12$ and note that q^n gives redundant information when $n = 6, 7, 8, 9, 11, 12$. Therefore,

$$c_6 = 0, \tag{2.3}$$

$$c_5 = 0, \tag{2.4}$$

$$c_4 = 0, \tag{2.5}$$

$$c_3 = 0, \tag{2.6}$$

$$c_1 + 3c_2 = 0, \tag{2.7}$$

$$-2c_1 + 4c_2 = 0. \tag{2.8}$$

For example, by equating coefficients for $n = 1$, we obtained that $c_6 = 0$ and $c_4 = 0$ was obtained from equating coefficients for $n = 3$. From equations (2.7) and (2.8), we obtain a system of equations and solving the system we obtained that $c_1 = c_2 = 0$. Hence, we have shown that the 6 monomials are linearly independent over $\mathbb{C}$. □

Now that we have shown that the 6 monomials are linearly independent, we present the following theorem which allows us to conclude 6 monomials represent a basis for $M_1(\Gamma(5))$.

Theorem 2.1.5. *The* 6 *monomials are modular forms of weight one for* $\Gamma(5)$.

Proof. This follows from Theorem 1.2.7 □

We also need the dimension formula for the vector space of weight one forms on $\Gamma(5)$.

Theorem 2.1.6. $\dim M_1(\Gamma_1(5)) = 6$.

Proof. The proof appears in [7, 10, p.91]. □

Using the fact that the 6 monomials are linearly independent and using Theorems 2.1.5 and 2.1.6 we obtained the following corollary.

Corollary 2.1.6.1. *A basis for* $M_1(\Gamma(5))$ *is* $\{A^5, A^4B, A^3B^2, A^2B^3, AB^4, B^5\}$.

By combining the results of Theorems 2.1.2, 2.1.5, and 1.2.5, we obtain representations for $A^5(q^{1/5})$ and $B^5(q^{1/5})$ as linear combinations of the basis in the last Corollary.

Theorem 2.1.7.

$$A^5(q^{1/5}) = A^5(q) - 3A^4(q)B(q) + 4A^3(q)B^2(q) - 2A^2(q)B^3(q) + A(q)B^4(q), \tag{2.9}$$

$$B^5(q^{1/5}) = B^5(q) + 3B^4(q)A(q) + 4B^3(q)A^2(q) + 2B^2(q)A^3(q) + B(q)A^4(q). \tag{2.10}$$

Proof. By Theorem 2.1.2, the functions $A^5(q)$ and $B^5(q)$ are elements of $M_1(\Gamma_1(5))$ and the 6 monomials are elements of $M_1(\Gamma(5))$ by Theorem 2.1.5. Hence, by Theorem 1.2.5 we can express $A^5(q^{1/5})$ as a linear combination of the 6 functions as follows

$$A^5(q^{1/5}) = c_1 A^5(q) + c_2 B^5(q) + c_3 A^4(q)B(q) + c_4 A^3(q)B^2(q) + c_5 A^2(q)B^3(q) + c_6 A(q)B^4(q),$$

where $c_k \in \mathbb{C}$ for $k = 1, 2, \cdots, 6$. To determine the restrictions on c_k, we equated coefficients for $q^{n/5}$ for $n = 1, 2, \cdots, 12$. As a result, we obtained the following

$$c_6 = 1, \tag{2.11}$$

$$c_5 = -2, \tag{2.12}$$

$$c_4 = 4, \tag{2.13}$$

$$c_3 = -3, \tag{2.14}$$

$$c_1 + 3c_2 = 1, \tag{2.15}$$

$$2c_6 = 2, \tag{2.16}$$

$$c_5 = -2, \tag{2.17}$$

$$-c_3 = 3, \tag{2.18}$$

$$-2c_1 + 4c_2 = -2, \tag{2.19}$$

$$2c_6 = 2, \tag{2.20}$$

$$c_5 = -2. \tag{2.21}$$

Note that when $n = 6, 7, 8, 9, 11,$ and 12, $q^{n/5}$ does not provided new information. From equations (2.15) and (2.19), we get that $c_1 = 1$ and $c_2 = 0$. Thus, equation (2.9) is satisfied. Similarly, using the same process, we can show that equation (2.10) is satisfied. □

2.2 Application to Partition Functions

In this section, we are going to start by proving

$$A^5(q)B^5(q) = \frac{q(q^5;q^5)_\infty^5}{(q;q)_\infty}, \tag{2.22}$$

by using the Theorem 1.2.1, the Jacobi Triple Product Identity. Next we are going to dissect equation (2.22) into components modulo five and show that each is a power series with integer coefficients congruent to 0 $\pmod 5$. This will prove the congruence (2.1).

Theorem 2.2.1.

$$A^5(q)B^5(q) = \frac{q(q^5;q^5)_\infty^5}{(q;q)_\infty}.$$

Proof. First, we find a representation for $A(q)$ as an infinite product. We let $a = -q$ and $b = -q^4$ in Theorem 1.2.1 to obtain

$$f(-q,-q^4) = \sum_{n=-\infty}^{\infty} (-q)^{n(n+1)/2}(-q^4)^{n(n-1)/2} = (q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty.$$

Thus,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} (-1)^{\frac{n^2+n+n^2-n}{2}} q^{\frac{n^2+n+4n^2-4n}{2}} &= (q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty \\ \sum_{n=-\infty}^{\infty} (-1)^{n^2} q^{\frac{5n^2-3n}{2}} &= (q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty.\end{aligned}$$

Since, n^2 and n have the same parity, the expression above simplifies to

$$\sum_{n=-\infty}^{\infty} (-1)^{n} q^{\frac{5n^2-3n}{2}} = (q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty.$$

Hence,

$$f(-q,-q^4) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2-3n}{2}} = (q;q^5)_\infty (q^4;q^5)_\infty (q^5;q^5)_\infty.$$

Therefore, by the Jacobi Triple Product Identity

$$A(q) = q^{1/5}(q;q)_\infty^{-3/5} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2-3n}{2}} = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty (q^5;q^5)_\infty}{(q;q)_\infty^{3/5}}.$$

To find the representation for $B(q)$, we let $a = -q^2$ and $b = -q^3$, so by the Jacobi Triple Product Identity

$$f(-q^2,-q^3) = \sum_{n=-\infty}^{\infty} (-q^2)^{n(n+1)/2} (-q^3)^{n(n-1)/2} = (q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty.$$

Thus,

$$\sum_{n=-\infty}^{\infty} (-1)^{\frac{n^2+n+n^2-n}{2}} q^{\frac{2n^2+2n+3n^2-3n}{2}} = (q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty,$$

and so

$$\sum_{n=-\infty}^{\infty} (-1)^{n^2} q^{\frac{5n^2-n}{2}} = (q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty.$$

Hence,

$$f(-q^2,-q^3) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2-n}{2}} = (q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty.$$

Therefore, by the Jacobi Triple Product Identity

$$B(q) = (q;q)_\infty^{-3/5} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2-n}{2}} = \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty}{(q;q)_\infty^{3/5}}.$$

Now that we have found product representations for $A(q)$ and $B(q)$, we express $A^5(q)B^5(q)$ in terms of infinite products to get

$$\begin{aligned}
A^5(q)B^5(q) &= \left(q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty^{3/5}}\right)^5 \cdot \left(\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty^{3/5}}\right)^5 \\
&= q\frac{(q;q^5)_\infty^5(q^4;q^5)_\infty^5(q^5;q^5)_\infty^5(q^2;q^5)_\infty^5(q^3;q^5)_\infty^5(q^5;q^5)_\infty^5}{(q;q)_\infty^6} \\
&= q\frac{(q;q)_\infty^5(q^5;q^5)_\infty^5}{(q;q)_\infty^5(q;q)_\infty} \\
&= q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}.
\end{aligned}$$

Thus, we have proven that

$$A^5(q)B^5(q) = \frac{q(q^5;q^5)_\infty^5}{(q;q)_\infty}.$$

□

Substituting $q^{1/5}$ for q in (2.22) implies

$$A^5(q^{1/5})B^5(q^{1/5}) = \frac{q^{1/5}(q;q)_\infty^5}{(q^{1/5};q^{1/5})_\infty}.$$

The denominator is the generating function for partitions where q is replaced by $q^{1/5}$. Hence,

$$\begin{aligned}
A^5(q^{1/5})B^5(q^{1/5}) &= q^{1/5}(q;q)_\infty^5\sum_{n=0}^{\infty}p(n)q^{n/5} \\
&= (q;q)_\infty^5\sum_{n=0}^{\infty}p(n)q^{\frac{n}{5}+\frac{1}{5}} \\
&= (q;q)_\infty^5\sum_{n=0}^{\infty}p(n)q^{\frac{n+1}{5}} \\
&= (q;q)_\infty^5\sum_{n=1}^{\infty}p(n-1)q^{n/5}.
\end{aligned}$$

Divide both sides of the above identity by $(q;q)_\infty^5$ to obtain

$$\frac{A^5(q^{1/5})B^5(q^{1/5})}{(q;q)_\infty^5} = \sum_{n=1}^{\infty} p(n-1)q^{n/5}. \tag{2.23}$$

Now we dissect the right side of (2.23) and obtain

$$\begin{aligned}\sum_{n=1}^{\infty} p(n-1)q^{n/5} = q&\sum_{n=0}^{\infty} p(5n+4)q^{5n/5} \\ &+\sum_{n=0}^{\infty} p(5n)q^{(5n+1)/5} \\ &+\sum_{n=0}^{\infty} p(5n+1)q^{(5n+2)/5} \\ &+\sum_{n=0}^{\infty} p(5n+2)q^{(5n+3)/5} \\ &+\sum_{n=0}^{\infty} p(5n+3)q^{(5n+4)/5}.\end{aligned}$$

On the right side of equation (2.23) extract every fifth term of the series in $q^{1/5}$ starting with $(q^{1/5})^1$, to get

$$q\sum_{n=0}^{\infty} p(5n+4)q^{5n/5} = q\sum_{n=0}^{\infty} p(5n+4)q^{n}. \tag{2.24}$$

In the next theorem, we find the dissection components that correspond to equation (2.24).

Theorem 2.2.2.

$$q\sum_{n=0}^{\infty} p(5n+4)q^{n} = \frac{5q(q^5;q^5)_\infty^5}{(q;q)_\infty^6}$$

Proof. Substituting equations (2.9) and (2.10) into the numerator on the left side of equation (2.23), we obtain

$$
\begin{aligned}
A^5(q^{1/5})B^5(q^{1/5}) &= \Big(A^5(q)-3A^4(q)B(q)+4A^3(q)B^2(q)-2A^2(q)B^3(q)+A(q)B^4(q)\Big)\\
&\quad\times\Big(B^5(q)+3B^4(q)A(q)+4B^3(q)A^2(q)+2B^2(q)A^3(q)+B(q)A^4(q)\Big)\\
&= 5A^5(q)B^5(q)+6A^4(q)B^6(q)+4A^3(q)B^7(q)-A^8(q)B^2(q)\\
&\quad+A^9(q)B(q)-6A^6(q)B^4(q)+A^2(q)B^8(q)+A(q)B^9(q).
\end{aligned}
\tag{2.25}
$$

We dissect the right side of equation (2.25) and obtain the following

$$
\begin{aligned}
\Big(A^5(q^{1/5})B^5(q^{1/5})\Big)_0 &= 5A^5(q)B^5(q),\\
\Big(A^5(q^{1/5})B^5(q^{1/5})\Big)_1 &= A(q)B^9(q)-6A^6(q)B^4(q),\\
\Big(A^5(q^{1/5})B^5(q^{1/5})\Big)_2 &= A^2(q)B^8(q),\\
\Big(A^5(q^{1/5})B^5(q^{1/5})\Big)_3 &= 4A^3(q)B^7(q)-A^8(q)B^2(q),\\
\Big(A^5(q^{1/5})B^5(q^{1/5})\Big)_4 &= 6A^4(q)B^6(q)+A^9(q)B(q).
\end{aligned}
$$

By Theorem 1.2.3,

$$
\frac{1}{(q;q)_\infty^5}\Big(A^5(q^{1/5})B^5(q^{1/5})\Big)_0 = \left(\frac{A^5(q^{1/5})B^5(q^{1/5})}{(q;q)_\infty^5}\right)_0,
$$

so

$$
\left(\frac{A^5(q^{1/5})B^5(q^{1/5})}{(q;q)_\infty^5}\right)_0 = \frac{1}{(q;q)_\infty^5}5A^5(q)B^5(q).
$$

Therefore, by equating the dissection components, we get

$$
\frac{5A^5(q)B^5(q)}{(q;q)_\infty^5} = q\sum_{n=0}^{\infty} p(5n+4)q^{5n/5}.
$$

Thus, from Theorem 2.2.1 we obtain

$$q\sum_{n=0}^{\infty} p(5n+4)q^n = \frac{5q(q^5;q^5)_\infty^5}{(q;q)_\infty^6}.$$

□

In the next theorem, we are going to show that the coefficients of the of the zero dissection are congruent to 0 (mod 5)

Theorem 2.2.3. *For any natural number n,*

$$p(5n+4) \equiv 0 \pmod 5.$$

Proof. Since $\frac{1}{(q;q)_\infty^6}$ is the sixth power of the generating function for partitions Theorem 1.0.1, it is a power series in q with positive integer coefficients. Then

$$\frac{1}{(q;q)_\infty^6} = \sum_{n=0}^{\infty} S_n q^n, \qquad S_n \in \mathbb{Z}^+.$$

Hence,

$$\frac{5qA^5(q)B^5(q)}{(q;q)_\infty^6} = 5q(q^5;q^5)_\infty^5 \left(\sum_{n=0}^{\infty} S_n q^n\right).$$

Similarly, since $(q^5;q^5)_\infty^5$ is a power series in q with integer coefficients,

$$5q(q^5;q^5)_\infty^5 \left(\sum_{n=0}^{\infty} S_n q^n\right) = 5q\left(\sum_{n=0}^{\infty} l_n q^n\right)\left(\sum_{n=0}^{\infty} S_n q^n\right), \qquad l_n \in \mathbb{Z}$$

and by taking Product of the series, we obtain

$$5q\left(\sum_{n=0}^{\infty} l_n q^n\right)\left(\sum_{n=0}^{\infty} S_n q^n\right) = 5q\sum_{n=0}^{\infty} g_n q^n, \qquad g_n \in \mathbb{Z}.$$

Therefore,

$$q\sum_{n=0}^{\infty} p(5n+4)q^n = 5q\sum_{n=0}^{\infty} g_n q^n, \qquad g_n \in \mathbb{Z}.$$

Hence, by equating coefficients on each side

$$p(5n+4) = 5g_n, \qquad \forall n \geq 0.$$

Thus, we can conclude that

$$p(5n+4) \equiv 0 \pmod 5.$$

$\square$

CHAPTER III

LEVEL SEVEN

This chapter is dedicated to prove the following congruence

$$p(7n+5) \equiv 0 \pmod 7.$$

The congruence above will be proven in the same manner as the previous congruence. In order to do this, we need to introduce essential definitions and theorems. We begin by listing a library of relations connecting the T_i defined in Definition 1.2.6 with Ramanujan Theta functions introduce in Definition 1.2.1. Note that in this chapter, $N = 7$ will be used in Definition 1.2.6.

