Duffing-van der Pol type oscillator

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DUFFING-VAN DER POL TYPE OSCILLATOR

A Thesis

by

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ABSTRACT

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The nonlinear Duffing-van der Pol oscillator system is studied by means of the Lie symmetry reduction method and the Preller-Singer method. With the particular case of coefficients, this system has physical relevance as a simple model in certain flow-induced structural vibration problems. Under certain parametric conditions, we are concerned with the first integrals of the Duffing-van der Pol oscillator system. After making a series of variable transformations, we apply the Preller-Singer method and the Lie symmetry reduction method to obtain the first integrals of the simplified equations without complicated calculations.

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CHAPTER I

INTRODUCTION

Following the book of P.E. Hydon [11], I will give the brief introduction of Lie symmetry.

1.1 Symmetries of Planar Objects

In order to understand symmetries of differential equations, firstly we can consider the symmetries of planar objects. Roughly speaking, a symmetry of a geometrical object is a transformation whose action leaves the object apparently unchanged. For instance, consider the result of rotating an equilateral triangle anticlockwise about its centre. After a rotation of $2\pi/3$, the triangle looks the same as it did before the rotation, so this transformation is a symmetry. Rotations of $4\pi/3$ and 2π are also symmetries of the equilateral triangle. In fact, rotating by 2π is equivalent to doing nothing, because each point is mapped to itself. The transformation mapping each point to itself is a symmetry of any geometrical object: it is called the trivial symmetry. In summary, a transformation is a symmetry if it satisfies the following:

(S1) The transformation preserves the structure,

(S2) The transformation is a diffeomorphism,

(S3) The transformation maps the object to itself.

Henceforth, we restrict attention to transformation satisfying (S1) and (S2). Such transformations are symmetries if they also satisfy (S3), which is called the symmetry condition.

1.2 Symmetry Condition

For simplicity, we shall consider only ODEs of the form

$$y^{(n)} = \omega(x, y, y', \cdots, y^{(n-1)}), \qquad y^{(k)} \equiv \frac{d^k y}{dx^k}. \tag{1}$$

It is assumed that ω is (locally) a smooth function of all of its arguments. We begin by stating the symmetry condition and examining some of its consequences. A symmetry of equation (1) is a diffeomorphism that maps the set of solutions of the ODE to itself. Any diffeomorphism,

$$\Gamma : (x, y) \mapsto (\hat{x}, \hat{y}),$$

maps smooth planar curves to smooth planar curves. This action of Γ on the plane induces an action on the derivatives $y^{(k)}$, which is the mapping

$$\Gamma : (x, y, y', \cdots, y^{(n)}) \mapsto \left(\hat{x}, \hat{y}, \hat{y}', \cdots, \hat{y}^{(n)}\right),$$

where

$$y^{(k)} = \frac{d^k \hat{y}}{d\hat{x}^k}, \qquad k = 1, \cdots, n.$$

This mapping is called the nth prolongation of Γ. The functions $\hat{y}^{(k)}$ are calculated recursively (using the chain rule) as follows:

$$\hat{y}^{(k)} = \frac{d\hat{y}^{(k-1)}}{d\hat{x}} = \frac{D_x \hat{y}^{k-1}}{D_x \hat{x}}, \qquad \hat{y}^{(0)} \equiv \hat{y}. \tag{2}$$

Here D_x is the total derivative with respect to x:

$$D_x = \partial_x + y' \partial_y + y'' \partial_{y'} + \cdots.$$

The symmetry condition for the ODE (1) is

$$\hat{y}^{(n)} = \omega(\hat{x}, \hat{y}, \hat{y}', \cdots, \hat{y}^{(n-1)}), \quad \textit{when equation } (1) \textit{ holds}, \tag{3}$$

where the functions $\hat{y}^{(k)}$ are given by (2).

For almost all OEDs, the symmetry condition (3) is nonlinear. Lie symmetries are obtained by linearizing (3) about $\varepsilon = 0$. No such linearization is possible for discrete symmetries, which makes them hard to find. However, It is usually easy to find out whether or not a given diffeomorphism is a symmetry of a particular ODE. The trivial symmetry corresponding to $\varepsilon = 0$ leaves every point unchanged. Therefore, for ε sufficiently close to zero, the prolonged Lie symmetries are of the form

$$\begin{aligned}
&\hat{x} = x + \varepsilon\xi + \mathbb{O}(\varepsilon^2), \\
&\hat{y} = y + \varepsilon\eta + \mathbb{O}(\varepsilon^2), \\
&\hat{y}^{(k)} = y^{(k)} + \varepsilon\eta^{(k)} + \mathbb{O}(\varepsilon^2), \;\; k \geq 1.
\end{aligned} \tag{4}$$

We substitute (4) into the symmetry condition (3); the $\mathbb{O}(\varepsilon)$ terms yield the linearized symmetry condition:

$$\eta^{(n)} = \xi\omega_x + \eta\omega_y + \eta^{(1)}\omega_{y'} + \cdots + \eta^{(n-1)}\omega_{y_{(n-1)}} \textit{ when equation } (1) \textit{ holds}. \tag{5}$$

The functions $\eta^{(k)}$ are calculated recursively from (2), as follows. For $k = 1$, we obtain

$$\begin{aligned}
\hat{y}^{(1)} &= \frac{D_x\hat{y}}{D_x\hat{x}} = \frac{y' + \varepsilon D_x\eta + \mathbb{O}(\varepsilon^2)}{1 + \varepsilon D_x\xi + \mathbb{O}(\varepsilon^2)} \\
&= y' + \varepsilon(D_x\eta - y' D_x\xi) + \mathbb{O}(\varepsilon^2).
\end{aligned}$$

Therefore, from (4), we have

$$\eta^{(1)} = D_x\eta - y' D_x\xi. \tag{6}$$

Similarly, we get

$$\hat{y}^{(k)} = \frac{y^{(k)} + \varepsilon D_x \eta^{(k-1)} + \mathbb{O}(\varepsilon^2)}{1 + \varepsilon D_x \xi + \mathbb{O}(\varepsilon^2)},$$

and hence we have

$$\eta^{(k)}(x, y, y', \cdots, y^{(k)}) = D_x\eta^{(k-1)} - y^{(k)} D_x\xi. \tag{7}$$

The function ξ, η and $\eta^{(k)}$ can all be written in terms of the characteristic, $Q = \eta - y'\xi$, as follows:

$$\begin{aligned}
&\xi = -Qy', \\
&\eta = Q - y'Q_{y'} \\
&\eta^{(k)} = D_x^k Q - y^{(k+1)} Q_{y'},\ k \geq 1.
\end{aligned}$$

In order to find the symmetry group G admitted by a differential equation with infinitesimal operator

$$X = \xi\partial_x + \eta\partial_y.$$

We introduce the prolonged infinitesimal generator

$$X^{(n)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \cdots + \eta^{(n)}\partial_{y^{(n)}}.$$

We can use the prolonged infinitesimal generator to write the linearized symmetry condition (5) in a compact form:

$$X^{(n)}\left(y^{(n)} - \omega(x, y, y', \cdots, y^{(n-1)})\right) = 0 \ \textit{when equation}\ (1)\ \textit{holds}.$$

1.3 The Determining Equations for Lie Point Symmetries

Every symmetry that we have met so far is a diffeomorphism of the form

$$(\hat{x}, \hat{y}) = (\hat{x}(x,y),\ \hat{y}(x,y)).$$

This type of diffeomorphism is called a point transformation; any point transformation that is also a symmetry is called a point symmetry. For now, we restrict attention to point symmetries.

