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Recommended Citation

Fu, Bin, Yumei Huo, and Hairong Zhao. "Sublinear Approximation Schemes for Scheduling Precedence Graphs of Bounded Depth." arXiv preprint arXiv:2302.00133 (2023).

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Sublinear Approximation Schemes for Scheduling Precedence Graphs of Bounded Depth

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Abstract

We study the classical scheduling problem on parallel machines where the precedence graph has the bounded depth h . Our goal is to minimize the maximum completion time. We focus on developing approximation algorithms that use only sublinear space or sublinear time. We develop the first one-pass streaming approximation schemes using sublinear space when all jobs' processing times differ no more than a constant factor c and the number of machines m is at most $\frac{2n\epsilon}{3hc}$. This is so far the best approximation we can have in terms of m , since no polynomial time approximation better than $\frac{4}{3}$ exists when $m = \frac{n}{3}$ unless P=NP. The algorithms are then extended to the more general problem where the largest αn jobs have no more than c factor difference. We also develop the first sublinear time algorithms for both problems. For the more general problem, when $m \leq \frac{\alpha n \epsilon}{20c^2 \cdot h}$, our algorithm is a randomized $(1 + \epsilon)$ -approximation scheme that runs in sublinear time. This work not only provides an algorithmic solution to the studied problem under big data environment, but also gives a methodological framework for designing sublinear approximation algorithms for other scheduling problems.

1. Introduction

Big data and cloud computing play a huge role nowadays in our digital society. Each day a large amount of data is generated and collected by a variety of programs and applications. These large sets of data, which are referred as “big data”, are hard to peruse or query on a regular computer. On the other hand, cloud computing provides a platform for processing big data efficiently on the “cloud” where the “cloud” is usually a set of high-powered servers from one of many providers. The “cloud” can view and query large data sets much more quickly than a standard computer could. Big data and cloud computing together provide the solutions for the companies with big data but limited resources, a dilemma encountered by many companies in manufacturing and service industries.

Two decades ago, researchers in the area of statistics, graph theory, etc. started to investigate the sublinear approximation algorithms that uses only sublinear space or sublinear time, namely sublinear space algorithms or sublinear time algorithms. With more and more data being generated and stored away in the data center, and higher and higher dimension of computation being required and performed remotely on the “cloud” in various applications, sublinear algorithms become a new paradigm in computing to solve the problems under big data and cloud computing. Unlike the traditional data model where all the data can be stored and retrieved locally and one can hope to get the exact answers, the goal of sublinear space and sublinear time algorithms in general, is to obtain reasonably good approximate answers without storing or scanning the entire input.

Sublinear space algorithms are also called streaming algorithms, which process the input where some or all of the data is not available for random access in the local computers but rather arrives as a sequence of items and can be examined in only a few passes (typically just one). Early research on streaming algorithms dealt with simple statistics of the input data streams, such as the median [21], the number of distinct elements [11], or frequency moments [2]. Recently, many effective streaming algorithms have been designed for a range of problems in statistics, optimization, and graph algorithms (see surveys by Muthukrishnan [22] and McGregor [19]).

Sublinear time algorithms target at giving good approximations after inspecting only a very small portion of the input. Usually, this is achieved by using randomization. Sublinear time algorithms have been derived for many computational problems, for example, checking polygon

intersections [4], approximating the average degree in a graph [10, 14], estimating the cost of a minimum spanning tree [5, 8, 7], finding geometric separators [12], and property testing [13, 15], etc. Developing sublinear time algorithms not only speeds up problem solving process, but also reveals some interesting properties of computation, especially the power of randomization.

This paper aims at designing both types of sublinear approximation algorithms for the classical parallel machine scheduling problem subject to precedence constraints. We hope that our algorithms not only provide algorithmic solutions for this specific problem under big data and cloud computing environment, but also provide a framework and insight for solving other scheduling problems under big data and cloud computing environment which are encountered by many companies in the manufacturing and service industries.

Formally our problem is to schedule n jobs on m identical parallel machines where there are precedence constraints between jobs. The jobs are all available for processing at time 0 and labeled as $1, 2, \dots, n$. Each job j , $1 \leq j \leq n$, has a processing time p_j . The jobs have precedence constraints, \prec , such that $i \prec j$ represents that job j cannot start until job i finishes. The jobs and their precedence constraints can be described by a directed acyclic graph (DAG), $G = (V, E)$, where V is a set of vertices representing the jobs and E is a set of directed arcs representing the precedence constraints among the jobs. We assume that there are no transitive edges in G . If there is a directed arc $\langle i, j \rangle$ in E , then we have the precedence constraint $i \prec j$, and we say that job i is the immediate predecessor of job j and j is the immediate successor of job i . We consider non-preemptive schedules, i.e. a job cannot be interrupted once it is started. Given a schedule S , let C_j be the completion time of job j in S , then the makespan of the schedule S is $C_{max} = \max_{1 \leq j \leq n} C_j$. The goal is to find the minimum makespan. Using the three field notation, the problem can be denoted as $P \mid prec \mid C_{max}$ when the number of machines m is arbitrary, and be denoted as $P_m \mid prec \mid C_{max}$ when m is fixed.

A lot of research has been done on this classical scheduling problem. For arbitrary precedence, when $m = 2$ and jobs have unit processing time, i.e., $P_2 \mid prec, p_j = 1 \mid C_{max}$, Coffman and Graham [6] gave an optimal polynomial time algorithm in 1972. In 1978, Lenstra and Kan [17] showed when jobs have unit processing time, the problem with arbitrary precedence constraints and arbitrary m , $P \mid prec, p_j = 1 \mid C_{max}$, is strongly NP-hard. When the jobs' processing times are either 1 or 2, Lenstra and Kan [17] and Ullman [25] independently showed that the problem $P_2 \mid prec, p_j = 1, 2 \mid$

C_{max} is strongly NP hard. However, the complexity of the problem $P_3 \mid prec, p_j = 1 \mid C_{max}$ remains open. Graham [16] showed that list scheduling is a $(2 - \frac{1}{m})$ -approximation for the problem with arbitrary m and arbitrary job processing times, i.e., $P \mid prec \mid C_{max}$. In 2011, Svensson [24] showed that assuming a new, possibly stronger, version of the unique games conjecture (introduced by Bansal and Khot [3]), it is NP-hard to approximate the scheduling problem, $P \mid prec, p_j = 1 \mid C_{max}$, within any factor strictly less than 2. This result improves the inapproximability of $4/3$ by Lenstra and Rinnooy Kan [17].

Due to the importance and the hardness of the problem, a lot of research focused on various types of precedence constraints. One type of precedence constraints studied in literature is precedence graphs with bounded height where the height is the number of vertices on the longest path. In 1978, Lenstra and Kan [17] showed that for arbitrary number of machines m , the problem is NP-hard even if the precedence constrained graph has bounded height and the jobs have unit processing time. In 1984, Doleva and Warmuth [9] developed an optimal algorithm for this problem when m is fixed and the running time of the algorithm is $n^{h(m-1)+1}$. In 2006, Aho and Mäkinen [1] considered a special case where both the height of the graph and the maximum degree are bounded, and jobs have unit processing time. They showed that for large n , the optimal schedule has makespan $\lceil n/m \rceil$ and can be scheduled using modified critical path rule. This result is in fact a special case of the one studied by Dolev and Warmuth [9]. For more related results, one can refer to the survey by Prot and Bellenguez-Morineau [23] on how the structure of precedence constraints may change the complexity of the scheduling problems.

In this paper, we focus on the problem where the precedence graph has bounded depth h . The depth of a job j , denoted as dp_j , is the number of jobs on the longest directed path ending at j in G . It is easy to see that the maximum depth of the jobs in G is equal to the height of the graph G . Given a precedence graph, one can easily compute the depth dp_j of each job j . In addition, we assume that the processing times of the jobs are constrained. We first consider the case that the processing times of the jobs vary from one to another within a factor c , i.e. $p_{max} \leq c \cdot p_{min}$ where $p_{max} = \max_{1 \leq j \leq n} \{p_j\}$, $p_{min} = \min_{1 \leq j \leq n} \{p_j\}$, and c is a constant integer. Using the three-field notation, we denote this problem as $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$. We then consider the more general case where the largest αn jobs have no more than c factor difference for some constant where $0 < \alpha \leq 1$. For a given set of n jobs, let $[j]$ be the j -th smallest job.

Then $p_{[1]}$ is the smallest job, and $p_{[n]}$ is the largest job. We denote this more generalized problem as $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$. Our goal is to develop sublinear approximation algorithms, that is, approximation algorithms using only sublinear time or sublinear space, for these two versions of precedence constrained scheduling problems.