Theorem 3.0.1.

$$\frac{T_3^5 T_5^3}{T_1} = q \frac{f\left(-q^2,-q^5\right)}{f^2\left(-q^3,-q^4\right)}(q^7;q^7)_\infty^3, \tag{3.1}$$

$$\frac{T_1^3 T_5^5}{T_3} = q \frac{f\left(-q,-q^6\right)}{f^2\left(-q^2,-q^5\right)}(q^7;q^7)_\infty^3, \tag{3.2}$$

$$\frac{T_1^5 T_3^3}{T_5} = \frac{f\left(-q^3,-q^4\right)}{f^2\left(-q,-q^6\right)}(q^7;q^7)_\infty^3, \tag{3.3}$$

$$\frac{T_1^3 T_3^5}{T_5} = q^{1/7} \frac{f\left(-q^2,-q^5\right)}{f^2\left(-q,-q^6\right)}(q^7;q^7)_\infty^3, \tag{3.4}$$

$$T_1 T_3 T_5^5 = q^{8/7} \frac{f\left(-q,-q^6\right)}{f\left(-q^2,-q^5\right) f\left(-q^3,-q^4\right)}(q^7;q^7)_\infty^3, \tag{3.5}$$

$$\frac{T_3^3 T_5^5}{T_1} = q^{9/7} \frac{f\left(-q,-q^6\right)}{f^2\left(-q^3,-q^4\right)}(q^7;q^7)_\infty^3, \tag{3.6}$$

$$T_1^5 T_3 T_5 = q^{2/7} \frac{f\left(-q^3,-q^4\right)}{f\left(-q,-q^6\right) f\left(-q^2,-q^5\right)}(q^7;q^7)_\infty^3, \tag{3.7}$$

$$T_1^3 T_3^3 T_5 = \frac{q^{3/7}}{f\left(-q,-q^6\right)}(q^7;q^7)_\infty^3, \tag{3.8}$$

$$\frac{T_1^5T_5^3}{T_3} = q^{4/7}\frac{f\left(-q^3,-q^4\right)}{f^2\left(-q^2,-q^5\right)}(q^7;q^7)_\infty^3, \tag{3.9}$$

$$T_1T_3^5T_5 = q^{4/7}\frac{f\left(-q^2,-q^5\right)}{f\left(-q,-q^6\right)f\left(-q^3,-q^4\right)}(q^7;q^7)_\infty^3, \tag{3.10}$$

$$T_1^3T_3T_5^3 = \frac{q^{5/7}}{f\left(-q^2,-q^5\right)}(q^7;q^7)_\infty^3, \tag{3.11}$$

$$T_1T_3^3T_5^3 = \frac{q^{6/7}}{f\left(-q^3,-q^4\right)}(q^7;q^7)_\infty^3. \tag{3.12}$$

We distinguish three of the quotients above and subsequently refer to them as:

$$x := \frac{T_3^5T_5^3}{T_1}, \quad y := \frac{T_1^3T_5^5}{T_3}, \quad \text{and} \quad z := \frac{T_1^5T_3^3}{T_5}.$$

3.1 Results

The purpose of this section is to prove that the coefficients of power series expansion for the zero dissection class of $x(q^{1/7})y(q^{1/7})z(q^{1/7})$ are integers congruent to 0 (mod 7). This will be used to prove equation (1.2). To do this, we must ensure that x, y and z are linearly independent and form a basis for the modular form of weight one for $\Gamma_1(7)$. We also need to show that the 12 quotients (3.1)-(3.12) are linearly independent and the functions represent a basis for the modular form of weight one for $\Gamma(7)$. This leads to the following theorem.

Theorem 3.1.1. *The functions* $x = \frac{T_3^5T_5^3}{T_1}$, $y = \frac{T_1^3T_5^5}{T_3}$ *and* $z = \frac{T_1^5T_3^3}{T_5}$ *are linearly independent over* $\mathbb{C}$.

Proof. Let

$$x(q) = \frac{q\,f\left(-q^2,-q^5\right)}{f^2\left(-q^3,-q^4\right)}(q^7;q^7)_\infty^3,$$
$$y(q) = \frac{q\,f\left(-q,-q^6\right)}{f^2\left(-q^2,-q^5\right)}(q^7;q^7)_\infty^3,$$
$$z(q) = \frac{f\left(-q^3,-q^4\right)}{f^2\left(-q,-q^6\right)}(q^7;q^7)_\infty^3.$$

The functions $x(q)$, $y(q)$ and $z(q)$ are linearly independent if the only solution to

$$k_1 \cdot x(q) + k_2 \cdot y(q) + k_3 \cdot z(q) = 0, \tag{3.13}$$

is $k_i = 0$ for $i = 1,2,3$. Let

$$G(q) = k_1 \frac{q f\left(-q^2,-q^5\right)}{f^2\left(-q^3,-q^4\right)} + k_2 \frac{q f\left(-q,-q^6\right)}{f^2\left(-q^2,-q^5\right)} + k_3 \frac{f\left(-q^3,-q^4\right)}{f^2\left(-q,-q^6\right)}.$$

The coefficients in the q-expansion of $G(q)$ are linear equations in k_1, k_2, k_3. The restrictions on the coefficients were obtained by equating coefficients of q^n on both sides of $G(q)$. The first equation $k_1 + k_2 + 2k_3 = 0$ was obtained from equating coefficients of q, $-k_2 + 3k_3 = 0$ was obtained from equating coefficients of q^2, and from equating coefficients of q^3 we get $-k_1 + 2k_2 + 3k_3$=0. We used Mathematica to derive these coefficients. The corresponding code is contained in Appendix C. Hence, we obtained the following system of equations,

$$\begin{aligned} k_1 + k_2 + 2k_3 &= 0, \\ -k_2 + 3k_3 &= 0, \\ -k_1 + 2k_2 + 3k_3 &= 0. \end{aligned}$$

Hence,

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 3 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

By row reducing, we obtained that the only solution to the system is $k_1 = k_2 = k_3 = 0$. Therefore, the functions $x(q)$, $y(q)$ and $z(q)$ are linearly independent over $\mathbb{C}$. $\square$

The preceding theorems are needed to conclude the following corollary.

Theorem 3.1.2. *The functions x, y, and z are elements of* $M_1(\Gamma_1(7))$.

Proof. The theorem is proven in [15] □

Theorem 3.1.3. $\dim M_1(\Gamma_1(7)) = 3$.

Proof. A proof can be given using dimension formulas from [7, 10, p.91]. □

Using the above theorems we obtain the following corollary.

Corollary 3.1.3.1. *A basis for* $M_1(\Gamma_1(7))$ *is* $\{x,y,z\}$.

In the following theorem we will prove that the 12 quotients (3.1)-(3.12) are linearly independent.

Theorem 3.1.4. *The 12 quotients, equations* (3.1)-(3.12)*, are linearly independent over* $\mathbb{C}$.

Proof. Let

$$\begin{aligned} G(q) = {} & k_1 \frac{q^{2/7} f(-q^3,-q^4)}{f(-q,-q^6) f(-q^2,-q^5)} + k_2 \frac{q^{3/7}}{f(-q,-q^6)} + k_3 \frac{q^{5/7}}{f(-q^2,-q^5)} + k_4 \frac{q^{6/7}}{f(-q^3,-q^4)} \\ & + k_5 \frac{q f(-q^2,-q^5)}{f^2(-q^3,-q^4)} + k_6 \frac{q^{9/7} f(-q,-q^6)}{f^2(-q^3,-q^4)} + k_7 \frac{q f(-q,-q^6)}{f^2(-q^2,-q^5)} \\ & + k_8 \frac{q^{4/7} f(-q^3,-q^4)}{f^2(-q^2,-q^5)} + k_9 \frac{f(-q^3,-q^4)}{f^2(-q,-q^6)} + k_{10} \frac{q^{1/7} f(-q^2,-q^5)}{f^2(-q,-q^6)} \\ & + k_{11} \frac{q^{8/7} f(-q,-q^6)}{f(-q^2,-q^5) f(-q^3,-q^4)} + k_{12} \frac{q^{4/7} f(-q^2,-q^5)}{f(-q,-q^6) f(-q^3,-q^4)}. \end{aligned}$$

To show that that the set of quotients (3.1)-(3.12), which forms a linear combination defining $G(q)$, are linearly independent, we need to show that the only possible solution for $G(q) = 0$ is $k_i = 0$ for $i = 1,\ldots,12$. By equating coefficients on both sides of $G(q)$, we obtained the restrictions for k_i. Next, we equate coefficients only for $n = 1,2,\ldots,21$; note that q^n gives redundant information for $n = 10,12,13,15,\ldots,20$. We used Mathematica to derive these coefficients. The corresponding code is contained in Appendix C. Hence,

$$k_{10} = 0, \tag{3.14}$$

$$k_1 = 0, \tag{3.15}$$

$$k_2 = 0, \tag{3.16}$$

$$k_8 + k_{12} = 0, \tag{3.17}$$

$$k_3 = 0, \tag{3.18}$$

$$k_4 = 0, \tag{3.19}$$

$$k_5 + k_7 + 2k_9 = 0, \tag{3.20}$$

$$2k_{10} + k_{11} = 0, \tag{3.21}$$

$$k_1 + k_6 = 0, \tag{3.22}$$

$$k_{12} = 0, \tag{3.23}$$

$$-k_7 + 3k_9 = 0, \tag{3.24}$$

$$-k_5 + 2k_7 + 3k_9 = 0. \tag{3.25}$$

For example, when we equated coefficients for $n = 1$, we obtained $k_{10} = 0$, and when we equated coefficients for $n = 4$ we got $k_8 + k_{12} = 0$ and $k_5 + k_7 + 2k_9 = 0$ was obtained when we equated coefficient for $n = 7$. Now from equation (3.23) we have that $k_{12} = 0$, which implies that $k_8 = 0$ by equation (3.17). Also from equation (3.15) $k_1 = 0$, so using equation (3.22) we have that $k_6 = 0$. From equation (3.14), $k_{10} = 0$, which implies that $k_{11} = 0$ from equation (3.21). Notice that from equation (3.24) $k_7 = 3k_9$, then using equation (3.20) we get

$$\begin{aligned} k_5 + k_7 + 2k_9 &= 0 \\ \implies k_5 + 3k_9 + 2k_9 &= 0 \\ \implies k_5 &= -5k_9. \end{aligned}$$

By using equation (3.25), we have

$$-k_5 + 2k_7 + 3k_9 = 0$$

$$\implies -(-5k_9)+2(3k_9)+3k_9=0$$

$$\implies k_9=0.$$

Since $k_7=3k_9$ and $k_5=-5k_9$ we have that $k_7=k_5=0$ is also true. Hence, we have that $G(q)=0$ only when the coefficients $k_i=0$ for $i=1,\dots,12$. Therefore, we have shown that the 12 quotients (3.1)-(3.12) are linearly independent over $\mathbb{Z}$. $\square$

We now present the subsequent theorem and lemma, which are essential for Corollary 3.1.6.1.

Theorem 3.1.5. *The following functions are modular forms on* $\Gamma(7)$ *of weight one*

$$q\,\frac{f\left(-q^2,-q^5\right)}{f^2\left(-q^3,-q^4\right)}(q^7;q^7)^3_\infty,\; q\,\frac{f\left(-q,-q^6\right)}{f^2\left(-q^2,-q^5\right)}(q^7;q^7)^3_\infty,\; q^{1/7}\,\frac{f\left(-q^2,-q^5\right)}{f^2\left(-q,-q^6\right)}(q^7;q^7)^3_\infty,$$
$$q^{8/7}\,\frac{f\left(-q,-q^6\right)}{f\left(-q^2,-q^5\right)f\left(-q^3,-q^4\right)}(q^7;q^7)^3_\infty,\; \frac{q^{6/7}}{f\left(-q^3,-q^4\right)}(q^7;q^7)^3_\infty,\; \frac{f\left(-q^3,-q^4\right)}{f^2\left(-q,-q^6\right)}(q^7;q^7)^3_\infty,$$
$$q^{2/7}\,\frac{f\left(-q^3,-q^4\right)}{f\left(-q,-q^6\right)f\left(-q^2,-q^5\right)}(q^7;q^7)^3_\infty,\; \frac{q^{3/7}}{f\left(-q,-q^6\right)}(q^7;q^7)^3_\infty,\; q^{4/7}\,\frac{f\left(-q^3,-q^4\right)}{f^2\left(-q^2,-q^5\right)}(q^7;q^7)^3_\infty,$$
$$q^{4/7}\,\frac{f\left(-q^2,-q^5\right)}{f\left(-q,-q^6\right)f\left(-q^3,-q^4\right)}(q^7;q^7)^3_\infty,\; \frac{q^{5/7}}{f\left(-q^2,-q^5\right)}(q^7;q^7)^3_\infty,\; q^{9/7}\,\frac{f\left(-q,-q^6\right)}{f^2\left(-q^3,-q^4\right)}(q^7;q^7)^3_\infty.$$

Proof. The proof follows from the fact that the twelve functions above agree with the twelve quotients of T functions (3.1)-(3.12). That these quotients, in turn are modular forms of weight one for $\Gamma(7)$ follows from Theorem 1.2.7. $\square$

Lemma 3.1.6. $\dim M_1(\Gamma(7))=12$.

Proof. A proof can be given using the dimension formulas from [7, 10, p.91]. $\square$

The following corollary follows from the fact that the twelve quotients (3.1)-(3.12) are elements of $M_1(\Gamma(7))$ which has dimension 12 and the fact that the quotients are linearly independent by Theorem 3.1.4.

Corollary 3.1.6.1. *The quotients* (3.1)-(3.12) *in Theorem 3.0.1 represent a basis for* $M_1(\Gamma(7))$.

Now that we have shown that $x(\tau)$, $y(\tau)$, and $z(\tau)$ and the twelve quotients are linearly independent and form a basis of their respective modular forms, we need to find the representations of $x(q^{1/7})$, $y(q^{1/7})$, and $z(q^{1/7})$ in term of the previously constructed basis of weight one forms for $\Gamma(7)$.