To find the Lie point symmetries of an ODE (1), we must first calculate $\eta^{(k)}, k = 1, \cdots, n$. The functions ξ and η depend upon x and y only. and therefore (6) and (7) give the following results.

$$\begin{aligned}
\eta^{(1)} &= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2; && (8)\\
\eta^{(2)} &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 \\
&\quad + \{\eta_y - 2\xi_x - 3\xi_y y'\}y''; && (9)\\
\eta^{(3)} &= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy})y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 \\
&\quad - \xi_{yyy}y'^4 + 3\{\eta_{xy} - \xi_{xx} + (\eta_{yy} - 3\xi_{xy})y' - 2\xi_{yy}y'^2\}y'' \\
&\quad - 3\xi_y y''^2 + \{\eta_y - 3\xi_x - 4\xi_y y'\}y''.
\end{aligned}$$

The number of terms in $\eta^{(k)}$ increases exponentially with k, so computer algebra is recommended for the study of high-order ODEs.

So now we restrict our attention on second-order ODEs

$$y'' = \omega(x, y, y').$$

The linearized symmetry condition is obtained by substituting (8) and (9) into (5) and then replacing y'' by $\omega(x,y,y')$. This gives

$$\eta_{xx}+(2\eta_{xy}-\xi_{xx})y'+(\eta_{yy}-2\xi_{xy})y'^2-\xi_{yy}y'^3+\{\eta_y-2\xi_x-3\xi_y y'\}\omega$$
$$=\xi\omega_x+\eta\omega_y+\{\eta_x+(\eta_y-\xi_x)y'-\xi_y y'^2\}\omega_{y'}. \tag{10}$$

Although equation (10) looks complicated, in some cases it can be solved without much trouble. Both ξ and η are independent of y', and therefore (10) can be decomposed into a system of PDEs, which are the *determining equations* for the Lie point symmetries.

CHAPTER II

DUFFING-VAN DER POL TYPE OSCILLATOR

2.1 Introduction

In this paper, we consider a general nonlinear oscillator system of the form

$$\ddot{y}+(\delta+\beta y^{m})\dot{y}-\mu y+\alpha y^{m+1}=0, \tag{11}$$

where an over-dot represents differentiation with respect to the independent variable x, and all coefficients δ, β, μ and α are real. It is referred as to the Duffing–van der Pol–type oscillator. since the choices $\alpha=0$ and $m=2$ lead equation (11) to the van der Pol oscillator

$$\ddot{y}+(\delta+\beta y^{2})\dot{y}-\mu y=0, \tag{12}$$

which was originally discovered by the Dutch electrical engineer van der Pol in electrical circuits [19, 20]. The choices $\beta=0$ and $m=2$ lead equation (11) to the damped Duffing equation [6, 10]

$$\ddot{y}+\delta\dot{y}-\mu y+\alpha y^{3}=0. \tag{13}$$

When $\beta=0$ and $m=1$, equation (11) becomes the damped Helmholtz oscillator [1, 17]

$$\ddot{y}+\delta\dot{y}-\mu y+\alpha y^{2}=0. \tag{14}$$

It is well known that there are a great number of theoretical works to deal with equations (12)–(14)

[10, 13], and applications of these three equations and the related equations can be seen in quite a few scientific areas [9].

In the present paper, we wish to show that under certain parametric conditions some first integrals of oscillator system (11) can be established.

2.2 Determining Equation System

Firstly, we consider the oscillator equation as following form:

$$\ddot{y} = -(\delta + \beta y^m)\dot{y} + \mu y - \alpha y^{m+1} = F(x, y, y'). \tag{15}$$

To investigate the integrability of this equation, the Lie theory of differential equations will be used [11]. However, it should be noted that the integrability of a differential equation can also be analyzed by means of Divisor theorem method [8]. The Lie theory is used in this work because this approach, except giving information about when the equation is integrable, allows the problem to be reduced to canonical variables which makes the integration of the equation in a more general and easier way [1].

It can be seen in [11] that in order to find the symmetry group G admitted by a differential equation with infinitesimal operator

$$X = \eta(x, y)\frac{\partial}{\partial y} + \xi(x, y)\frac{\partial}{\partial x},$$

it is needed to find an infinitesimal operator X_1 such that

$$X_1(\ddot{y} + (\delta + \beta y^m)\dot{y} - \mu y + \alpha y^{m+1}) = 0. \tag{16}$$

The operator X_1 is

$$X_1 = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + A(x,y,\dot{y})\frac{\partial}{\partial \dot{y}} + B(x,y,\dot{y},\ddot{y})\frac{\partial}{\partial \ddot{y}},$$

where $A(x,y,\dot{y})$ and $B(x,y,\dot{y},\ddot{y})$ are defined as follows [1]:

$$\begin{aligned} A(x,y,\dot{y}) &= \eta_x + \dot{y}(\eta_y - \xi_x) - \dot{y}^2\xi_y, \\ B(x,y,\dot{y},\ddot{y}) &= \eta_{xx} + \dot{y}(2\eta_{xy} - \xi_{xx}) + \dot{y}^2(\eta_{yy} - 2\xi_{xy}) - \\ &\quad \dot{y}^3\xi_{yy} + \ddot{y}(\eta_y - 2\xi_x - 3\dot{y}\xi_y). \end{aligned}$$

All $\xi(x,y)$ and $\eta(x,y)$ that verify equation (16) generate infinitesimal operators X as in equation (16) which comprise the symmetries of the differential equation. Also, it is known that one symmetry can be used to reduce by one the order of a differential equation. Thus, Duffing-van der Pol oscillator will be integrated only if $\xi(x,y)$ and $\eta(x,y)$ are such that they generate two linearly independent infinitesimal operators [1].

Following the procedure to determine the symmetries of a differential equation mentioned in the former section, equation (16) reads

$$\begin{aligned} &\eta_{xx} + (2\eta_{xy} - \xi_{xx})y'_x + (\eta_{yy} - 2\xi_{xy})(y'_x)^2 - \xi_{yy}(y'_x)^3 = \\ &(2\xi_x - \eta_y + 3\xi_y y'_x)F + \xi F_x + \eta F_y + [\eta_x + (\eta_y - \xi_x)y'_x - \xi_y(y'_x)^2]F_{y'_x}. \end{aligned} \tag{17}$$

Although equation (17) looks complicated, it is commonly easy to solve. Both ξ and η are independent of y' and therefore equation (17) can be decomposed into a system of PDEs, which are the determining equations for the Lie point symmetries. The procedure will be illustrated as follows: As ξ and η are independent of y', the linearized symmetry condition splits into the following system of determining equations:

$$
\begin{aligned}
[y']^0 &: \eta_{xx} = (\mu\eta - \delta\eta_x) + (2\mu\xi_x - \mu\eta_y)y - \\
&\quad ((m+1)\alpha\eta + \beta\eta_x)\, y^m + (\eta_y - 2\xi_x)\alpha y^{m+1}, && (18)\\
[y']^1 &: 2\eta_{xy} - \xi_{xx} = -\delta\xi_x + 3\mu\xi_y y - m\beta\eta y^{m-1} - \beta\xi_x y^m - 3\alpha\xi_y y^{m+1}, && (19)\\
[y']^2 &: \eta_{yy} - 2\xi_{xy} = -2\delta\xi_y - 2\beta\xi_y y^m, && (20)\\
[y']^3 &: \xi_{yy} = 0. && (21)
\end{aligned}
$$