1.1. New Contributions

In this work, we develop two types of sublinear approximation algorithms for the classical parallel machine scheduling problems where the precedence graph has bounded depth h and the processing times of jobs are constrained. Specifically, our contributions are listed as follows:

- i. We develop two streaming approximation schemes for the problem $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ depending on whether c , h , and each job's depth are known or not. The algorithms are then extended to solve the more general problem where the largest αn jobs have no more than c factor difference for some constant α , $0 < \alpha \leq 1$, $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$.
- ii. We develop the first randomized approximation schemes that uses only sublinear time for both problems. In particular, for the more general problem, $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, when $m \leq \frac{\alpha n \epsilon}{20c^2 \cdot h}$, our algorithm is a randomized $(1 + \epsilon)$ -approximation scheme that runs in time $O(\frac{c^4 h^2 m^2}{\alpha^3 \epsilon^6} \log^2(\frac{cn}{\epsilon}) \log(\frac{h}{\epsilon} \log(\frac{cn}{\epsilon})))$.
- iii. Our approximation results greatly complement the in-approximability results of the studied problems. When $m = \frac{n}{3}$, even if $h = 3$ and $c = 1$, the problems cannot be approximated within a factor of $\frac{4}{3}$ in polynomial time unless $P=NP$ (see Section 2 for reference). Surprisingly, our results show that when m is a little bit smaller, i.e., upper bounded by n times a factor that depends on ϵ , h and α , then the problems admit polynomial time approximation schemes. For example, if $m \leq \frac{n}{15}$, $h = 3$ and $c = 1$, then there is a polynomial time 1.3-approximation for $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$.
- iv. We provide a methodological framework for designing sublinear approximation algorithms that can be used for solving other scheduling problems. The framework starts with generating the “sketch of input”, which is a summarized description of the input jobs, then computes an

approximate value of the optimal criterion, and finally generates the “sketch of schedule”, a succinct description of a schedule that achieves the approximate value.

We introduce the concept of “sketch of schedule” for the applications where not only an approximate value, but also a schedule associated with the approximate value is needed. As illustrated in the paper, we can use the “sketch of schedule” to easily generate a real schedule when the complete jobs information is read.

The paper is organized as follows. In Section 2, we give the complexity of the studied scheduling problems. In Section 3, we present the streaming algorithms for our problems. In Section 4, we design the randomized sublinear time algorithms for our problems. Finally, we draw the concluding remarks in Section 5.

2. Complexity

From the introduction, we know that if the jobs have unit processing time, then $P_m \mid prec, dp_j \leq h, p_j = 1 \mid C_{max}$ is solvable in $O(n^{h(m-1)+1})$ time which is polynomial if m is constant (see [9] for reference); however, the problem with arbitrary m , $P \mid prec, dp_j \leq h, p_j = 1 \mid C_{max}$, is NP-hard in the strong sense even if $h = 3$ (see [17] for reference). In this section, we first show that if we allow jobs to have different processing times, then even for fixed m , the problem becomes NP-hard.

Theorem 1. *The problem $P_m \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ is NP-hard.*

Proof: We will reduce even-odd partition problem to a restricted even-odd partition problem, and then reduce the restricted even-odd partition problem to $P_2 \mid p_{max} \leq c \cdot p_{min} \mid C_{max}$, which implies that $P_m \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ is NP-hard.

Even-odd partition: there is a set of $2n$ integers $B = \{b_i, 1 \leq i \leq 2n\}$ such that $b_i < b_{i+1}$ for all $1 \leq i < 2n$, is there a partition of B into B_1 and B_2 such that B_1 and hence B_2 contains exactly one of $\{b_{2i-1}, b_{2i}\}$ for each $1 \leq i \leq n$, and $\sum_{b_i \in B_1} b_i = \sum_{b_i \in B_2} b_i$?

Restricted even-odd partition: Given a set of $2n$ integers $D = \{d_i, 1 \leq i \leq 2n\}$ such that $d_i < d_{i+1}$ for all $1 \leq i < 2n$, and $d_{2n} \leq cd_1$ for some constant $c > 1$, is there a partition of D into D_1 and D_2 such that D_1 and hence D_2 contains exactly one of $\{d_{2i-1}, d_{2i}\}$ for each $1 \leq i \leq n$, and $\sum_{d_i \in D_1} d_i = \sum_{d_i \in D_2} d_i$?

Given an arbitrary instance $B = \{b_i, 1 \leq i \leq 2n\}$ of the even-odd partition problem, we can reduce it to an instance of the restricted even-odd partition problem $D = \{d_i, 1 \leq i \leq 2n\}$ as follows. Without loss of generality, we can assume that $b_{2n} > c \cdot b_1$. Let Y be the integer such that $Y \geq \frac{b_{2n} - cb_1}{c-1}$, i.e. $b_{2n} \leq c \cdot b_1 + (c-1)Y$. For each $1 \leq i \leq 2n$, let $d_i = b_i + Y$. It is easy to see that $d_{2n} = b_{2n} + Y \leq cb_1 + c \cdot Y = c \cdot d_1$. It is trivial to show that there is a solution to instance B if and only if there is a solution for instance D . Thus the restricted even-odd partition is also NP-hard. The restricted even-odd partition can be easily reduced to the scheduling problem $P_2 \mid p_{max} \leq c \cdot p_{min} \mid C_{max}$, which implies that $P_m \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ is NP-hard. ■

The next theorem is showing the in-approximability of our problems. In the strong NP-hardness proof of $P \mid prec, p_j = 1 \mid C_{max}$ in [17], the scheduling instance created from the instance of clique problem has a precedence graph of height 3, and there is a schedule of the $n = 3m$ jobs with makespan of 3 if and only if there is a solution to the clique instance. This implies if an approximation algorithm can generate a schedule with approximation ratio less than $4/3$, it must be optimal, which is impossible unless $P=NP$.

Theorem 2. *Given any $\epsilon > 0$, unless $P=NP$, there is no polynomial time $(4/3 - \epsilon)$ -approximation algorithm for $P \mid prec, dp_j \leq h, p_j = 1 \mid C_{max}$ even if $h = 3$.*

Despite the in-approximability result from Theorem 2, in the next two sections, we will develop approximation schemes that use only sublinear space or sublinear time for our studied problems when m is upper bounded by n times a factor.

3. Streaming Algorithms using Sublinear Space

At the conceptual level, our streaming algorithms have the following two stages:

Stage 1: Generate and store a sketch of the input while reading the input stream.

Stage 2: Compute an approximation of the optimal value based on the sketch of the input.

Roughly speaking, the sketch of the input is a summary of the input jobs which requires only sublinear space. Instead of storing the accurate processing times of the jobs, we map each job's

processing time into the range of $[(1 + \delta)^u, (1 + \delta)^{u+1})$ where δ is a parameter. Thus we only need to store the number of jobs that are mapped in each range for each depth. We then use the rounded processing time for each job to obtain the approximation of the optimal makespan. A formal definition of the sketch for our problems is given below.

Definition 3. For a given parameter δ , and an instance of the problem $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ or $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, the **sketch of the input** with respect to δ , denoted as $SKJ_\delta = \{(d, u, n_{d,u})\}$, consists of a set of tuples, $(d, u, n_{d,u})$, where $n_{d,u}$ is the number of jobs with the depth d and the processing time in the range of $[(1 + \delta)^u, (1 + \delta)^{u+1})$.

The size of the sketch, which is the number of tuples $(d, u, n_{d,u})$, may be different for different problems and different types of stream input. In some cases, for example, we disregard the jobs with small processing time.

In the following subsection, we will first present our streaming algorithms for the problem $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$. For the stream input, we consider both the case where c , h and dp_j , $1 \leq j \leq n$, are given and the case where these information is not directly given. We will then adapt our algorithms to the more general problem $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$.

3.1. Streaming Approximation Schemes for $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$

3.1.1. The parameters c , h , and dp_j are known

We study the problem under data stream model assuming c , h , and dp_j are known. The jobs are given via stream and each job j is described by a pair (p_j, dp_j) , where p_j and dp_j are job j 's processing time and depth, respectively. Without loss of generality, we can assume $p_j \in [1, c]$ in this case. The algorithm is simple: scan the jobs from the stream and generate the sketch of the input, $SKJ_\delta = \{(d, u, n_{d,u})\}$, where $n_{d,u}$ is the number of jobs with the depth d and the processing time in the range of $[(1 + \delta)^u, (1 + \delta)^{u+1})$; for each d , $1 \leq d \leq h$, compute the length of the time interval where all the jobs with the depth d can be feasibly scheduled, and then return the total length of these intervals. The complete algorithm is given in Streaming-Algorithm1.

Theorem 4. For any ϵ , when $m \leq \frac{2n\epsilon}{3 \cdot h \cdot c}$, Streaming-Algorithm1 is a one-pass streaming approximation scheme for $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ that uses $O(\frac{h \log c}{\epsilon})$ space, $O(1)$ update

Algorithm Streaming-Algorithm1

Input: Parameters ϵ , m , c and h Stream input: (p_j, dp_j) , $1 \leq j \leq n$.

Output: An approximate value of the optimal makespan

```
1: let  $\delta = \frac{\epsilon}{3}$ ,  $k = \lfloor \log_{1+\delta} c \rfloor$ 
2: read the input stream and generate the sketch of the input  $SKJ_\delta$ :
3:   initialize the input sketch:  $SKJ_\delta = \{(d, u, n_{d,u}) : 1 \leq d \leq h, 0 \leq u \leq k, n_{d,u} = 0\}$ 
4:   for each job  $j$  with  $(p_j, dp_j)$  in the stream input do
5:     update  $n_{d,u} = n_{d,u} + 1$  where  $d = dp_j$  and  $u = \lfloor \log_{1+\delta} p_j \rfloor$ 
6:   end for
7: compute the approximate makespan
8:   let  $rp_k = c$ 
9:   for each  $u$ ,  $0 \leq u < k$ 
10:    let  $rp_u = (1 + \delta)^{u+1}$ 
11:   for each  $d$ ,  $1 \leq d \leq h$  do
12:     let  $A_d = \frac{1}{m} \sum_{u=0}^k (n_{d,u} \cdot rp_u)$ 
13:   end for
14:    $A = \sum_{d=1}^h (\lfloor A_d \rfloor + c)$ 
15: return  $A$ 
```

time for each job in the stream, and $O(\frac{h \log c}{\epsilon})$ time to return the approximate makespan.