Definition 3.1.1. *Denote*

$$\alpha_1 = f(-q^3, -q^4),$$

$$\alpha_2 = f(-q, -q^6),$$

and

$$\alpha_3 = f(-q^2, -q^5).$$

Theorem 3.1.7. *Let α_1, α_2, and α_3 be defined by Definition 3.1.1, then*

$$x(q^{1/7}) = (q;q)_\infty^3\Bigg(-\frac{3q^{6/7}}{\alpha_1} - \frac{q^{3/7}}{\alpha_2} - \frac{2q^{9/7}\alpha_2}{\alpha_1^2} - \frac{q^{4/7}\alpha_1}{\alpha_3^2} + \frac{2q^{5/7}}{\alpha_3} + \frac{2q^{8/7}\alpha_2}{\alpha_1\alpha_3} + \frac{q\,\alpha_3}{\alpha_1^2} + \frac{q^{1/7}\alpha_3}{\alpha_2^2} + \frac{3q^{4/7}\alpha_3}{\alpha_1\alpha_2}\Bigg), \tag{3.26}$$

$$y(q^{1/7}) = (q;q)_\infty^3\Bigg(-\frac{q^{6/7}}{\alpha_1} + \frac{2q^{3/7}}{\alpha_2} + \frac{q^{9/7}\alpha_2}{\alpha_1^2} - \frac{2q^{4/7}\alpha_1}{\alpha_3^2} + \frac{q\,\alpha_2}{\alpha_3^2} - \frac{q^{2/7}\alpha_1}{\alpha_2\alpha_3} - \frac{3q^{8/7}\alpha_2}{\alpha_1\alpha_3} + \frac{3q^{5/7}}{\alpha_3} + \frac{q^{1/7}\alpha_3}{\alpha_2^2}\Bigg), \tag{3.27}$$

$$z(q^{1/7}) = (q;q)_\infty^3\Bigg(\frac{3q^{3/7}}{\alpha_2} + \frac{q^{4/7}\alpha_1}{\alpha_3^2} + \frac{2q^{6/7}}{\alpha_1} + \frac{q^{9/7}\alpha_2}{\alpha_1^2} + \frac{\alpha_1}{\alpha_2^2} + \frac{3q^{2/7}\alpha_1}{\alpha_2\alpha_3} + \frac{q^{4/7}\alpha_3}{\alpha_1\alpha_2} + \frac{q^{5/7}}{\alpha_3} + \frac{2q^{1/7}\alpha_3}{\alpha_2^2}\Bigg). \tag{3.28}$$

Proof. We know from Theorem 3.1.2, that the functions $x(q)$, $y(q)$ and $z(q)$ are elements of $M_1(\Gamma_1(7))$. Similarly, from Theorem 3.1.5 that the 12 quotients (3.1)-(3.12) are elements of $M_1(\Gamma(7))$. By Theorem 1.2.5, since $x(q) \in M_1(\Gamma_1(7))$ and by replacing q with $q^{1/7}$, we have that $x(q^{1/7}) \in M_1(\Gamma(7))$. Therefore, we can write $x(q^{1/7})$ as linear combination of the twelve basis elements ,(3.1)-(3.12), of $M_1(\Gamma(7))$ as follows

$$
\begin{aligned}
x(q^{1/7}) = \Bigg(& k_1 \frac{q^{2/7}\,\alpha_1}{\alpha_2\alpha_3} + k_2 \frac{q^{3/7}}{\alpha_2} + k_3 \frac{q^{5/7}}{\alpha_3} + k_4 \frac{q^{6/7}}{\alpha_1} + k_5 \frac{q\,\alpha_3}{f^2(-q^3,-q^4)} + k_6 \frac{q^{9/7}\,\alpha_2}{f^2(-q^3,-q^4)} \\
& + k_7 \frac{q\,\alpha_2}{f^2(-q^2,-q^5)} + k_8 \frac{q^{4/7}\,\alpha_1}{f^2(-q^2,-q^5)} + k_9 \frac{\alpha_1}{f^2(-q,-q^6)} + k_{10} \frac{q^{1/7}\,\alpha_3}{f^2(-q,-q^6)} \\
& + k_{11} \frac{q^{8/7}\,\alpha_2}{\alpha_3\alpha_1} + k_{12} \frac{q^{4/7}\,\alpha_3}{\alpha_2\alpha_1} \Bigg) (q;q)_\infty^3,
\end{aligned}
$$

where $k_i \in \mathbb{C}$ for $i = 1, \dots, 12$. In order to determine restrictions on k_i, we equated the coefficients for $q^{n/7}$ where $n = 1, 2, \dots, 21$. We used Mathematica to derive these coefficients. The corresponding code is contained in Appendix D. Hence, we obtain the following

$$k_{10} = 1, \tag{3.29}$$

$$k_1 = 0, \tag{3.30}$$

$$k_2 = -1, \tag{3.31}$$

$$k_8 + k_{12} = 2, \tag{3.32}$$

$$k_3 = 2, \tag{3.33}$$

$$k_4 = -3, \tag{3.34}$$

$$k_5 + k_7 + 2k_9 = 1, \tag{3.35}$$

$$2k_{10} + k_{11} = 3, \tag{3.36}$$

$$k_1 + k_6 = -2, \tag{3.37}$$

$$k_2 = -1, \tag{3.38}$$

$$k_{12} = 3, \tag{3.39}$$

$$-k_7 + 3k_9 = 0, \tag{3.40}$$

$$2k_{10} - k_{11} = 1, \tag{3.41}$$

$$2k_1 - k_6 = 2, \tag{3.42}$$

$$k_2 = -1, \tag{3.43}$$

$$2k_8 = -2, \tag{3.44}$$

$$k_3 = 2, \tag{3.45}$$

$$-k_5 + 2k_7 + 3k_9 = -1. \tag{3.46}$$

Notice that equating the coefficients of $q^{n/7}$ for $n = 10, 12, 13, 15, \cdots, 20$ does not provide new information. However, using equations (3.32) and (3.39) we able to show that $k_8 = -1$. Also using equations (3.35), (3.40) and (3.46) we obtain that $k_5 = 1$ and $k_7 = k_9 = 0$. Thus, we have shown that (3.26) is satisfied. Similarly, we repeat this process to for solve $y(q^{1/7})$ and $z(q^{1/7})$, so we have that (3.27) and (3.28) are satisfied. □

Now that we found the representations $x(q^{1/7})$, $y(q^{1/7})$, and $z(q^{1/7})$, we can show that the coefficients of the zero dissection of the product of the three functions mentioned, are congruent to 0 (mod 7).

Theorem 3.1.8. *The zero dissection of $x(q^{1/7})\,y(q^{1/7})\,z(q^{1/7})$ is a power series in q with integer coefficients that are each congruent to* 0 (mod 7).

Proof. Since $x(q)$ $y(q)$, and $z(q)$ are weight one forms on $\Gamma_1(7)$, the product $x(q)y(q)z(q)$ is a weight 3 form, refer to Lemma 1.2.6, on $\Gamma_1(7)$. By Theorem 1.2.7, $x(q^{1/7})y(q^{1/7})z(q^{1/7})$ is a weight 3 form on $\Gamma(7)$; therefore it has a representation in terms of the dissection components. Using Theorem 3.1.7 we obtained

$$\begin{aligned}
x(q^{1/7})\,y(q^{1/7})\,z(q^{1/7}) = (q;q)_\infty^9 \Bigg(& -\frac{2q^{27/7}\alpha_2^3}{\alpha_1^6} + q^{12/7}\left(\frac{\alpha_3^3}{\alpha_1^3\alpha_2^3} + \frac{2\alpha_1^3}{\alpha_3^6} - \frac{16}{\alpha_1\alpha_2^2} + \frac{14\alpha_1}{\alpha_2\alpha_3^3}\right) \\
& + \frac{7q^{26/7}\alpha_2^3}{\alpha_1^5\alpha_3} + q^{11/7}\left(\frac{8\alpha_3^2}{\alpha_1^2\alpha_2^3} - \frac{17\alpha_1^2}{\alpha_2\alpha_3^4} + \frac{15}{\alpha_2^2\alpha_3}\right) \\
& + \frac{17q^{23/7}\alpha_2^2}{\alpha_1^4\alpha_3} + q^2\left(\frac{7\alpha_3^2}{\alpha_1^3\alpha_2^2} - \frac{28}{\alpha_1\alpha_2\alpha_3} + \frac{14\alpha_1}{\alpha_3^4}\right) \\
& + \frac{q^{2/7}\alpha_1\alpha_3^2}{\alpha_2^6} + q^{6/7}\left(\frac{\alpha_1^3}{\alpha_2^3\alpha_3^3} + \frac{7\alpha_3^3}{\alpha_1\alpha_2^5} - \frac{3\alpha_1}{\alpha_2^4}\right) \\
& + \frac{2q^{4/7}\alpha_1\alpha_3}{\alpha_2^5} + q^{15/7}\left(-\frac{\alpha_1^2\alpha_2}{\alpha_3^6} - \frac{15}{\alpha_1^2\alpha_2} - \frac{5}{\alpha_3^3}\right)
\end{aligned}$$

$$
\begin{aligned}
&+q^{18/7}\left(-\frac{5}{\alpha_1^3}-\frac{11\alpha_2}{\alpha_1\alpha_3^3}\right)+q^{19/7}\left(\frac{\alpha_2^2}{\alpha_3^5}-\frac{3\alpha_3}{\alpha_1^4}+\frac{13\alpha_2}{\alpha_1^2\alpha_3^2}\right)\\
&+q^{3}\left(\frac{7\alpha_2^2}{\alpha_1^2\alpha_3^3}-\frac{14\alpha_2}{\alpha_1^4}\right)+q^{10/7}\left(\frac{7\alpha_1^3}{\alpha_2\alpha_3^5}+\frac{14\alpha_3}{\alpha_1\alpha_2^3}-\frac{13\alpha_1}{\alpha_2^2\alpha_3^2}\right)\\
&+q^{24/7}\left(\frac{\alpha_2^3}{\alpha_1^3\alpha_3^3}-\frac{8\alpha_2^2}{\alpha_1^5}\right)+q^{13/7}\left(\frac{16}{\alpha_2\alpha_3^2}-\frac{13\alpha_3}{\alpha_1^2\alpha_2^2}-\frac{8\alpha_1^2}{\alpha_3^5}\right)\\
&+q\left(\frac{14\alpha_3}{\alpha_2^4}-\frac{7\alpha_1^2}{\alpha_2^3\alpha_3^2}\right)+q^{16/7}\left(\frac{15}{\alpha_1\alpha_3^2}-\frac{11\alpha_3}{\alpha_1^3\alpha_2}+\frac{2\alpha_1\alpha_2}{\alpha_3^5}\right)\\
&+q^{22/7}\left(\frac{2\alpha_2\alpha_3}{\alpha_1^5}-\frac{8\alpha_2^2}{\alpha_1^3\alpha_3^2}\right)+q^{20/7}\left(\frac{\alpha_3^2}{\alpha_1^5}+\frac{14\alpha_2}{\alpha_1^3\alpha_3}-\frac{5\alpha_2^2}{\alpha_1\alpha_3^4}\right)\\
&+q^{5/7}\left(\frac{8\alpha_3^2}{\alpha_2^5}-\frac{5\alpha_1^2}{\alpha_2^4\alpha_3}\right)+q^{8/7}\left(\frac{5\alpha_1^3}{\alpha_2^2\alpha_3^4}+\frac{17\alpha_3^2}{\alpha_1\alpha_2^4}-\frac{11\alpha_1}{\alpha_2^3\alpha_3}\right)\\
&+q^{25/7}\left(\frac{\alpha_2^2\alpha_3}{\alpha_1^6}-\frac{5\alpha_2^3}{\alpha_1^4\alpha_3^2}\right)+q^{9/7}\left(\frac{5\alpha_3^3}{\alpha_1^2\alpha_2^4}-\frac{8\alpha_1^2}{\alpha_2^2\alpha_3^3}+\frac{5}{\alpha_2^3}\right)\\
&+q^{3/7}\left(\frac{2\alpha_3^3}{\alpha_2^6}-\frac{\alpha_1^2}{\alpha_2^5}\right)+q^{17/7}\left(\frac{5\alpha_3^2}{\alpha_1^4\alpha_2}-\frac{16}{\alpha_1^2\alpha_3}+\frac{3\alpha_2}{\alpha_3^4}\right)\Bigg).
\end{aligned}
$$

By selecting monomials that contributed to the desired dissection class, we were able to dissect the right side of the above equation to obtain

$$
\begin{aligned}
\left(x(q^{1/7})y(q^{1/7})z(q^{1/7})\right)_0 &= (q;q)_\infty^9\Bigg(q\left(\frac{14\alpha_3}{\alpha_2^4}-\frac{7\alpha_1^2}{\alpha_2^3\alpha_3^2}\right)+q^2\left(\frac{7\alpha_3^2}{\alpha_1^3\alpha_2^2}-\frac{28}{\alpha_1\alpha_2\alpha_3}+\frac{14\alpha_1}{\alpha_3^4}\right)\\
&\quad+q^3\left(\frac{7\alpha_2^2}{\alpha_1^2\alpha_3^3}-\frac{14\alpha_2}{\alpha_1^4}\right)\Bigg),\\
\left(x(q^{1/7})y(q^{1/7})z(q^{1/7})\right)_1 &= (q;q)_\infty^9\Bigg(q^{8/7}\left(\frac{5\alpha_1^3}{\alpha_2^2\alpha_3^4}+\frac{17\alpha_3^2}{\alpha_1\alpha_2^4}-\frac{11\alpha_1}{\alpha_2^3\alpha_3}\right)\\
&\quad+q^{15/7}\left(-\frac{\alpha_1^2\alpha_2}{\alpha_3^6}-\frac{15}{\alpha_1^2\alpha_2}-\frac{5}{\alpha_3^3}\right)\Bigg),\\
\left(x(q^{1/7})y(q^{1/7})z(q^{1/7})\right)_2 &= (q;q)_\infty^9\Bigg(\frac{q^{2/7}\alpha_1\alpha_3^2}{\alpha_2^6}+q^{9/7}\left(\frac{5\alpha_3^3}{\alpha_1^2\alpha_2^4}-\frac{8\alpha_1^2}{\alpha_2^2\alpha_3^3}+\frac{5}{\alpha_2^3}\right)\\
&\quad+q^{16/7}\left(\frac{15}{\alpha_1\alpha_3^2}-\frac{11\alpha_3}{\alpha_1^3\alpha_2}+\frac{2\alpha_1\alpha_2}{\alpha_3^5}\right)+\frac{17q^{23/7}\alpha_2^2}{\alpha_1^4\alpha_3}\Bigg).
\end{aligned}
$$

The same process will allow us to write $\left(x(q^{1/7})y(q^{1/7})z(q^{1/7})\right)_j$ for $0 \le j \le 6$. Now, note that we simplified the zero dissection class by factoring, and we have

$$7q\,(q;q)_\infty^9\left(\frac{2\,\alpha_3}{\alpha_2^4}-\frac{\alpha_1^2}{\alpha_2^3\alpha_3^2}-\frac{2\,q^2\,\alpha_2}{\alpha_1^4}+\frac{q^2\,\alpha_2^2}{\alpha_1^2\alpha_3^3}+\frac{q\,\alpha_3^2}{\alpha_1^3\alpha_2^2}-\frac{4\,q}{\alpha_1\alpha_2\alpha_3}+\frac{2\,q\,\alpha_1}{\alpha_3^4}\right).$$

Thus, we have shown that the coefficients of the zero dissection class of $x(q^{1/7})y(q^{1/7})z(q^{1/7})$ are congruent to 0 (mod 7). □

Theorem 3.1.9.

$$x(q)y(q)z(q)=q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}.$$

Proof. First, we find the representation for $x(q)$ as an infinite product.Using Theorem 1.2.1, Jacobi Triple Product Identity, and letting $a=-q^2$ and $b=-q^5$ we obtain

$$f(-q^2,-q^5)=(q^2;q^7)_\infty(q^5;q^7)_\infty(q^7;q^7)_\infty. \tag{3.47}$$

Also, letting $a=-q^3$ and $b=-q^4$ we have that

$$f(-q^3,-q^4)=(q^3;q^7)_\infty(q^4;q^7)_\infty(q^7;q^7)_\infty. \tag{3.48}$$

Substituting equations (3.47) and (3.48) into equation (3.1) we get

$$\begin{aligned}x(q)&=q\frac{f(-q^2,-q^5)}{f(-q^3,-q^4)}(q^7;q^7)_\infty^3\\&=q\frac{(q^2;q^7)_\infty(q^5;q^7)_\infty(q^7;q^7)_\infty}{(q^3;q^7)_\infty^2(q^4;q^7)_\infty^2(q^7;q^7)_\infty^2}(q^7;q^7)_\infty^3.\end{aligned}$$