From the condition in equation (21), it is obvious that

$$\xi = a(x)y + b(x). \tag{22}$$

and this result in equation (20) implies that

$$\eta = a'(x)y^2 - \delta a(x)y^2 - \frac{2\beta a(x)}{(m+1)(m+2)}y^{m+2} + c(x)y + d(x), m \neq -1, m \neq -2, \tag{23}$$

where $a(x), b(x), c(x), d(x)$ are arbitrary functions. If both results are used in equation (18), this is a polynomial of $2m+2$ degree in $[y]$ which is zero if and only if the following equations are verified:

$$
\begin{aligned}
&[y^{2m+2}] : \beta^2 a' - \alpha\beta a = 0,\\
&[y^{m+2}] : \frac{2\beta\mu a}{m+2} + \frac{2\beta\delta a'}{(m+1)(m+2)} + \\
&\qquad \frac{2\beta a''}{(m+1)(m+2)} - \alpha(m+1)a' + \alpha(m-1)a\delta - \beta a'' + \beta\delta a' = 0,\\
&[y^{m+1}] : \alpha c m + c'\beta + 2\alpha b' = 0,\\
&[y^m] : (m+1)\alpha d + d'\beta = 0, && (24)\\
&[y^2] : a''' - a'\mu - \delta^2 a' - \delta a\mu = 0,
\end{aligned}
$$

$$[y^1]: c'' + \delta c' - 2b'\mu = 0,$$
$$[y^0]: d'' - d\mu + \delta d' = 0.$$

If both results are used in equation (19), this is a polynomial of $2m+1$ degree in [y] which is zero if and only if the following equations are verified:

$$\begin{aligned}
&[y^{2m+1}]: \frac{2ma\beta^2}{(m+1)(m+2)} = 0, \\
&[y^{m+1}]: -(m+1)a'\beta + ma\delta\beta - 3\alpha a = -\frac{4\beta}{m+1}a', \\
&[y^m]: -cm\beta - \beta b' = 0, \\
&[y^{m-1}]: -md\beta = 0, \\
&[y^1]: 3a'' - 3\delta a' - 3a\mu = 0, \\
&[y^0]: 2c' - b'' = -\delta b'.
\end{aligned} \tag{25}$$

Here we restrict $\alpha \neq 0, \quad \beta \neq 0$. These two determining equation systems imply that $a(x) = 0$ and $d(x) = 0$. Then The determining system about $b(x)$ and $c(x)$ are obtained as follows:

$$cm + b' = 0, \tag{26}$$
$$c'\beta + \alpha b' = 0, \tag{27}$$
$$2c' - b'' + \delta b' = 0, \tag{28}$$
$$c'' + \delta c' - 2b'\mu = 0. \tag{29}$$

From equations (26) and (27), $b(x)$ and $c(x)$ are obtained as follows:

$$b = \frac{-c_0}{\alpha}\beta e^{\frac{\alpha m}{\beta}x} + b_0, \tag{30}$$
$$c = c_0 e^{\frac{\alpha m}{\beta}x}, \tag{31}$$

where b_0 and c_0 are arbitrary constants. Here there are only two options to verify all conditions: The first one is when $c_0 \equiv 0$. In this case b_0 can be arbitrary constant, and this means that:

$$\xi = 1, \qquad \eta = 0.$$

Hence, only one infinitesimal operator is obtained, namely $\chi_1 = \partial x$.

The second option in order to get two symmetries is assuming $c_0 \neq 0$. Substituting equations (31) and (30) into equation (28), we obtain one condition:

$$m = \frac{\delta\beta}{\alpha} - 2. \tag{32}$$

then, substituting equations (31) and (30) into equation (29), we obtain another condition:

$$\frac{\alpha^2 m}{\beta^2} = -\frac{\alpha\delta}{\beta} - 2\mu. \tag{33}$$

After combining equations (32) and (33), parametric condition is obtained as follows:

$$\delta = \frac{\alpha}{\beta} - \frac{\mu\beta}{\alpha}. \tag{34}$$

Because b_0 and c_0 are arbitrary constants, for our convenience, we may assume $b_0 = 0$ and $c_0 = 1$. Then we have:

$$b = -\frac{1}{\alpha}\beta e^{\frac{\alpha m}{\beta}x}, \qquad c = e^{\frac{\alpha m}{\beta}x},$$

which is equivalent to

$$\xi = -\frac{1}{\alpha}\beta e^{\frac{\alpha m}{\beta}x}, \qquad \eta = e^{\frac{\alpha m}{\beta}x}y.$$

After combining the first choice, thus two infinitesimal generators are found, namely

$$\chi_1 = \partial x, \qquad \chi_2 = -\frac{1}{\alpha}\beta e^{\frac{\alpha m}{\beta}x}\partial x + e^{\frac{\alpha m}{\beta}x}y\partial y.$$

Every infinitesimal generator is of the form:

$$\chi = c_1\chi_1 + c_2\chi_2,$$

where χ_1 is a homothety operator and χ_2 is a translation operator.

In conclusion, only when it is verified that $\delta = \frac{\alpha}{\beta} - \frac{\mu\beta}{\alpha}$, the oscillator is completely integrable. Otherwise, the oscillator is only partially integrable and there is no way to write down the solution in terms of known functions.

2.3 Reduction to Canonical Variables

We know that if an ordinary differential equation admits an infinitesimal generator, then there exists a pair of variables:

$$t = f(x,y), \qquad u = g(x,y),$$

called canonical variables, with f and $g(g \neq 0)$ being arbitrary particular solutions of the first-order linear partial differential equations [7]

$$\xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y} = \chi, \tag{35}$$

$$\xi(x,y)\frac{\partial g}{\partial x} + \eta(x,y)\frac{\partial g}{\partial y} = 0, \tag{36}$$

where χ is a nonzero constant and can be chosen arbitrarily. Suppose that the general solution of the characteristic equation

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)},$$

has the form $U(x,y) = C$, where C is arbitrary, then the general solutions of (35) and (36) can be expressed by

$$f(x,y) = \chi \int \frac{dx}{\xi^*(x,U)} + \Phi_1(U), \tag{37}$$

$$g(x,y) = \Phi_2(U), U = U(x,y), \tag{38}$$

where $\Phi_1(U)$ and $\Phi_2(U)$ are the arbitrary functions, $\xi^*(x,U(x,y)) \equiv \xi(x,y)$, and U in the integral is regarded as a parameter later. Choosing $\chi = m$ in (35) and using (37) and (38), we obtain a particular solution:

$$f(x,y) = e^{-\frac{\alpha m}{\beta}x}, \qquad g(x,y) = ye^{\frac{\alpha}{\beta}x}. \tag{39}$$

Since $t = f(x,y)$ and $u = g(x,y)$, formula (39) is equivalent to the parametric form:

$$x = -\frac{\beta}{\alpha m}\ln t, \qquad y = ut^{\frac{1}{m}}. \tag{40}$$

By this nonlinear transformation, we have:

$$\begin{aligned} \frac{\partial y}{\partial x} &= \frac{\partial y}{\partial t}\frac{\partial t}{\partial x}, \\ &= -\frac{\alpha m}{\beta}u'_t t^{\frac{m+1}{m}} - \frac{\alpha}{\beta}ut^{\frac{1}{m}}, \end{aligned} \tag{41}$$

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \frac{\partial y'_t}{\partial t}\frac{\partial t}{\partial x}, \\ &= \frac{\alpha^2 m^2}{\beta^2}u''_t t^{\frac{2m+1}{m}} + \frac{\alpha^2 m(m+2)}{\beta^2}t^{\frac{m+1}{m}}u'_t + \frac{\alpha^2}{\beta^2}t^{\frac{1}{m}}u. \end{aligned} \tag{42}$$

Substituting equations (41) and (42) into equation (15), we obtain

$$\begin{aligned} & t^{2m+1}(\frac{\alpha^2 m^2}{\beta^2}u'' - \alpha m u' u^m) + t^{m+1}(\frac{\alpha^2 m(m+2)}{\beta^2}u' - \frac{\delta \alpha m}{\beta}u' - \alpha u^{m+1} + u^{m+1}) \\ & + t(\frac{\alpha^2}{\beta^2}u - \frac{\alpha \delta u}{\beta} - \mu u) = 0. \end{aligned} \tag{43}$$

Here we restrict $\alpha = 1$. Under the parametric condition (34), equation (43) changes into autonomous equation:

$$\frac{m}{\beta^2}u''_{tt} = u'_t u^m, \tag{44}$$

which is easily integrated :

$$u'_t = \frac{\beta^2}{m(m+1)}u^{m+1} + I. \tag{45}$$

Now we do the reverse transformation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} \\ &= -\frac{\beta}{m}y' e^{\frac{m+1}{\beta}x} - \frac{1}{m}ye^{\frac{m+1}{\beta}x}. \end{aligned} \tag{46}$$

Substituting equations (46) into (45), we obtain the first integral of equation (15):

$$\left(y' + \frac{1}{\beta}y + \frac{\beta}{m+1}y^{m+1}\right)e^{\frac{m+1}{\beta}x} = I. \tag{47}$$

CHAPTER III

SPECIAL CASES

Following the previous procedure, we can apply this method into some special cases of Duffing-van der Pol type oscillator.

