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On hypergroups of order at most 6

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ON HYPERGROUPS OF ORDER AT MOST 6

A Thesis

by

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ABSTRACT

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This thesis surveys recent results on hypergroups as defined by Frédéric Marty in [3] and [4] and their relation to association schemes as presented in [5]. We show that every association scheme is a hypergroup. Then, we compile a few general results on hypergroups needed for our investigation of hypergroups with three, four and six elements. From [1] and [7], we give examples of hypergroups that do not come from finite schemes and from no scheme at all.

Our main result occurs when considering hypergroups S with six elements that have a non-normal closed subset T of order 2 with three cosets. Since such class of hypergroups is too large to be completely described, we investigate a subclass $\mathcal{S}$ determined in [7]. We found that at least four hypergroups in this class come from finite schemes. For such purposes, we use the Hanaki-Miyamoto Classification of Small Association Schemes; cf. [8].

DEDICATION

I dedicate this work to my family, who have helped me in every situation in life and who want me to excel wherever I am. Love you!

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CHAPTER I

INTRODUCTION

This thesis surveys recent results on hypergroups as defined by the French mathematician Frédéric Marty in [3] and [4] and their relation to association schemes as presented in [5]. Chapter 2 is dedicated to the definitions and notation that we will use in the work; additionally, we show that every association scheme is a hypergroup. Chapter 3 introduces hypergroups, with the emphasis on their closed subsets. Chapter 4 completely classifies hypergroups with three elements. Chapter 5 classifies non-symmetric hypergroups with four elements. Among the results in the former classification, Chapter 6 describes hypergroups which cannot be realized as finite association schemes, but are realized as infinite association schemes and even from no scheme at all. They are the *Tomaselli* and *Regensburg hypergroups*. It is interesting to see this result in connection with the fact that a finite association scheme with less than six elements must be commutative; cf. [2]. The main result of our work occurs in Chapter 7, where we classify four non-commutative hypergroups of six elements with a non-normal closed subset of two elements with three cosets in the hypergroup.

CHAPTER II

DEFINITIONS AND NOTATION

We start with the definition of a hypergroup.

Let S be a set, and let $\varphi : S \times S \to \mathcal{P}(S) \setminus \{\emptyset\}$ be a map. For any two elements s and t in S we let $\varphi(s,t) := st$. Such a map is called the *hyperoperation on* S. Notice that, in contrast with groups, the image of a pair of elements under φ produces *a non-empty subset of* S and not an element of it. Let P and Q be two non-empty subsets of S. We call the union of the sets pq, where p and q are elements of P and Q, respectively, the *complex product of P and Q*. Whenever one of the subsets of the product is a singleton $\{s\}$, we write s instead of $\{s\}$. Then, S is called a *hypergroup*, if the following conditions hold:

H1 For any three elements p, q and r in S, we have $(pq)r = p(qr)$.

H2 There exists an element e in S such that for all elements $s \in S$, we have $se = \{s\}$.

H3 For any element s in S, there exists an element s^* in S such that for any three elements p, q, and r in S with $r \in pq$, we have $p \in rq^*$ and $q \in p^*r$.

Condition H1 is the *associativity* condition. The element e in H2 is called a *neutral element of* S. In Chapter 3 we show that a hypergroup possesses exactly one neutral element. Condition H3 involves a new operation *: $S \to S$, where any three elements p, q, and r in S with $r \in pq$ satisfy $p \in rq^*$ and $q \in p^*r$. The map is called an *inverse function of* S. We also show that * is unique. From the definition of the hyperoperation on S, if $|\varphi(p,q)| = 1$, the reader can replace the * symbol with $^{-1}$ to obtain the usual $r = pq \iff p = rq^{-1} \iff q = p^{-1}r$ (and in so doing work with group terminology!). We call S *commutative* if $st = ts$ for every elements s and t in S.

Let R be a non-empty subset of S. We let $R^* := \{r \in S \mid r^* \in R\}$ and say that a non-empty subset R of S is *closed in* S if $R^*R \subseteq R$. Equivalently, R is closed if and only if for any two elements p and q in R, we have $p^*q \subseteq R$ and that occurs if and only if for each element r in R, we have $r^* \in R$ and, for any two elements p and q in R, $pq \subseteq R$. The notion of a closed subset emerges from the idea to collect all elements together such that their hyperproducts as well as their inverses are back in the set. The reader may recall such motivation from the definition of a subgroup. We call a subset R of S *symmetric* if $r = r^*$ for every element r in R.

We will now introduce concepts related to our investigation on closed subsets. An element s in S is called *thin* if $\{1\} = s^*s$. If a subset T of S contains only thin elements, we say T is *thin*. If S possesses exactly two closed subsets, namely $\{1\}$ and itself, then S is called *primitive*. Let l be an element of $S \setminus \{1\}$. Then, if $\{1, l\}$ is closed, we say that l is an *involution in* S. What does it mean that $\{1, l\}$ is closed? We can take the definition and show that $\{1, l\}^*\{1, l\} \subseteq \{1, l\}$; however, it may seem more practical to show that 1^* and l^* are back in $\{1, l\}$ (in fact, we will show that $1^* = 1$). If l is thin, we call l a *thin involution*.

Let R be a closed subset of S and pick any element s in S. Then the set $sR = \bigcup_{r \in R} sr$ is called a *left coset of* R *represented by* s and the set $Rs = \bigcup_{r \in R} rs$ is called the *right coset of* R *represented by* s. The set of all left cosets of R in S will be denoted by S/R, so $S/R = \{sR \mid s \in S\}$.

We shall now provide a framework for the concept of wreath products. Let R be a closed subset of S. For any element s in S define $R_s := \{q \in S \mid sq = \{r\}\}$. For each element s in S, we set $s^R := RsR$ and $S//R := \{q^R \mid q \in Q\}$. Suppose S_1 and S_2 are hypergroups of S. Then S is called a *wreath product* of S_1 and S_2, denoted by $S = S_1 \wr S_2$, if R is a closed subset of S such that

$$R \cong S_1 \text{ and } S//R \cong S_2,$$

with the condition that $R \subseteq R_s$ for every s in $S \setminus R$.

Let P and Q be non-empty subsets of S. We let $N_P(Q) = \{p \in P : Qp \subseteq pQ\}$. The latter set is called the *normalizer of Q in P*. If $Q = N_P(Q)$, we say that P is *normal in Q*. Now, assume R is a closed subset of S. If $S = N_S(R)$, then R is a *normal closed subset of* S. We will see that normal closed subsets are essential to classify hypergroups of order 6.

We now give the definition of an association scheme.

Let X be a set. We denote by 1_X the set of all pairs (x, x) where x is an element of X. If r is any relation on X, we define r^* to be the set of all pairs (y, z) such that (z, y) is in r. For every element x in X we let $xr := \{y \in X : (x, y) \in r\}$.

Now, let S be a partition of $X \times X$ such that $1_X \in S$ and $\emptyset \notin S$ and for every element $s \in S$ we have $s^* \in S$. Then the set S is called an *association scheme on* X if for every three relations p, q and r in S with $r \in pq$ there exists a cardinal number $a_{pqr} \neq 0$ such that for any two elements $y \in X$ and $z \in yr$ we have $|yp \cap zq^*| = a_{pqr}$. The cardinal numbers in a scheme S are called the *structure constants of* S. We refer to the number of elements in S as the *rank* of S and the number of elements in its underlying set X as the *order* of X. The letter S stands for an association scheme for the rest of this chapter.

Lemma 2.1 *Let p and q be elements of S. Then we have the following.*

(i) *The numbers a_{1pq}, δ_{pq} and a_{p1q} are equal.*

(ii) *If s is an element of S, then $a_{pqs} = a_{q^*p^*s^*}$.*

(iii) *If t and u are elements of S, then we have*

$$\sum_{s \in S} a_{pqs} a_{stu} = \sum_{s \in S} a_{psu} a_{qts}.$$

Proof. (i) Note that δ_{pq} is the Kronecker delta. The result is obtained by the definition of the structure constants of S.

(ii) Suppose y is an element in X and that $z \in ys$. Then we get $|yp \cap zq^*| = a_{pqs}$. Since $z \in ys$ we obtain that $y \in zs^*$. Thus, $|zq^* \cap yp^{**}| = a_{q^*p^*s^*}$, and, since $p^{**} = p$, it follows that $a_{pqs} = a_{q^*p^*s^*}$.

(iii) Suppose y is an element in X and that $z \in yu$. Let us count the pairs $(v, w) \in (yp \times zt^*)$ in two ways such that $w \in vq$. Together with the definition of the structure constants, we obtain our result. □

Suppose s is in S. Let $n_s := a_{ss^*1}$. We call n_s the *valency of* s. Since $s \neq \emptyset$, we have $1 \leq n_s$.

Lemma 2.2 *For any three elements* p, q, *and* r *in* S, *we have* $a_{psq}n_q = a_{qs^*p}n_p$.

Proof. We use Lemma 2.1 (iii) for s, q^* and 1 instead of q, t and u, respectively. We get that $a_{psq}n_q = n_p a_{sq^*p^*}$; cf. (i). Since by Lemma 2.1 (ii) $a_{sq^*p^*} = a_{qs^*p}$, we have $a_{psq}n_q = a_{qs^*p}n_p$. □

If p and q are relations of S, let $pq := \{r \in S \mid a_{pqr} \neq 0\}$. The latter is called the *complex multiplication on* S, and it defines a hyperoperation on S. Let P and Q be non-empty subsets of S. We let PQ be the union of the sets pq, where p and q come from their respective sets P and Q. As before, if p is a relation in S and Q is a non-empty subset of S, we write pQ instead of $\{p\}Q$. We write Pq instead of $P\{q\}$ if P is a non-empty subset of S and p a relation in S.

Lemma 2.3 *Let* p, q *and* r *be relations in* S. *Then* $(pq)r = p(qr)$.

Proof. We show only one inclusion, that is, $(pq)r \subseteq p(qr)$. The other containment is obtained in a similar fashion. Suppose s is a relation in $(pq)r$. We need to show that $s \in p(qr)$.

Since $s \in (pq)r$, there exists an element $t \in pq$ with $a_{trs} \neq 0$. Then, as $t \in pq$, we also have $a_{pqt} \neq 0$. Thus, the number $a_{pqt}a_{trs} \neq 0$.

Knowing that $a_{pqt}a_{trs} \neq 0$ and by Lemma 2.1 (iii), there exists a relation u in S such that $a_{pus}a_{qru} \neq 0$. Since $a_{pus}a_{qru}$ is non-zero, we get that $a_{pus} \neq 0$ and $a_{qru} \neq 0$. Since $a_{pus} \neq 0$ we get that $s \in pu$, and since $a_{qru} \neq 0$ we have $u \in qr$. Thus $s \in p(qr)$. □

Lemma 2.4 *Suppose* p, q *and* r *are relations in* S *such that* $r \in pq$. *Then* $p \in rq^*$ *and* $q \in p^*r$.