Similarly, to find the representation of $y(q)$ as an infinite product we let $a=-q$ and $b=-q^6$ and we get

$$f(-q,-q^6)=(q;q^7)_\infty(q^6;q^7)_\infty(q^7;q^7)_\infty. \tag{3.49}$$

Therefore, substituting equations (3.47) and (3.49) into equation (3.2) we obtain

$$
\begin{aligned}
y(q) &= q\frac{f(-q,-q^6)}{f(-q^2,-q^5)}(q^7;q^7)_\infty^3 \\
&= q\frac{(q;q^7)_\infty(q^6;q^7)_\infty(q^7;q^7)_\infty}{(q^2;q^7)_\infty^2(q^5;q^7)_\infty^2(q^7;q^7)_\infty^2}(q^7;q^7)_\infty^3.
\end{aligned}
$$

Finally, we find the representations of $z(q)$ as an infinite product. We substitute equations (3.48) and (3.49) into (3.3) and we obtain

$$
\begin{aligned}
z(q) &= \frac{f(-q^3,-q^4)}{f(-q,-q^6)}(q^7;q^7)_\infty^3 \\
&= \frac{(q^3;q^7)_\infty(q^4;q^7)_\infty(q^7;q^7)_\infty}{(q;q^7)_\infty^2(q^6;q^7)_\infty^2(q^7;q^7)_\infty^2}(q^7;q^7)_\infty^3.
\end{aligned}
$$

Then taking the product of $x(q)$, $y(q)$, and $z(q)$ we obtain

$$
\begin{aligned}
x(q)(q)z(q) &= \left(q\frac{(q^2;q^7)_\infty(q^5;q^7)_\infty(q^7;q^7)_\infty}{(q^3;q^7)_\infty^2(q^4;q^7)_\infty^2(q^7;q^7)_\infty^2}(q^7;q^7)_\infty^3\right) \\
&\quad \times \left(q\frac{(q;q^7)_\infty(q^6;q^7)_\infty(q^7;q^7)_\infty}{(q^2;q^7)_\infty^2(q^5;q^7)_\infty^2(q^7;q^7)_\infty^2}(q^7;q^7)_\infty^3\right) \\
&\quad \times \left(\frac{(q^3;q^7)_\infty(q^4;q^7)_\infty(q^7;q^7)_\infty}{(q;q^7)_\infty^2(q^6;q^7)_\infty^2(q^7;q^7)_\infty^2}\right) \\
&= q^2\frac{(q^2;q^7)_\infty(q^5;q^7)_\infty(q;q^7)_\infty(q^6;q^7)_\infty(q^3;q^7)_\infty(q^4;q^7)_\infty(q^7;q^7)_\infty^{12}}{(q^3;q^7)_\infty^2(q^4;q^7)_\infty^2(q^2;q^7)_\infty^2(q^5;q^7)_\infty^2(q;q^7)_\infty^2(q^6;q^7)_\infty^2(q^7;q^7)_\infty^6} \\
&= q^2\frac{(q;q)_\infty(q^7;q^7)_\infty^{11}}{(q;q)_\infty^2(q^7;q^7)_\infty^4} \\
&= q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}.
\end{aligned}
$$

□

Theorem 3.1.10.

$$
\sum_{n=0}^{\infty} p(7n+5)q^n = 7q\,(q;q)_\infty^2\left(\frac{2\,\alpha_3}{\alpha_2^4} - \frac{\alpha_1^2}{\alpha_2^3\alpha_3^2} - \frac{2\,q^2\,\alpha_2}{\alpha_1^4} + \frac{q^2\,\alpha_2^2}{\alpha_1^2\alpha_3^3} + \frac{q\,\alpha_3^2}{\alpha_1^3\alpha_2^2} - \frac{4\,q}{\alpha_1\alpha_2\alpha_3} + \frac{2\,q\,\alpha_1}{\alpha_3^4}\right).
$$

Proof. In the last theorem we showed that

$$x(q)y(q)z(q) = q^2 \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty},$$

substituting q with $q^{1/7}$ we obtain

$$x(q^{1/7})y(q^{1/7})z(q^{1/7}) = q^{2/7} \frac{(q;q)_\infty^7}{(q^{1/7};q^{1/7})_\infty}.$$

Note that the denominator is the generating function for partitions in the variable $q^{1/7}$. Then,

$$\begin{aligned} x(q^{1/7})y(q^{1/7})z(q^{1/7}) &= q^{2/7}(q;q)_\infty^7 \sum_{n=0}^{\infty} p(n)q^{n/7} \\ &= (q;q)_\infty^7 \sum_{n=0}^{\infty} p(n)q^{(n+2)/7} \\ &= (q;q)_\infty^7 \sum_{n=2}^{\infty} p(n-2)q^{n/7}. \end{aligned}$$

By dividing both sides by $(q;q)_\infty^7$ we obtain

$$\frac{x(q^{1/7})y(q^{1/7})z(q^{1/7})}{(q;q)_\infty^7} = \sum_{n=2}^{\infty} p(n-2)q^{n/7}. \tag{3.50}$$

We now decompose equation (3.50) into

$$\begin{aligned} \frac{x(q^{1/7})y(q^{1/7})z(q^{1/7})}{(q;q)_\infty^7} = &\sum_{n=0}^{\infty} p(7n+5)q^{7n/7} \\ &+ \sum_{n=0}^{\infty} p(7n+6)q^{(7n+1)/7} \\ &+ \sum_{n=0}^{\infty} p(7n)q^{(7n+2)/7} \\ &+ \sum_{n=0}^{\infty} p(7n+1)q^{(7n+3)/7} \\ &+ \sum_{n=0}^{\infty} p(7n+2)q^{(7n+4)/7} \end{aligned}$$

$$+\sum_{n=0}^{\infty} p(7n+3)q^{(7n+5)/7}$$
$$+\sum_{n=0}^{\infty} p(7n+4)q^{(7n+6)/7}.$$

Therefore,

$$\left(\frac{x(q^{1/7})y(q^{1/7})z(q^{1/7})}{(q;q)_\infty^7}\right)_0 = \frac{1}{(q;q)_\infty^7}\left(x(q^{1/7})y(q^{1/7})z(q^{1/7})\right)_0 = \sum_{n=0}^{\infty} p(7n+5)q^n.$$

In Theorem 3.1.8, we showed that

$$\left(x(q^{1/7})y(q^{1/7})z(q^{1/7})\right)_0 = 7q\,(q;q)_\infty^9\left(\frac{2\,\alpha_3}{\alpha_2^4} - \frac{\alpha_1^2}{\alpha_2^3\alpha_3^2} - \frac{2\,q^2\,\alpha_2}{\alpha_1^4} + \frac{q^2\,\alpha_2^2}{\alpha_1^2\alpha_3^3} + \frac{q\,\alpha_3^2}{\alpha_1^3\alpha_2^2} - \frac{4\,q}{\alpha_1\alpha_2\alpha_3} + \frac{2\,q\,\alpha_1}{\alpha_3^4}\right).$$

Thus,

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7q\,(q;q)_\infty^2\left(\frac{2\,\alpha_3}{\alpha_2^4} - \frac{\alpha_1^2}{\alpha_2^3\alpha_3^2} - \frac{2\,q^2\,\alpha_2}{\alpha_1^4} + \frac{q^2\,\alpha_2^2}{\alpha_1^2\alpha_3^3} + \frac{q\,\alpha_3^2}{\alpha_1^3\alpha_2^2} - \frac{4\,q}{\alpha_1\alpha_2\alpha_3} + \frac{2\,q\,\alpha_1}{\alpha_3^4}\right).$$

□

Theorem 3.1.11.

$$p(7n+5) \equiv 0 \ (mod\ 7).$$

Proof. This follows Theorem 3.1.10 and Theorem 1.0.2 and the last Theorem. □

3.2 Other Congruences

In this section, we demonstrate two things. First the coefficients of $x^2(q^{1/7})$, $y^2(q^{1/7})$ and $z^2(q^{1/7})$ for terms with index congruent to n (mod 7), where $n = 3,5,6$, are congruent to 0 (mod 14). Second, that the coefficients of the j dissection, where $j = 0,1,\cdots,6$, for the different combinations of $x(q^{1/7})$, $y(q^{1/7})$ and $z(q^{1/7})$ are congruent to $0 \pmod{n}$ where $n = 7, 21, 49$. But first we need to introduce the following theorem.

Theorem 3.2.1. *Klein's Relation*

$$q\frac{f^2(-q,-q^6)}{f(-q^3,-q^4)}+\frac{f^2(-q^3,-q^4)}{f(-q^2,-q^5)}-\frac{f^2(-q^2,-q^5)}{f(-q,-q^6)}=0.$$

Proof. The proof of this theorem is given in [17, 20, p.300]. □

The next theorems will show that the coefficients of $x^2(q^{1/7})$, $y^2(q^{1/7})$ and $z^2(q^{1/7})$ for terms with index congruent to $n \pmod 7$, where $n = 3, 5, 6$, are congruent to $0 \pmod{14}$.

Theorem 3.2.2. *Let*

$$x^2(q) = \sum_{n=0}^{\infty} a_n q^n, \qquad a_n \in \mathbb{Z}.$$

Then a_{7n+3} is congruent to $0 \pmod{14}$.

Proof. In Theorem 3.1.7, we found the representation for $x(q^{1/7})$ and by taking the second power we obtained

$$\begin{aligned} x^2(q^{1/7}) = (q;q)_\infty^6 \Bigg(& -\frac{2q^{2/7}\alpha_3}{\alpha_2^3} + q^{8/7}\left(\frac{11\alpha_3^2}{\alpha_1^2\alpha_2^2} - \frac{10}{\alpha_2\alpha_3} + \frac{\alpha_1^2}{\alpha_3^4}\right) + q^{5/7}\left(\frac{6\alpha_3^2}{\alpha_1\alpha_2^3} - \frac{2\alpha_1}{\alpha_2^2\alpha_3}\right) \\ & + q^{9/7}\left(\frac{20}{\alpha_1\alpha_2} - \frac{4\alpha_1}{\alpha_3^3}\right) + q^{10/7}\left(\frac{10}{\alpha_3^2} - \frac{24\alpha_3}{\alpha_1^2\alpha_2}\right) + \frac{14q^{15/7}\alpha_2}{\alpha_1^3} \\ & + q^{11/7}\left(\frac{6\alpha_3^2}{\alpha_1^3\alpha_2} - \frac{16}{\alpha_1\alpha_3}\right) + q^{12/7}\left(\frac{23}{\alpha_1^2} - \frac{2\alpha_2}{\alpha_3^3}\right) - \frac{4q^{17/7}\alpha_2^2}{\alpha_1^3\alpha_3} \\ & + q^{13/7}\left(\frac{8\alpha_2}{\alpha_1\alpha_3^2} - \frac{18\alpha_3}{\alpha_1^3}\right) + q\left(\frac{2\alpha_1}{\alpha_2\alpha_3^2} - \frac{12\alpha_3}{\alpha_1\alpha_2^2}\right) + \frac{5q^{6/7}}{\alpha_2^2} + \frac{q^{2/7}\alpha_3^2}{\alpha_2^4} \\ & + q^{16/7}\left(\frac{\alpha_2^2}{\alpha_1^2\alpha_3^2} - \frac{4\alpha_2\alpha_3}{\alpha_1^4}\right) + \frac{4q^{18/7}\alpha_2^2}{\alpha_1^4} + q^2\left(\frac{\alpha_3^2}{\alpha_1^4} - \frac{14\alpha_2}{\alpha_1^2\alpha_3}\right)\Bigg). \end{aligned} \tag{3.51}$$

Then, by dissecting the right side of equation (3.51), we obtained

$$
\begin{aligned}
\left(x^2(q^{1/7})\right)_0 &= (q;q)_\infty^6 \left(q\left(\frac{2\alpha_1}{\alpha_2\alpha_3^2} - \frac{12\alpha_3}{\alpha_1\alpha_2^2}\right) + q^2\left(\frac{\alpha_3^2}{\alpha_1^4} - \frac{14\alpha_2}{\alpha_1^2\alpha_3}\right)\right),\\
\left(x^2(q^{1/7})\right)_1 &= (q;q)_\infty^6 \left(q^{8/7}\left(\frac{11\alpha_3^2}{\alpha_1^2\alpha_2^2} - \frac{10}{\alpha_2\alpha_3} + \frac{\alpha_1^2}{\alpha_3^4}\right) + \frac{14q^{15/7}\alpha_2}{\alpha_1^3}\right),\\
\left(x^2(q^{1/7})\right)_2 &= (q;q)_\infty^6 \Bigg(-\frac{2q^{2/7}\alpha_3}{\alpha_2^3} + \frac{q^{2/7}\alpha_3^2}{\alpha_2^4} + q^{9/7}\left(\frac{20}{\alpha_1\alpha_2} - \frac{4\alpha_1}{\alpha_3^3}\right)\\
&\qquad + q^{16/7}\left(\frac{\alpha_2^2}{\alpha_1^2\alpha_3^2} - \frac{4\alpha_2\alpha_3}{\alpha_1^4}\right)\Bigg),\\
\left(x^2(q^{1/7})\right)_3 &= (q;q)_\infty^6 \left(q^{10/7}\left(\frac{10}{\alpha_3^2} - \frac{24\alpha_3}{\alpha_1^2\alpha_2}\right) - \frac{4q^{17/7}\alpha_2^2}{\alpha_1^3\alpha_3}\right)
\end{aligned}
$$

We dissect the right side of equation (3.51) by selecting the monomials that contributed to a corresponding dissection class. Using this process, we were able to find the dissection for each congruence class. In particular, simplifying the 3 dissection class by factoring we get

$$
\frac{2q^{10/7}(q;q)_\infty^6\left(-\frac{12\alpha_3^3}{\alpha_1^2\alpha_2} - \frac{2\alpha_2^2\alpha_3 q}{\alpha_1^3} + 5\right)}{\alpha_3^2}
$$

By applying Klein's relation, recall Theorem 3.2.1, we obtained

$$
\begin{aligned}
&\left(\frac{14q^{10/7}\left(\alpha_1^2\alpha_2 - 2\alpha_3^3\right)}{\alpha_1^2\alpha_2\alpha_3^2} + \left(-\frac{4q^{10/7}}{\alpha_1^2\alpha_3}\right)\left(\frac{\alpha_1^2}{\alpha_3} - \frac{\alpha_3^2}{\alpha_2} + \frac{\alpha_2^2 q}{\alpha_1}\right)\right)(q;q)_\infty^6\\
&= \frac{14q^{10/7}(q;q)_\infty^6\left(\alpha_1^2\alpha_2 - 2\alpha_3^3\right)}{\alpha_1^2\alpha_2\alpha_3^2}.
\end{aligned}
$$

The coefficients have a common factor of 14 . □

The proofs for the following theorems are similar to the proof of Theorem 3.2.2.