Proof: First we consider the complexity. The space complexity is dominated by the sketch for which we can use a two dimensional array of size $h \cdot k = O(\frac{h \log c}{\epsilon})$. It is easy to see that the update time for each job is $O(1)$. Finally, it takes $O(h \cdot k) = O(\frac{h \log c}{\epsilon})$ time to compute and return the approximate value A .

Now we consider the approximation ratio of the algorithm. Let I be the input instance, and let I' be the instance corresponding to the sketch SKJ_δ which consists of $n_{d,u}$ jobs that have processing time rp_u for each d, u . Alternatively, we can also view I' being obtained from I by rounding up the processing time of each job. Let C_{max}^* and C'_{max} be the optimal makespan for the instance I and I' , respectively. It is easy to see that $C_{max}^* \leq C'_{max} \leq (1 + \delta)C_{max}^*$. In the following, we prove that the returned value from Streaming-Algorithm1, A , satisfies the inequality, $C_{max}^* \leq A \leq (1 + \epsilon)C_{max}^*$.

First, we show that A is an upper bound of C'_{max} . Consider a schedule S for the jobs of instance I' which schedules the jobs as follows: the jobs at each depth d are scheduled using list scheduling rule (i.e. schedule the jobs one by one in the given order to the machine that is available at the earliest time), and jobs with depth $d + 1$ can start only after all jobs with depth d complete. It is easy to see that in S the jobs at a depth d are scheduled into an interval of length at most

$\lfloor \frac{1}{m} \sum_{u=0}^k (n_{d,u} \cdot rp_u) \rfloor + c = \lfloor A_d \rfloor + c$. Therefore, the makespan of the feasible schedule S is at most $\sum_{d=1}^h (\lfloor A_d \rfloor + c) = A$, which implies that $A \geq C'_{max}$, where C'_{max} is the optimal makespan for the instance I' .

On the other hand, it is obvious that $\sum_{d=1}^h A_d = \sum_{d=1}^h (\frac{1}{m} \sum_{u=0}^k (n_{d,u} \cdot rp_u))$ is a lower bound of C'_{max} , and $C^*_{max} \leq C'_{max} \leq (1 + \delta)C^*_{max}$. Thus,

$$A = \sum_{d=1}^h (\lfloor A_d \rfloor + c) \leq \left(\sum_{d=1}^h A_d \right) + h \cdot c \leq C'_{max} + h \cdot c \leq (1 + \delta)C^*_{max} + h \cdot c.$$

Since $C^*_{max} \geq \frac{n}{m}$, when $m \leq \frac{2n\epsilon}{3 \cdot h \cdot c}$, we have $h \cdot c \leq \frac{2\epsilon}{3} \frac{n}{m} \leq \frac{2\epsilon}{3} C^*_{max}$. Therefore,

$$A \leq (1 + \delta)C^*_{max} + \frac{2\epsilon}{3} C^*_{max} = (1 + \frac{\epsilon}{3})C^*_{max} + \frac{2\epsilon}{3} C^*_{max} \leq (1 + \epsilon)C^*_{max}.$$

In summary, we have $C^*_{max} \leq C'_{max} < A \leq (1 + \epsilon)C^*_{max}$, and this completes the proof. ▀

Recall the inapproximability result of the problem $P \mid prec, dp_j \leq h, p_j = 1 \mid C_{max}$ from Theorem 2, which tells us no approximation better than $\frac{4}{3}$ is possible in polynomial time even if the height is bounded and all jobs have unit processing time unless $P=NP$. Our result from Theorem 4 surprisingly shows that if m is bounded by a fraction of n then we can get a $(1 + \epsilon)$ -approximation even if the processing times are slightly different. For example, if $m \leq \frac{n}{15}$, $h = 3$ and $c = 1$, then we can get a 1.3-approximation.

Theorem 4 also shows that Stream-Algorithm1 only takes constant time to read and process each job in the stream input, and then constant time and constant space to return an approximation of the makespan if the parameters are known. In some cases, however, we may not know the exact value of c , but we are given an estimate \hat{c} of the parameter c . In these cases, we can still apply Stream-Algorithm1 by using \hat{c} . As long as $\frac{\hat{c}}{c}$ is a constant, we still have a $(1 + \epsilon)$ -approximation with the same space and time complexity.

3.1.2. The parameters c , h , and dp_j , are unknown

In this subsection, we consider the case that the parameters c (or the estimate \hat{c}) and h are not known, furthermore, the depth of the jobs are not given directly as in the previous section. Instead, the stream input consists of all the jobs (j, p_j) in arbitrary order followed by all the arcs $\langle i, j \rangle$ of

the precedence graph in topological order.

In this case, we need to compute and update both the depth of each job and the sketch of the input dynamically as we read the input. We use a B-tree to maintain the sketch tuples of the input $(d, u, n_{d,u})$ where (d, u) is the key. We define a linear order to compare two tuples (d_1, u_1, n_{d_1, u_1}) and (d_2, u_2, n_{d_2, u_2}) , we say $(d_1, u_1, n_{d_1, u_1}) < (d_2, u_2, n_{d_2, u_2})$ if 1) $u_1 < u_2$, or 2) $u_1 = u_2$ and $d_1 < d_2$. Additionally we use an array B to store the jobs' information: for each job j with the processing time p_j , we maintain a pair (dp_j, u_j) , where dp_j represents its current depth, and $u_j = \lfloor \log_{1+\delta} p_j \rfloor$.

When each job (j, p_j) arrives in the stream input, we update job j 's entry in the array B such that $dp_j = 1$ and $u_j = \lfloor \log_{1+\delta} p_j \rfloor$, then create and insert a node $(1, u_j, n_{1, u_j})$ into the tree. Simultaneously we update the smallest processing time p_{min} and the largest processing time p_{max} . After all the jobs are read in, we can get the final p_{min} and p_{max} and compute $c = \lceil p_{max}/p_{min} \rceil$.

When each arc $\langle i, j \rangle$, which indicates job i is the direct predecessor of job j , arrives in the stream input, we access job i 's and j 's entries in the array to obtain their keys (d_i, u_i) and (d_j, u_j) , and compute $dp_j = \max(d_j, d_i + 1)$. If $dp_j > d_j$, we will update the node (d_j, u_j, n_{d_j, u_j}) by setting $n_{d_j, u_j} = n_{d_j, u_j} - 1$ or delete this node if n_{d_j, u_j} becomes 0; and then update the node $(dp_j, u_j, n_{dp_j, u_j})$ by setting $n_{dp_j, u_j} = n_{dp_j, u_j} + 1$ or insert a new node if the node with the key (dp_j, u_j) does not exist in the tree. The job j 's entry in the array B is also updated with (dp_j, u_j) . After all the arcs are read in, we can get the sketch of the stream input and the largest depth h . The complete algorithm is given in Streaming-Algorithm2.

Theorem 5. *If parameters c and h are not known, both the jobs and the precedence graph in topological order are input via the stream, for any ϵ , when $m \leq \frac{2n\epsilon}{3 \cdot h \cdot c}$, Streaming-Algorithm2 is a one-pass streaming approximation scheme for $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ that uses $O(n)$ space, takes $O(\log(\frac{h}{\epsilon} \log c))$ update time for processing each job and each arc in the stream, and $O(\frac{h \log c}{\epsilon})$ time to return the approximate makespan.*

Proof: The main difference of Streaming-Algorithm2 and Streaming-Algorithm1 is the implementation. The analysis for approximation ratio is similar to Theorem 4. We will use the same notations as in the proof of Theorem 4. So C_{max}^* is the optimal makespan of the input instance, C'_{max} is the optimal makespan for the instance I' corresponding to the sketch SKJ_δ . We can construct a schedule S for I' whose makespan is at most $\sum_{d=1}^h (\lfloor A_d \rfloor + p_{max}) = A$, which implies

Algorithm Streaming-Algorithm2

Input: Parameters ϵ , m Stream input: the set of jobs in arbitrary order, (j, p_j) , $1 \leq j \leq n$, followed by

the set of arcs of the precedence graph in topological order Output: An approximate value of the optimal makespan

- 1: create an empty B-tree T and an array B of size n
 - 2: initialize $p_{min} = \infty$, $p_{max} = 1$, $h = 1$
 - 3: let $\delta = \frac{\epsilon}{3}$
 - 4: read the input stream and generate the sketch of the input SKJ_δ :
 - 5: **for** each job j with (p_j, dp_j) in the stream input **do**
 - 6: let $u = \lfloor \log_{1+\delta} p_j \rfloor$
 - 7: $B[j] = (1, u)$
 - 8: **if** there is a node $(1, u, n_{1,u})$ in the tree T **then**
 - 9: update this node by setting $n_{1,u} = n_{1,u} + 1$
 - 10: **else**
 - 11: create and insert a node $(1, u, 1)$ into T
 - 12: **end if**
 - 13: **if** $p_{min} > p_j$, $p_{min} = p_j$
 - 14: **if** $p_{max} < p_j$, $p_{max} = p_j$
 - 15: **end for**
 - 16: **for** each arc $\langle i, j \rangle$, in the stream input **do**
 - 17: let $(d_i, u_i) = B[i]$ and $(d_j, u_j) = B[j]$
 - 18: **if** $d_i + 1 > d_j$ **then**
 - 19: $dp_j = d_i + 1$
 - 20: $B[j] = (dp_j, u_j)$
 - 21: update the node (d_j, u_j, n_{d_j, u_j}) in T as below
 - 22: $n_{d_j, u_j} = n_{d_j, u_j} - 1$
 - 23: **if** $n_{d_j, u_j} = 0$, delete this node
 - 24: **if** the node with the key (dp_j, u_j) does not exist in the tree **then**
 - 25: insert a new node $(dp_j, u_j, 1)$
 - 26: **else**
 - 27: update the node $(dp_j, u_j, n_{dp_j, u_j})$ in T by setting $n_{dp_j, u_j} = n_{dp_j, u_j} + 1$
 - 28: **end if**
 - 29: **end if**
 - 30: **if** $h < dp_j$, set $h = dp_j$
 - 31: **end for**
 - 32: traverse all the nodes $(d, u, n_{d,u})$ in T
 - 33: let $SKJ_\delta = \{(d, u, n_{d,u})\}$
 - 34: compute the approximate makespan
 - 35: let $u_- = \lfloor \log_{1+\delta} p_{min} \rfloor$ and $u_+ = \lfloor \log_{1+\delta} p_{max} \rfloor$
 - 36: let $rp_{u_+} = p_{max}$
 - 37: **for** each $u_- \leq u < u_+$
 - 38: let $rp_u = (1 + \delta)^{u+1}$
 - 39: **for** each d
 - 40: let $A_d = \frac{1}{m} \sum_{u=u_-}^{u_+} (n_{d,u} \cdot rp_u)$
 - 41: let $A = \sum_{d=1}^h (\lfloor A_d \rfloor + p_{max})$
 - 42: return A
-

that $A \geq C'_{max}$.