Proof. From $r \in pq$, we get that $a_{pqr} \neq 0$. Then, Lemma 2.2 tells us that $a_{rq^*p} \neq 0$ and therefore $p \in rq^*$. Since $a_{pqr} \neq 0$ and by Lemma 2.1 (ii), we also get that $a_{q^*p^*r^*} \neq 0$. As a result, Lemma 2.2 tells us that $a_{r^*pq^*} \neq 0$ and then using Lemma 2.1 (ii) we obtain that $a_{p^*rq} \neq 0$ so $q \in p^*r$. □

By Lemma 2.1 (i), Lemma 2.3 and Lemma 2.4, we obtain that every scheme S is a hypergroup with respect to complex multiplication. That means we can investigate a scheme S purely on its algebraic axioms, and in so doing, separate S from its geometry on its underlying set X.

Thus if we look at the class of all association schemes isomorphic to hypergroups, we will see that such schemes provide a powerful framework to classify many of the hypergroups in our research.

Throughout several chapters in our work we classify hypergroups that arise from the Hanaki-Miyamoto Classification of Association Schemes with Small Vertices. Thus, for any scheme $HM_n(m)$ in such classification, the letter n represents the counting number of the scheme and m the rank of the scheme. Isomorphism classes that are labeled with *Tbd* are yet to be determined.

CHAPTER III

THEORY OF HYPERGROUPS

In this chapter, we discuss results from the general theory of hypergroups, based on [6] and [7]. We will use the definitions from the previous chapter without further mention. For the remainder of this chapter, the set S will stand for a hypergroup.

Lemma 3.1 *Suppose e is a neutral element of S and * is an inverse function on S. Then $e \in s^*s$* for every element $s \in S$.

Proof. From H2 we know that $\{s\} = se$. Then, by H3, $e \in s^*s$. □

Lemma 3.2 *Suppose * is an inverse function on S. Then $s = s^{**}$ for every element s in S.*

Proof. By Lemma 3.1 we know that $e \in s^*s$. Then, by H3, $s \in (s^*)^*e = s^{**}e$. From Condition H2 we obtain that $s^{**}e = \{s^{**}\}$. Thus, $s = s^{**}$. □

Lemma 3.3 *Let e be a neutral element of S and suppose s is an element of S. Then $es = \{s\}$.*

Proof. Since es is a non-empty subset of S, pick any element r in es. Then, by H3, we obtain that $e \in rs^*$, which is equivalent to $s^* \in r^*e = \{r^*\}$. Thus $s^* = r^*$, and

$$s = (s^*)^* = (r^*)^* = r.$$

Since $s \in es$ for *any* s, we get that es contains exactly one element, namely s. Thus $es = \{s\}$. □

Proposition 3.4 *There is exactly one neutral element e of S.*

Proof. Suppose e and e' are any two neutral elements of S. From H2 and Lemma 3.3 we obtain $\{e\} = e'e = \{e'\}$. Thus, $e' = e$. □

Lemma 3.5 *Let * be any inverse function of S. We have $e^* = e$.*

Proof. By Lemma 3.1 we obtain $e \in e^*e$. From H2, we know that $e^*e = \{e^*\}$. Thus, $e = e^*$. $\square$

Proposition 3.4 showed us that there exists only one neutral element uniquely determined from S. We saw that such element behaves the same way as in group theory. From now on, we will denote the neutral element of S by 1. Similarly, the following lemma and proposition will show us that there is only one inverse function * of S.

Lemma 3.6 *Let p and q be elements of S. Then $1 \in pq$ if and only if $p = q^*$.*

Proof. Let us assume that 1 is in pq. Then, by H3, we obtain that $p \in 1 \cdot q^* = \{q^*\}$. Thus, $p \in \{q^*\}$ and that means $p = q^*$. Now, assume that $p = q^*$. By Lemma 3.1, we know $1 \in q^*q$. Thus, we have $1 \in pq$. $\square$

Proposition 3.7 *There exists exactly one inverse function of S.*

Proof. Suppose * and $^\#$ are two inverse functions on S and pick an element s in S. We need to show that $s^* = s^\#$. From Lemma 3.1 we know that $1 \in s^*s$. Thus, as $^\#$ is also assumed to be an inverse function on S, by Lemma 3.6 we obtain that $1 \in s^*s$ if and only if $s^* = s^\#$. Thus, we get that $^* =^\#$. $\square$

Lemma 3.8 *Suppose p and q are elements of S. If r is an element of S such that $r \in pq$, then we have $r^* \in q^*p^*$.*

Proof. By H3 we obtain $r \in pq \iff p \in rq^* \iff q^* \in r^*p \iff r^* \in q^*p^*$. $\square$

Lemma 3.9 *Suppose s is an element of S. Then we have the following:*

(i) *The sets $s \in ss$, $s \in s^*s$, $s \in ss^*$ and $s^* \in ss^*$ are pair-wise equivalent.*

(ii) *If $ss = \{s^*\}$. Then $s^*s = ss^*$.*

Proof. (i) We obtain all the results from H3.

(ii) Since $ss = \{s^*\}$, by H1 be obtain $s^*s = (ss)s = s(ss) = ss^*$. $\square$

Lemma 3.10 *Suppose p and q be elements of S. Then we have the following:*

(i) *If $qp = \{q^*\}$, then $p^*qp = \{q\}$.*

(ii) *If $p \in q^*q \cap qq \cap qq^*$, then we have $\{q, q^*\} \subseteq pq \cap pq^*$.*

Proof. (i) We have $p^*(qp) = p^*q^* = \{q\}$; cf. Lemma 3.8.

(ii) Since $p \in qq^*$, we have $q \in pq$. From $p \in qq$ we obtain that $q^* \in pq$ and $q \in pq^*$. Finally, since $p \in q^*q$, we obtain $q^* \in pq^*$. □

Lemma 3.11 *Suppose p and q are elements of a hypergroup S. Assume q is thin. Then we obtain that $|pq| = 1$.*

Proof. Let r be an element of pq. Then, by H3, we obtain that $p \in rq^*$. Since we are assuming that q is thin, we get that $pq \subseteq rq^*q = r \cdot 1 = \{r\}$. □

We shall now discuss results from the notion a of closed subset R of a hypergroup S.

Lemma 3.12 *Suppose R is a closed subset of S. We obtain the following results:*

(i) *The neutral element 1 is in R.*

(ii) *The set R is $*$-invariant, that is, $R^* = R$.*

(iii) *We obtain $RR = R$.*

Proof. (i) From the definition of a closed subset of S we know that R is non-empty and that $R^*R \subseteq R$. Then $r^*r \subseteq R$ for any r in R. Thus, by Lemma 3.1, we have 1 is in R.

(ii) Since 1 is in R, we can use H2 in the complex product

$$R^* = R^* \cdot 1 \subseteq R^*R \subseteq R.$$

So $R^* \subseteq R$. If we $*$ all the elements of both sets, we obtain that $R^{**} \subseteq R^*$ and by Lemma 3.2 we get $R^{**} = R$, so $R \subseteq R^*$. Thus, $R^* = R$.

(iii) Knowing that 1 is in R, we obtain $R = R \cdot 1 \subseteq RR$.

On the other hand, from part (ii) and Lemma 3.1 we get that $RR = R^*R \subseteq R$. Thus, we have $RR = R$. □

Lemma 3.13 *Suppose Q and R are closed subsets of S. We have the following:*

(i) *If s and t are elements of S and $s \in QtR$, then $QtR \subseteq QsR$.*

(ii) *The set of all elements QsR, where $s \in S$, forms a partition of S.*

Proof. (i) Since we are assuming that $s \in QtR$, there exist elements $q \in Q$ and $r \in R$ such that $s \in qtr$. Thus, there exists an element $w \in qt$ such that $s \in wr$. Knowing that $w \in qt$ and by H3, we obtain $t \in q^*w$ and since $s \in wr$, we also get $w \in sr^*$. Then since we assumed Q and R are closed, we obtain $t \in q^*sr^* \subseteq QsR$. Thus, $QtR \subseteq QsR$.

(ii) Lemma 3.12 (i) shows that 1 is in Q and in R. Thus, for every element s in S we have $s \in QsR$; therefore, $S = QsR$.

Assume s and t are elements of S and suppose that the intersection of QsR and QtR is not empty. Then we have an element $w \in QsR \cap QtR$, and that means $w \in QsR$. Thus, $QwR \subseteq QsR$. Part (i) shows us that $QsR \subseteq QwR$. So we have $QsR = QwR$. The same reasoning shows that $QtR = QwR$. So $QsR = QtR$. □

Lemma 3.14 *Suppose P is a subset of S and that P contains the neutral element of S. Then P is closed in S if and only if $S/P = \{sP \mid s \in S\}$ forms a partition on S.*

Proof. We first assume P is closed. If we let $Q = \{1\}$ from Lemma 3.13 (ii), we obtain that $S/P = \{sP \mid s \in S\}$ is a partition of S.

On the other hand, suppose S/P forms a partition on S. For any element p in P, Lemma 3.1 shows us that $1 \in p^*p \subseteq P^*P$. Seeing that 1 is in P and that S/P is a partition of S, we get that $P^*P = P$. Since the element p was arbitrary, it follows that P is closed in S. □

Lemma 3.15 Suppose R is a closed subset of S and that $p \in S$ with $pR = \{p\}$. If there exists an element $q \in p^*R$ such that $q^* \in p^*R$, then $p^* = p = q$.

Proof. Since $q \in p^*R$, we note $q \in 1 \cdot p^*R \subseteq 1 \cdot qR$. Then by Lemma 3.13 (ii), $1 \cdot p^*R = 1 \cdot qR$. Thus, $p^*R = qR$ and as $p^* \in p^*R$, we have $p^* \in qR$. Now, as $q^* \in p^*R$, by Lemma 3.8 we get that $q \in Rp$.

Using Lemma 3.13 (ii) again, we obtain that

$$p^* \in qR \subseteq RqR = RpR = Rp,$$

and $p^* \in Rp \cdot 1 = Rp^* \cdot 1$, so $Rp = Rp^*$. Thus, $p \in Rp^*$. By Lemma 3.8, we have $p^* \in pR = \{p\}$. Thus, $p^* = p$ and from our assumption that $q \in p^*R$, we get that $q \in pR = \{p\}$. □

We shall now provide sufficient conditions for normality. Then we will show that whenever a subset T of S is not closed, the set S/T possesses either 3 or 4 elements. We will use the former cardinality for our investigation in Chapter 7.