Theorem 3.2.3. *Let*

$$
y^2(q) = \sum_{n=0}^{\infty} a_n q^n, \qquad a_n \in \mathbb{Z}.
$$

Then a_{7n+6} *is congruent to* $0 \pmod{14}$.

Theorem 3.2.4. *Let*

$$z^2(q) = \sum_{n=0}^{\infty} a_n q^n, \qquad a_n \in \mathbb{Z}.$$

Then a_{7n+5} *is congruent to* $0 \pmod{14}$

Finally, the following theorems will demonstrate that the coefficients of the j dissection for the different products of $x(q^{1/7})$, $y(q^{1/7})$ and $z(q^{1/7})$ are congruent to $0 \pmod{n}$ where $n = 7, 21,$ or 49.

Theorem 3.2.5. *Let*

$$x^3(q) = \sum_{n=0}^{\infty} a_n q^n, \qquad a_n \in \mathbb{Z}.$$

Then a_{7n+1} *is congruent to* $0 \pmod{7}$ *and* a_{7n+4} *is congruent to* $0 \pmod{21}$.

Proof. First, we show that the coefficients of the one dissection of $x^3(q^{1/7})$ are congruent to $0 \pmod{7}$. Using Theorem 3.1.7, we can write $x(q^{1/7})$ a combination of the quotients (3.1)-(3.12) and taking the third power we get

$$\begin{aligned} x^3(q^{1/7}) = (q;q)_\infty^9 \Bigg(& q^{12/7}\left(\frac{45\alpha_3^3}{\alpha_1^3\alpha_2^3} - \frac{120}{\alpha_1\alpha_2^2} + \frac{21\alpha_1}{\alpha_2\alpha_3^3} - \frac{\alpha_1^3}{\alpha_3^6}\right) + q^{6/7}\left(\frac{9\alpha_3^3}{\alpha_1\alpha_2^5} - \frac{3\alpha_1}{\alpha_2^4}\right) \\ & + q^{11/7}\left(-\frac{93\alpha_3^2}{\alpha_1^2\alpha_2^3} + \frac{54}{\alpha_2^2\alpha_3} - \frac{3\alpha_1^2}{\alpha_2\alpha_3^4}\right) + q^{8/7}\left(\frac{6\alpha_1}{\alpha_2^3\alpha_3} - \frac{27\alpha_3^2}{\alpha_1\alpha_2^4}\right) \\ & + q^{13/7}\left(\frac{180\alpha_3}{\alpha_1^2\alpha_2^2} - \frac{72}{\alpha_2\alpha_3^2} + \frac{6\alpha_1^2}{\alpha_3^5}\right) + q^{10/7}\left(\frac{66\alpha_3}{\alpha_1\alpha_2^3} - \frac{15\alpha_1}{\alpha_2^2\alpha_3^2}\right) \\ & + q^{20/7}\left(-\frac{45\alpha_3^2}{\alpha_1^5} + \frac{144\alpha_2}{\alpha_1^3\alpha_3} - \frac{3\alpha_2^2}{\alpha_1\alpha_3^4}\right) + q^{19/7}\left(\frac{183\alpha_3}{\alpha_1^4} - \frac{105\alpha_2}{\alpha_1^2\alpha_3^2}\right) \\ & + q^{2}\left(-\frac{153\alpha_3^2}{\alpha_1^3\alpha_2^2} + \frac{159}{\alpha_1\alpha_2\alpha_3} - \frac{21\alpha_1}{\alpha_3^4}\right) + q^{22/7}\left(\frac{75\alpha_2\alpha_3}{\alpha_1^5} - \frac{45\alpha_2^2}{\alpha_1^3\alpha_3^2}\right) \\ & + q^{16/7}\left(\frac{240\alpha_3}{\alpha_1^3\alpha_2} - \frac{117}{\alpha_1\alpha_3^2} + \frac{3\alpha_1\alpha_2}{\alpha_3^5}\right) + q^{23/7}\left(\frac{63\alpha_2^2}{\alpha_1^4\alpha_3} - \frac{6\alpha_2\alpha_3^2}{\alpha_1^6}\right) \end{aligned} \tag{3.52}$$

$$+q^{17/7}\left(-\frac{123\alpha_3^2}{\alpha_1^4\alpha_2}+\frac{201}{\alpha_1^2\alpha_3}-\frac{18\alpha_2}{\alpha_3^4}\right)+q^{24/7}\left(\frac{\alpha_2^3}{\alpha_1^3\alpha_3^3}-\frac{48\alpha_2^2}{\alpha_1^5}\right)$$
$$+q^3\left(\frac{\alpha_3^3}{\alpha_1^6}-\frac{144\alpha_2}{\alpha_1^4}+\frac{18\alpha_2^2}{\alpha_1^2\alpha_3^3}\right)+q^{25/7}\left(\frac{12\alpha_2^2\alpha_3}{\alpha_1^6}-\frac{6\alpha_2^3}{\alpha_1^4\alpha_3^2}\right)$$
$$+q^{9/7}\left(\frac{30\alpha_3^3}{\alpha_1^2\alpha_2^4}-\frac{31}{\alpha_2^3}+\frac{3\alpha_1^2}{\alpha_2^2\alpha_3^3}\right)+\frac{q^{3/7}\alpha_3^3}{\alpha_2^6}-\frac{3\,q^{5/7}\alpha_3^2}{\alpha_2^5}$$
$$+q^{15/7}+\left(\frac{30\alpha_3^3}{\alpha_1^4\alpha_2^2}-\frac{231}{\alpha_1^2\alpha_2}+\frac{53}{\alpha_3^3}\right)-\frac{8\,q^{27/7}\alpha_2^3}{\alpha_1^6}+\frac{9\,q\,\alpha_3}{\alpha_2^4}$$
$$\left.+q^{18/7}\left(\frac{9\alpha_3^3}{\alpha_1^5\alpha_2}-\frac{246}{\alpha_1^3}+\frac{54\alpha_2}{\alpha_1\alpha_3^3}\right)+\frac{12\,q^{26/7}\alpha_2^3}{\alpha_1^5\alpha_3}\right).$$

We dissected the right side of equation (3.52), by selecting monomials that contribute to the desire dissection, we obtained

$$\left(x^3(q^{1/7})\right)_0=(q;q)_\infty^9\left(\frac{9\,q\,\alpha_3}{\alpha_2^4}+q^2\left(-\frac{153\alpha_3^2}{\alpha_1^3\alpha_2^2}+\frac{159}{\alpha_1\alpha_2\alpha_3}-\frac{21\alpha_1}{\alpha_3^4}\right)\right.$$
$$\left.+q^3\left(\frac{\alpha_3^3}{\alpha_1^6}-\frac{144\alpha_2}{\alpha_1^4}+\frac{18\alpha_2^2}{\alpha_1^2\alpha_3^3}\right)\right),$$
$$\left(x^3(q^{1/7})\right)_1=(q;q)_\infty^9\left(q^{8/7}\left(\frac{6\alpha_1}{\alpha_2^3\alpha_3}-\frac{27\alpha_3^2}{\alpha_1\alpha_2^4}\right)+q^{15/7}\left(\frac{30\alpha_3^3}{\alpha_1^4\alpha_2^2}-\frac{231}{\alpha_1^2\alpha_2}+\frac{53}{\alpha_3^3}\right)\right.$$
$$\left.+q^{22/7}\left(\frac{75\alpha_2\alpha_3}{\alpha_1^5}-\frac{45\alpha_2^2}{\alpha_1^3\alpha_3^2}\right)\right),$$
$$\left(x^3(q^{1/7})\right)_2=(q;q)_\infty^9\left(q^{9/7}\left(\frac{30\alpha_3^3}{\alpha_1^2\alpha_2^4}-\frac{31}{\alpha_2^3}+\frac{3\alpha_1^2}{\alpha_2^2\alpha_3^3}\right)+q^{16/7}\left(\frac{240\alpha_3}{\alpha_1^3\alpha_2}-\frac{117}{\alpha_1\alpha_3^2}+\frac{3\alpha_1\alpha_2}{\alpha_3^5}\right)\right.$$
$$\left.+q^{23/7}\left(\frac{63\alpha_2^2}{\alpha_1^4\alpha_3}-\frac{6\alpha_2\alpha_3^2}{\alpha_1^6}\right)\right).$$

The missing dissections can be found by using the same process above. Hence, we have that one dissection is

$$-\frac{q^{8/7}(q;q)_\infty^9}{\alpha_1^5\alpha_2^4\alpha_3^3}\Big(-6\alpha_2\alpha_3^2\alpha_1^6+27\alpha_3^5\alpha_1^4+45\alpha_2^6\alpha_3\alpha_1^2q^2-75\,q^2\,\alpha_2^5\alpha_3^4-53\,q\,\alpha_2^4\alpha_1^5$$
$$+231\,q\,\alpha_2^3\alpha_3^3\alpha_1^3-30\,q\,\alpha_2^2\alpha_3^6\alpha_1\Big).$$

By applying Klein's relation we obtain

$$\begin{aligned}
&(q;q)_\infty^9\Bigg(\frac{7\,q^{8/7}\left(-54\alpha_1^3\alpha_3^5+51\alpha_1^5\alpha_2\alpha_3^2+15\alpha_2^2\alpha_3^6q+14\alpha_1^4\alpha_2^4q\right)}{\alpha_1^4\alpha_2^4\alpha_3^3}\\
&\qquad+\left(-\frac{3q^{8/7}\left(117\alpha_3^2\alpha_1^3+5\,q\,\alpha_2^2\left(3\alpha_1^2\alpha_2-5\alpha_3^3\right)\right)}{\alpha_1^4\alpha_2^3\alpha_3^2}\right)\left(\frac{\alpha_1^2}{\alpha_3}-\frac{\alpha_3^2}{\alpha_2}+\frac{\alpha_2^2q}{\alpha_1}\right)\Bigg)\\
&=\frac{7\,q^{8/7}(q;q)_\infty^9\left(-54\alpha_1^3\alpha_3^5+51\alpha_1^5\alpha_2\alpha_3^2+15\alpha_2^2\alpha_3^6q+14\alpha_1^4\alpha_2^4q\right)}{\alpha_1^4\alpha_2^4\alpha_3^3}.
\end{aligned}$$

Therefore, we have shown that the coefficients for the third dissection of $x^3(q^{1/7})$ is congruent to $0 \pmod 7$. Similarly, repeating the same process we can prove that the four dissection of $x^3(q^{1/7})$ is congruent to $0 \pmod{21}$. □

The following theorems can be proven by applying the same process used to prove the above theorem.

Theorem 3.2.6. *Let*

$$y^3(q)=\sum_{n=0}^{\infty}a_nq^n,\qquad a_n\in\mathbb{Z}.$$

Then a_{7n+1} *is congruent to* $0 \pmod{21}$ *and* a_{7n+2} *is congruent to* $0 \pmod 7$.

Theorem 3.2.7. *Let*

$$z^3(q)=\sum_{n=0}^{\infty}a_nq^n,\qquad a_n\in\mathbb{Z}.$$

Then a_{7n+2} *is congruent to* $0 \pmod{21}$ *and* a_{7n+4} *is congruent to* $0 \pmod 7$.

Theorem 3.2.8. *Let*

$$x^2(q)y(q)=\sum_{n=0}^{\infty}a_nq^n,\qquad a_n\in\mathbb{Z}.$$

Then a_{7n} *is congruent to* $0 \pmod{49}$ *and* a_{7n+5} *is congruent to* $0 \pmod 7$.

Theorem 3.2.9. *Let*

$$y^2(q)x(q) = \sum_{n=0}^{\infty} a_n q^n, \qquad a_n \in \mathbb{Z}.$$

Then a_{7n} *and* a_{7n+2} *are congruent to* $0 \pmod 7$.

Theorem 3.2.10. *Let*

$$x^2(q)z(q) = \sum_{n=0}^{\infty} a_n q^n, \qquad a_n \in \mathbb{Z}.$$

Then a_{7n} *and* a_{7n+1} *are congruent to* $0 \pmod 7$.

Theorem 3.2.11. *Let*

$$z^2(q)x(q) = \sum_{n=0}^{\infty} a_n q^n, \qquad a_n \in \mathbb{Z}.$$

Then a_{7n} *and* a_{7n+6} *are congruent to* $0 \pmod 7$.

Theorem 3.2.12. *Let*

$$y^2(q)z(q) = \sum_{n=0}^{\infty} a_n q^n, \qquad a_n \in \mathbb{Z}.$$

Then a_{7n} *is congruent to* $0 \pmod{49}$ *and* a_{7n+3} *is congruent to* $0 \pmod 7$.

Theorem 3.2.13. *Let*

$$z^2(q)y(q) = \sum_{n=0}^{\infty} a_n q^n, \qquad a_n \in \mathbb{Z}.$$

Then a_{7n} *and* a_{7n+4} *are congruent to* $0 \pmod 7$.

Using a new approach, we were able to prove Ramanujan's congruences with a parallel justification. By applying the same techniques, we proved new congruences for a more general class of series that include those used to prove Ramanujan's congruences.