It is obvious that $C'_{max} \geq \sum_{d=1}^h A_d$. Thus,

$$A = \sum_{d=1}^h (\lfloor A_d \rfloor + p_{max}) \leq \left(\sum_{d=1}^h A_d \right) + h \cdot p_{max} \leq C'_{max} + h \cdot p_{max} \leq (1 + \delta)C^*_{max} + h \cdot p_{max}.$$

Since $p_{max} \leq c \cdot p_{min}$ and $C^*_{max} \geq \frac{n \cdot p_{min}}{m}$, we have $h \cdot p_{max} \leq h \cdot c \cdot p_{min} \leq h \cdot c \cdot \frac{m}{n} C^*_{max}$. when $m \leq \frac{2n\epsilon}{3 \cdot h \cdot c}$, we get $h \cdot p_{max} \leq \frac{2\epsilon}{3} C^*_{max}$. Therefore,

$$A \leq (1 + \delta)C^*_{max} + \frac{2\epsilon}{3} C^*_{max} = (1 + \frac{\epsilon}{3})C^*_{max} + \frac{2\epsilon}{3} C^*_{max} = (1 + \epsilon)C^*_{max}.$$

In summary, we have $C^*_{max} \leq C'_{max} \leq A \leq (1 + \epsilon)C^*_{max}$

Now we analyze the complexity of Streaming-Algorithm2, the number of nodes in B-tree T is at most $O(h \log_{1+\delta} \lceil \frac{p_{max}}{p_{min}} \rceil) = O(h \log_{1+\delta} c) = O(\frac{h}{\epsilon} \log c)$. So when each job or arc is read from the stream input, the corresponding update time for search, insertion or update operation on the B-tree is always $O(\log(h \log_{1+\delta} c)) = O(\log(\frac{h}{\epsilon} \log c))$. After the input is read in, it takes additional $O(\frac{h}{\epsilon} \log c)$ time to traverse B-tree and compute the approximation of the optimal value. The stream input size is $O(n + e)$, where n is the number of jobs and e is the number of arcs of the precedence graph. Streaming-Algorithm2, uses only $O(n)$ space to store the array B and the tree T , which is sublinear considering the number of arcs usually has $e = O(n^{1+\beta})$, $0 < \beta \leq 1$ in a dense graph. █

3.2. Streaming Approximation Algorithms for $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$

In this section, we consider the more general case where the largest αn jobs have no more than c factor difference for some constant $0 < \alpha \leq 1$. Apparently, the problem $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ is the special case where $\alpha = 1$. Following the same procedure of our streaming algorithms, we need to compute the sketch of the input $SKJ_\delta = \{(d, u, n_{d,u})\}$. However, different from the case $\alpha = 1$, i.e., the problem $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$, for which there are only constant number $O(\frac{h \log c}{\epsilon})$ of entries in the sketch of the input, for the problem $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, if we consider all jobs in the sketch there may be a

very large number of entries in the sketch of the input since p_{max} may be very large compared with p_{min} . We will show in the following that when we generate the sketch of the input we can ignore those small jobs whose processing time is less than $\frac{p_{max}}{n^2}$ and still get a good approximation of the optimal makespan using only sublinear space.

3.2.1. The parameters c , h , and dp_j , are known

We study the streaming algorithm for $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$ when the parameters c , h , and dp_j for all $1 \leq j \leq n$ are known. As mentioned above, the jobs with processing time less than $\frac{p_{max}}{n^2}$ will not be included in the sketch SKJ_δ . Specifically, $SKJ_\delta = \{(d, u, n_{d,u}) : u_- \leq u \leq u_+, 1 \leq d \leq h\}$, where $u_- = \lfloor \log_{1+\delta} \frac{p_{max}}{n^2} \rfloor$, and $u_+ = \lfloor \log_{1+\delta} p_{max} \rfloor$. Without loss of generality, we assume p_{max} is not known until all input is read. So p_{max} in our algorithm represents the current maximum processing time of the jobs that we have read so far. We use a B -tree to store all the considered tuples, $(d, u, n_{d,u})$. When a job j with (p_j, dp_j) arrives, if $p_j < \frac{p_{max}}{n^2}$, we skip this job and continue to read the next job. Otherwise, let $d = dp_j$ and $u = \lfloor \log_{1+\delta} p_j \rfloor$, and we update B -tree as follows: if $(d, u, n_{d,u})$ exists in the tree, update this node with $(d, u, n_{d,u} + 1)$; otherwise, insert a new node $(d, u, 1)$. To limit the number of nodes in the tree, whenever a new node is inserted, we check the node with the smallest u , $(d', u', n_{d',u'})$, if $u' < \lfloor \log_{1+\delta} \frac{p_{max}}{n^2} \rfloor$, we delete the smallest node. The final sketch of the input SKJ_δ includes only the tuples $(d, u, n_{d,u})$ from the B -tree such that $u_- \leq u \leq u_+$. We present our algorithm formally in Streaming-Algorithm3.

Theorem 6. *When $m \leq \frac{2n\alpha\epsilon}{3(h+1)\cdot c}$, Streaming-Algorithm3 is a streaming approximation scheme for the problem $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$ that uses $O(\frac{h}{\epsilon} \log n)$ space, takes $O(\log \frac{h}{\epsilon} + \log \log n)$ update time for each job in the stream, and $O(\frac{h}{\epsilon} \log n)$ time to return an approximate value that is at most $(1 + \epsilon)$ times the optimal makespan.*

Proof: We first analyze the approximation ratio. Let I be the given instance. Let I' be the instance obtained from I by rounding up all the jobs with the processing times greater than or equal to $\frac{p_{max}}{n^2}$, i.e. for each job j in I , if $p_j \geq \frac{p_{max}}{n^2}$, we round it up to rp_u where $u = \lfloor \log_{1+\delta} p_j \rfloor$; otherwise, we keep it same as before. Let C_{max}^* and C'_{max} be the optimal makespan for I and I' respectively. Let I'' be the instance corresponding to the sketch SKJ_δ . Apparently I'' can be obtained from I' by removing the small jobs whose processing time is less than $\frac{p_{max}}{n^2}$. Let

Algorithm Streaming-Algorithm3

Input: Parameters $\epsilon, m, n, \alpha, c$ and h

Stream input: (p_j, dp_j) for all jobs $1 \leq j \leq n$.

Output: An approximate value of the optimal makespan

```
1: let  $\delta = \frac{\epsilon}{3}$ 
2: create an empty B-tree  $T$ 
3: initialize  $p_{max} = 1$ 
4: read the input stream and generate the sketch of the input  $SKJ_\delta$ :
5:   for each job  $j$  with  $(j, p_j)$  in the stream input do
6:     if  $p_j < \frac{p_{max}}{n^2}$  then
7:       skip this job and continue the next job
8:     else
9:       if  $p_{max} < p_j$ ,  $p_{max} = p_j$ 
10:      let  $d = dp_j$ ,  $u = \lceil \log_{1+\delta} p_j \rceil$ , and update B-tree as follows:
11:      if node  $(d, u, n_{d,u})$  exists in the tree then
12:        update the node with  $n_{d,u} = n_{d,u} + 1$ 
13:      else
14:        insert a new node  $(d, u, 1)$ 
15:        let  $(d', u', n_{d',u'})$  be the node with the smallest  $u$ 
16:        if  $u' < \log_{1+\delta} \frac{p_{max}}{n^2}$ , delete  $(d', u', n_{d',u'})$  from the tree
17:      end if
18:    end if
19:  end for
20:  let  $u_- = \lceil \log_{1+\delta} \frac{p_{max}}{n^2} \rceil$ ,  $u_+ = \lceil \log_{1+\delta} p_{max} \rceil$ 
21:  traverse  $T$  and generate the sketch using only the nodes with  $u_- \leq u \leq u_+$ 
22:   $SKJ_\delta = \{(d, u, n_{d,u}) : 1 \leq d \leq h, u_- \leq u \leq u_+\}$ 
23: compute the approximate makespan
24:   let  $rp_{u_+} = p_{max}$ 
25:   for each  $u_- \leq u < u_+$ 
26:     let  $rp_u = (1 + \delta)^{u+1}$ 
27:   for each  $d$ 
28:     let  $A_d = \frac{1}{m} \sum_{u=u_-}^{u_+} (n_{d,u} \cdot rp_u)$ 
29:   let  $A = (\sum_{d=1}^h (\lfloor A_d \rfloor + p_{max})) + \lceil \frac{p_{max}}{n} \rceil$ 
30: return  $A$ 
```

C''_{max} be the optimal makespan for I'' . Then we have $C''_{max} \geq \sum_{d=1}^h A_d$. It is easy to see that $C''_{max} \leq C'_{max} \leq (1 + \delta)C^*_{max}$, and $C^*_{max} \leq C'_{max} \leq C''_{max} + n \cdot \frac{p_{max}}{n^2} = C''_{max} + \frac{p_{max}}{n}$.