Lemma 3.16 *Suppose s is an element in S. Then each of the following imply that s is in $N_S(R)$:*

(i) *The set $(sR)^*$ is in S/R.*

(ii) *The set sR is $*$-invariant.*

(iii) *We obtain $sR = \{s\}$ and $s^*R = \{s^*\}$.*

Proof. (i) Since s is an element in S, it follows that $s \in sR$. Thus, by Lemma 3.8, $s^* \in (sR)^*$. On the other hand, we also know that $s^* \in s^*R$. Thus, by Lemma 3.14, we obtain that S/R is a partition of S, so $(sR)^* = s^*R$. Then, using Lemma 3.8 again, we see that $sR = Rs$. Thus, $s \in N_S(R)$.

(ii) Since $sR \in S/R$ and $(sR)^* = sR$, we see that $(sR)^* \in S/R$. By (i), the result follows.

(iii) We have $sR = \{s\}$. Using Lemma 3.8 we have $(sR)^* = \{s^*\} = s^*R$. So $(sR)^* \in S/R$ and from (i) we obtain that $s \in N_S(R)$. □

Corollary 3.17. *The following imply that R is normal in S:*

(i) *Every $s \in S \setminus R$ satisfies $(sR)^* \in S/R$.*

(ii) *For any $s \in S \setminus R$, we have $(sR)^* = sR$.*

(iii) *The set $S \setminus R$ is symmetric.*

(iv) *The set S/R possesses exactly two elements.*

(v) *The cardinality of S/R is exactly 1 more than the cardinality of $S \setminus R$.*

Proof. (i) By Lemma 3.16 (i), for every $s \in S$, $(sR)^*$ is in S/R. Thus, in particular, $(sR)^*$ is in S/R for elements $s \in S \setminus R$. The result follows.

(ii) It follows from Lemma 3.16 (ii).

(iii) Let $s \in S \setminus R$. Then $s = s^*$. Thus, by (ii) we obtain our result.

(iv) We know R is one of the cosets of S/R. Since $|S/R| = 2$, then $S \setminus R$ is the other coset. Noticing that $R^* = R$ and therefore $(S \setminus R)^* = S \setminus R$, the proof follows from (ii).

(v) Since $|S/R| = |S \setminus R| + 1$, we know that for any $s \in S \setminus R$, the coset $sR = \{s\}$. Thus, by Lemma 3.16 (iii) R is normal in S. □

Lemma 3.18 *Suppose $|S/R| = |S \setminus R|$. If $S \setminus R$ possesses exactly two non-symmetric elements, then R is normal in S.*

Proof. Since $|S/R| = |S \setminus R|$ and by removing $\{R\}$ from S/R we find that the resulting set contains exactly one uniquely determined coset of cardinality 2; the rest of the cosets possess only one element. Let p and q be the elements of such coset. If $\{p,q\}^* = \{p,q\}$, then by Corollary 3.17 (i) R is normal in S. Suppose $\{p,q\}^* \neq \{p,q\}$. Then we have either $p^* \notin \{p,q\}$ or $q^* \notin \{p,q\}$. Assume, without loss of generality, that $p^* \notin \{p,q\}$. Since every coset apart from $\{p,q\}$ is a singleton, we get that $p^*R = \{p^*\}$. Then, by Lemma 3.15 we have a contradiction (switch p^* with p). Thus $\{p,q\}^* = \{p,q\} \in S/R$. We saw that any coset apart from $\{p,q\}$ and R is a singleton and therefore symmetric. Thus, $(sR)^* \in S/R$ for any elements $s \in S \setminus R$. Then by Corollary 3.17 (i), R is normal in S. □

Theorem 3.19 *If $|S \setminus R| \leq 3$, then R is normal in S.*

Proof. (Case 1) Suppose $|S/R| = 1$. Then $S = R$ and R is normal in S.

(Case 2) Assume $|S/R| = 2$. Then for every $s \in S \setminus R$, we have $Rs = S \setminus R = sR$. It follows that R is normal.

(Case 3) Suppose $3 \leq |S/R|$. Then we get $|S/R| \leq |S \setminus R| + 1 \leq 4$. Thus, we obtain that $|S/R| = |S \setminus R| = 3$ or $|S/R| = |S \setminus R| + 1$.

Suppose $|S/R| = |S \setminus R|$. Then if each $s \in S \setminus R$ is symmetric, then by Corollary 3.17 (iii) R is normal in S; on the other hand, if not every element is symmetric, there exist exactly two non-symmetric elements. Thus, by Lemma 3.18, R is normal in S.

Now, assume $|S/R| = |S \setminus R| + 1$. Then, by Corollary 3.17 (v) we obtain our desired result. Thus, R is normal in S. $\square$

We have seen conditions for a closed subset R to be normal in S. The following results help us investigate, in particular, non-normal closed subsets of a hypergroup S with six elements, which is the content of Chapter 7. For the remainder of the chapter, the set T denotes a non-normal closed subset of S.

Lemma 3.20 Suppose T is not a normal subset of S and $|S \setminus T| = 4$. We have the following:

(i) *The cardinality of S/T is either 3 or 4.*

(ii) *Let $|S/T| = 3$. Then $S \setminus T$ possesses elements p, q and r such that p and q are symmetric, $r^* \neq r$ and $S/T = \{T, \{p, r\}, \{q, r^*\}\}$.*

(iii) *Let $|S/T| = 4$. Then $S \setminus T$ possesses elements p and q such that $p \neq p^*$, $q \neq q^*$ and $S/T = \{T, \{p^*\}, \{p, q\}, \{q^*\}\}$.*

Proof. (i) By the contrapositive of Corollary 3.17 (iv), if T is not normal, then $3 \leq |S/T|$. Together with our assumption that $|S \setminus T| = 4$, we have $|S/T| \leq 5$ (when all the elements in $S \setminus T$ are symmetric).

If $|S/T| = 5$, then by Corollary 3.17 (v), T is normal in S, so we obtain a contradiction.

(ii) If there exists an element $s \in S$ such that $sT = \{s\}$, then there exists and element $r \in s^*T$ such that $r^* \in s^*T$. Then using Lemma 3.15 we obtain that $s^* = s$.

Thus, if there exists a coset of T in S different from T that is a singleton, then all the cosets are $*$-invariant, so by Corollary 3.17 (ii), T is normal, and that is a contradiction. As a result, the two cosets in $(S/T) \setminus \{T\}$ must contain exactly two elements.

Now, we know that if the four elements in $S \setminus T$ are symmetric, then by Corollary 3.17 (iii), T is normal in S. Thus, there exists an element $r \in S \setminus T$ such that $r^* \neq r$.

Suppose $\{r, r^*\}$ is a coset of T in S. Then $\{r, r^*\}^* = \{r, r^*\}$ forces the remaining coset in $(S/T) \setminus \{T\}$ to be $*$-invariant. Consequently, T is normal, a contradiction. Thus, both elements cannot be in the same coset. Let p and q be elements in $S \setminus T$ such that $\{p, r\}$ and $\{q, r^*\}$ are cosets of T in S.

Suppose $q = p^*$. Then $\{p, r\}^* = \{q, r^*\} \in S \setminus T$ and by Corollary 3.17 (i), T is normal, a contradiction. Thus, p and q must be symmetric, and $S/T = \{T, \{p, r\}, \{q, r^*\}\}$.

(iii) We are assuming that $|S/T| = |S \setminus T| = 4$. Suppose every element in $S \setminus T$ is symmetric, then by Corollary 3.17 (iii), T is normal in S. Now, suppose $S \setminus T$ possesses exactly two elements that are symmetric. Then by Lemma 3.18, T is normal in S. Thus, none of the elements in $S \setminus T$ are symmetric. Then there exists elements p and q in $S \setminus T$ such that $S \setminus T = \{p, p^*, q, q^*\}$.

From $|S/T| = |S \setminus T| = 4$, there exists a coset of T in S different from T that possesses exactly two elements, and the rest are singletons. Assume, without loss of generality, that $\{p\}$ is not a coset of T in S. Then one of the sets $\{p, p^*\}$, $\{p, q\}$ and $\{p, q^*\}$ needs to be the coset in S/T.

Suppose $\{p, p^*\}$ is in S/T. Then by Corollary 3.17 (ii), T is normal. Thus, $\{p, q\}$ and $\{p, q^*\}$ are the possible sets for the coset in S/T. Assume, without loss of generality, that $\{p, q\}$ is in S/T. Then $p^*T = \{p*\}$, $pT = \{p, q\}$ and $q^*T = \{q^*\}$. $\square$

Corollary 3.21 *If S possesses exactly six elements and T is not normal in S, then $|T| = 2$ and $|S/T| \in \{3, 4\}$.*

Proof. Using Theorem 3.19, $|T| = 2$ (note that $|T| \neq 1$; otherwise, $T = \{1\}$, contrary to the assumption that T is not normal). The second statement follows from Lemma 3.20 (i). $\square$

Involutions play a role in our investigation of hypergroups of order 6 with a non-normal closed subset of two elements. We use the element l as notation for an involution in S.

Lemma 3.22 *For each involution l in S the following holds:*

(i) *We have $l^* = l$.*

(ii) *If l is not thin, then $ll = \{1, l\}$.*

Proof. (i) From the definition of an involution we know $1 \neq l$ and $\{1, l\}$ is closed. Then, using Lemma 3.12 (ii) we get that $\{1, l\}^* = \{1, l\}$. Thus $l^* \in \{1, l\}$ and $l^* \neq 1$. It follows that $l^* = l$.

(ii) By Lemma 3.1 we know $1 \in l^*l$; however, as l is not thin, we get that $l^*l \neq \{1\}$. Since $\{1, l\}$ is closed, the have $l^*l \subseteq \{1, l\}$. Thus, $l^*l = \{1, l\}$. Using (i) we obtain that $ll = \{1, l\}$. □

Lemma 3.23 *Suppose p and q are elements in S with $p \neq q$ and such that $p\{1, l\} = \{p, q\}$. Then we have the following.*

(i) *We have $q \in pl$ and $pl \subseteq \{p, q\}$.*

(ii) *The statements $pl = \{q\}$ and $l \notin p^*p$ are equivalent.*

Proof. (i) We know $q \in p\{1, l\}$. Then we have $q \in p \cdot 1 \cup pl$. From $q \notin p \cdot 1$ we get that $q \in pl$. Thus, as $pl \subseteq p\{1, l\}$, we obtain our result.