3.1 Duffing Type Oscillator

Assume $\alpha \neq 0$ and $\beta \equiv 0$, the equation (11) changes into the following form:

$$\ddot{y} + \delta \dot{y} - \mu y + \alpha y^{m+1} = 0. \tag{48}$$

Here we restrict $m \neq 0$ and $m \neq 1$. By the previous result, the system about $b(x)$ and $c(x)$ is obtained as follows:

$$cm + 2b' = 0, \tag{49}$$

$$2c' - b'' + \delta b' = 0, \tag{50}$$

$$c'' + \delta c' - 2b'\mu = 0. \tag{51}$$

From equations (49) and (50), we can assume $b(x)$ and $c(x)$ are obtained as follows:

$$b = \frac{-c_0(m+4)}{2\delta} e^{\frac{\delta m}{m+4}x} + b_0, \tag{52}$$

$$c = c_0 e^{\frac{\delta m}{m+4}x}, \tag{53}$$

where b_0 and c_0 are arbitrary constants. Here there are also two options to verify all conditions:

The first one is when $c_0 \equiv 0$. This means that:

$$\xi = 1, \qquad \eta = 0.$$

Hence, one infinitesimal operator is obtained, namely $\chi_1 = \partial x$.

The second option in order to get two symmetries is assuming $c_0 \neq 0$. Substituting equations (53) and (52) into equation (51), we obtain the parametric condition:

$$\mu = -\frac{2m+4}{(m+4)^2}\delta^2. \tag{54}$$

For our convenience, we choose $b_0 = 0$ and $c_0 = 1$. Then we have:

$$b = \frac{-(m+4)}{2\delta} e^{\frac{\delta m}{m+4}x}, \qquad c = e^{\frac{\delta m}{m+4}x}.$$

Combining $a(x) = 0, d(x) = 0$, consequently we obtain:

$$\xi = \frac{-(m+4)}{2\delta} e^{\frac{\delta m}{m+4}x}, \qquad \eta = e^{\frac{\delta m}{m+4}x} y.$$

Therefore infinitesimal generators are found, namely

$$\chi = \frac{-(m+4)}{2\delta} e^{\frac{\delta m}{m+4}x} \partial x + e^{\frac{\delta m}{m+4}x} y \partial y.$$

In conclusion, only when it is verified that $\mu = -\frac{2m+4}{(m+4)^2}\delta^2$, the oscillator is completely integrable.

Following the previous procedure, choosing $\chi = \frac{m}{2}$ in (35) and using (37) and (38), we obtain a particular solution:

$$f(x,y) = e^{-\frac{\delta m}{m+4}x}, \qquad g(x,y) = y e^{\frac{2\delta}{m+4}x}. \tag{55}$$

Since $t = f(x,y)$ and $u = g(x,y)$, formula (55) is equivalent to the parametric form:

$$x = -\frac{m+4}{\delta m}\ln t, \qquad y = ut^{\frac{2}{m}}.$$

By this nonlinear transformation, we have:

$$\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{\partial y}{\partial t}\frac{\partial t}{\partial x}, \\
&= -\frac{\delta m}{m+4}u_t' t^{\frac{m+2}{m}} - \frac{2\delta}{m+4}ut^{\frac{2}{m}}, \qquad (56) \\
\frac{\partial^2 y}{\partial x^2} &= \frac{\partial y_t'}{\partial t}\frac{\partial t}{\partial x}, \\
&= \frac{\delta^2 m^2}{(m+4)^2}u_t'' t^{\frac{2(m+1)}{m}} + \frac{\delta^2 m}{(m+4)}t^{\frac{2+m}{m}}u_t' + \frac{4\delta^2}{(m+4)^2}t^{\frac{2}{m}}u. \qquad (57)
\end{aligned}$$

Substituting equations (56) and (57) into equation (15), we obtain

$$\begin{aligned}
&\left(\frac{4\delta^2}{(m+4)^2}u - \frac{2\delta^2}{m+4}u - \mu u\right)t^{\frac{2}{m}} + \\
&\left(\frac{m(m+2)}{(m+4)^2}\delta^2 u_t' + \frac{2m}{(m+4)^2}\delta^2 u_t' - \frac{m\delta^2}{m+4}u_t'\right)t^{\frac{m+2}{m}} + \\
&\left(\left(\frac{m\delta}{m+4}\right)^2 u_t'' + \alpha u^{m+1}\right)t^{\frac{2m+2}{m}} = 0. \qquad (58)
\end{aligned}$$

Under the parametric condition (54), equation (58) change into:

$$\frac{m^2\delta^2}{(m+4)^2}u_{tt}'' = -\alpha u^{m+1},$$

which is easily integrated:

$$(u_t')^2 = -\frac{2(m+4)^2}{m^2\delta^2}\frac{\alpha}{m+2}u^{m+2} + I. \qquad (59)$$

Now we do the reverse transformation:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t}, \\
&= \left(-\frac{m+4}{\delta m}y' - \frac{2y}{m}\right) e^{\frac{(m+2)}{m+4}\delta x}.
\end{aligned} \tag{60}
$$

Substituting equations (60) into (59), we obtain the first integral of equation (48):

$$
\left(\frac{(m+4)^2}{(\delta m)^2}(y')^2 + \frac{4}{m^2}y^2 + \frac{4(m+4)}{m^2\delta}yy' + \frac{2\alpha(m+4)^2}{m^2\delta^2(m+2)}y^{m+2}\right) e^{\frac{2\delta(m+2)}{m+4}x} = I, \tag{61}
$$

under the condition

$$
\mu = -\frac{2m+4}{(m+4)^2}\delta^2.
$$

Base on this result, now we consider one famous special case of it.

Case: Duffing Equation

The choices $\beta = 0$ and $m = 2$ lead equation (11) to the damped Duffing equation [6, 10]

$$
\ddot{y} + \delta\dot{y} - \mu y + \alpha y^3 = 0. \tag{62}
$$

Substituting $m = 2$ into equation (61), we obtain the first integral of equation (62):

$$
\left(\frac{9}{\delta^2}(y')^2 + y^2 + \frac{6}{\delta}yy' + \frac{9\alpha}{2\delta^2}y^4\right) e^{\frac{4}{3}\delta x} = I, \tag{63}
$$

under the condition $\mu = -\frac{2}{9}\delta^2$.

3.2 Helmholtz Oscillator

In this section we assume $\beta \equiv 0$ and $m \equiv 1$ in equation (11), we can obtain the Holmholtz oscillator,

$$\ddot{y} + \delta \dot{y} - \mu y + \alpha y^2 = 0, \tag{64}$$

which is a simple nonlinear oscillator with quadratic nonlinearity [1]. Under the condition $\beta \equiv 0$ and $m \equiv 1$, from the equations (22) and (23), it is obvious that

$$\xi = a(x)y + b(x), \tag{65}$$

and

$$\eta = a'(x)y^2 - \delta a(x)y^2 + c(x)y + d(x), m \neq -1, m \neq -2, \tag{66}$$

where $a(x), b(x), c(x), d(x)$ are arbitrary functions. If both results are used in equation (18) This is a polynomial of 3 degree in $[y]$ which is zero if and only if the following equations are verified:

$$\begin{aligned}
&[y^3] : -2\alpha a' + \alpha a \delta = 0, \\
&[y^2] : \alpha c + 2\alpha b' = 0, \\
&[y^1] : 2\alpha d + c'' + \delta c' - 2b' \mu = 0, \\
&[y^0] : d'' - d\mu + \delta d' = 0.
\end{aligned}$$