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APPENDIX A

APPENDIX A

LEVEL FIVE LINEARLY INDEPENDENT CODE

$$A[\mathrm{q\_}] = \frac{q^{1/5}\left(\prod_{n=0}^{50}\left(1-q^{5n+1}\right)\right)\left(\prod_{n=0}^{50}\left(1-q^{5n+4}\right)\right)}{\left(\prod_{n=1}^{50}(1-q^{n})\right)^{3/5}}\left(\prod_{n=1}^{50}\left(1-q^{5n}\right)\right);$$

$$B[\mathrm{q\_}] = \frac{\left(\prod_{n=0}^{50}\left(1-q^{5n+2}\right)\right)\left(\prod_{n=0}^{50}\left(1-q^{5n+3}\right)\right)}{\left(\prod_{n=1}^{50}(1-q^{n})\right)^{3/5}}\left(\prod_{n=1}^{50}\left(1-q^{5n}\right)\right);$$

$\text{Series}\left[c_1(A[q])^5+c_2(B[q])^5,\{q,0,3\}\right]$

$c_2+(c_1+3c_2)\,q+(-2c_1+4c_2)\,q^2+(4c_1+2c_2)\,q^3+O[q]^4$

$\text{Solve}\left[\left\{\text{SeriesCoefficient}\left[c_1(A[q])^5+c_2(B[q])^5,\{q,0,1\}\right]==0,\right.\right.$

$\left.\left.\text{SeriesCoefficient}\left[c_1(A[q])^5+c_2(B[q])^5,\{q,0,2\}\right]==0\right\},\{c_1,c_2\}\right]$

$\{\{c_1\to 0,c_2\to 0\}\}$

$\text{SeriesCoefficient}\left[c_1(A[q])^5+c_2(B[q])^5,\{q,0,1\}\right]$

$\text{SeriesCoefficient}\left[c_1(A[q])^5+c_2(B[q])^5,\{q,0,2\}\right]$

c_1+3c_2

$-2c_1+4c_2$

$G[\mathrm{q\_}]=c_1(A[q])^5+c_2(B[q])^5+c_3\,(A[q])^4*B[q]+c_4\,(A[q])^3*(B[q])^2$

$+c_5\,(A[q])^2*(B[q])^3+c_6\,A[q]*(B[q])^4/.q\to q^5;$

$\text{Solve}[\{\text{SeriesCoefficient}[G[q],\{q,0,1\}]==0,\text{SeriesCoefficient}[G[q],\{q,0,2\}]==0,$

$\text{SeriesCoefficient}[G[q],\{q,0,3\}]==0,\text{SeriesCoefficient}[G[q],\{q,0,4\}]==0,$

$\text{SeriesCoefficient}[G[q],\{q,0,5\}]==0,\text{SeriesCoefficient}[G[q],\{q,0,6\}]==0,$

$\text{SeriesCoefficient}[G[q],\{q,0,7\}]==0,\text{SeriesCoefficient}[G[q],\{q,0,8\}]==0,$

$\text{SeriesCoefficient}[G[q],\{q,0,9\}]==0,\text{SeriesCoefficient}[G[q],\{q,0,10\}]==0,$

$\text{SeriesCoefficient}[G[q],\{q,0,11\}]==0,\text{SeriesCoefficient}[G[q],\{q,0,12\}]==0\},$

$\{c_1,c_2,c_3,c_4,c_5,\ c_6\}]$

$\{\{c_1 \to 0, c_2 \to 0, c_3 \to 0, c_4 \to 0, c_5 \to 0, c_6 \to 0\}\}$

SeriesCoefficient$[G[q], \{q, 0, 1\}]$

SeriesCoefficient$[G[q], \{q, 0, 2\}]$

SeriesCoefficient$[G[q], \{q, 0, 3\}]$

SeriesCoefficient$[G[q], \{q, 0, 4\}]$

SeriesCoefficient$[G[q], \{q, 0, 5\}]$

SeriesCoefficient$[G[q], \{q, 0, 6\}]$

SeriesCoefficient$[G[q], \{q, 0, 7\}]$

SeriesCoefficient$[G[q], \{q, 0, 8\}]$

SeriesCoefficient$[G[q], \{q, 0, 9\}]$

SeriesCoefficient$[G[q], \{q, 0, 10\}]$

SeriesCoefficient$[G[q], \{q, 0, 11\}]$

SeriesCoefficient$[G[q], \{q, 0, 12\}]$

c_6

c_5

c_4

c_3

$c_1 + 3c_2$

$2c_6$

c_5

0

$-c_3$

$-2c_1 + 4c_2$

$2c_6$

c_5

APPENDIX B

APPENDIX B

LEVEL FIVE COEFFICIENTS

$$A[\text{q\_}]=\frac{q^{1/5}\left(\prod_{n=0}^{50}\left(1-q^{5n+1}\right)\right)\left(\prod_{n=0}^{50}\left(1-q^{5n+4}\right)\right)}{\left(\prod_{n=1}^{50}(1-q^n)\right)^{3/5}}\left(\prod_{n=1}^{50}\left(1-q^{5n}\right)\right);$$

$$B[\text{q\_}]=\frac{\left(\prod_{n=0}^{50}\left(1-q^{5n+2}\right)\right)\left(\prod_{n=0}^{50}\left(1-q^{5n+3}\right)\right)}{\left(\prod_{n=1}^{50}(1-q^n)\right)^{3/5}}\left(\prod_{n=1}^{50}\left(1-q^{5n}\right)\right);$$

$$G[\text{q\_}]=c_1(A[q])^5+c_2(B[q])^5+c_3\ (A[q])^4*B[q]+c_4\ (A[q])^3*(B[q])^2$$

$$+c_5\ (A[q])^2*(B[q])^3+c_6\ A[q]*(B[q])^4;$$

$$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{1}{5}\right\}\right]==\text{SeriesCoefficient}\left[\left(A\left[q^{\frac{1}{5}}\right]\right)^5,\left\{q,0,\tfrac{1}{5}\right\}\right]$$

$$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{2}{5}\right\}\right]==\text{SeriesCoefficient}\left[\left(A\left[q^{\frac{1}{5}}\right]\right)^5,\left\{q,0,\tfrac{2}{5}\right\}\right]$$

$$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{3}{5}\right\}\right]==\text{SeriesCoefficient}\left[\left(A\left[q^{\frac{1}{5}}\right]\right)^5,\left\{q,0,\tfrac{3}{5}\right\}\right]$$

$$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{4}{5}\right\}\right]==\text{SeriesCoefficient}\left[\left(A\left[q^{\frac{1}{5}}\right]\right)^5,\left\{q,0,\tfrac{4}{5}\right\}\right]$$

$$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{5}{5}\right\}\right]==\text{SeriesCoefficient}\left[\left(A\left[q^{\frac{1}{5}}\right]\right)^5,\left\{q,0,\tfrac{5}{5}\right\}\right]$$

$$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{6}{5}\right\}\right]==\text{SeriesCoefficient}\left[\left(A\left[q^{\frac{1}{5}}\right]\right)^5,\left\{q,0,\tfrac{6}{5}\right\}\right]$$

$$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{7}{5}\right\}\right]==\text{SeriesCoefficient}\left[\left(A\left[q^{\frac{1}{5}}\right]\right)^5,\left\{q,0,\tfrac{7}{5}\right\}\right]$$

$$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{8}{5}\right\}\right]==\text{SeriesCoefficient}\left[\left(A\left[q^{\frac{1}{5}}\right]\right)^5,\left\{q,0,\tfrac{8}{5}\right\}\right]$$

$$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{9}{5}\right\}\right]==\text{SeriesCoefficient}\left[\left(A\left[q^{\frac{1}{5}}\right]\right)^5,\left\{q,0,\tfrac{9}{5}\right\}\right]$$

$$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{10}{5}\right\}\right]==\text{SeriesCoefficient}\left[\left(A\left[q^{\frac{1}{5}}\right]\right)^5,\left\{q,0,\tfrac{10}{5}\right\}\right]$$

$$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{11}{5}\right\}\right]==\text{SeriesCoefficient}\left[\left(A\left[q^{\frac{1}{5}}\right]\right)^5,\left\{q,0,\tfrac{11}{5}\right\}\right]$$

$$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{12}{5}\right\}\right]==\text{SeriesCoefficient}\left[\left(A\left[q^{\frac{1}{5}}\right]\right)^5,\left\{q,0,\tfrac{12}{5}\right\}\right]$$

$c_6 == 1$

$c_5 == -2$

$c_4 == 4$

$c_3 == -3$

$c_1 + 3c_2 == 1$

$2c_6 == 2$

$c_5 == -2$

True

$-c_3 == 3$

$-2c_1 + 4c_2 == -2$

$2c_6 == 2$

$c_5 == -2$

$$\text{SeriesCoefficient}\left[G[q], \{q, 0, \tfrac{1}{5}\}\right] == \text{SeriesCoefficient}\left[\left(B\left[q^{\frac{1}{5}}\right]\right)^5, \{q, 0, \tfrac{1}{5}\}\right]$$

$$\text{SeriesCoefficient}\left[G[q], \{q, 0, \tfrac{2}{5}\}\right] == \text{SeriesCoefficient}\left[\left(B\left[q^{\frac{1}{5}}\right]\right)^5, \{q, 0, \tfrac{2}{5}\}\right]$$

$$\text{SeriesCoefficient}\left[G[q], \{q, 0, \tfrac{3}{5}\}\right] == \text{SeriesCoefficient}\left[\left(B\left[q^{\frac{1}{5}}\right]\right)^5, \{q, 0, \tfrac{3}{5}\}\right]$$

$$\text{SeriesCoefficient}\left[G[q], \{q, 0, \tfrac{4}{5}\}\right] == \text{SeriesCoefficient}\left[\left(B\left[q^{\frac{1}{5}}\right]\right)^5, \{q, 0, \tfrac{4}{5}\}\right]$$

$$\text{SeriesCoefficient}\left[G[q], \{q, 0, \tfrac{5}{5}\}\right] == \text{SeriesCoefficient}\left[\left(B\left[q^{\frac{1}{5}}\right]\right)^5, \{q, 0, \tfrac{5}{5}\}\right]$$

$$\text{SeriesCoefficient}\left[G[q], \{q, 0, \tfrac{6}{5}\}\right] == \text{SeriesCoefficient}\left[\left(B\left[q^{\frac{1}{5}}\right]\right)^5, \{q, 0, \tfrac{6}{5}\}\right]$$

$$\text{SeriesCoefficient}\left[G[q], \{q, 0, \tfrac{7}{5}\}\right] == \text{SeriesCoefficient}\left[\left(B\left[q^{\frac{1}{5}}\right]\right)^5, \{q, 0, \tfrac{7}{5}\}\right]$$

$$\text{SeriesCoefficient}\left[G[q], \{q, 0, \tfrac{8}{5}\}\right] == \text{SeriesCoefficient}\left[\left(B\left[q^{\frac{1}{5}}\right]\right)^5, \{q, 0, \tfrac{8}{5}\}\right]$$

$$\text{SeriesCoefficient}\left[G[q], \{q, 0, \tfrac{9}{5}\}\right] == \text{SeriesCoefficient}\left[\left(B\left[q^{\frac{1}{5}}\right]\right)^5, \{q, 0, \tfrac{9}{5}\}\right]$$

$$\text{SeriesCoefficient}\left[G[q], \{q, 0, \tfrac{10}{5}\}\right] == \text{SeriesCoefficient}\left[\left(B\left[q^{\frac{1}{5}}\right]\right)^5, \{q, 0, \tfrac{10}{5}\}\right]$$

$$\text{SeriesCoefficient}\left[G[q], \{q, 0, \tfrac{11}{5}\}\right] == \text{SeriesCoefficient}\left[\left(B\left[q^{\frac{1}{5}}\right]\right)^5, \{q, 0, \tfrac{11}{5}\}\right]$$

$$\text{SeriesCoefficient}\left[G[q], \{q, 0, \tfrac{12}{5}\}\right] == \text{SeriesCoefficient}\left[\left(B\left[q^{\frac{1}{5}}\right]\right)^5, \{q, 0, \tfrac{12}{5}\}\right]$$

$c_6 == 3$

$c_5 == 4$

$c_4 == 2$

$c_3 == 1$

$c_1 + 3c_2 == 3$

$2c_6 == 6$

$c_5 == 4$

True

$-c_3 == -1$

$-2c_1 + 4c_2 == 4$

$2c_6 == 6$

$c_5 == 4$

APPENDIX C

APPENDIX C

LEVEL SEVEN LINEARLY INDEPENDENT CODE

$f[\text{h\_},\text{m\_}]=\sum_{n=-100}^{100}h^{n(n+1)/2}m^{n(n-1)/2};$

$$G[\text{q\_}]=k_1q^{2/7}\frac{f[-q^3,-q^4]}{f[-q,-q^6]f[-q^2,-q^5]}+k_2q^{3/7}\frac{1}{f[-q,-q^6]}+k_3q^{5/7}\frac{1}{f[-q^2,-q^5]}+k_4q^{6/7}\frac{1}{f[-q^3,-q^4]}$$
$$+k_5q\frac{f[-q^2,-q^5]}{f[-q^3,-q^4]^2}+k_6q^{9/7}\frac{f[-q,-q^6]}{f[-q^3,-q^4]^2}+k_7q\frac{f[-q,-q^6]}{f[-q^2,-q^5]^2}+k_8q^{4/7}\frac{f[-q^3,-q^4]}{f[-q^2,-q^5]^2}+k_9\frac{f[-q^3,-q^4]}{f[-q,-q^6]^2}$$
$$+k_{10}q^{1/7}\frac{f[-q^2,-q^5]}{f[-q,-q^6]^2}+k_{11}q^{8/7}\frac{f[-q,-q^6]}{f[-q^2,-q^5]f[-q^3,-q^4]}+k_{12}q^{4/7}\frac{f[-q^2,-q^5]}{f[-q,-q^6]f[-q^3,-q^4]}/.q\to q^7;$$

Solve[{SeriesCoefficient[G[q], {q, 0, 1}] == 0, SeriesCoefficient[G[q], {q, 0, 2}] == 0,

SeriesCoefficient[G[q], {q, 0, 3}] == 0}, {k_5, k_7, k_9}]

$\{\{k_5\to 0, k_7\to 0, k_9\to 0\}\}$

SeriesCoefficient[G[q], {q, 0, 1}]

SeriesCoefficient[G[q], {q, 0, 2}]

SeriesCoefficient[G[q], {q, 0, 3}]

$k_5+k_7+2k_9$

$-k_7+3k_9$

$-k_5+2k_7+3k_9$

Solve[{SeriesCoefficient[G[q], {q, 0, 1}] == 0, SeriesCoefficient[G[q], {q, 0, 2}] == 0,

SeriesCoefficient[G[q], {q, 0, 3}] == 0, SeriesCoefficient[G[q], {q, 0, 4}] == 0,

SeriesCoefficient[G[q], {q, 0, 5}] == 0, SeriesCoefficient[G[q], {q, 0, 6}] == 0,

SeriesCoefficient[G[q], {q, 0, 7}] == 0, SeriesCoefficient[G[q], {q, 0, 8}] == 0,

SeriesCoefficient[G[q], {q, 0, 9}] == 0, SeriesCoefficient[G[q], {q, 0, 10}] == 0,

SeriesCoefficient[G[q], {q, 0, 11}] == 0, SeriesCoefficient[G[q], {q, 0, 12}] == 0,

SeriesCoefficient[G[q], {q, 0, 13}] == 0, SeriesCoefficient[G[q], {q, 0, 14}] == 0,

SeriesCoefficient[$G[q]$, {q, 0, 21}] == 0}, {$k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, k_{11}, k_{12}$}]

$\{\{k_1 \to 0, k_2 \to 0, k_3 \to 0, k_4 \to 0, k_5 \to 0, k_6 \to 0, k_7 \to 0, k_8 \to 0, k_9 \to 0, k_{10} \to 0,$

$k_{11} \to 0, k_{12} \to 0\}\}$

SeriesCoefficient[$G[q]$, {q, 0, 1}]

SeriesCoefficient[$G[q]$, {q, 0, 2}]

SeriesCoefficient[$G[q]$, {q, 0, 3}]

SeriesCoefficient[$G[q]$, {q, 0, 4}]

SeriesCoefficient[$G[q]$, {q, 0, 5}]

SeriesCoefficient[$G[q]$, {q, 0, 6}]

SeriesCoefficient[$G[q]$, {q, 0, 7}]

SeriesCoefficient[$G[q]$, {q, 0, 8}]

SeriesCoefficient[$G[q]$, {q, 0, 9}]

SeriesCoefficient[$G[q]$, {q, 0, 10}]

SeriesCoefficient[$G[q]$, {q, 0, 11}]

SeriesCoefficient[$G[q]$, {q, 0, 12}]

SeriesCoefficient[$G[q]$, {q, 0, 13}]

SeriesCoefficient[$G[q]$, {q, 0, 14}]

SeriesCoefficient[$G[q]$, {q, 0, 15}]

SeriesCoefficient[$G[q]$, {q, 0, 16}]

SeriesCoefficient[$G[q]$, {q, 0, 17}]

SeriesCoefficient[$G[q]$, {q, 0, 18}]

SeriesCoefficient[$G[q]$, {q, 0, 19}]

SeriesCoefficient[$G[q]$, {q, 0, 20}]

SeriesCoefficient[$G[q]$, {q, 0, 21}]