(ii) Using (i) we obtain that $pl = \{q\}$ if and only if $p \notin pl$. Then, by H3, the latter result is equivalent to $l \in p^*p$. □

CHAPTER IV

HYPERGROUPS OF ORDER 3

In this chapter, we determine all hypergroups with three elements. In [7], it is shown that there exist exactly ten isomorphism classes of such hypergroups, one of which is *not* realized as a finite association scheme.

In this chapter, the letter S stands for a hypergroup of exactly three elements.

Lemma 4.1 *Let R be a closed subset of S such that $|R| + 1 = |S|$. Then S is a wreath product of R and a hypergroup of order 2.*

Proof. From $|R| + 1 = |S|$ we obtain that $|S \setminus R| = 1$ and we denote the resulting element by s. Suppose r is an element of R and that $R \cap sr$ is a non-empty subset of S. Then choose an element $t \in sr$. We obtain that $s \in tr^* \subseteq R$ (recall that t and r are in R, so their products and inverses are there as well). But $s = S \setminus R$, which is a contradiction. Thus, $R \cap sr$ is empty. As a result, for every element $r \in R$, we have $sr = \{s\}$, because the set does not intersect with the rest of S (which is R), and we have $R \subseteq R_s$. Then, by Lemma 3.13 (ii), we obtain that $|S//R| = 2$. □

Theorem 4.2 *If a hypergroup S with three elements is not primitive, then it is a wreath product of two hypergroups of order 2.*

Proof. Since we are assuming that S is not primitive, there exists a closed subset R different from S and $\{1\}$. It follows that $|R| = 2$. Then, by Lemma 4.1, we obtain our result. □

Assume R is closed in S and that S is a wreath product of R and $S//R$. In the following classes, the hypergroup $S = \{1, r, s\}$ and $R = \{1, r\}$. As a result, we have $r^* = r$ because R is closed in S. Thus, we also have $s^* = s$.

Corollary 4.3 *Suppose $S = \{1, r, s\}$ is not primitive and $R = \{1, r\}$. Then S possesses one of the following isomorphism classes.*

1	1	r	s
1	$\{1\}$	$\{r\}$	$\{s\}$
r	$\{r\}$	$\{1\}$	$\{s\}$
s	$\{s\}$	$\{s\}$	R

Table 1: *Tbd*

2	1	r	s
1	$\{1\}$	$\{r\}$	$\{s\}$
r	$\{r\}$	$\{1\}$	$\{s\}$
s	$\{s\}$	$\{s\}$	S

Table 2: $HM_2(6, 8)$

3	1	r	s
1	$\{1\}$	$\{r\}$	$\{s\}$
r	$\{r\}$	R	$\{s\}$
s	$\{s\}$	$\{s\}$	R

Table 3: $HM_3(6, 8)$

4	1	r	s
1	$\{1\}$	$\{r\}$	$\{s\}$
r	$\{r\}$	R	$\{s\}$
s	$\{s\}$	$\{s\}$	S

Table 4: $HM_2(9)$

Proof. The result follows immediately from Theorem 4.2. □

In Example 1, both R and $S//R$ are thin. Example 2 has R as thin while $S//R$ as not thin. In Example 3, R is not thin, but $S//R$ is thin. Example 4 has R and $S//R$ not thin. Examples 2, 3 and 4 arise from finite association schemes obtained from the Hanaki-Miyamoto Classification of Association Schemes with Small Vertices. Note that both Examples 2 and 3 have more than one realization as a finite scheme. All of the examples above arise from finite association schemes. We go through all the different cases S can have to find the rest of the isomorphism classes.

Theorem 4.4 *Let S be a hypergroup with three elements and assume S is primitive and also symmetric. Let p and q be the unique elements in $S \setminus \{1\}$. We have $pq = \{p, q\}$.*

Proof. Since S is primitive, p is not an involution. Thus, $\{1, p\}^*\{1, p\} \nsubseteq \{1, p\}$, and $pp = p^*p \notin \{1, p\}$ because all the other elements in the complex product are in $\{1, p\}$. So pp must contain q. Thus $p \in p^*q$ and since p is symmetric, $p \in pq$. Similarly, we get that $q \in qp$. Then by Lemma 3.9, $pq = qp$. Thus, $q \in pq$. Since $p \neq q^*$, Lemma 3.5 shows us that $1 \notin pq$. Thus, $pq = \{p, q\}$. □

Corollary 4.5 *Suppose S is primitive and symmetric. Then S possesses one of the following isomorphism classes.*

5	1	p	q
1	$\{1\}$	$\{p\}$	$\{q\}$
p	$\{p\}$	$\{1,q\}$	$\{p,q\}$
q	$\{q\}$	$\{p,q\}$	$\{1,p\}$

Table 5: $HM_2(5)$

6	1	p	q
1	$\{1\}$	$\{p\}$	$\{q\}$
p	$\{p\}$	$\{1,q\}$	$\{p,q\}$
q	$\{q\}$	$\{p,q\}$	S

Table 6: *Petersen Graph*

7	1	p	q
1	$\{1\}$	$\{p\}$	$\{q\}$
p	$\{p\}$	S	$\{p,q\}$
q	$\{q\}$	$\{p,q\}$	S

Table 7: $HM_3(9)$

Proof. The result follows immediately from Theorem 4.4. $\square$

Note that $S = \{1, p, q\}$ and that $p^* = p$ and $q^* = q$. Example 5 is isomorphic to the Schurian Scheme that comes from a subgroup of two elements in a dihedral group of order 10, and Example 4 is the scheme that comes from the Petersen graph. Both Example 5 and Example 7 come from the Hanaki-Miyamoto Classification of Association Schemes; cf. [8].

We consider the last case where we obtain the rest of the isomorphism classes of S. Suppose S is not symmetric.

Theorem 4.6 *If S is not symmetric, then S is thin or there exists an idempotent element in S.*

Proof. Since we assumed that S is not symmetric, theres exists an element $s \in S$ such that $s^* \neq s$. Then, as S contains exactly three elements, we have $S = \{1, s, s^*\}$. Because s is not symmetric, Lemma 3.6 shows us that $1 \notin ss$.

Because s^*s is $*$-invariant and $1 \in s^*s$, we obtain that $s^*s = \{1\}$ or $s^*s = S$. If s is thin, then $s \notin ss$ and together with $1 \notin ss$ we obtain that $ss = \{s^*\}$. From Lemma 4.4 we obtain that $ss^* = s^*s = \{1\}$. Thus, s^* is also thin, and as a result, S is thin.

Suppose now that $s^*s = S$. Then $s \in s^*s$ and $s^* \in s^*s$. Since $s \in s^*s$ we have $s \in ss$, and since $s^* \in s^*s$ we obtain $s \in ss^*$. Then, by the symmetricity of ss^*, we get that from $s \in ss^*$ we have $s^* \in ss^*$. Thus, as $1 \in ss^*$, we have $ss^* = S$. $\square$

From Theorem 4.6 we obtain the last isomorphism classes of a hypergroup with three elements. Notice that our hypergroup $S = \{1, s, s^*\}$ and $s^* \neq s$.

Corollary 4.7 *Suppose $S = \{1, s, s^*\}$ is not symmetric. Then S possesses one of the following isomorphism classes.*

8	1	s	s^*
1	$\{1\}$	$\{s\}$	$\{s^*\}$
s^*	$\{s^*\}$	$\{1\}$	$\{s\}$
s	$\{s\}$	$\{s^*\}$	$\{1\}$

Table 8: *Group* C_3

9	1	s	s^*
1	$\{1\}$	$\{s\}$	$\{s^*\}$
s^*	$\{s^*\}$	S	$\{s^*\}$
s	$\{s\}$	$\{s\}$	S

Table 9: *Infinite scheme*

10	1	s	s^*
1	$\{1\}$	$\{s\}$	$\{s^*\}$
s^*	$\{s^*\}$	S	$\{s, s^*\}$
s	$\{s\}$	$\{s, s^*\}$	S

Table 10: $HM_2(7)$

Proof. The result follows immediately from Theorem 4.6. $\square$

Example 8 is the thin hypergroup of three elements, and it is isomorphic to the Cyclic Group of order 3. Example 9 does not arise from a finite association scheme, but an from infinite one, where its underlying set $X = \mathbb{Q}$ and $s =<$; cf. [4]. Example 10 comes from the finite scheme $HM_2(7)$.

CHAPTER V

NON-SYMMETRIC HYPERGROUPS OF ORDER 4

In this chapter we classify all non-symmetric hypergroups with four elements. There exist thirty seven isomorphism classes of such hypergroups. Five of them are not commutative and therefore do not arise from finite association schemes; cf. [7].

Throughout this chapter, S denotes a non-symmetric hypergroup with four elements. The letters r and r^* denote the two non-symmetric elements of S, and s denotes the only symmetric element of S different from 1. Thus, $S = \{1, r, r^*, s\}$.

We work with the set r^*r. Since r^*r is $*$-invariant, the inverse of any element in r^*r is back in r^*r. Thus, we obtain only the following cases:

$$r^*r = \{1\}, \quad r^*r = \{1, s\}, \quad r^*r = \{1, r, r^*\}, \quad \text{or} \quad r^*r = S.$$

We first assume $r^*r = \{1\}$. We will see that there exist three isomorphism classes of hypergroups in this case. Note that by definition, r is thin in this case.

Lemma 5.1 *Suppose $r^*r = \{1\}$. Then we have the following:*

(i) *We obtain $rr^* = \{1\}$.*

(ii) *We obtain $rr = \{r^*\}$ or $rr = \{1\}$.*

(iii) *If $rr = \{r^*\}$, then $sr = \{s\}$, $sr^* = \{s\}$ and $\{1, r, r^*\} \subseteq ss$.*

(iv) *If $rr = \{s\}$, then $sr = \{r^*\}$, $sr^* = \{r\}$ and $ss = \{1\}$.*

Proof. (i) Since $r \notin r^*r$, it follows that $r^* \notin rr^*$ and that $r \notin rr^*$. Thus, $rr^* \subseteq \{1, s\}$. Suppose $s \in rr^*$. Then we have $sr \subseteq rr^*r = \{r\}$, and, since $s \notin sr$, it follows by Lemma 3.8 and H3 that $s \notin rs$.

On the other hand, since $s \notin r^*r$ we get that $r \notin rs$ and, as $r \neq s$, we obtain that $1 \notin rs$; cf. Lemma 3.6. Thus, $rs = \{r^*\}$ and $rr = rsr = r^*r = \{1\}$. Contradiction, for $r^* \neq r$. □

(ii) Since $r \notin r^*r$ and by H3, we have $r \notin rr$. From our assumption that r is thin and by Lemma 3.11 (ii), we get that rr is a singleton. Since r is thin, we also have that $1 \notin rr$. Thus, $rr = \{r^*\}$ or $rr = \{s\}$.