As before, we can construct a schedule S for I'' using list scheduling rule to schedule the jobs depth by depth starting with $d = 1$. To get a schedule for all jobs in I' based on S , for each depth d , we can simply insert into S all the small jobs of this depth onto the first machine after all big jobs of depth d finish and before the first big job of $d + 1$ starts. Let the new schedule be S' . Apparently the makespan of S' is at least C^*_{max} and at most $A = \sum_{d=1}^h (\lfloor A_d \rfloor + p_{max}) + \lceil \frac{p_{max}}{n} \rceil$. Thus, we have $A \geq C^*_{max}$ and

$$A = \left(\sum_{d=1}^h (\lfloor A_d \rfloor + p_{max}) \right) + \lceil \frac{p_{max}}{n} \rceil \leq C''_{max} + h \cdot p_{max} + \lceil \frac{p_{max}}{n} \rceil. \quad (1)$$

Since the largest αn jobs have no more than c factor difference, each of the largest αn jobs has processing time at least $\frac{p_{max}}{c}$. Thus, we have

$$C^*_{max} \geq \alpha \cdot n \cdot \frac{p_{max}}{c} \cdot \frac{1}{m} = \frac{\alpha n}{c \cdot m} p_{max},$$

which implies $p_{max} \leq \frac{c \cdot m}{\alpha \cdot n} C^*_{max}$. If we plug this into inequality (1), we get

$$\begin{aligned} A &\leq C''_{max} + h \cdot p_{max} + \lceil \frac{p_{max}}{n} \rceil \\ &\leq C''_{max} + (h + 1) \cdot p_{max} \\ &\leq C''_{max} + (h + 1) \cdot \frac{c \cdot m}{\alpha \cdot n} C^*_{max} \\ &\leq (1 + \delta) C^*_{max} + \frac{(h+1) \cdot c \cdot m}{\alpha n} C^*_{max} \\ &\leq (1 + \delta + \frac{(h+1) \cdot c \cdot m}{\alpha n}) C^*_{max} \\ &\leq (1 + \frac{\epsilon}{3} + \frac{(h+1) \cdot c \cdot m}{\alpha n}) C^*_{max}. \end{aligned}$$

If $m \leq \frac{2n\alpha\epsilon}{3(h+1) \cdot c}$, we have $A \leq (1 + \frac{\epsilon}{3} + \frac{2\epsilon}{3}) C^*_{max} = (1 + \epsilon) C^*_{max}$.

Now we consider the complexity. The space complexity is dominated by the B-tree. As the way it is implemented, each time a node is inserted into the tree, if there is a node $(d, u, n_{d,u})$ such that $u < \log_{1+\delta} \frac{p_{max}}{n^2}$, the smallest such node will be deleted from the tree. In this way, the number of nodes in the tree is at most $h \cdot \log_{1+\delta} n^2 = O(\frac{h}{\epsilon} \log n)$. For each job in the stream input,

a constant number of tree operations are needed, and thus the update time for processing each job is $O(\log(\frac{h}{\epsilon} \log n) = O(\log(\frac{h}{\epsilon}) + \log \log n)$ time. After reading all the jobs, the computation of the approximation is bounded by the size of the sketch which is $O(\frac{h}{\epsilon} \log n)$. ■

3.2.2. The parameters c , h , and dp_j are unknown

We consider the problem $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$ when the parameters c , h , and dp_j are not known. In this case, the stream input would include the jobs followed by the arcs. As in Streaming-Algorithm 2, we use a B-tree to store the sketch information, and an array to store the jobs' information. Both the array and the tree are updated when we read the jobs and arcs from the stream input. The streaming algorithm will be similar to Streaming-Algorithm2 but with some nodes for small processing times excluded as in Streaming-Algorithm3. Using similar arguments as in the proof of Theorem 5 and 6, we can get the following theorem.

Theorem 7. *If parameters c and h are not known, the jobs, and the precedence graph in topological order are input via the stream, for any ϵ , when $m \leq \frac{2n\alpha\epsilon}{3(h+1)\cdot c}$, there is a streaming approximation scheme for the problem $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$ that uses $O(n)$ space, takes $O(\log \frac{h}{\epsilon} + \log \log n)$ update time for each job in the stream, and $O(\frac{h}{\epsilon} \log n)$ time to return the approximate value.*

3.3. The Sketch of the Schedule

All the streaming algorithms we have presented so far return an approximate value of the optimal makespan. This may be sufficient for some scheduling and planning applications. However, in many other applications, it would be desirable to have a schedule whose makespan is the approximate value. In the traditional data model, a schedule explicitly or implicitly specifies for each job on which machine and in what time interval it is scheduled. This means we need at least $\Omega(n)$ time complexity and space complexity to describe a schedule. For the big data model, we introduce the concept of **sketch of a schedule** which is a condensed description of a schedule using only sublinear space. In the following, we first give a formal definition for the **sketch of a schedule**, then we show that our previous algorithms can compute simultaneously not only an approximate value, but also the **sketch of a schedule**, and finally we show how the sketch can be used to

generate a real schedule that achieves the approximate value when the jobs are scanned in the second pass.

Definition 8. For the problems $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{max} \mid C_{max}$ and $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, the **sketch of a schedule** describes a feasible schedule and consists of a set of time instants t_d , $1 \leq d \leq h$, such that all the jobs of depth d can be scheduled during the interval $[t_{d-1}, t_d)$ for $1 \leq d \leq h$ where $t_0 = 0$. Mathematically we denote the **sketch of a schedule** as $SKS = \{t_d : 1 \leq d \leq h\}$.

For the problem $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$, by the proof of Theorem 4 we know all the jobs of depth d can be feasibly scheduled during an interval of length $\lfloor A_d \rfloor + c$, which implies that $SKS = \{t_d : 1 \leq d \leq h\}$ where $t_d = \sum_{k=1}^d (\lfloor A_k \rfloor + c)$ for all $1 \leq d \leq h$, is the sketch of a schedule that can be computed by Streaming-Algorithm1.

Lemma 9. For the problem $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$, Streaming-Algorithm1 can compute a sketch of a schedule $SKS = \{t_d : 1 \leq d \leq h\}$ where $t_d = \sum_{k=1}^d (\lfloor A_k \rfloor + c)$ for all $1 \leq d \leq h$.

Based on the sketch of the schedule, if we scan all the jobs in the second time, we can generate a feasible schedule using the Algorithm SketchToSchedule.

By lemma 9 and the Algorithm SketchToSchedule, we have the following theorem.

Theorem 10. For $P \mid prec, dp_j \leq h, p_{min} \leq c \cdot p_{max} \mid C_{max}$, given any $0 < \epsilon < 1$, when $m \leq \frac{2n\epsilon}{3 \cdot h \cdot c}$, Streaming-Algorithm1 can compute a sketch of the schedule SKS which can be applied to Algorithm SketchToSchedule to generate a feasible schedule with the makespan at most $(1+\epsilon)$ times the optimal makespan.

Proof: Since the total length of the jobs at depth d after rounding is $m \cdot A_d$, and the largest processing time is c , it is easy to see that Algorithm SketchToSchedule generates a feasible schedule of these jobs during the interval $[t_{d-1}, t_d]$, where $t_d = t_{d-1} + \lfloor A_d \rfloor + c$. The final schedule of all n jobs has the makespan at most $t_h = \sum_{d=1}^h \lfloor A_d \rfloor + c$, which is at most by $(1+\epsilon)C_{max}^*$ by the proof of Theorem 4. ▀

Similarly, Streaming-Algorithm2 can compute a sketch of a schedule $SKS = \{t_d : 1 \leq d \leq h\}$ where $t_d = \sum_{k=1}^d (\lfloor A_k \rfloor + p_{max})$ for all $1 \leq d \leq h$, and we have the following theorem.

Algorithm SketchToSchedule

Input: $SKS = \{t_d : 1 \leq d \leq h\}$ Stream input: (p_j, dp_j) for all jobs $1 \leq j \leq n$.Output: a feasible schedule S of all the jobs

```
1: for each  $d, 1 \leq d \leq h$  do
2:   let  $curM_d$  and  $curT_d$  be the machine and time where next job with depth  $d$  will be scheduled.
3:    $curM_d = 1; curT_d = t_{d-1}$ 
4: end for
5: read the job stream input and generate the schedule:
6:   for each job  $(p_j, dp_j)$  do
7:     let  $d = dp_j$ 
8:     if  $curT_d + p_j \leq t_d$  then
9:       schedule job  $j$  at time  $curT_d$  on machine  $curM_d$ 
10:      set  $curT_d = curT_d + p_j$ 
11:     else
12:       schedule job  $j$  at time  $t_{d-1}$  on machine  $curM_d + 1$ 
13:       set  $curM_d = curM_d + 1$  and  $curT_d = t_{d-1} + p_j$ 
14:     end if
15:   end for
```

Theorem 11. For $P \mid prec, dp_j \leq h, p_{min} \leq c \cdot p_{max} \mid C_{max}$, given any $0 < \epsilon < 1$, when $m \leq \frac{2n\epsilon}{3 \cdot h \cdot c}$ Streaming-Algorithm2 can compute a sketch of a schedule SKS which can be applied to Algorithm SketchToSchedule to generate a feasible schedule with the makespan at most $(1+\epsilon)$ times the optimal makespan.