If both results are used in equation (19) This is a polynomial of 2 degree in $[y]$ which is zero if and only if the following equations are verified:

$$\begin{aligned}
&[y^2] : -3\alpha a = 0, \\
&[y^1] : 3a'' - 3\delta a' - 3a\mu = 0, \\
&[y^0] : 2c' - b'' = -\delta b'.
\end{aligned}$$

These two determining systems imply that $a(x) = 0$, Then we obtain a system about $b(x)$ and $c(x)$

$$c + 2b' = 0, \tag{67}$$

$$2c' - b'' + \delta b' = 0, \tag{68}$$

$$2\alpha d + c'' + \delta c' - 2b'\mu = 0 \tag{69}$$

$$d'' + d\mu + \delta d' = 0. \tag{70}$$

The equations (67) and (68) imply that

$$c = c_0 e^{\frac{\delta}{5}x},$$
$$b = -\frac{5}{2\delta} c_0 e^{\frac{\delta}{5}x} + b_0,$$

where b_0 and c_0 are constant. When this result is used in equation (69), it is obtained that

$$d = -\frac{1}{2\alpha}\left(\frac{6}{25}\delta^2 + \mu\right)c$$

and finally, this result in equation(70) means that

$$-\frac{1}{2\alpha}c\left(\frac{6}{25}\delta^2 + \mu\right)\left(\frac{6}{25}\delta^2 - \mu\right) = 0 \tag{71}$$

If $c_0 \equiv 0$, in this case b_0 can be arbitrary constant, and this means that:

$$\xi = 1, \qquad \eta = 0.$$

Hence, only one infinitesimal operator is obtained, namely $\chi_1 = \partial x$.

If we assume $c_0 \neq 0$ we can deduce two different first integrals for the Helmholtz oscillator immediately:

Case 1. when $\mu = \frac{6}{25}r^2$, because b_0 and c_0 are arbitrary constants, for our convenience. assume $b_0 = 0$ and $c_0 = 1$ we have:

$$b = \frac{-5}{2\delta}e^{\frac{\delta}{5}x}, \quad c = e^{\frac{\delta}{5}x}, \quad d = \frac{-6}{25\alpha}\delta^2 e^{\frac{\delta}{5}x}.$$

Then, we obtain

$$\xi = \frac{-5}{2\delta}e^{\frac{\delta}{5}x}, \qquad \eta = e^{\frac{\delta}{5}x}\left(y - \frac{6\delta^2}{25\alpha}\right).$$

Therefore two infinitesimal generators are found, namely

$$\chi_1 = \partial x, \qquad \chi_2 = \frac{-5}{2\delta}e^{\frac{\delta}{5}x}\partial x + e^{\frac{\delta}{5}x}\left(y - \frac{6\delta^2}{25\alpha}\right)\partial y.$$

In conclusion, when it is verified that $\mu = \frac{6}{25}r^2$, the oscillator is completely integrable.

Following the previous procedure, choosing $\chi = \frac{1}{2}$ in (35) and using (37) and (38), we obtain a particular solution:

$$f(x,y) = e^{-\frac{\delta}{5}x}, \qquad g(x,y) = e^{\frac{2\delta}{5}x}\left(y - \frac{6\delta^2}{25\alpha}\right). \tag{72}$$

Since $t = f(x,y)$ and $u = g(x,y)$, formula (72) is equivalent to the parametric form:

$$x = -\frac{5}{\delta}\ln t, \qquad y = ut^2 + \frac{6\delta^2}{25\alpha}.$$

By this nonlinear transformation, we have:

$$\begin{aligned} \frac{\partial y}{\partial x} &= \frac{\partial y}{\partial t}\frac{\partial t}{\partial x}, \\ &= -\frac{\delta}{5}u_t' t^3 - \frac{2\delta}{5}ut^2, \end{aligned} \tag{73}$$

$$\begin{aligned}\frac{\partial^2 y}{\partial x^2} &= \frac{\partial y_t'}{\partial t}\frac{\partial t}{\partial x},\\ &= \frac{\delta^2}{25}u_t''t^4+\frac{\delta^2}{5}u_t't^3+\frac{4\delta^2}{25}ut^2. \end{aligned} \tag{74}$$

Substituting equations (73) and (74) into equation (15), we obtain

$$\left(\frac{6\delta^2}{25}u''+\alpha u^2\right)t^4+\left(\frac{6\delta^2}{25}u-\mu u\right)t^2+\frac{36\delta^4}{625\alpha}\mu=0, \tag{75}$$

Under the parametric condition $\mu=\frac{6\delta^2}{25}$, equation (75) changes into:

$$\frac{\delta^2}{25}u_{tt}''=-\alpha u^2,$$

which is easily integrated:

$$(u_t')^2=-\frac{50\alpha}{3\delta^2}u^3+I. \tag{76}$$

Now we do the reverse transformation:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t},\\ &= \left(-\frac{5}{\delta}y'-2y+\frac{12\delta^2}{25\alpha}\right)e^{\frac{3\delta}{5}x}. \end{aligned} \tag{77}$$

Substituting equations (77) into (76), we obtain the first integral of equation (64):

$$\left(\frac{24\delta^2}{25\alpha}y-8y^2+\frac{50\alpha}{3\delta^2}y^3+\frac{20}{\delta}yy'-\frac{24\delta}{5\alpha}y'+\frac{25}{\delta^2}(y')^2\right)e^{\frac{6\delta}{5}x}=I, \tag{78}$$

under the condition

$$\mu=\frac{6}{25}\delta^2.$$

Case 2. If we choose $\mu=-\frac{6}{5}\delta^2$. Base on the same procedure, we can obtain the first integral of equation (48):

$$\left(\frac{25}{\delta^2}(y')^2+4y^2+\frac{20}{\delta}yy'+\frac{50\alpha}{3\delta^2}y^3\right)e^{\frac{6}{5}\delta x}=I, \tag{79}$$

under the condition

$$\mu=-\frac{6}{25}\delta^2.$$

In the paper [8] and [1], they obtained the similar solutions.

3.3 van der Pol Oscillator

Assume $\alpha\equiv 0, \quad \beta\neq 0$, the equation (11) change into the following form

$$\ddot{y}+(\delta+\beta y^2)\dot{y}-\mu y=0. \tag{80}$$

It is referred as to the van der Pol oscillator, we also obtain the determining system about $b(x)$ and $c(x)$ as

$$\begin{aligned}
&c'\beta=0,\\
&cm+b'=0,\\
&2c'-b''+\delta b'=0,\\
&c''+\delta c'-2b'\mu=0.
\end{aligned}$$

Observing this system, we only have:

$$b=b_0, \qquad c=0.$$

This means only one infinitesimal operator is obtained, namely $X=\partial_x$, and as a consequence the differential equation is partially integrable.

CHAPTER IV

PRELLE–SINGER METHOD FOR SOLVING SECOND-ORDER ODES

In this chapter, in order to present our results in a straightforward way, we start our attention by briefly reviewing the Prelle–Singer procedure for solving second-order ODEs developed by Duarte et al. [5] and Chandrasekar et al. [3].