(iii) We have $rr = \{r^*\}$. Thus, $s \notin rr$ so $r \notin r^*s$. Then, by Lemma 3.8, $r^* \notin sr$. Since by (i) $s \notin rr^*$, we obtain that $r \notin sr$, and together with $1 \notin sr$, we have $sr^* = s$.

Since $s \notin rr$, we also obtain that $r \notin sr^*$. From our assumption that $r^*r = \{1\}$, we get that $s \notin r^*r$ and therefore $r^* \notin sr^*$. Together with $1 \notin sr^*$, we have $sr^* = \{s\}$.

As $s \in sr$, we obtain $r \in ss$. Then, since ss is $*$-invariant, we have $\{1, r, r^*\} \subseteq ss$.

(iv) Knowing that $rr = \{s\}$, we obtain that $r^* \in sr$. Since r is thin and by Lemma 3.11, we get that sr consists of exactly one element. Thus, $sr = \{r^*\}$.

Since $rr = \{s\}$, we also get that $r \in r^*s$. By (i), we saw that r^* is thin. Thus, $sr^* = \{r\}$.

Since $s \in rr$ and $sr = \{r^*\}$, we get that $ss \subseteq srr = r^*r = \{1\}$. Thus, $ss = \{1\}$. □

Corollary 5.2 *Suppose $r^*r = \{1\}$. Then S possesses one of the following isomorphism classes.*

1	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1\}$	$\{r\}$	$\{s\}$
r	$\{r\}$	$\{r^*\}$	$\{1\}$	$\{s\}$
s	$\{s\}$	$\{s\}$	$\{s\}$	$\{1, r, r^*\}$

Table 11: $HM_4(6)$

2	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1\}$	$\{r\}$	$\{s\}$
r	$\{r\}$	$\{r^*\}$	$\{1\}$	$\{s\}$
s	$\{s\}$	$\{s\}$	$\{s\}$	S

Table 12: $HM_4(9), HM_6(12, 15, 18), HM_5(21)$

3	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1\}$	$\{s\}$	$\{r\}$
r	$\{r\}$	$\{s\}$	$\{1\}$	$\{r^*\}$
s	$\{s\}$	$\{s\}$	$\{s\}$	$\{1\}$

Table 13: *Group* C_4

Proof. We obtain our results immediately from Lemma 5.1. □

Example 1 and Example 2 are wreath products of a cyclic group of order 3 with a thin and non-thin involution, respectively. Example 3 comes from C_4.

Now we assume $r^*r = \{1, s\}$.

Lemma 5.3 *Suppose $r^*r = \{1, s\}$. We have the following:*

(i) *We obtain $rr^* = \{1, s\}$.*

(ii) *We obtain $rr = \{r^*\}$, $rr = \{s\}$, or $rr = \{r^*, s\}$.*

(iii) *If $rr = \{r^*\}$, then $sr = \{r\}, sr^* = \{r^*\}$ and $ss \subseteq \{1, s\}$.*

(iv) *If $rr = \{s\}$, then $\{r, r^*\} \subseteq sr \cap sr^*$ and $s \in ss$. Additionally, $s \in sr$, $s \in sr^*$ and $\{r, r^*\} \subseteq ss$ are equivalent.*

(v) *If $rr = \{r^*, s\}$, then $sr = \{r, r^*, s\} = sr^*$ and $\{1, r, r^*\} \subseteq ss$.*

Proof. (i) Since $r \notin r^*r$, it follows that $r^* \notin rr^*$ and $r \notin rr^*$. As a result, $rr^* \subseteq \{1, s\}$. Suppose $s \notin rr^*$. It follows that $rr^* = \{1\}$. Then, using Lemma 5.2 (i) for r^* instead of r and r instead of r^* we get that $r^*r = \{1\}$. But we assumed $r^*r = \{1, s\}$, so we obtain a contradiction. Thus, $rr^* = \{1, s\}$.

(ii) We are assuming that $r^*r = \{1, s\}$. Thus, $r \notin r^*r$ and therefore $r \notin rr$. Together with $1 \notin rr$, we obtain that $rr \subseteq \{r^*, s\}$.

(iii) Since $rr = \{r^*\}$ and $s \in r^*r$, we have $sr \subseteq r^*rr = r^*r^* = \{r\}$. Thus, $sr = \{r\}$.

From (i) we know $s \in rr^*$. As a result, $sr^* \subseteq rr^*r^* = rr = \{r^*\}$. Thus, $sr^* = \{r^*\}$.

We know $s \notin sr$. Then $r \notin ss$ and $r^* \notin ss$. So we have $ss \subseteq \{1, s\}$.

(iv) Using (i) and Lemma 3.10 (ii) we obtain that $\{r, r^*\} \subseteq sr \cap sr^*$. Since $s \in rr$, we obtain $r \in rs$. Together with $rr = \{s\}$, we have $s \in rr \subseteq rrs = ss$. The equivalence statement is obtained from the definition of a hypergroup.

(v) From (i) and Lemma 3.10 (ii) we obtain that $\{r, r^*\} \subseteq sr \cap sr^*$. Suppose that $s \notin sr$. It follows that $sr = \{r, r^*\}$. Thus, $r \in sr^* \subseteq srr = rr \cup r^*r = \{1, r^*, s\}$, which is a contradiction. Finally, by H3, we obtain that $s \in sr$, $s \in sr^*$ and that $\{r, r^*\} \subseteq ss$. □

From $r^*r = \{1, s\}$, we obtain the next six isomorphism classes of hypergroups.

Corollary 5.4 *Suppose $r^*r = \{1, s\}$. Then S possesses one of the following isomorphism classes.*

4	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, s\}$	$\{r\}$	$\{r^*\}$
r	$\{r\}$	$\{r^*\}$	$\{1, s\}$	$\{r\}$
s	$\{s\}$	$\{r\}$	$\{r^*\}$	$\{1\}$

Table 14: $HM_6(6)$

5	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, s\}$	$\{r\}$	$\{r^*\}$
r	$\{r\}$	$\{r^*\}$	$\{1, s\}$	$\{r\}$
s	$\{s\}$	$\{r\}$	$\{r^*\}$	$\{1, s\}$

Table 17: $HM_6(9), HM_{10}(21), HM_{11}(15), HM_{13}(12), HM_{14}(18)$

6	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, s\}$	$\{s\}$	$\{r, r^*\}$
r	$\{r\}$	$\{s\}$	$\{1, s\}$	$\{r, r^*\}$
s	$\{s\}$	$\{r, r^*\}$	$\{r, r^*\}$	$\{1, s\}$

Table 15: *Tbd*

7	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, s\}$	$\{s\}$	$\{r, r^*, s\}$
r	$\{r\}$	$\{s\}$	$\{1, s\}$	$\{r, r^*, s\}$
s	$\{s\}$	$\{r, r^*, s\}$	$\{r, r^*, s\}$	S

Table 18: *Tbd*

8	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, s\}$	$\{r, s\}$	$\{r, r^*, s\}$
r	$\{r\}$	$\{r^*, s\}$	$\{1, s\}$	$\{r, r^*, s\}$
s	$\{s\}$	$\{r, r^*, s\}$	$\{r, r^*, s\}$	$\{1, r, r^*\}$

Table 16: *Tbd*

9	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, s\}$	$\{r, s\}$	$\{r, r^*, s\}$
r	$\{r\}$	$\{r^*, s\}$	$\{1, s\}$	$\{r, r^*, s\}$
s	$\{s\}$	$\{r, r^*, s\}$	$\{r, r^*, s\}$	S

Table 19: *Tbd*

Proof. Our results immediately follow from Lemma 5.3. □

Example 4 and Example 5 are found in the Hanaki-Miyamoto Classification of Schemes. As before, notice that Example 5 is isomorphic to several schemes on finite sets. Current research is undergoing to determine the underlying structure, if any, of Examples 6 through 9; cf. [7].

Now, we consider the third case, when $r^*r = \{1, r, r^*\}$.

From Lemma 3.9 (i) we obtain that $r^*r = \{1, r, r^*\}$ implies $\{1, r, r^*\} \subseteq rr^*$. If we assume $rr^* = \{1, r, r^*\}$, then we obtain nine isomorphism classes of hypergroups. On the other hand, if we assume that $rr^* = S$, we obtain five isomorphism classes. Let us first assume the former.

Lemma 5.5 *Suppose $r^*r = \{1, r, r^*\} = rr^*$ and $s \notin rr$. Then we have the following.*

(i) *We obtain $sr = \{s\}$ and $sr^* = \{s\}$.*

(ii) *We obtain $rr = \{r\}$ or $rr = \{r, r^*\}$.*

(iii) *We obtain $ss = \{1, r, r^*\}$ or $ss = S$.*

Proof. (i) Since $s \notin rr^*$, we get that $r \notin sr$, and since $s \notin rr$, it follows that $r^* \notin sr$. Together with $1 \notin sr$ we obtain that $sr = \{s\}$.

Since $s \notin rr$, we get that $r \notin sr^*$, and as $s \notin r^*r$, we obtain that $r^* \notin sr^*$. Then, together with $1 \notin sr$, we have $sr^* = \{s\}$.

(ii) Since $r \in r^*r$, we get that $r \in rr$. Then, together with $s \notin rr$ and $1 \notin rr$, we obtain that $rr \subseteq \{r, r^*\}$.

(iii) Using (i) we obtain that $s \in sr$ implies $r \in ss$. Then, by Lemma 3.8 and by the symmetricity of s we have $r^* \in ss$. It follows that $\{1, r, r^*\} \subseteq ss$. □

Corollary 5.6 *Suppose $r^*r = \{1, r, r^*\} = rr^*$. Then S possesses one of the following isomorphism classes.*

10	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r^*\}$	$\{s\}$
r	$\{r\}$	$\{r\}$	$\{1, r, r^*\}$	$\{s\}$
s	$\{s\}$	$\{s\}$	$\{s\}$	$\{1, r, r^*\}$

Table 20: *Tbd*

11	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r^*\}$	$\{s\}$
r	$\{r\}$	$\{r\}$	$\{1, r, r^*\}$	$\{s\}$
s	$\{s\}$	$\{s\}$	$\{s\}$	S

Table 22: *Tbd*

12	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r, r^*\}$	$\{s\}$
r	$\{r\}$	$\{r, r^*\}$	$\{1, r, r^*\}$	$\{s\}$
s	$\{s\}$	$\{s\}$	$\{s\}$	$\{1, r, r^*\}$

Table 21: $HM_6(14)$

13	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r, r^*\}$	$\{s\}$
r	$\{r\}$	$\{r, r^*\}$	$\{1, r, r^*\}$	$\{s\}$
s	$\{s\}$	$\{s\}$	$\{s\}$	S

Table 23: $HM_8(21)$

Proof. The result follows immediately from Lemma 5.5. $\square$

Notice that in Corollary 5.6, the set $\{1, r, r^*\}$ is closed in S and that S is a wreath product of $\{1, r, r^*\}$ and a hypergroup of cardinality 2. In Example 10 and Example 11, r is idempotent, and in Example 12 and Example 13, $rr = \{r, r^*\}$ implies that the closed subset $\{1, r, r^*\} = HM_3(7)$. From Example 12, $ss = \{1, r, r^*\}$ implies that $S//\{1, r, r^*\}$ is thin, and from Example 13, $ss = S$ implies $S//\{1, r, r^*\}$ that is not thin. Together with Example 1 and Example 2 of this section, these examples are wreath products of Examples 8, 9 and 10 of Chapter 4, where $|S| = 3$.