For the problem $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, Streaming-Algorithm3 gives an approximate value of the optimal makespan. However, the small jobs from depth d are not considered when we calculate A_d , so the sketch of the schedule is slightly different from previous problem. We will show that in this case, the sketch of a schedule is given by $SKS = \{t_d : t_d = t_{d-1} + (\lfloor A_d \rfloor + p_{max} + \lceil \frac{p_{max}}{n} \rceil), 1 \leq d \leq h\}, t_0 = 0$.

Theorem 12. For the problem $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, given any $0 < \epsilon < 1$, when $m \leq \frac{2n\alpha\epsilon}{3(h+1) \cdot c}$, Streaming-Algorithm3 can compute a sketch of the schedule $SKS = \{t_d : t_d = t_{d-1} + (\lfloor A_d \rfloor + p_{max} + \lceil \frac{p_{max}}{n} \rceil), 1 \leq d \leq h\}, t_0 = 0$, and based on SKS , Algorithm SketchToSchedule can generate a feasible schedule with the makespan at most $(1+\epsilon)$ times the optimal makespan.

Proof: From the proof of Theorem 6, we know that the interval with the length $\lfloor A_d \rfloor + p_{max}$ can feasibly fit in all the jobs of depth d and with the process time at least $\frac{p_{max}}{n^2}$. If we add additional length of $\lceil n \cdot \frac{p_{max}}{n^2} \rceil = \lceil \frac{p_{max}}{n} \rceil$ to the interval, we can guarantee that both the large jobs and the small

jobs of depth d can be fit in. Hence, $SKS = \{t_d : t_d = t_{d-1} + (\lfloor A_d \rfloor + p_{max} + \lceil \frac{p_{max}}{n} \rceil), 1 \leq d \leq h\}$, $t_0 = 0$, describes a feasible schedule such that all the jobs of depth d can be scheduled during the interval $[t_{d-1}, t_d]$. Based on the sketch SKS , we can use Algorithm SketchToSchedule to generate a feasible schedule with the makespan at most

$$t_h = \sum_{d=1}^h (\lfloor A_d \rfloor + p_{max} + \lceil \frac{p_{max}}{n} \rceil) \leq \left(\sum_{d=1}^h \lfloor A_d \rfloor \right) + (h+1)p_{max}.$$

From the proof of Theorem 6, we know $t_h \leq C''_{max} + (h+1)p_{max} \leq (1+\epsilon)C^*_{max}$. ■

In summary, if we can read the input in two passes, based on the sketch of the schedule produced by all our streaming algorithms, the Algorithm SketchToSchedule can generate a feasible schedule with the makespan at most $(1+\epsilon)$ times the optimal value.

Theorem 13. *For the problems $P \mid prec, dp_j \leq h, p_{[min]} \leq c \cdot p_{max} \mid C_{max}$, and $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, when $m \leq \frac{2n\epsilon}{3-h-c}$ and $m \leq \frac{2n\alpha\epsilon}{3(h+1)\cdot c}$, respectively, there exist streaming approximation schemes that can return an approximate value and a sketch of a schedule in one pass, and output a schedule for each job in constant time in the second pass.*

4. Randomized Sublinear Time Algorithm

In the previous section, we studied the streaming algorithms, which scan all the input data, generate a sketch of the input data, and use it to compute an approximate value of the optimal makespan and a sketch of a schedule that describes a feasible schedule with the approximated makespan. In this section, we study a different computing paradigm, sublinear time algorithms which are also inspired by the boost of multitude of data in manufacturing and service industry. For sublinear time algorithms, our goal is to compute an approximate value of the optimal solution by considering only a fraction of the input data. As most sublinear time algorithms, our algorithms are randomized. Like streaming algorithms, our sublinear time algorithms also use the sketch of the input to compute the approximate value and the sketch of the schedule. The concept of the sketch of the input and the sketch of the schedule are similar to the ones that we defined in the streaming algorithms. However, since we do not read all input, the sketches are not accurate. We call them **estimated sketch of the input**, and **estimated sketch of the schedule**.

The **estimated sketch of the input** is an estimated summary of the n input jobs that is computed based on the sketch of n' sample jobs. The sample size n' is determined by the approximation ratio ϵ , and other parameters. We will show that with appropriate sample size, the estimated sketch of the input can give a good approximation of the accurate sketch of the input with high probability, and thus can give a good approximation of the optimal makespan. Formally, the **estimated sketch of the input** is defined as follows:

Definition 14. For a given parameter δ , and an instance of the problem $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ or $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, the **estimated sketch of the input** with respect to δ , is denoted as $\widehat{SKJ}_\delta = \{(d, u, \hat{e}_{d,u})\}$ where $\hat{e}_{d,u}$ is the estimated number of jobs with the depth d and the processing time in the range of $[(1+\delta)^u, (1+\delta)^{u+1})$.

Similarly, the **estimated sketch of a schedule** is a concise description of a schedule. Based on the estimated sketch of the schedule, with high probability, we can generate a feasible schedule with the makespan of at most $(1+\epsilon)$ times the optimal makespan. Formally the **estimated sketch of a schedule** is defined as follows:

Definition 15. For the problems $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{max} \mid C_{max}$ and $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, the **estimated sketch of a schedule** describes a schedule and consists of a set of time instants t_d , $1 \leq d \leq h$, such that all the jobs of depth d are scheduled during the interval $[t_{d-1}, t_d)$ for $1 \leq d \leq h$ where $t_0 = 0$. Mathematically we denote the **estimated sketch of a schedule** as $\widehat{SKS} = \{t_d : 1 \leq d \leq h\}$.

At the conceptual level, our sublinear time algorithms have the following three steps:

Step 1: Compute the sample size n' that is sublinear in n but is sufficient for computing an estimated sketch of the input jobs that is close to the accurate sketch for the original input data.

Step 2: Sample n' jobs uniformly at random from the input, find the sketch of the sampled jobs, $SKJ'_\delta = \{(d, u, n'_{d,u})\}$, and calculate the estimated sketch of all n jobs, $\widehat{SKJ}_\delta = \{(d, u, \hat{e}_{d,u})\}$.

Step 3: Based on the estimated sketch of the input, compute an approximation of the optimal value and an estimate sketch of a schedule.

In the following, we first develop a sublinear time algorithm for $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ and then adapt it to solve the general problem $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$.

4.1. Sublinear Time Algorithm for $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$

As before, each job j is represented by a pair (p_j, dp_j) . Without loss of generality, we assume that $1 \leq p_j \leq c$ for all $1 \leq j \leq n$. Our algorithm mainly consists of the three steps as described above: (1) compute the sample size n' ; (2) sample n' jobs uniformly at random, find the sketch of the sampled jobs, $SKJ'_\delta = \{(d, u, n'_{d,u})\}$, where $n'_{d,u}$ is the number of sampled jobs with the depth d and the processing time in the range of $[(1 + \delta)^u, (1 + \delta)^{u+1})$, and calculate the estimated sketch of the input for n jobs, $\widehat{SKJ}_\delta = \{(d, u, \hat{e}_{d,u})\}$ such that $\hat{e}_{d,u} = \frac{n}{n'} n'_{d,u}$, and $\hat{e}_{d,u} \geq 2\tau(n, h, c)$, where $\tau(n, h, c)$ is determined by some parameters; (3) compute an approximation of the optimal value. The algorithm is formally described in Randomized-Algorithm1.

Now we give the performance analysis for the above algorithm. The time complexity is dominated by the sampling operation. Thus we have the following lemma.

Lemma 16. *The running time of the algorithm is $O(\frac{c^2 h^2 \log^2 c}{\epsilon^6} \log(\frac{h}{\epsilon} \log c) \cdot m^2)$.*

Proof: The algorithm takes n' random samples and the processing time for each sampled job is $O(1)$. So the running time of the algorithm is $O(n') = O(\frac{1}{\beta^2} \cdot \ln \frac{2}{\gamma}) = O(\frac{c^2 h^2 k^2 m^2}{\epsilon^4} \log(hk)) = O(\frac{c^2 h^2 \log^2 c}{\epsilon^6} \log(\frac{h}{\epsilon} \log c) \cdot m^2)$. ▀

From now on we focus on the accuracy analysis for our algorithm. Since in our analysis we use the bounds that Ma [18] has obtained based on the well-known Chernoff bounds (see [20]), and the union bound from probability theory, we list them in the following for reference.

Lemma 17 (Lemma 3 in Ma [18]). *Let X_1, \dots, X_n be n independent random 0-1 variables and $X = \sum_{i=1}^n X_i$.*

i. If X_i takes 1 with probability at most p for $i = 1, \dots, n$, then for any $\beta > 0$, $\Pr(X > pn + \beta n) < e^{-\frac{1}{3}n\beta^2}$.

ii. If X_i takes 1 with probability at least p for $i = 1, \dots, n$, then for any $\beta > 0$, $\Pr(X < pn - \beta n) < e^{-\frac{1}{2}n\beta^2}$.