SeriesCoefficient[$G[q]$, {q, 0, 22}]

k_{10}

k_1

k_2

$k_8 + k_{12}$

k_3

k_4

$k_5 + k_7 + 2k_9$

$2k_{10} + k_{11}$

$k_1 + k_6$

k_2

k_{12}

0

0

$-k_7 + 3k_9$

$2k_{10} - k_{11}$

$2k_1 - k_6$

k_2

$2k_8$

k_3

0

$-k_5 + 2k_7 + 3k_9$

$2k_{10} + k_{11}$

APPENDIX D

APPENDIX D

LEVEL SEVEN COEFFICIENTS

$f[\text{h\_},\text{m\_}]=\sum_{n=-100}^{100} h^{n(n+1)/2} m^{n(n-1)/2};$

$G[\text{q\_}]=k_1 q^{2/7}\frac{f[-q^3,-q^4]}{f[-q,-q^6]f[-q^2,-q^5]}+k_2 q^{3/7}\frac{1}{f[-q,-q^6]}+k_3 q^{5/7}\frac{1}{f[-q^2,-q^5]}+k_4 q^{6/7}\frac{1}{f[-q^3,-q^4]}$
$+k_5 q\frac{f[-q^2,-q^5]}{f[-q^3,-q^4]^2}+k_6 q^{9/7}\frac{f[-q,-q^6]}{f[-q^3,-q^4]^2}+k_7 q\frac{f[-q,-q^6]}{f[-q^2,-q^5]^2}+k_8 q^{4/7}\frac{f[-q^3,-q^4]}{f[-q^2,-q^5]^2}+k_9\frac{f[-q^3,-q^4]}{f[-q,-q^6]^2}$
$+k_{10} q^{1/7}\frac{f[-q^2,-q^5]}{f[-q,-q^6]^2}+k_{11} q^{8/7}\frac{f[-q,-q^6]}{f[-q^2,-q^5]f[-q^3,-q^4]}+k_{12} q^{4/7}\frac{f[-q^2,-q^5]}{f[-q,-q^6]f[-q^3,-q^4]};$

$x[\text{q\_}]=q\frac{f[-q^2,-q^5]}{f[-q^3,-q^4]^2}\left(\prod_{n=1}^{50}\left(1-q^{7n}\right)\right)^3;$

$y[\text{q\_}]=q\frac{f[-q,-q^6]}{f[-q^2,-q^5]^2}\left(\prod_{n=1}^{50}\left(1-q^{7n}\right)\right)^3;$

$z[\text{q\_}]=\frac{f[-q^3,-q^4]}{f[-q,-q^6]^2}\left(\prod_{n=1}^{50}\left(1-q^{7n}\right)\right)^3;$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{1}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{1}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{2}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{2}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{3}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{3}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{4}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{4}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{5}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{5}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{6}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{6}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{7}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{7}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{8}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{8}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{9}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{9}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{10}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{10}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{11}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{11}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{12}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{12}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\tfrac{13}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\tfrac{13}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{14}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{14}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{15}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{15}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{16}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{16}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{17}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{17}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{18}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{18}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{19}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{19}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{20}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{20}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{21}{7}\right\}\right]==\text{SeriesCoefficient}\left[x\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{21}{7}\right\}\right]$

$k_{10} == 1$

$k_1 == 0$

$k_2 == -1$

$k_8 + k_{12} == 2$

$k_3 == 2$

$k_4 == -3$

$k_5 + k_7 + 2k_9 == 1$

$2k_{10} + k_{11} == 3$

$k_1 + k_6 == -2$

$k_2 == -1$

$k_{12} == 3$

True

True

$-k_7 + 3k_9 == 0$

$2k_{10} - k_{11} == 1$

$2k_1 - k_6 == 2$

$k_2 == -1$

$2k_8 == -2$

$k_3 == 2$

True

$-k_5+2k_7+3k_9==-1$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{1}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{1}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{2}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{2}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{3}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{3}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{4}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{4}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{5}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{5}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{6}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{6}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{7}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{7}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{8}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{8}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{9}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{9}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{10}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{10}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{11}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{11}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{12}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{12}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{13}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{13}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{14}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{14}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{15}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{15}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{16}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{16}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{17}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{17}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{18}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{18}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{19}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{19}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{20}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{20}{7}\right\}\right]$

$\textbf{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{21}{7}\right\}\right]==\textbf{SeriesCoefficient}\left[y\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{21}{7}\right\}\right]$

$k_{10}==1$

$k_1==-1$

$k_2==2$

$k_8+k_{12}==-2$

$k_3 == 3$

$k_4 == -1$

$k_5 + k_7 + 2k_9 == 1$

$2k_{10} + k_{11} == -1$

$k_1 + k_6 == 0$

$k_2 == 2$

$k_{12} == 0$

True

True

$-k_7 + 3k_9 == -1$

$2k_{10} - k_{11} == 5$

$2k_1 - k_6 == -3$

$k_2 == 2$

$2k_8 == -4$

$k_3 == 3$

True

$-k_5 + 2k_7 + 3k_9 == 2$

SeriesCoefficient $\left[G[q], \left\{q, 0, \frac{1}{7}\right\}\right]$ **==SeriesCoefficient** $\left[z\left[q^{\frac{1}{7}}\right], \left\{q, 0, \frac{1}{7}\right\}\right]$

SeriesCoefficient $\left[G[q], \left\{q, 0, \frac{2}{7}\right\}\right]$ **==SeriesCoefficient** $\left[z\left[q^{\frac{1}{7}}\right], \left\{q, 0, \frac{2}{7}\right\}\right]$

SeriesCoefficient $\left[G[q], \left\{q, 0, \frac{3}{7}\right\}\right]$ **==SeriesCoefficient** $\left[z\left[q^{\frac{1}{7}}\right], \left\{q, 0, \frac{3}{7}\right\}\right]$

SeriesCoefficient $\left[G[q], \left\{q, 0, \frac{4}{7}\right\}\right]$ **==SeriesCoefficient** $\left[z\left[q^{\frac{1}{7}}\right], \left\{q, 0, \frac{4}{7}\right\}\right]$

SeriesCoefficient $\left[G[q], \left\{q, 0, \frac{5}{7}\right\}\right]$ **==SeriesCoefficient** $\left[z\left[q^{\frac{1}{7}}\right], \left\{q, 0, \frac{5}{7}\right\}\right]$

SeriesCoefficient $\left[G[q], \left\{q, 0, \frac{6}{7}\right\}\right]$ **==SeriesCoefficient** $\left[z\left[q^{\frac{1}{7}}\right], \left\{q, 0, \frac{6}{7}\right\}\right]$

SeriesCoefficient $\left[G[q], \left\{q, 0, \frac{7}{7}\right\}\right]$ **==SeriesCoefficient** $\left[z\left[q^{\frac{1}{7}}\right], \left\{q, 0, \frac{7}{7}\right\}\right]$

SeriesCoefficient $\left[G[q], \left\{q, 0, \frac{8}{7}\right\}\right]$ **==SeriesCoefficient** $\left[z\left[q^{\frac{1}{7}}\right], \left\{q, 0, \frac{8}{7}\right\}\right]$

SeriesCoefficient $\left[G[q], \left\{q, 0, \frac{9}{7}\right\}\right]$ **==SeriesCoefficient** $\left[z\left[q^{\frac{1}{7}}\right], \left\{q, 0, \frac{9}{7}\right\}\right]$

SeriesCoefficient $\left[G[q], \left\{q, 0, \frac{10}{7}\right\}\right]$ **==SeriesCoefficient** $\left[z\left[q^{\frac{1}{7}}\right], \left\{q, 0, \frac{10}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{11}{7}\right\}\right]==\text{SeriesCoefficient}\left[z\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{11}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{12}{7}\right\}\right]==\text{SeriesCoefficient}\left[z\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{12}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{13}{7}\right\}\right]==\text{SeriesCoefficient}\left[z\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{13}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{14}{7}\right\}\right]==\text{SeriesCoefficient}\left[z\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{14}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{15}{7}\right\}\right]==\text{SeriesCoefficient}\left[z\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{15}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{16}{7}\right\}\right]==\text{SeriesCoefficient}\left[z\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{16}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{17}{7}\right\}\right]==\text{SeriesCoefficient}\left[z\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{17}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{18}{7}\right\}\right]==\text{SeriesCoefficient}\left[z\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{18}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{19}{7}\right\}\right]==\text{SeriesCoefficient}\left[z\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{19}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{20}{7}\right\}\right]==\text{SeriesCoefficient}\left[z\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{20}{7}\right\}\right]$

$\text{SeriesCoefficient}\left[G[q],\left\{q,0,\frac{21}{7}\right\}\right]==\text{SeriesCoefficient}\left[z\left[q^{\frac{1}{7}}\right],\left\{q,0,\frac{21}{7}\right\}\right]$

$k_{10} == 2$

$k_1 == 3$

$k_2 == 3$

$k_8 + k_{12} == 2$

$k_3 == 1$

$k_4 == 2$

$k_5 + k_7 + 2k_9 == 2$

$2k_{10} + k_{11} == 4$

$k_1 + k_6 == 4$

$k_2 == 3$

$k_{12} == 1$

True

True

$-k_7 + 3k_9 == 3$

$2k_{10} - k_{11} == 4$

$2k_1 - k_6 == 5$

$k_2 == 3$

$2k_8 == 2$

$k_3 == 1$

True

$-k_5 + 2k_7 + 3k_9 == 3$

APPENDIX E

APPENDIX E

LEVEL SEVEN MATHEMATICA CODE

«Notation`

Symbolize [__]

$$F[c_1:\_,c_2:\_,c_3:\_,c_4:\_,c_5:\_,c_6:\_,c_7:\_,c_8:\_,c_9:\_,c_{10}:\_,c_{11}:\_,c_{12}:\_]=$$
$$\left(c_1 q^{2/7}\frac{f[-q^3,-q^4]}{f[-q,-q^6]f[-q^2,-q^5]}+c_2 q^{3/7}\frac{1}{f[-q,-q^6]}+c_3 q^{5/7}\frac{1}{f[-q^2,-q^5]}+c_4 q^{6/7}\frac{1}{f[-q^3,-q^4]}\right.$$
$$+c_5 q\frac{f[-q^2,-q^5]}{f[-q^3,-q^4]^2}+c_6 q^{9/7}\frac{f[-q,-q^6]}{f[-q^3,-q^4]^2}+c_7 q\frac{f[-q,-q^6]}{f[-q^2,-q^5]^2}+c_8 q^{4/7}\frac{f[-q^3,-q^4]}{f[-q^2,-q^5]^2}+c_9\frac{f[-q^3,-q^4]}{f[-q,-q^6]^2}$$
$$\left.+c_{10} q^{1/7}\frac{f[-q^2,-q^5]}{f[-q,-q^6]^2}+c_{11} q^{8/7}\frac{f[-q,-q^6]}{f[-q^2,-q^5]f[-q^3,-q^4]}+c_{12} q^{4/7}\frac{f[-q^2,-q^5]}{f[-q,-q^6]f[-q^3,-q^4]}\right)(\mathrm{qq}^3);$$

$$x[0]=F[a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}]/.\{a_1\to 0,a_2\to 0,a_3\to 0,$$
$$a_4\to 0,a_5\to 1,a_6\to 0,a_7\to 0,a_8\to 0,a_9\to 0,a_{10}\to 0,a_{11}\to 0,a_{12}\to 0\};$$

$$x[1]=F[a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}]/.\{a_1\to 0,a_2\to 0,a_3\to 0,$$
$$a_4\to 0,a_5\to 0,a_6\to 0,a_7\to 0,a_8\to 0,a_9\to 0,a_{10}\to 1,a_{11}\to 1,a_{12}\to 0\};$$

$$x[2]=F[a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}]/.\{a_1\to 0,a_2\to 0,a_3\to 0,$$
$$a_4\to 0,a_5\to 0,a_6\to -2,a_7\to 0,a_8\to 0,a_9\to 0,a_{10}\to 0,a_{11}\to 0,a_{12}\to 0\};$$

$$x[3]=F[a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}]/.\{a_1\to 0,a_2\to -1,a_3\to 0,$$
$$a_4\to 0,a_5\to 0,a_6\to 0,a_7\to 0,a_8\to 0,a_9\to 0,a_{10}\to 0,a_{11}\to 0,a_{12}\to 0\};$$

$$x[4]=F[a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}]/.\{a_1\to 0,a_2\to 0,a_3\to 0,$$
$$a_4\to 0,a_5\to 0,a_6\to 0,a_7\to 0,a_8\to -1,a_9\to 0,a_{10}\to 0,a_{11}\to 0,a_{12}\to 3\};$$

$$x[5]=F[a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}]/.\{a_1\to 0,a_2\to 0,a_3\to 2,$$
$$a_4\to 0,a_5\to 0,a_6\to 0,a_7\to 0,a_8\to 0,a_9\to 0,a_{10}\to 0,a_{11}\to 0,a_{12}\to 0\};$$

$$x[6]=F[a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}]/.\{a_1\to 0,a_2\to 0,a_3\to 0,$$

$a_4 \to -3, a_5 \to 0, a_6 \to 0, a_7 \to 0, a_8 \to 0, a_9 \to 0, a_{10} \to 0, a_{11} \to 0, a_{12} \to 0\}$;

$y[0] = F[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}] /. \{a_1 \to 0, a_2 \to 0, a_3 \to 0,$
$a_4 \to 0, a_5 \to 0, a_6 \to 0, a_7 \to 1, a_8 \to 0, a_9 \to 0, a_{10} \to 0, a_{11} \to 0, a_{12} \to 0\}$;
$y[1] = F[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}] /. \{a_1 \to 0, a_2 \to 0, a_3 \to 0,$
$a_4 \to 0, a_5 \to 0, a_6 \to 0, a_7 \to 0, a_8 \to 0, a_9 \to 0, a_{10} \to 1, a_{11} \to -3, a_{12} \to 0\}$;
$y[2] = F[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}] /. \{a_1 \to -1, a_2 \to 0, a_3 \to 0,$
$a_4 \to 0, a_5 \to 0, a_6 \to 1, a_7 \to 0, a_8 \to 0, a_9 \to 0, a_{10} \to 0, a_{11} \to 0, a_{12} \to 0\}$;
$y[3] = F[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}] /. \{a_1 \to 0, a_2 \to 2, a_3 \to 0,$
$a_4 \to 0, a_5 \to 0, a_6 \to 0, a_7 \to 0, a_8 \to 0, a_9 \to 0, a_{10} \to 0, a_{11} \to 0, a_{12} \to 0\}$;
$y[4] = F[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}] /. \{a_1 \to 0, a_2 \to 0, a_3 \to 0,$
$a_4 \to 0, a_5 \to 0, a_6 \to 0, a_7 \to 0, a_8 \to -2, a_9 \to 0, a_{10} \to 0, a_{11} \to 0, a_{12} \to 0\}$;
$y[5] = F[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}] /. \{a_1 \to 0, a_2 \to 0, a_3 \to 3,$
$a_4 \to 0, a_5 \to 0, a_6 \to 0, a_7 \to 0, a_8 \to 0, a_9 \to 0, a_{10} \to 0, a_{11} \to 0, a_{12} \to 0\}$;
$y[6] = F[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}] /. \{a_1 \to 0, a_2 \to 0, a_3 \to 0,$
$a_4 \to -1, a_5 \to 0, a_6 \to 0, a_7 \to 0, a_8 \to 0, a_9 \to 0, a_{10} \to 0, a_{11} \to 0, a_{12} \to 0\}$;