Lemma 5.7 *Suppose $r^*r = \{1, r, r^*\} = rr^*$ and that $s \in rr$. Then we have the following.*

(i) *We obtain $sr = \{r^*\}$ or $\{r^*, s\}$.*

(ii) *We obtain $rr = \{r, s\}$ or $\{r, r^*, s\}$.*

(iii) *If $sr = \{r^*\}$, then $rr = \{r, r^*, s\}$, $sr^* = \{r\}$ and $ss = \{1\}$.*

(iv) *If $sr = \{r^*s\}$, then $sr^* = \{r, s\}$ and $\{1, r, r^*\} \subseteq ss$.*

Proof. (i) From our assumption that $s \in rr$, we get that $r^* \in rs$ and since $s \notin r^*r$, we obtain $r \notin rs$. Together with $1 \notin rs$, we have $rs = \{r^*\}$ or $rs = \{r^*, s\}$.

(ii) From our assumption that $r \in r^*r$, we get that $r \in rr$, and from $s \in rr$ and $1 \notin rr$ we then obtain that $rr = \{r, s\}$ or $rr = \{r, r^*, s\}$.

(iii) Suppose that $rr \neq \{r, r^*, s\}$. From (ii), we have $rr = \{r, s\}$. Since $sr = \{r^*\}$, we obtain that $r \in r^*r = srr = sr \cup ss = \{r^*\} \cup ss$. Thus, $r \in ss$ and $s \in sr$, a contradiction.

Since $s \notin r^*r$ we get that $r^* \notin sr^*$, and as $s \notin sr$ we have $s \notin sr^*$. Together with $1 \notin sr^*$, we obtain that $sr^* = \{r^*\}$. From $sr = \{r^*\}$, we have $s \notin sr$. The latter implies that $r \notin ss$ and that is equivalent to $r^* \notin ss$. Since $ss \subseteq srr = r^*r = \{1, r, r^*\}$, it follows that $ss = \{1\}$.

(iv) Since $r^* \in sr$, we obtain that $r \in sr^*$, and as $s \in sr$, we get that $s \in sr^*$. Knowing that $s \notin r^*r$ we get that $r^* \notin sr^*$. From $s \in sr$ we get that $r \in ss$. Then, by the symmetricity of s and Lemma 3.8, we get that $r^* \in ss$. It follows that $\{1, r, r^*\} \subseteq ss$. $\square$

Corollary 5.8 *Suppose $r^*r = \{1, r, r^*\} = rr^*$ and that $s \in rr$. Then S possesses one of the following isomorphism classes.*

14	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r, r^*, s\}$	$\{r\}$
r	$\{r\}$	$\{r, r^*, s\}$	$\{1, r, r^*\}$	$\{r^*\}$
s	$\{s\}$	$\{r^*\}$	$\{r\}$	$\{1\}$

Table 24: $HM_6(8), HM_{11}(16)$

15	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r^*, s\}$	$\{r, s\}$
r	$\{r\}$	$\{r, s\}$	$\{1, r, r^*\}$	$\{r^*, s\}$
s	$\{s\}$	$\{r^*, s\}$	$\{r, s\}$	$\{1, r, r^*\}$

Table 27: *Tbd*

16	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r^*, s\}$	$\{r, s\}$
r	$\{r\}$	$\{r, s\}$	$\{1, r, r^*\}$	$\{r^*, s\}$
s	$\{s\}$	$\{r^*, s\}$	$\{r, s\}$	S

Table 25: *Tbd*

17	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r, r^*, s\}$	$\{r, s\}$
r	$\{r\}$	$\{r, r^*, s\}$	$\{1, r, r^*\}$	$\{r^*, s\}$
s	$\{s\}$	$\{r^*, s\}$	$\{r, s\}$	$\{1, r, r^*\}$

Table 28: *Tbd*

18	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r, r^*, s\}$	$\{r, s\}$
r	$\{r\}$	$\{r, r^*, s\}$	$\{1, r, r^*\}$	$\{r^*, s\}$
s	$\{s\}$	$\{r^*, s\}$	$\{r, s\}$	S

Table 26: *Tbd*

Proof. The result follows immediately from Lemma 5.7. □

Lemma 5.9 *Suppose $r^*r = \{1, r, r^*\}$ and that $rr^* = S$. Then we have the following:*

(i) *The set $\{r, s\} \subseteq sr, r \in rr, r^* \notin sr^*$ and $\{1, r, r^*\} \subseteq ss$.*

(ii) *If $rr = \{r\}$, then $sr = \{r, s\}, sr^* = \{s\}$ and $ss = S$.*

(iii) *If $rr \neq \{r\}$, then $s \in rr$, $sr = \{r, r^*, s\}$ and $sr^* = \{r, s\}$.*

Proof. (i) From $r \in rr^*$ we have $r \in rr$, and as $s \in rr^*$, we get $r \in sr$. From $s \notin r^*r$ we have $r^* \notin sr^*$. Suppose $s \notin sr$. Then $s \notin rs$. Together with $r \notin rs$, we obtain $rs = \{r^*\}$. By Lemma 3.10 (i), we obtain $srs = \{r\}$. Note that $r^* \notin sr$ implies $s \notin rr^*$. Thus, $r^* \in sr$, $r^*s \subseteq srs = \{r\}$ and $sr = \{r^*\}$. But $r \in sr$, so $r^* = r$, a contradiction. Thus, $s \in sr$, $s \in sr^*$ and $\{r, r^*\} \subseteq ss$.

(ii) Since $s \notin rr$, we obtain that $r^* \notin sr$. Then, by (i), we get that $sr = \{r, s\}$. From $s \notin rr$, we also obtain that $r \notin sr^*$, and by (i), we know $r^* \notin sr^*$. Together with $1 \notin sr^*$, it follows that $sr^* = \{s\}$. From $r \in sr$ we get that $r^* \in r^*s$. Then $s \in sr^* \subseteq sr^*s = ss$. Including the last result in (i), we get that $ss = S$.

(iii) Suppose $s \notin rr$. From (i) we get that $r \in rr$ but since $r =\neq rr$ and $1 \notin rr$, we have $r^* \in rr$. From (i) we also get that $r \in sr$. Then $r^* \in rr \subseteq srr \subseteq sr \cup sr^*$. But $s \notin rr$ implies $r^* \notin sr$ and $s \notin r^*r$ implies $r^* \notin sr^*$, a contradiction. So $s \in rr$. Thus, we obtain that $r^* \in sr$. Then, by (i) we know $\{r, s\} \subseteq sr$. Together with $1 \notin sr$ we obtain $sr = \{r, r^*, s\}$. On the other hand, $s \in rr$ implies $r \in sr^*$. By (i) we have $r^* \notin sr^*$ but $s \in sr^*$. Together with $1 \notin sr^*$, we get that $sr^* = \{r, s\}$. $\square$

Corollary 5.10 *Suppose $r^*r = \{1, r, r^*\}$ and that $rr^* = S$. Then S possesses one of the following isomorphism classes.*

19	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r^*\}$	$\{r^*, s\}$
r	$\{r\}$	$\{r\}$	S	$\{s\}$
s	$\{s\}$	$\{r, s\}$	$\{s\}$	S

Table 29: *Tbd*

20	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r^*, s\}$	$\{r, r^*, s\}$
r	$\{r\}$	$\{r, s\}$	S	$\{r^*, s\}$
s	$\{s\}$	$\{r, r^*, s\}$	$\{r, s\}$	$\{1, r, r^*\}$

Table 32: *Tbd*

21	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r, r^*, s\}$	$\{r, r^*, s\}$
r	$\{r\}$	$\{r, r^*, s\}$	S	$\{r^*, s\}$
s	$\{s\}$	$\{r, r^*, s\}$	$\{r, s\}$	$\{1, r, r^*\}$

Table 30: *Tbd*

22	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r^*, s\}$	$\{r, r^*, s\}$
r	$\{r\}$	$\{r, s\}$	S	$\{r^*, s\}$
s	$\{s\}$	$\{r, r^*, s\}$	$\{r, s\}$	S

Table 33: *Tbd*

23	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	$\{1, r, r^*\}$	$\{r, r^*, s\}$	$\{r, r^*, s\}$
r	$\{r\}$	$\{r, r^*, s\}$	S	$\{r^*, s\}$
s	$\{s\}$	$\{r, r^*, s\}$	$\{r, s\}$	S

Table 31: *Tbd*

Proof. The result follows immediately from Lemma 5.9. □

Several of the hypergroups in Examples 19 through 23 are described in more detail in the next chapter. Example 19 is called the *Tomaselli hypergroup*, and Examples 20 through 23 are called the *Regensburg hypergroups*. The Tomaselli hypergroup does not have a realization on a finite scheme, but on an infinite scheme, and the Regensburg hypergroups in Examples 20 through 22 do not come from association schemes at all; cf. [1].

We now investigate the last case, when $r^*r = S$. Thus, by Lemma 1.2.3 (i), we get that $\{1, r, r^*\} \subseteq rr^*$. If $rr^* = \{1, r, r^*\}$, we fall in the previous case (we interchange r and r^*). Thus, we only consider the case when $rr^* = S$.

Lemma 5.11 *Suppose $r^*r = S = rr^*$. Then we have the following:*

(i) *We obtain $r \in rs = sr$.*

(ii) *The statements $s \in rr$ and $r^* \in sr$ are equivalent.*

(iii) *The statements $r \in ss$ and $s \in sr$ are equivalent.*

(iv) *We obtain $rr = \{r\}, rr = \{r, r^*\}, rr = \{r, s\}$ or $rr = \{r, r^*, s\}$.*

(v) *If $ss = \{1\}$, then $rr \subseteq \{r, r^*\}, sr = \{r\}$ and $sr^* = \{r^*\}$.*

Proof. (i) Since $s \in r^*r$, we get that $r \in rs$. Also, from $s \in rr^*$, we have $r \in sr$. Then, since $r^* \in sr$ if and only if $r^* \in rs$, and $s \in sr$ if and only if $s \in rs$, we get that $rs = sr$, for both sets contain exactly every element of S except from 1.