Consider the second-order ODE of the rational form

$$\frac{d^2y}{dx^2} = \phi(x, y, y') = \frac{P(x, y, y')}{Q(x, y, y')}, \quad P,\ Q \in \mathbb{C}[x, y, y']. \tag{81}$$

where y' denotes differentiation with respect to x, P and Q are polynomials in x, y and y' with coefficients in the complex field. Suppose that equation (15) admits a first integral $I(x, y, y') = C$, with C constant on the solutions, so we have the total differential

$$dI = I_x dx + I_y dy + I_{y'} dy' = 0, \tag{82}$$

where the subscript denotes partial differentiation with respect to the corresponding variable. On the solution, since $y'dx = dy$ and equation (81) is equivalent to $\frac{P}{Q}dx = dy'$, adding a null term $S(x, y, y')y'dx - S(x, y, y')dy$ to both side yields

$$\left(\frac{P}{Q} + Sy'\right) dx - Sdy - dy' = 0. \tag{83}$$

From (82) and (83), one can see that on the solutions, the corresponding coefficients of (82) and (83) should be proportional. There exists a proper integrating factor $R(x, y, y')$ for expression (83), such that on the solutions

$$dI = R(\phi + Sy')dx - SRdy - Rdy' = 0. \tag{84}$$

Comparing the corresponding terms in (82) and (84), we have

$$\begin{aligned} I_x &= R(\phi + Sy'), \\ I_y &= -SR, \\ I_{y'} &= -R, \end{aligned} \tag{85}$$

and the compatibility conditions $I_{xy} = I_{yx},\ I_{xy'} = I_{y'x}$ and $I_{yy'} = I_{y'y}$. Using these three compatibility conditions respectively, we obtain three equivalent equations as follows:

$$\begin{aligned} D[S] &= -\phi_y + S\phi_{y'} + S^2, \\ D[R] &= -R(S + \phi_{y'}), \\ R_y &= R_{y'}S + S_{y'}R, \end{aligned} \tag{86}$$

where D is an differential operator

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial y'}.$$

For the given expression of ϕ, one can solve the first equation of (86) for S. Substituting S into the second equation of (86) one can get an explicit form for R by solving it. Once a compatible solution R and S satisfying the extra constraint (the third equation of (86)) is derived, integrating

(85), from (82) one may obtain a first integral of motion as follows

$$
\begin{aligned}
I(x,y,y') \quad = \quad & \int R(\phi+Sy')dx - \int [RS + \frac{\partial}{\partial y}\int R(\phi+Sy')dx]dy - \int \{R + \\
& \frac{\partial}{\partial y'}(\int R(\phi+Sy')dx - \int [RS + \frac{\partial}{\partial y}\int R(\phi+Sy')dx]dy\}dy'.
\end{aligned} \tag{87}
$$

4.1 Nonlinear Transformations

In this subsection, in order to avoid doing complicated computations, we will make a series of nonlinear transformations to equation (11). For our convenience, we assume $\alpha = 1$ in equation (11) (this can be easily obtained by re-scaling parameters of equation (11). Namely, we consider the oscillator equation:

$$\ddot{y} + (\delta + \beta y^m)\dot{y} - \mu y + y^n = 0. \tag{88}$$

Firstly, we make the natural logarithm transformation:

$$x = -\frac{1}{\delta}\ln\tau, \tag{89}$$

that is

$$\frac{\partial\tau}{\partial x} = -\delta e^{-x\delta} = -\delta\tau.$$

After substituting the following two derivatives into equation (88):

$$
\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{\partial y}{\partial \tau} * \frac{\partial \tau}{\partial x} = -\delta\tau\frac{\partial y}{\partial \tau}, \\
\frac{\partial^2 y}{\partial \xi^2} &= \delta^2\tau\frac{\partial y}{\partial \tau} + \delta^2\tau^2\frac{\partial^2 y}{\partial \tau^2},
\end{aligned}
$$

then it becomes

$$\delta^2\tau^2\frac{\partial^2 y}{\partial\tau^2}-\beta\delta\tau y^m\frac{\partial y}{\partial\tau}-\mu y+y^n=0. \tag{90}$$

Further, we take the variable transformation as:

$$q=\tau^\kappa, \qquad y=\tau^{-\frac{1}{2}(\kappa-1)}H(q), \tag{91}$$

A direction calculation gives

$$\begin{aligned}\frac{\partial u}{\partial\tau}&=-\frac{1}{2}(\kappa-1)q^{-\frac{\kappa+1}{2\kappa}}H(q)+\kappa q^{\frac{\kappa-1}{2\kappa}}\frac{\partial H}{\partial q},\\ \frac{\partial^2 u}{\partial\tau^2}&=\frac{1}{4}(\kappa^2-1)q^{-\frac{\kappa+3}{2\kappa}}H(q)+\kappa^2 q^{\frac{3(\kappa-1)}{2\kappa}}\frac{\partial^2 H}{\partial q^2}.\end{aligned}$$

After substituting the above equalities into equation (90), we obtain

$$\frac{\partial^2 H}{\partial q^2}=\frac{\beta}{\delta\kappa}q^{\frac{m-\kappa(m+2)}{2\kappa}}H^m\frac{\partial H}{\partial q}-\frac{1}{\delta^2\kappa^2}q^{\frac{-(3+n)\kappa+n-1}{2\kappa}}H^n-\frac{1}{2}\frac{(\kappa-1)\beta}{\delta\kappa^2}H^{m+1}q^{\frac{m-\kappa(m+4)}{2\kappa}}, \tag{92}$$

where an over-dot represents differentiation with respect to the independent variable q, and

$$\kappa^2=\frac{4\mu}{\delta^2}+1. \tag{93}$$

4.2 Force-Free Duffing-van der Pol Oscillator

We know that the choices $m=2$ and $n=3$ lead equation (11) to the standard form of the Duffing-van der Pol oscillator equation, whose autonomous version (force-free) is:

$$\ddot{y}+(\delta+\beta y^2)\dot{y}-\mu y+y^3=0. \tag{94}$$

Equation (23) arises in a model describing the propagation of voltage pulses along a neuronal axon and has recently received much attention from many authors. A vast amount of literature exists on this equation; for details and applications, see [12, 14] and references therein.

From equation (93), we can see that if take $n = 3$ and $m = 2$, then equation (93) can be reduced to a simple form

$$\frac{\partial^2 H}{\partial q^2} = Aq^p H^2 \frac{\partial H}{\partial q} + Bq^{p-1}H^3, \tag{95}$$

where

$$\begin{aligned} p &= \frac{1}{\kappa} - 2, \qquad A = \frac{\beta}{\delta\kappa}, \\ B &= -\frac{1}{\delta^2\kappa^2} - \frac{(\kappa-1)\beta}{2\delta\kappa^2}. \end{aligned}$$

Choosing $\phi(q, H, H') = Aq^p H^2 \frac{\partial H}{\partial q} + Bq^{p-1}H^3$ and following the procedure in Section 2, we obtain three determining equations:

$$S_q + \dot{H}S_H + \phi S_{\dot{H}} = -2Aq^p H\dot{H} + (ASq^p - 3Bq^{p-1})H^2 + S^2, \tag{96}$$

$$R_q + R_H\dot{H} + \phi R_{\dot{H}} = -RS - RAq^p H^2, \tag{97}$$

$$R_H = R_{\dot{H}}S + S_{\dot{H}}R. \tag{98}$$

In general, it is not easy to solve system (86) and get exact solutions (S, R) in the explicit forms. But in our case of (96)-(98) we may seek an ansatz for S and R of the form:

$$S = \frac{a(q,H) + b(q,H)\dot{H}}{c(q,H) + d(q,H)\dot{H}}, \qquad R = e(q,H) + f(q,H)\dot{H}, \tag{99}$$

where a, b, c, d and e, f are functions of q, H to be determined. Substituting S into equation (96), we get the equation system:

$$
\begin{aligned}
&[\dot{H}]^0 : -3Bc^2H^2q^{p-1} + Aacq^pH^2 + a^2 = a_qc - ac_q + bcBH^3q^{p-1} - adBH^3q^{p-1},\\
&[\dot{H}]^1 : -2Ac^2q^pH - 6BcdH^2q^{p-1} + 2Aadq^pH^2 + 2ab\\
&\qquad\qquad = a_qd + b_qc - ad_q - bc_q + a_Hc - ac_H,\\
&[\dot{H}]^2 : -4Acdq^pH - 3Bd^2H^2q^{p-1} + Abdq^pH^2 + b^2\\
&\qquad\qquad = b_qd - bd_q + a_Hd + b_Hc - ad_H - bc_H,\\
&[\dot{H}]^3 : -2Ad^2q^pH = b_Hd - bd_H.
\end{aligned}
$$

Substituting S and R into equation (97), we obtain another equation system:

$$
\begin{aligned}
&[\dot{H}]^0 : e_qc + BcfH^3q^{p-1} = -ae - Aceq^pH^2,\\
&[\dot{H}]^1 : f_qc + e_Hc + 2Afcq^pH^2 + e_qd + BfdH^3q^{p-1}\\
&\qquad\qquad = -be - Adeq^pH^2 - af,\\
&[\dot{H}]^2 : f_Hc + f_qd + e_Hd + 2Afdq^pH^2 = -bf,\\
&[\dot{H}]^3 : f_Hd = 0.
\end{aligned}
$$

Under the parametric condition

$$\delta = \frac{3}{\beta} - \frac{\mu\beta}{3}, \tag{100}$$

we solve the above two nonlinear systems for a nontrivial solution with the aid of Maple, and the corresponding forms of S and R reads:

$$S = -\frac{1}{q} - \frac{\beta}{\delta\kappa} q^{\frac{1-2\kappa}{\kappa}} H^2, \quad R = e^{\ln q}, \tag{101}$$

which also satisfies equation (98).