Algorithm Randomized-Algorithm1

Input: Parameters: m, c, h, ϵ Jobs: (p_j, dp_j) , $1 \leq j \leq n$, $1 \leq dp_j \leq h$

Output: An approximation of the optimal makespan

- 1: compute the sample size n'
 - 2: let $\delta = \frac{\epsilon}{20}$, and $k = \lfloor \log_{1+\delta} c \rfloor$
 - 3: let $p = \frac{5\delta}{2c \cdot h \cdot k \cdot m}$, and $\beta = \delta p$
 - 4: let $n' = \frac{3}{\beta^2} \cdot \ln \frac{2}{\gamma}$, where $\gamma = \frac{1}{10hk}$
 - 5: sample n' jobs uniformly at random, and compute the sketch of the sampled jobs $SKJ'_\delta = \{(d, u, n'_{d,u})\}$
 - 6: compute the estimated sketch of all jobs \widehat{SKJ}_δ
 - 7: let $\tau(n, h, c) = n \cdot p$
 - 8: $\widehat{SKJ}_\delta = \emptyset$
 - 9: **for** each $(d, u, n'_{d,u}) \in SKJ'_\delta$ **do**
 - 10: let $\hat{e}_{d,u} = n \cdot \frac{n'_{d,u}}{n'}$
 - 11: **if** $\hat{e}_{d,u} > 2\tau(n, h, c)$
 - 12: $\widehat{SKJ}_\delta = \widehat{SKJ}_\delta \cup \{(d, u, \hat{e}_{d,u})\}$
 - 13: **end for**
 - 14: compute the estimated makespan
 - 15: let $rp_k = c$
 - 16: **for** each u , $0 \leq u < k$
 - 17: let $rp_u = (1 + \delta)^{u+1}$
 - 18: **for** each d , $1 \leq d \leq h$ **do**
 - 19: let $\hat{A}_d = \frac{1}{m} \sum_{u=0}^k (\hat{e}_{d,u} \cdot rp_u)$, where $(d, u, \hat{e}_{d,u}) \in \widehat{SKJ}_\delta$
 - 20: **end for**
 - 21: let $\hat{A} = \sum_{d=1}^h (\lfloor \hat{A}_d \rfloor + c)$
 - 22: return \hat{A}
-

Fact 18 (Union bound). Let E_1, E_2, \dots, E_m be m events that may not be independent, we have the inequality

$$\Pr(E_1 \cup E_2 \dots \cup E_m) \leq \Pr(E_1) + \Pr(E_2) + \dots + \Pr(E_m).$$

We will use Lemma 17 and Fact 18 to show that $\hat{e}_{d,u}$, computed by Randomized-Algorithm1, is a good estimate of the exact number of jobs with the depth d and processing time in the range of $[(1 + \delta)^u, (1 + \delta)^{u+1})$, $n_{d,u}$. Specifically, we have that with high probability: (1) if $n_{d,u}$ is at least $\tau(n, h, c)$, then our estimate, $\hat{e}_{d,u}$, is in the range of $[(1 - \delta)n_{d,u}, (1 + \delta)n_{d,u}]$; and (2) if $n_{d,u} < \tau(n, h, c)$, our estimated number of jobs, $\hat{e}_{d,u}$, is no more than $2\tau(n, h, c)$.

Lemma 19. For any d, u , let $\hat{e}_{d,u}$ be the value computed by Randomized-Algorithm1, then we have:

- (i) If $n_{d,u} \geq \tau(n, h, c)$, $\Pr((1 - \delta)n_{d,u} \leq \hat{e}_{d,u} \leq (1 + \delta)n_{d,u}) \geq 1 - \gamma$; and
- (ii) If $n_{d,u} < \tau(n, h, c)$, $\Pr(\hat{e}_{d,u} \leq 2\tau(n, h, c)) \geq 1 - \gamma$.

Proof: Let X_i denote the indicator random variable for the event that the i -th sample job has depth d , and the processing time is in $[(1 + \delta)^u, (1 + \delta)^{u+1})$. Then $n'_{d,u} = \sum_{i=1}^{n'} X_i$. Since n' jobs are sampled uniformly at random from n jobs, we have $\Pr(X_i = 1) = \frac{n_{d,u}}{n}$. For convenience, we let $p_0 = \frac{n_{d,u}}{n}$. By line 7 of our algorithm, $p = \frac{\tau(n, h, c)}{n}$.

We first prove (i): if $n_{d,u} \geq \tau(n, h, c)$, $\Pr((1 - \delta)n_{d,u} \leq \hat{e}_{d,u} \leq (1 + \delta)n_{d,u}) \geq 1 - \gamma$. It is sufficient to show that $\Pr(\hat{e}_{d,u} \leq (1 - \delta)n_{d,u}) \leq \frac{\gamma}{2}$ and $\Pr(\hat{e}_{d,u} \geq (1 + \delta)n_{d,u}) \leq \frac{\gamma}{2}$. By line 10 of the algorithm, $\hat{e}_{d,u} = n \cdot \frac{n'_{d,u}}{n'}$, thus we have

$$\begin{aligned} \Pr(\hat{e}_{d,u} \leq (1 - \delta)n_{d,u}) &= \Pr\left(n \cdot \frac{n'_{d,u}}{n'} \leq (1 - \delta)n_{d,u}\right) \\ &= \Pr\left(n'_{d,u} \leq (1 - \delta) \frac{n_{d,u}}{n} \cdot n'\right) \\ &= \Pr(n'_{d,u} \leq (1 - \delta)p_0 n') \\ &= \Pr(n'_{d,u} \leq (p_0 - \delta p_0)n') \end{aligned}$$

If $n_{d,u} \geq \tau(n, h, c)$, then $\Pr(X_i = 1) = p_0 = \frac{n_{d,u}}{n} \geq \frac{\tau(n, h, c)}{n} = p$. Using this fact, and applying Lemma 17 for the variable $n'_{d,u}$, $n'_{d,u} = \sum_{i=1}^{n'} X_i$, we get

$$\Pr(n'_{d,u} \leq (p_0 - \delta p_0)n') \leq e^{-\frac{1}{2}n'(\delta p_0)^2} \leq e^{-\frac{1}{2}n'(\delta p)^2} \leq e^{-\frac{1}{2}n'\beta^2} \leq \frac{\gamma}{2},$$

which means that $\Pr(\hat{e}_{d,u} \leq (1 - \delta)n_{d,u}) \leq \frac{\gamma}{2}$. Similarly, we have

$$\begin{aligned}
\Pr(\hat{e}_{d,u} \geq (1 + \delta)n_{d,u}) &= \Pr(n'_{d,u} \geq (p_0 + \delta p_0)n') \\
&\leq e^{-\frac{1}{3}n'(\delta p_0)^2} \\
&\leq e^{-\frac{1}{3}n'(\delta p)^2} \\
&\leq e^{-\frac{1}{3}n'\beta^2} \\
&\leq \frac{\gamma}{2}.
\end{aligned}$$

Next, we show (ii): if $n_{d,u} < \tau(n, h, c)$, $\Pr(\hat{e}_{d,u} \leq 2\tau(n, h, c)) \geq 1 - \gamma$. As for (i), we prove that $\Pr(\hat{e}_{d,u} > 2\tau(n, h, c)) \leq \gamma$. By line 7 of the algorithm, $\tau(n, h, c) = n \cdot p$, and $\hat{e}_{d,u} = n \cdot \frac{n'_{d,u}}{n'}$.

$$\Pr(\hat{e}_{d,u} > 2\tau(n, h, c)) = \Pr(\hat{e}_{d,u} > 2np) = \Pr(n'_{d,u} > 2n'p) \leq \Pr(n'_{d,u} > (p + \delta p)n').$$

If $n_{d,u} < \tau(n, h, c)$, then $\Pr(X_i = 1) = \frac{n_{d,u}}{n} \leq \frac{\tau(n, h, c)}{n} = p$. Using this fact, and applying Lemma 17 for the variable $n'_{d,u}$, $n'_{d,u} = \sum_{i=1}^{n'} X_i$, we get

$$\Pr(n'_{d,u} > (p + \delta p)n') \leq e^{-\beta^2 \frac{n'}{3}} \leq \frac{\gamma}{2},$$

which implies that $\Pr(\hat{e}_{d,u} > 2\tau(n, h, c)) \leq \frac{\gamma}{2} < \gamma$. This completes the proof. ■

Lemma 19 tells us that the estimated sketch of input \widehat{SKJ}_δ approximates the exact sketch of input SKJ_δ very well. Based on this, we will show that the estimated makespan, \hat{A} , computed from the estimated sketch, is a good approximation of the optimal makespan. For the ease of our analysis and proof later, we summarize all the symbols we use in the following:

- I : the input instance for the algorithm
- $SKJ_\delta = \{(d, u, n_{d,u})\}$: the exact sketch of all jobs in I where $n_{d,u}$ is the number of jobs in I with the depth d and the processing time in the range of $[(1 + \delta)^u, (1 + \delta)^{u+1})$ for all $1 \leq d \leq h$ and $1 \leq u \leq k$
- I_{round} : the instance corresponding to the sketch SKJ_δ with the rounded processing times for all the jobs, that is, for each $(d, u, n_{d,u}) \in SKJ_\delta$, there are $n_{d,u}$ jobs at depth d whose

processing times are rp_u

- I_{big} : the instance obtained from the instance I_{round} by removing the jobs corresponding to $(d, u, n_{d,u})$ where $n_{d,u} < 3\tau(n, h, c)$ for all $1 \leq d \leq h$ and $1 \leq u \leq k$
- $\widehat{SKJ}_\delta = \{(d, u, \hat{e}_{d,u})\}$: the estimated sketch for the jobs in I , which is computed by Randomized-Algorithm1, and where $\hat{e}_{d,u}$ is the estimated value for $n_{d,u}$. Note that only the tuples with $\hat{e}_{d,u} > 2\tau(n, h, c)$ are included in \widehat{SKJ}_δ .
- \hat{I} : the instance corresponding to the estimated sketch $\widehat{SKJ}_\delta = \{(d, u, \hat{e}_{d,u})\}$, that is, for each $(d, u, \hat{e}_{d,u}) \in \widehat{SKJ}_\delta$, there are $\hat{e}_{d,u}$ jobs at depth d whose processing times are rp_u
- (d, u) -group of an instance: the group of all the jobs in the instance with depth d and processing time rp_u

We first compare the optimal makespan of instance \hat{I} and that of instance I_{big} .