(i), (ii) They follow from H3.

(iv) From $r \in r^*r$ we get that $r \in rr$. Then, as $1 \notin rr$, we have our result.

(v) Suppose $ss = \{1\}$. By using (i), we obtain that $sr \subseteq srs \subseteq rss = \{r\}$. Since $r^* \notin sr$, by (ii) we get that $s \notin rr$. Then, as $1 \notin rr$, we have $rr \subseteq \{r, r^*\}$.

Suppose $s \in sr$. Then $r \in ss$, a contradiction. Now, suppose $r^* \in sr$. Then $s \in rr$, a contradiction. Together with $1 \notin sr$, we have $sr = \{r\}$. Thus, by (i), $rs = \{r\}$, and that implies $sr^* = \{r^*\}$. □

Corollary 5.12 *Suppose $r^*r = S = rr^*$. Then S possesses one of the following isomorphism classes.*

24	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r^*\}$	$\{r^*\}$
r	$\{r\}$	$\{r\}$	S	$\{r\}$
s	$\{s\}$	$\{r\}$	$\{r^*\}$	$\{1\}$

Table 34: *Tbd*

26	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r^*\}$	$\{r^*\}$
r	$\{r\}$	$\{r\}$	S	$\{r\}$
s	$\{s\}$	$\{r\}$	$\{r^*\}$	$\{1,s\}$

Table 35: *Tbd*

28	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r^*,s\}$	$\{r,r^*\}$
r	$\{r\}$	$\{r,s\}$	S	$\{r,r^*\}$
s	$\{s\}$	$\{r,r^*\}$	$\{r,r^*\}$	$\{1,s\}$

Table 36: *Tbd*

30	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r^*\}$	$\{r^*,s\}$
r	$\{r\}$	$\{r\}$	S	$\{r,s\}$
s	$\{s\}$	$\{r,s\}$	$\{r^*,s\}$	$\{1,r,r^*\}$

Table 37: *Tbd*

32	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r^*,s\}$	$\{r,r^*,s\}$
r	$\{r\}$	$\{r,s\}$	S	$\{r,r^*,s\}$
s	$\{s\}$	$\{r,r^*,s\}$	$\{r,r^*,s\}$	$\{1,r,r^*\}$

Table 38: *Tbd*

34	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r^*\}$	$\{r^*,s\}$
r	$\{r\}$	$\{r\}$	S	$\{r,s\}$
s	$\{s\}$	$\{r,s\}$	$\{r^*,s\}$	S

Table 39: *Tbd*

36	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r^*,s\}$	$\{r,r^*,s\}$
r	$\{r\}$	$\{r,s\}$	S	$\{r,r^*,s\}$
s	$\{s\}$	$\{r,r^*,s\}$	$\{r,r^*,s\}$	S

Table 40: *Tbd*

25	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r,r^*\}$	$\{r^*\}$
r	$\{r\}$	$\{r,r^*\}$	S	$\{r\}$
s	$\{s\}$	$\{r\}$	$\{r^*\}$	$\{1\}$

Table 41: $HM_5(14)$

27	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r,r^*\}$	$\{r^*\}$
r	$\{r\}$	$\{r,r^*\}$	S	$\{r\}$
s	$\{s\}$	$\{r\}$	$\{r^*\}$	$\{1,s\}$

Table 42: $HM_7(21)$

29	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r,r^*,s\}$	$\{r,r^*\}$
r	$\{r\}$	$\{r,r^*,s\}$	S	$\{r,r^*\}$
s	$\{s\}$	$\{r,r^*\}$	$\{r,r^*\}$	$\{1,s\}$

Table 43: $HM_{18}(16), HM_{19}(16)$

31	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r,r^*\}$	$\{r^*,s\}$
r	$\{r\}$	$\{r,r^*,s\}$	S	$\{r,s\}$
s	$\{s\}$	$\{r,s\}$	$\{r^*,s\}$	$\{1,r,r^*\}$

Table 44: *Tbd*

33	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r,r^*,s\}$	$\{r,r^*,s\}$
r	$\{r\}$	$\{r,r^*,s\}$	S	$\{r,r^*,s\}$
s	$\{s\}$	$\{r,r^*,s\}$	$\{r,r^*,s\}$	$\{1,r,r^*\}$

Table 45: *Tbd*

35	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r,r^*\}$	$\{r^*,s\}$
r	$\{r\}$	$\{r,r^*\}$	S	$\{r,s\}$
s	$\{s\}$	$\{r,s\}$	$\{r^*,s\}$	S

Table 46: *Tbd*

37	1	r	r^*	s
1	$\{1\}$	$\{r\}$	$\{r^*\}$	$\{s\}$
r^*	$\{r^*\}$	S	$\{r,r^*,s\}$	$\{r,r^*,s\}$
r	$\{r\}$	$\{r,r^*,s\}$	S	$\{r,r^*,s\}$
s	$\{s\}$	$\{r,r^*,s\}$	$\{r,r^*,s\}$	S

Table 47: *Tbd*

Proof. The result follows from Lemma 5.11. □

As from previous isomorphism classes, current research is going on to determine the structure of most of the hypergroups in this case.

We classified all the isomorphism classes of non-symmetric hypergroups with exactly four elements. We see that among them, five classes of non-commutative hypergroups do not come from any association scheme on a finite set. Four out of five of such hypergroups are described in the next chapter.

CHAPTER VI

TOMASELLI AND REGENSBURG HYPERGROUPS

In the present chapter, we describe the non-symmetric hypergroups given in Chapter 5 that do not arise from finite association schemes. We will see that Example 19 does come from an association scheme; however, the underlying set of the scheme is infinite. This example is called the *Tomaselli hypergroup*. We will also see that Examples 20, 21 and 22 do not come from association schemes. Together with Example 23, these hypergroups are called the *Regensburg hypergroups*. The results of this chapter are due to the collaborative work of Christopher French and Paul-Hermann Zieschang in [1] and [7], respectively. For the remainder of the chapter, the set S stands for an association scheme.

We begin by constructing the scheme that is isomorphic to the *Tomaselli* hypergroup. Let X be the set of all finite sequences in $\mathbb{Q}$. For every two sequences $x = \{x_m\}$ and $y = \{y_n\}$ in X, the relation $r \in S$ is defined to be the set of all elements (x, y) such that the following hold:

(i) *We have* $m \leq n$*.*

(ii) *We have* $x_i = y_i$ *for every* i *with* $1 \leq i < m$*.*

(iii) *We have* $x_m \leq y_m$*.*

(iv) *The sequences* x *and* y *are not the same.*

By definition, the set r^* collects the elements (y, x) in $X \times X$ where $(x, y) \in r$. Furthermore, let $s := (X \times X) \setminus \{1, r, r^*\}$. Then, it follows that $s^* = s$.

Lemma 6.1 *Suppose* $S = \{1, r, r^*, s\}$*. Then,* S *is an association scheme on* X*.*

Proof. We refer to French [1], private communication. □

Corollary 6.2 *The hypergroup of order 4 in Example 19 comes from an association scheme on an infinite set.*

Proof. The result follows immediately from Lemma 6.3. □

Lemma 6.3 *Suppose S is an association scheme and let s and r be elements in S with $s \in rr^*$ and $ss \cap sr = \{r^*\}$. If s is symmetric, then r is also symmetric.*

Proof. Suppose X possesses elements v and w with $(v, w) \in s$. Since $w \in vs$ and $s \in rr^*$ we get that $w \in rr^*$. That means there exists an element $y \in vr$ such that $w \in yr^*$.
Note that since we are assuming $r^* \in ss$ and s is symmetric, $r \in ss$. Since $y \in vr$ and $r \in ss$ we have $y \in vss$. Then there exists an element $x \in vs$ such that $y \in xs$. In a similar fashion we see that there exists an element $z \in ws$ such that $y \in zs$.

From $v \in ws$ and the symmetricity of s we obtain that $v \in ws$. Together with $x \in vs$ we get that $x \in wss$. Additionally, since $y \in wr$ and $x \in ys$ we get that $x \in wrs$. Then, from our assumption that $ss \cap rs = \{r^*\}$, we get that $x \in wr^*$. A similar construction shows that $z \in vr^*$. Now, as $y \in zs$ and $x \in ys$ we obtain that $x \in zss$, and from $v \in zr$ and $x \in vs$ we get that $x \in zrs$. Then $x \in zr^*$. The same reasoning gives us $z \in xr^*$, and it forces $r^* = r$. □

Corollary 6.4 *The hypergroups of order 4 in Examples 20 and 21 do not arise from schemes.*

Proof. From the hypermultiplication tables of Examples 20 and 21, we see that both hypergroups satisfy the assumption of Lemma 6.3; however, r is not symmetric. □

Lemma 6.5 *Suppose S is a scheme on a set X. Assume that s and r are in S such that*

$$rs = \{r^*, s\}, \quad r^*r^* \cap rr = \{s\}, \quad sr^* \cap rs = \{s\} \quad \textit{and} \quad r^*r \cap rs = \{r^*\}.$$

If x, y and z are elements in X such that $(x, y) \in s, (y, z) \in s$ and $(z, x) \in s$, then $xr \cap yr^* \subseteq zs$.

Proof. Let $u \in xr \cap yr^*$. We need to show that $u \in zs$. Since $y \in ur$ and $z \in ys$, we get that $z \in urs$. Then together with our assumption that $rs = \{r^*, s\}$, we obtain that $z \in ur^*$ or $z \in us$. Let us assume $z \notin us$. We obtain that $z \in ur^*$ and that s is not the same as r^*.

Since $x \in ys$ and $s \in rr$ we obtain that $x \in yrr$. Then there exists an element $v \in yr$ such that $x \in vr$. Observing that $x \in ur^*$ and $v \in xr^*$ we get that $v \in urr^*$. Similarly, as $y \in ur$ and $v \in yr$, it follows that $v \in urr$. From our assumption that $r^*r^* \cap rr = \{s\}$ we get that $v \in us$.

Since $u \in vs$ and $z \in ur^*$, we get that $z \in vsr^*$. Furthermore, from $x \in vr$ and $z \in xs$ we obtain that $x \in vrs$. Since we assumed that $sr^* \cap rs = \{s\}$, it follows that $z \in vs$.

We know $z \in ys$ and $s \in rr$. Then, $z \in yrr$. Thus, there exists $w \in yr$ such that $z \in wr$. Knowing that $y \in vr^*$ and $w \in yr$ we obtain $w \in vrr^*$. Additionally, as $z \in rs$ and $w \in zr^*$ we have $w \in vsr^*$. Then, as $r^*r \cap sr^* = \{r\}$, we get that $w \in vr$.