Substituting the solution set (101) into formula (87), we can obtain the first integral of equation (94) immediately:

$$\left(\frac{1}{2}\kappa\delta+\frac{1}{2}\delta-\frac{3}{\beta}\right)q^{\frac{1-\kappa}{2\kappa}}H-\kappa\delta q^{\frac{\kappa+1}{2\kappa}}\dot{H}+\frac{\beta}{3}q^{\frac{3-3\kappa}{2\kappa}}H^3=I_1e^{\frac{3\ln q}{\kappa\beta\delta}}. \tag{102}$$

Using the inverse transformations (89) and (91), and changing to the original variables, we obtain that under the parametric condition (100), the Duffing-van der Pol equation (94) has the first integral of the form

$$\left[\dot{y}+\left(\delta-\frac{3}{\beta}\right)y+\frac{\beta}{3}y^3\right]e^{\frac{3\xi}{\beta}}=I_1. \tag{103}$$

It is remarkable that in ([3], pp.2467), ([4], pp.4528) and ([18], pp.1936), authors studied the first integral of the oscillator equation (94) by the Lie symmetry method etc. and claimed that the nontrivial first integral exists only for the parametric choice

$$\delta=\frac{4}{\beta},\qquad \mu=-\frac{3}{\beta^2}. \tag{104}$$

However, in view of our condition (100) and formula (103), it shows that our parametric constraint (100) is weaker than the corresponding ones described in the literature [3, 4, 18], and the first integral presented in [3, 4, 18] is just a particular case of (103).

4.3 Duffing-van der Pol-Type Oscillator

In this subsection, we extend the technique used in the preceding subsection to a more general oscillator equation in the case of $n=m+1$, that is

$$\ddot{y}+(\delta+\beta y^m)\dot{y}-\mu y+y^{m+1}=0, \tag{105}$$

where an over-dot still denotes differentiation with respect to x. Note that the choice $n=m+1$

leads equation (93) to a simple form

$$\frac{\partial^2 H}{\partial q^2} = \frac{\beta}{\delta\kappa} q^p H^m \frac{\partial H}{\partial q} + \left(-\frac{1}{\delta^2\kappa^2} - \frac{(\kappa-1)\beta}{2\delta\kappa^2}\right) H^{m+1} q^{p-1}, \tag{106}$$

where

$$p = \frac{m-\kappa(m+2)}{2\kappa}.$$

For the notational convenience, we denote that

$$A = \frac{\beta}{\delta\kappa}, \qquad B = -\frac{1}{\delta^2\kappa^2} - \frac{(\kappa-1)\beta}{2\delta\kappa^2},$$

then equation (106) becomes

$$\ddot{H} = Aq^p H^m \dot{H} + BH^{m+1} q^{p-1}. \tag{107}$$

Choosing $\phi(q,H,H') = Aq^p H^m \frac{\partial H}{\partial q} + Bq^{p-1} H^{m+1}$ and following the procedure in Section 2, we obtain three determining equations:

$$S_q + \dot{H} S_H + \phi S_{\dot{H}} = -mAq^p H^{m-1} \dot{H} + (ASq^p - (m+1)Bq^{p-1})H^m + S^2, \tag{108}$$

$$R_q + R_H \dot{H} + \phi R_{\dot{H}} = -RS - RAq^p H^m, \tag{109}$$

$$R_H = R_{\dot{H}} S + S_{\dot{H}} R. \tag{110}$$

Here we use the same ansatz for S and R as given in (29). Substituting S into equation (108), we get the equation system:

$$
\begin{aligned}
&[\dot{H}]^0 : -(m+1)Bc^2H^mq^{p-1} + Aacq^pH^m + a^2 \\
&\qquad = a_qc - ac_q + bcBH^{m+1}q^{p-1} - adBH^{m+1}q^{p-1}, \\
&[\dot{H}]^1 : -mAc^2q^pH^{m-1} - 2(m+1)BcdH^mq^{p-1} + Aadq^pH^m + Aq^pH^mbc + 2ab \\
&\qquad = a_qd + b_qc - ad_q - bc_q + a_Hc - ac_H + bcAq^pH^m - adAq^pH^m, \\
&[\dot{H}]^2 : -2mAcdq^pH^{m-1} - (m+1)Bd^2H^mq^{p-1} + Abdq^pH^m + b^2 \\
&\qquad = b_qd - bd_q + a_Hd + b_Hc - ad_H - bc_H, \\
&[\dot{H}]^3 : -mAd^2q^pH^{m-1} = b_Hd - bd_H.
\end{aligned}
\tag{111, 112}
$$

Substituting S and R into equation (109), we obtain another equation system:

$$
\begin{aligned}
&[\dot{H}]^0 : e_qc + BcfH^{m+1}q^{p-1} = -ae - Aceq^pH^m, \\
&[\dot{H}]^1 : f_qc + e_Hc + 2Afcq^pH^m + e_qd + BfdH^{m+1}q^{p-1} \\
&\qquad = -be - Adeq^pH^m - af, \\
&[\dot{H}]^2 : f_Hc + f_qd + e_Hd + 2Afdq^pH^m = -bf, \\
&[\dot{H}]^3 : f_Hd = 0.
\end{aligned}
\tag{113}
$$

We solve the above two nonlinear systems (112) and (113) for a nontrivial solution with the aid of Maple, and find that under the parametric conditions

$$m = \frac{(1-\kappa)\beta\delta}{2} - 1, \qquad \kappa^2 = \frac{4\mu}{\delta^2} + 1, \tag{114}$$

the three determining equations (108)–(110) have the solution of the form

$$S = -\frac{1}{q} - \frac{\beta}{\delta\kappa}q^{\frac{m(1-\kappa)}{2\kappa}-1}H^m, \qquad R = e^{\ln q}. \tag{115}$$

After substitution of the solution set (115) into formula (87), we derive the first integral of equation (107) as follows

$$\kappa\delta H - \kappa\delta q\dot{H} + \frac{2}{\delta(1-\kappa)} q^{\frac{m(1-\kappa)}{2\kappa}} H^{m+1} = I,$$

where I is an arbitrary integration constant. By virtue of the inverse transformations (89) and (91), and changing to the original variables, we obtain that under the parametric condition (114), the Duffing-van der Pol–type equation (105) has the first integral of the form

$$\left[\dot{y} + \frac{\delta(\kappa+1)}{2} y + \frac{2}{\delta(1-\kappa)} y^{m+1}\right] e^{\frac{1}{2}\delta(1-\kappa)x} = I \tag{116}$$

It is remarkable that the first integral of the Duffing-van der Pol oscillator equation (94) obtained in Section 3.2 is just a particular case of formula (116). In the recently published Handbooks of ODEs such as [2, 15, 21], there are quite a few first integrals (conservation laws) collected for ordinary differential equations of the type $y'' = c_1 x^{l_1} y^{m_1} (y')^{k_1} + c_1 x^{l_2} y^{m_2} (y')^{k_2}$, but our formulas of first integrals of equation (105) or (106) described herein are not presented there.