$z[0] = F[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}] /. \{a_1 \to 0, a_2 \to 0, a_3 \to 0,$
$a_4 \to 0, a_5 \to 0, a_6 \to 0, a_7 \to 0, a_8 \to 0, a_9 \to 1, a_{10} \to 0, a_{11} \to 0, a_{12} \to 0\}$;
$z[1] = F[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}] /. \{a_1 \to 0, a_2 \to 0, a_3 \to 0,$
$a_4 \to 0, a_5 \to 0, a_6 \to 0, a_7 \to 0, a_8 \to 0, a_9 \to 0, a_{10} \to 2, a_{11} \to 0, a_{12} \to 0\}$;
$z[2] = F[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}] /. \{a_1 \to 3, a_2 \to 0, a_3 \to 0,$
$a_4 \to 0, a_5 \to 0, a_6 \to 1, a_7 \to 0, a_8 \to 0, a_9 \to 0, a_{10} \to 0, a_{11} \to 0, a_{12} \to 0\}$;
$z[3] = F[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}] /. \{a_1 \to 0, a_2 \to 3, a_3 \to 0,$
$a_4 \to 0, a_5 \to 0, a_6 \to 0, a_7 \to 0, a_8 \to 0, a_9 \to 0, a_{10} \to 0, a_{11} \to 0, a_{12} \to 0\}$;
$z[4] = F[a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}] /. \{a_1 \to 0, a_2 \to 0, a_3 \to 0,$
$a_4 \to 0, a_5 \to 0, a_6 \to 0, a_7 \to 0, a_8 \to 1, a_9 \to 0, a_{10} \to 0, a_{11} \to 0, a_{12} \to 1\}$;

$z[5]=F\left[a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}\right]/.\{a_1\to 0,a_2\to 0,a_3\to 1,$

$a_4\to 0,a_5\to 0,a_6\to 0,a_7\to 0,a_8\to 0,a_9\to 0,a_{10}\to 0,a_{11}\to 0,a_{12}\to 0\};$

$z[6]=F\left[a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}\right]/.\{a_1\to 0,a_2\to 0,a_3\to 0,$

$a_4\to 2,a_5\to 0,a_6\to 0,a_7\to 0,a_8\to 0,a_9\to 0,a_{10}\to 0,a_{11}\to 0,a_{12}\to 0\};$

K[s_, n_]:=Flatten[Table[Permutations[IntegerPartitions[s, {n}, {0, 1, 2, 3, 4, 5, 6}][[z]]],

{z, 1, Length[IntegerPartitions[s, {n}, {0, 1, 2, 3, 4, 5, 6}]]}], 1];

Two[q_, e_, d_]:=Expand[Sum[q[K[d, 2][[i]][[1]]]e[K[d, 2][[i]][[2]]],

{i, 1, Length[K[d, 2]]}]];

Three[q_, e_, r_, d_]:=Sum[q[K[d, 3][[i]][[1]]]e[K[d, 3][[i]][[2]]]r[K[d, 3][[i]][[3]]],

{i, 1, Length[K[d, 3]]}];

Simplify[Three[x, y, z, 0] + Three[x, y, z, 7] + Three[x, y, z, 14]]

$$7q\mathrm{qq}^9\left(\frac{2f[-q^2,-q^5]}{f[-q,-q^6]^4}-\frac{2q^2f[-q,-q^6]}{f[-q^3,-q^4]^4}+\frac{qf[-q^2,-q^5]^2}{f[-q,-q^6]^2f[-q^3,-q^4]^3}+\frac{q^2f[-q,-q^6]^2}{f[-q^2,-q^5]^3f[-q^3,-q^4]^2}-\frac{4q}{f[-q,-q^6]f[-q^2,-q^5]f[-q^3,-q^4]}+\frac{2qf[-q^3,-q^4]}{f[-q^2,-q^5]^4}-\frac{f[-q^3,-q^4]^2}{f[-q,-q^6]^3f[-q^2,-q^5]^2}\right)$$

$$\mathbf{7q\mathrm{qq}^9\left(\frac{2f[-q^2,-q^5]}{f[-q,-q^6]^4}-\frac{2q^2f[-q,-q^6]}{f[-q^3,-q^4]^4}+\frac{qf[-q^2,-q^5]^2}{f[-q,-q^6]^2f[-q^3,-q^4]^3}+\frac{q^2f[-q,-q^6]^2}{f[-q^2,-q^5]^3f[-q^3,-q^4]^2}-\frac{4q}{f[-q,-q^6]f[-q^2,-q^5]f[-q^3,-q^4]}+\frac{2qf[-q^3,-q^4]}{f[-q^2,-q^5]^4}-\frac{f[-q^3,-q^4]^2}{f[-q,-q^6]^3f[-q^2,-q^5]^2}\right)/.f[-q^3,-q^4]\to\alpha_1}$$

$$\mathbf{/.f[-q,-q^6]\to\alpha_2/.f[-q^2,-q^5]\to\alpha_3}$$

$$7q\mathrm{qq}^9\left(-\frac{2q^2\alpha_2}{\alpha_1^4}+\frac{2q\alpha_1}{\alpha_3^4}+\frac{q^2\alpha_2^2}{\alpha_1^2\alpha_3^3}-\frac{\alpha_1^2}{\alpha_2^3\alpha_3^2}-\frac{4q}{\alpha_1\alpha_2\alpha_3}+\frac{2\alpha_3}{\alpha_2^4}+\frac{q\alpha_3^2}{\alpha_1^3\alpha_2^2}\right)$$

Do[Print[Simplify[Two[$x, x, 0+n$] + Two[$x, x, 7+n$] + Two[$x, x, 14+n$],

$$\mathbf{\frac{f[-q^3,-q^4]^2}{f[-q^2,-q^5]}-\frac{f[-q^2,-q^5]^2}{f[-q,-q^6]}+\frac{qf[-q,-q^6]^2}{f[-q^3,-q^4]}==0\Big]\Big],\{n,0,6\}\Big]}$$

$$\frac{q\mathrm{qq}^6\left(qf[-q^2,-q^5]^4-\frac{26f[-q^2,-q^5]^3f[-q^3,-q^4]^3}{f[-q,-q^6]^2}+\frac{16f[-q^3,-q^4]^5}{f[-q,-q^6]}\right)}{f[-q^2,-q^5]^2f[-q^3,-q^4]^4}$$

$$\frac{q^{8/7}\mathrm{qq}^6\left(-\frac{24f[-q^2,-q^5]^3}{f[-q,-q^6]}+\frac{25f[-q^2,-q^5]^6}{f[-q,-q^6]^2f[-q^3,-q^4]^2}+f[-q^3,-q^4]^2\right)}{f[-q^2,-q^5]^4}$$

$$\frac{q^{2/7}\mathrm{qq}^6\left(\frac{26f[-q^2,-q^5]^5}{f[-q,-q^6]^4}-\frac{4qf[-q^2,-q^5]^6}{f[-q,-q^6]^2f[-q^3,-q^4]^3}-5qf[-q^3,-q^4]-\frac{25f[-q^2,-q^5]^2f[-q^3,-q^4]^2}{f[-q,-q^6]^3}\right)}{f[-q^2,-q^5]^3}$$

$$\frac{14q^{10/7}\mathrm{qq}^6\left(1-\frac{2f[-q^2,-q^5]^3}{f[-q,-q^6]f[-q^3,-q^4]^2}\right)}{f[-q^2,-q^5]^2}$$

$$\frac{2q^{4/7}\text{qq}^6\left(-\frac{f[-q^2,-q^5]^2}{f[-q,-q^6]^3}+\frac{5qf[-q^2,-q^5]^3}{f[-q,-q^6]f[-q^3,-q^4]^3}-\frac{10q}{f[-q^3,-q^4]}\right)}{f[-q^2,-q^5]}$$

$$-\frac{q^{5/7}\text{qq}^6\left(-29f[-q^2,-q^5]^5+2qf[-q,-q^6]^4f[-q^3,-q^4]+25f[-q,-q^6]f[-q^2,-q^5]^2f[-q^3,-q^4]^2\right)}{f[-q,-q^6]^3f[-q^2,-q^5]^3f[-q^3,-q^4]}$$

$$q^{6/7}\text{qq}^6\left(\frac{5}{f[-q,-q^6]^2}-\frac{18qf[-q^2,-q^5]}{f[-q^3,-q^4]^3}+\frac{8qf[-q,-q^6]}{f[-q^2,-q^5]^2f[-q^3,-q^4]}\right)$$

Do[Print[Simplify[Three[$x,x,x,0+n$] + Three[$x,x,x,7+n$] + Three[$x,x,x,14+n$],

$$\boldsymbol{\frac{f[-q^3,-q^4]^2}{f[-q^2,-q^5]}-\frac{f[-q^2,-q^5]^2}{f[-q,-q^6]}+\frac{qf[-q,-q^6]^2}{f[-q^3,-q^4]}==0\Big]\Big],\{n,0,6\}\Big]}$$

$$q\text{qq}^9\Big(\frac{-298qf[-q,-q^6]^2f[-q^2,-q^5]^6f[-q^3,-q^4]^2+330f[-q^2,-q^5]^5f[-q^3,-q^4]^5-39qf[-q,-q^6]^4f[-q^3,-q^4]^6}{f[-q,-q^6]^4f[-q^2,-q^5]^4f[-q^3,-q^4]^5}$$
$$+\frac{f[-q,-q^6]\left(qf[-q^2,-q^5]^9-321f[-q^2,-q^5]^2f[-q^3,-q^4]^7\right)}{f[-q,-q^6]^4f[-q^2,-q^5]^4f[-q^3,-q^4]^5}\Big)$$

$$\frac{7q^{8/7}\text{qq}^9\left(14q+\frac{15qf[-q^2,-q^5]^6}{f[-q,-q^6]^2f[-q^3,-q^4]^4}-\frac{54f[-q^2,-q^5]^5}{f[-q,-q^6]^4f[-q^3,-q^4]}+\frac{51f[-q^2,-q^5]^2f[-q^3,-q^4]}{f[-q,-q^6]^3}\right)}{f[-q^2,-q^5]^3}$$

$$q^{9/7}\text{qq}^9\Big(\frac{339f[-q^2,-q^5]^8f[-q^3,-q^4]^3-520f[-q,-q^6]f[-q^2,-q^5]^5f[-q^3,-q^4]^5+3qf[-q,-q^6]^5f[-q^3,-q^4]^6}{f[-q,-q^6]^4f[-q^2,-q^5]^5f[-q^3,-q^4]^5}$$
$$+\frac{f[-q,-q^6]^2\left(-6qf[-q^2,-q^5]^9+183f[-q^2,-q^5]^2f[-q^3,-q^4]^7\right)}{f[-q,-q^6]^4f[-q^2,-q^5]^5f[-q^3,-q^4]^5}\Big)$$

$$-q^{3/7}\text{qq}^9\Big(\frac{19q^2f[-q,-q^6]^7f[-q^3,-q^4]^3-488f[-q^2,-q^5]^7f[-q^3,-q^4]^3+752f[-q,-q^6]f[-q^2,-q^5]^4f[-q^3,-q^4]^5}{f[-q,-q^6]^6f[-q^2,-q^5]^4f[-q^3,-q^4]^3}$$
$$+\frac{f[-q,-q^6]^2\left(171qf[-q^2,-q^5]^8-265f[-q^2,-q^5]f[-q^3,-q^4]^7\right)}{f[-q,-q^6]^6f[-q^2,-q^5]^4f[-q^3,-q^4]^3}\Big)$$

$$21q^{11/7}\text{qq}^9\Big(\frac{-17f[-q^2,-q^5]^6f[-q^3,-q^4]^3+18f[-q,-q^6]f[-q^2,-q^5]^3f[-q^3,-q^4]^5}{f[-q,-q^6]^3f[-q^2,-q^5]^4f[-q^3,-q^4]^5}$$
$$+\frac{f[-q,-q^6]^2\left(qf[-q^2,-q^5]^7-3f[-q^3,-q^4]^7\right)+f[-q,-q^6]^2\left(qf[-q^2,-q^5]^7-3f[-q^3,-q^4]^7\right)}{f[-q,-q^6]^3f[-q^2,-q^5]^4f[-q^3,-q^4]^5}\Big)$$

$$-q^{5/7}\text{qq}^9\Big(\frac{435f[-q^2,-q^5]^8f[-q^3,-q^4]^3-570f[-q,-q^6]f[-q^2,-q^5]^5f[-q^3,-q^4]^5+qf[-q,-q^6]^5f[-q^3,-q^4]^6}{f[-q,-q^6]^5f[-q^2,-q^5]^6f[-q^3,-q^4]^3}$$
$$+\frac{-6f[-q,-q^6]^2\left(40qf[-q^2,-q^5]^9-23f[-q^2,-q^5]^2f[-q^3,-q^4]^7\right)}{f[-q,-q^6]^5f[-q^2,-q^5]^6f[-q^3,-q^4]^3}\Big)$$

$$q^{6/7}\text{qq}^9\Big(\frac{394f[-q^2,-q^5]^8f[-q^3,-q^4]^3-615f[-q,-q^6]f[-q^2,-q^5]^5f[-q^3,-q^4]^5+9qf[-q,-q^6]^5f[-q^3,-q^4]^6}{f[-q,-q^6]^5f[-q^2,-q^5]^5f[-q^3,-q^4]^4}$$
$$+\frac{f[-q,-q^6]^2\left(-53qf[-q^2,-q^5]^9+227f[-q^2,-q^5]^2f[-q^3,-q^4]^7\right)}{f[-q,-q^6]^5f[-q^2,-q^5]^5f[-q^3,-q^4]^4}\Big)$$

BIOGRAPHICAL SKETCH

Mayra C. Huerta was born in Mercedes, Texas on June 22, 1991. She graduated from Weslaco High School in 2010 and from the University of Texas Pan-American (UTPA) in 2015 with a Bachelor of Science in Applied Mathematics.

During her undergraduate and graduate education, she was a member of the Experimental Algebra and Geometry Lab (EAGL). As an EAGL member, she spent time doing community outreach by going to local high schools and presenting "Your Teachers are Lying to You!". She participated in the following presentations, Donna and Donna North high schools, San Benito Idea school, and PSJA Memorial high school, where she took the lead on the presentations at the San Benito Idea, Donna and Donna North High schools. She also participated in the special presentations for students at the freshmen orientation and for teachers from different local schools. During her graduate education, she was vice-president of her university's chapter of the Society for Industrial and Applied Mathematics (SIAM). Under her vice-presidency, SIAM chapter organized events that brought awareness to the STEM fields to the community. One example is the calculus review session that was held before every exam at the University of Texas Rio Grande Valley (UTRGV). Under her vice-presidency, she made SIAM the host organization for the pi day celebration, which is held every year. As an EAGL and SIAM members, she participated in a panel discussion in the nationally recognized Hispanic Engineering Science and Technology (HESTEC) held at UTRGV for a week once a year.

Mayra received a Masters of Science in Mathematics from UTRGV in May 2017. Her plans are to get a lecture position at a community college to mentor and inspire students. Her current email is mhuer10@gmail.com.