Since $v \in wr^*$ and $x \in vr$, we have $x \in wr^*r$. We also have $z \in wr$ and $x \in zs$ yielding $x \in wrs$. Then, together with $r^*r \cap rs = \{r^*\}$, we get that $x \in wr^*$. Thus we obtain that (u, w) is in $r^*r \cap rr \cap r^*r^* = r^*r \cap \{s\} = \emptyset$, a contradiction. Then $z \in us$, and that means $u \in zs$. □

Theorem 6.6 *Suppose S is an association scheme on a set X. Assume there exists elements s and r in S such that $s \in ss$ and*

$$rs = \{r^*, s\}, \quad r^*r^* \cap rr = \{s\}, \quad sr^* \cap rs = \{s\} \quad \text{and} \quad r^*r \cap rs = \{r^*\}.$$

Then our scheme is the singleton $\{1\}$.

Proof. From $r^* \in rs$ we obtain that $s \in rr$. Since $y \in xs$, we get that $y \in xrr$. Then, there exists an element $v \in xr$ with $y \in vr$. Noting that $v \in xr \cap yr^*$ we obtain by Lemma 6.2 that $v \in zs$. From $s \in rr$ and $z \in xs$ it follows that $z \in xrr$. That means there exists an element $w \in xr$ such that $z \in wr$. Thus, as $w \in xr \cap zr^*$. From Lemma 6.2 follows that $w \in ys$. Since $(v, w) \in rs \cap sr^* \cap r^*r = \{r^*\} \cap \{s\}$, we get that r^* and s must be the same. Thus we obtain $r = (r^*)^* = s^* = s$; therefore $S = \{1\}$. □

Corollary 6.7 *The hypergroup of order 4 in Example 22 does not come from an association scheme.*

Proof. The hypergroup satisfies the conditions stated in Theorem 6.6, but S is not $\{1\}$. □

CHAPTER VII

HYPERGROUPS OF ORDER 6 WITH AN INVOLUTION

In this chapter the letter S stands for a hypergroup with six elements and the letter T stands for a non-normal closed subset of S of cardinality 2 and three cosets in S. We found that at least four isomorphism classes of hypergroups in $\mathcal{S}$ arise from finite association schemes. Throughout the chapter, the set S and the set T remain with the conditions stated above.

Since $|T| = 2$, we get that the unique element in T different from 1 is an involution, and we denote it by l. From Corollary 3.17 (ii), we get that S possesses elements p, q and r such that p and q are symmetric but r is not symmetric, $S = \{1, l, r, r^*, p, q\}$ and $S/T = \{T, \{p, r\}, \{q, r^*\}\}$. Moreover, $rT = \{p, r\}$ and $r^*T = \{q, r^*\}$ because $1 \in T$ and therefore $r \in rT$ and $r^* \in r^*T$.

By Lemma 3.23 (i), we see that $p \in rl$ and $rl \subseteq \{p, r\}$; similarly, we obtain that $1 \in r^*l$ and that $r^*l \subseteq \{q, r^*\}$. Thus, we obtain the following cases:

(i) *We have* $rl = \{p\}$ *and* $r^*l = \{q\}$.

(ii) *We have* $rl = \{p\}$ *and* $r^*l = \{q, r^*\}$.

(iii) *We have* $rl = \{p, r\}$ *and* $r^*l = \{q\}$.

(iv) *We have* $rl = \{p, r\}$ *and* $r^*l = \{q, r^*\}$.

In Case (i) we find fourteen isomorphism classes of S. Case (ii) possesses exactly eleven isomorphism classes of S. Moreover, each of the hypergroups obtained in Case (iii) is isomorphic to a hypergroup in Case (ii). Case (iv) remains under investigation; cf. [7].

We give four isomorphism classes obtained from Case (i) and (ii) that come from association schemes. Examples 1, 2 and 3 are found in Case (i) while Example 4 comes from Case (ii).

Proposition 7.1 *Suppose that $rl = \{p\}$ and $r^*l = \{q\}$. Then S possesses one of the following isomorphism classes. Moreover, each isomorphism class arises from an association scheme on a finite set.*

1	1	l	r^*	r	p	q
1	$\{1\}$	$\{l\}$	$\{r^*\}$	$\{r\}$	$\{p\}$	$\{q\}$
l	$\{l\}$	$\{1\}$	$\{p\}$	$\{q\}$	$\{r^*\}$	$\{r\}$
r	$\{r\}$	$\{p\}$	$\{1,q\}$	$\{p,q\}$	$\{r,r^*\}$	$\{l,r^*\}$
r^*	$\{r^*\}$	$\{q\}$	$\{p,q\}$	$\{1,p\}$	$\{l,r\}$	$\{r,r^*\}$
p	$\{p\}$	$\{r\}$	$\{r,r^*\}$	$\{l,r^*\}$	$\{1,q\}$	$\{p,q\}$
q	$\{q\}$	$\{r^*\}$	$\{1,r\}$	$\{r,r^*\}$	$\{p,q\}$	$\{1,p\}$

Table 48: $HM_{10}(10)$

2	1	l	r^*	r	p	q
1	$\{1\}$	$\{l\}$	$\{r^*\}$	$\{r\}$	$\{p\}$	$\{q\}$
l	$\{l\}$	$\{1\}$	$\{p\}$	$\{q\}$	$\{r^*\}$	$\{r\}$
r	$\{r\}$	$\{p\}$	$\{1,r,r^*\}$	$\{r,r^*\}$	$\{p,q\}$	$\{l,p,q\}$
r^*	$\{r^*\}$	$\{q\}$	$\{r,r^*\}$	$\{1,r,r^*\}$	$\{l,p,q\}$	$\{p,q\}$
p	$\{p\}$	$\{r\}$	$\{p,q\}$	$\{l,p,q\}$	$\{1,r,r^*\}$	$\{r,r^*\}$
q	$\{q\}$	$\{r^*\}$	$\{l,p,q\}$	$\{p,q\}$	$\{r,r^*\}$	$\{1,r,r^*\}$

Table 49: $HM_{10}(14), HM_8(22), HM_{73}(30)$

3	1	l	r^*	r	p	q
1	$\{1\}$	$\{l\}$	$\{r^*\}$	$\{r\}$	$\{p\}$	$\{q\}$
l	$\{l\}$	$\{1\}$	$\{p\}$	$\{q\}$	$\{r^*\}$	$\{r\}$
r	$\{r\}$	$\{p\}$	$\{1,p,q\}$	$\{p,q\}$	$\{r,r^*\}$	$\{l,r,r^*\}$
r^*	$\{r^*\}$	$\{q\}$	$\{p,q\}$	$\{1,p,q\}$	$\{l,r,r^*\}$	$\{r,r^*\}$
p	$\{p\}$	$\{r\}$	$\{r,r^*\}$	$\{l,r,r^*\}$	$\{1,p,q\}$	$\{p,q\}$
q	$\{q\}$	$\{r^*\}$	$\{l,r,r^*\}$	$\{r,r^*\}$	$\{p,q\}$	$\{1,p,q\}$

Table 50: $HM_{37}(18), HM_{21}(26)$

Proof. See Theorem 2.3.1 in [7] and refer to Example 3, Example 6 and Example 7. Each of the previous isomorphism classes of S appear in the Hanaki-Miyamoto Classification of Association Schemes with Small Vertices; cf. [8]. $\square$

Now, we look at Case (ii), where $rl = \{p\}$ and $r^*l = \{q, r\}$. Recall that in Chapter 4 we considered a subset rr^* of S different from 1 and we obtained several cases of what rr^* may be. For the last isomorphism class of S we assume $rr^* = \{1, l, p\}$.

Proposition 7.2 *Suppose $rl = \{p\}$ and that $r^*l = \{q, r^*\}$. If $rr^* = \{1, l, p\}$, then S possesses one of the following isomorphism classes. Moreover, each isomorphism class arises from an association scheme on a finite set.*

4	1	l	r^*	r	p	q
1	$\{1\}$	$\{l\}$	$\{r^*\}$	$\{r\}$	$\{p\}$	$\{q\}$
l	$\{l\}$	T	$\{p, r^*\}$	$\{q\}$	$\{r^*\}$	$\{q, r\}$
r	$\{r\}$	$\{p, r\}$	$\{1, p, q\}$	$\{q, r^*\}$	$\{q\}$	$\{l, q, r, r^*\}$
r^*	$\{q\}$	$\{q\}$	$\{q, r\}$	$\{1, l, q\}$	$\{l, r^*\}$	$S \setminus T$
p	$\{p\}$	$\{r\}$	$\{q\}$	$\{l, r\}$	$\{1, p\}$	$\{q, r^*\}$
q	$\{q\}$	$\{q, r^*\}$	$\{l, q, r, r^*\}$	$S \setminus T$	$\{q, r\}$	S

Table 51: $HM_{19}(21)$

Proof. See Proposition 2.3.4 in [7]. Example 4 appears in the Hanaki Miyamoto Classification of Association Schemes; cf. [8]. $\square$

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BIOGRAPHICAL SKETCH

Jordy Cheyem Lopez was born in H. Matamoros, Tamaulipas, Mexico, on August 27, 1993, and he was the first child of Dr. Jose Maria López Valdez and Dr. Emilia Garcia Pineda, both of who have the teaching profession. He studied in the bilingual school Villa Freinet of the same city and then in Faith Christian Academy and Valley Christian High School in Brownsville, Texas. In the latter school, he became the Student Council President, Valedictorian and was accepted with full tuition aid to the Scorpion Scholars Program of The University of Texas at Brownsville and Texas Southmost College. Lopez studied mathematics in the 4 Plus 1 Program in Mathematics and received both the Bachelor and Master of Science in Mathematics from The University of Texas Rio Grande Valley, the successor of UT Brownsville, on May 13, 2016. Additionally, he earned a Minor in Art and focused on painting and drawing.

Among his areas of interest, Lopez studied hypergroups and their relationships to association schemes and buildings. He gave a presentation on the Baker-Campbell-Hausdorff Formula for Lie Groups and their Lie Algebras. He also participated in the weekly math seminars of the University. Lopez presented his thesis defense on hypergroups of order at most 6 on April 27, 2016.

While at UTB and UTRGV, Lopez worked as a tutor in various programs. He worked for the Math Department as a Teacher Assistant and helped in areas such as calculus, discrete math, modern algebra and statistics. He was also a tutor for the College of Science, Mathematics and Technology, the Learning Enrichment Program and Title V. Lopez participated in several art shows in Mexico and in the United States, and he received a Mexican consul's art recognition twice.

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