4.4 Solutions in the Parametric Form

In this subsection, by virtue of the first integral (116), we may choose a proper value for I_2 and consider three particular cases where exact solutions of the oscillator equation (105) can be expressed in the parametric forms.

Case 1: assume that $m \neq -1$ and $\kappa \neq -1$, and

$$\begin{aligned} m &= -\frac{2\kappa}{\kappa+1}, \\ \frac{\beta}{\delta\kappa} &= \frac{1}{\delta^2\kappa^2} + \frac{(\kappa-1)\beta}{2\delta\kappa^2}, \end{aligned} \tag{117}$$

where

$$\kappa^2 = \frac{4\mu}{\delta^2} + 1.$$

In this case, equation (106) takes the form:

$$\ddot{H} = Aq^{-m-2}H^m\dot{H} - AH^{m+1}q^{-m-3}. \tag{118}$$

From the first integral (116), taking $I_2 = 0$, we know that the solution of equation (118) can be expressed in the parametric form [15]:

$$\begin{aligned} q &= aC_1^m \left(\int \frac{dt}{1 \pm t^{m+1}} + C_2 \right)^{-1}, \\ H &= bC_1^{m+1} t \left(\int \frac{dt}{1 \pm t^{m+1}} + C_2 \right)^{-1}, \end{aligned} \tag{119}$$

where C_1 and C_2 are arbitrary constants, a and b are also arbitrary but satisfy

$$\frac{\beta}{\delta\kappa} = \mp(m+1)a^{m+1}b^{-m}. \tag{120}$$

Applying the inverse transformation of (91) to formula (119), namely

$$\tau = q^{\frac{1}{\kappa}}, \qquad H = y\tau^{\frac{1}{2}(\kappa-1)},$$

we have

$$\begin{aligned} \tau &= a^{\frac{1}{\kappa}} C_1^{\frac{m}{\kappa}} \left(\int \frac{dt}{1 \pm t^{m+1}} + C_2 \right)^{\frac{-1}{\kappa}}, \\ y &= \tau^{-\frac{1}{2}(\kappa-1)} bC_1^{m+1} t \left(\int \frac{dt}{1 \pm t^{m+1}} + C_2 \right)^{-1}. \end{aligned} \tag{121}$$

Further, applying the inverse transformation of (89) to formula (121), under the given parametric condition (117), we obtain the solution for equation (105) in the parametric form as follows:

$$x = \frac{-\ln\left(a^{\frac{1}{\kappa}} C_1^{\frac{m}{\kappa}} \left(\int \frac{dt}{1 \pm t^{m+1}} + C_2\right)^{\frac{-1}{\kappa}}\right)}{\delta},$$
$$y = e^{\frac{1}{2}\delta(\kappa-1)\xi} b C_1^{m+1} t \left(\int \frac{dt}{1 \pm t^{m+1}} + C_2\right)^{-1},$$

where a and b are arbitrary constants, and satisfy condition (120).

Case 2: assume that

$$m = -2, \quad \kappa = -2, \quad \beta\delta = -2. \tag{122}$$

So equation (106) takes the form:

$$\ddot{H} = Aq^{\frac{1}{2}} H^{-2} \dot{H} - AH^{-1} q^{-\frac{1}{2}}, \tag{123}$$

where $A = -\frac{\beta}{2\delta}$.

Using the first integral (116) again, we know that the solution of equation (123) can be expressed in the parametric form:

$$\begin{aligned} q &= aC_1^4 F^{-2}, \\ H &= bC_1^3 t^{-1} E F^{-2}, \end{aligned} \tag{124}$$

where a and b are also arbitrary but satisfy

$$\frac{\beta}{2\delta} = a^{\frac{-3}{2}} b^2, \tag{125}$$

and

$$E=\sqrt{t(t+1)}-\ln(\sqrt{t}+\sqrt{t+1})+C_2, \quad F=E\sqrt{\frac{t+1}{t}}-t. \tag{126}$$

Applying the inverse transformation of (91) to formula (124), namely

$$\tau=q^{-\frac{1}{2}}, \qquad H=y\tau^{-\frac{3}{2}},$$

we have

$$\begin{aligned} \tau &= a^{-\frac{1}{2}}C_1^{-2}F, \\ y &= \tau^{\frac{3}{2}}bC_1^3t^{-1}EF^{-2}. \end{aligned} \tag{127}$$

Further, applying the inverse transformation of (89) to formula (127), under the given parametric condition (122), we obtain the solution for equation (105) in the parametric form as follows:

$$\begin{aligned} x &= \frac{\ln\left(aC_1^4F^{-2}\right)}{2\delta}, \\ y &= e^{-\frac{3}{2}\delta\xi}bC_1^3t^{-1}EF^{-2}, \end{aligned}$$

where a and b are arbitrary constants, and satisfy condition (125).

Case 3: assume that

$$m=-3, \quad \kappa=-3, \quad \beta\delta=-1. \tag{128}$$

In this case, equation (106) takes the form:

$$\ddot{H}=AqH^{-3}\dot{H}-AH^{-2}, \tag{129}$$

where $A=-\frac{\beta}{3\delta}$.

We know that the solution of equation (129) can be expressed in the parametric form:

$$
\begin{aligned}
q &= aC_1^3F^{-1}\sqrt{\frac{t+1}{t}},\\
H &= bC_1^2F^{-1},
\end{aligned}
\tag{130}
$$

where F is the same as that in (126), C_1 and C_2 are arbitrary constants, a and b are also arbitrary but satisfy

$$
\frac{\beta}{3\delta} = 2a^{-2}b^3. \tag{131}
$$

Applying the inverse transformation of (91) to formula (130), namely

$$
\tau = q^{-\frac{1}{3}}, \qquad H = y\tau^{-2},
$$

we have

$$
\begin{aligned}
\tau &= a^{-\frac{1}{3}}C_1^{-1}F^{\frac{1}{3}}\left(\frac{t+1}{t}\right)^{-\frac{1}{6}},\\
y &= \tau^2 bC_1^2F^{-1}.
\end{aligned}
\tag{132}
$$

Further, applying the inverse transformation of (89) to formula (132), under the given parametric condition (128), we obtain the solution for equation (105) in the parametric form as follows:

$$
\begin{aligned}
x &= \frac{\ln\left(aC_1^3F^{-1}\sqrt{\frac{t+1}{t}}\right)}{3\delta},\\
y &= e^{-2\delta\xi}bC_1^2F^{-1},
\end{aligned}
$$

where a and b are arbitrary constants, and satisfy condition (131).

4.5 Conclusion

Finding first integrals (conservation laws) and exact solutions for various nonlinear differential equations has been an interesting subject in mathematical and physical communities. Since 1983, Prelle and Singer presented a deductive method for solving first–order ODEs that presents a solution in terms of elementary functions if such a solution exists. This technique has attracted many researchers from diverse groups and has been extended to autonomous systems of ODEs of higher dimensions for finding the first integrals and exact solutions under certain assumptions. From illustrative examples in these works, the obtained first integrals of autonomous systems are usually of rational or quasi-rational forms and searching for solution sets (S, R) usually involves complicated calculations. However, the generalization of this procedure to autonomous/nonautonomous systems of higher dimensions to find elementary first integrals in an effective manner is still an interesting and important subject.

In this chapter, we showed that under certain parametric conditions, some new first integrals of the Duffing–van der Pol–type oscillator equation (11) could be established. To reach our goal, we first made a series of nonlinear transformations to simplify equation (11) to a simple form, then by means of the Preller–Singer method we derived the first integral of the resultant equation. Through the inverse transformations we obtain the first integrals of the original oscillator equations. Finally, using the established first integral, we obtain exact solutions of equation (11) in the parametric forms.

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BIOGRAPHICAL SKETCH

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