Lemma 20. *Let $C_{max}^*(I_{big})$ and $C_{max}^*(\hat{I})$ be the optimal makespan for instances I_{big} and \hat{I} respectively, with probability of at least $\frac{9}{10}$, we have*

$$(1 - \delta)(C_{max}^*(I_{big}) - h \cdot c) < C_{max}^*(\hat{I}) \leq (1 + \delta)C_{max}^*(I_{big}) + \frac{15\delta n}{m} + h \cdot c. \quad (2)$$

Proof: From our definition of I_{big} and \hat{I} , we know that for any (d, u) -group included in I_{big} , we must have $n_{d,u} \geq 3\tau(n, h, c)$ and for any (d, u) -group included in \hat{I} , we must have $\hat{e}_{d,u} > 2\tau(n, h, c)$. We first show that with high probability the instance \hat{I} contains all jobs from instance I_{big} . Consider an arbitrary (d, u) -group from I_{big} , we must have $n_{d,u} \geq 3\tau(n, h, c)$, since $\delta = \frac{\epsilon}{20} < \frac{1}{20}$, we have $(1 - \delta)n_{d,u} \geq 2\tau(n, h, c)$. By Lemma 19, with the probability of at least $1 - \gamma$, we have $2\tau(n, h, c) < (1 - \delta)n_{d,u} \leq \hat{e}_{d,u} \leq (1 + \delta)n_{d,u}$. That means, with the probability of at least $1 - \gamma$, we have that any (d, u) -group in I_{big} is also included in \hat{I} . In other words, the probability that a (d, u) -group is in I_{big} but not in \hat{I} is less than γ . Since there are at most $h \cdot k$ (d, u) -groups, by Fact 18, the probability that some (d, u) -groups are in I_{big} but not in \hat{I} is at most $\gamma \cdot h \cdot k = \frac{1}{10}$. Therefore, considering all (d, u) -groups in I_{big} , we have that with probability at least $\frac{9}{10}$, all (d, u) -groups that are included in I_{big} are also included in \hat{I} .

A lower bound of $C_{max}^*(\hat{I})$ can be obtained by considering only those (d, u) -groups that are in I_{big} . To schedule the jobs in these groups from \hat{I} , one need an interval of length at least $\sum_d \frac{1}{m} \sum_u (\hat{e}_{d,u} \cdot rp_u) \geq \sum_d \frac{1}{m} \sum_u ((1 - \delta)n_{d,u} \cdot rp_u)$. So we have

$$C_{max}^*(\hat{I}) \geq \sum_{d=1}^h \left(\frac{1}{m} \sum_{u=0}^k ((1 - \delta)n_{d,u} \cdot rp_u) \right). \quad (3)$$

For all the jobs from I_{big} , we have:

$$C_{max}^*(I_{big}) \geq \sum_{d=1}^h \left(\frac{1}{m} \sum_{u=0}^k (n_{d,u} \cdot rp_u) \right), \quad (4)$$

and

$$C_{max}^*(I_{big}) \leq \sum_{d=1}^h \left(\frac{1}{m} \sum_{u=0}^k (n_{d,u} \cdot rp_u) + c \right) = \sum_{d=1}^h \left(\frac{1}{m} \sum_{u=0}^k (n_{d,u} \cdot rp_u) \right) + h \cdot c. \quad (5)$$

By inequalities (3) and (5) we have

$$C_{max}^*(\hat{I}) \geq (1 - \delta)(C_{max}^*(I_{big}) - h \cdot c). \quad (6)$$

Next, we consider the upper bound of $C_{max}^*(\hat{I})$. We split the jobs in \hat{I} into two parts: those (d, u) -groups that are in both I_{big} and \hat{I} , and those (d, u) -groups that are in \hat{I} but not in I_{big} . For the jobs in \hat{I} from the former, we need an interval of length at most $\sum_d (\frac{1}{m} (\sum_u (\hat{e}_{d,u} \cdot rp_u)) + c) \leq \sum_d (\frac{1}{m} (\sum_u ((1 + \delta)n_{d,u} \cdot rp_u)) + c)$ to schedule them; for the jobs from the latter (d, u) -groups, we note that each such group must correspond to a group in instance I where $n_{d,u} < 3\tau(n, h, c)$ and there are at most $h \cdot k$ such groups. By Lemma 19, with the probability of at least $1 - \gamma$, we have at most $\hat{e}_{d,u} \leq 6\tau(n, h, c)$ jobs in \hat{I} for each (d, u) -group in the latter type. Thus we can schedule these jobs in an interval of at most $6\tau(n, h, c) \cdot h \cdot k \cdot c$. Combining both types of groups and by inequality (4), we have

$$\begin{aligned} C_{max}^*(\hat{I}) &\leq \sum_{d=1}^h \left(\frac{1}{m} \left(\sum_{u=0}^k ((1 + \delta)n_{d,u} \cdot rp_u) \right) + c \right) + 6\tau(n, h, c) \cdot h \cdot k \cdot c \\ &\leq (1 + \delta)C_{max}^*(I_{big}) + h \cdot c + 6\tau(n, h, c) \cdot h \cdot k \cdot c. \end{aligned}$$

By line 7 of the algorithm, $\tau(n, h, c) = n \cdot p = \frac{5\delta n}{2c \cdot h \cdot k \cdot m}$, we have

$$C_{max}^*(\hat{I}) \leq (1 + \delta)C_{max}^*(I_{big}) + \frac{15\delta n}{m} + h \cdot c. \quad (7)$$

Therefore, from (6) and (7), we get

$$(1 - \delta)(C_{max}^*(I_{big}) - h \cdot c) \leq C_{max}^*(\hat{I}) \leq (1 + \delta)C_{max}^*(I_{big}) + \frac{15\delta n}{m} + h \cdot c.$$

■

The next lemma compares the optimal makespan of instance I_{round} and that of instance I and I_{big} .

Lemma 21. *Let $C_{max}^*(I)$ and $C_{max}^*(I_{round})$ be the optimal makespan for instances I and I_{round} respectively, we have the following inequalities:*

$$C_{max}^*(I) \leq C_{max}^*(I_{round}) \leq (1 + \delta)C_{max}^*(I). \quad (8)$$

$$C_{max}^*(I_{round}) - \frac{8\delta n}{m} \leq C_{max}^*(I_{big}) \leq C_{max}^*(I_{round}). \quad (9)$$

Proof: By our notation, I is the original instance of n jobs where a job j has processing time p_j and depth dp_j , and I_{round} is the instance from I after rounding up the jobs' processing time such that if $(1 + \delta)^u \leq p_j \leq (1 + \delta)^{u+1}$, then the rounded processing time is $rp_u \leq (1 + \delta)p_j$. It is easy to see that we have

$$C_{max}^*(I) \leq C_{max}^*(I_{round}) < (1 + \delta)C_{max}^*(I).$$

The instance I_{big} can be obtained from the instance I_{round} by removing those (d, u) -group jobs where $n_{d,u} < 3\tau(n, h, c)$. The total number of the jobs removed is at most $3\tau(n, h, c) \cdot h \cdot k$, and each of these jobs have processing time at most c . Therefore, we have $C_{max}^*(I_{round}) - 3\tau(n, h, c) \cdot h \cdot k \cdot c \leq C_{max}^*(I_{big}) \leq C_{max}^*(I_{round})$. Since $\tau(n, h, c) = \frac{5\delta n}{2c \cdot h \cdot k \cdot m}$, we get

$$C_{max}^*(I_{round}) - \frac{8\delta n}{m} \leq C_{max}^*(I_{big}) \leq C_{max}^*(I_{round}).$$

■

Combining all the lemmas we proved in this section, we can prove that the Randomized-Algorithm1 is an approximation scheme.

Theorem 22. *If $m \leq \frac{n\epsilon}{20h \cdot c}$, Randomized-Algorithm1 is a randomized $(1+\epsilon)$ -approximation scheme for $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ that runs in $O(\frac{c^2 h^2 \log^2 c}{\epsilon^6} \log(\frac{h}{\epsilon} \log c) \cdot m^2)$ time.*

Proof: The running time follows from Lemma 16. We focus on the approximation ratio. By our notation, \hat{I} is the instance corresponding to the estimated sketch $\widehat{SKJ}_\delta = \{(d, u, \hat{e}_{d,u})\}$ where $\hat{e}_{d,u}$ is the estimated value for $n_{d,u}$. Only the tuples with $\hat{e}_{d,u} > 2\tau(n, h, c)$ are included in \widehat{SKJ}_δ . By Randomized-Algorithm1, $\hat{A}_d = \frac{1}{m} \sum_{u=0}^k (\hat{e}_{d,u} \cdot r p_u)$ and $\hat{A} = \sum_{d=1}^h (\lfloor \hat{A}_d \rfloor + c)$. Following the same proof as in Theorem 4, we can get

$$C_{max}^*(\hat{I}) \leq \hat{A} \leq C_{max}^*(\hat{I}) + h \cdot c.$$

By inequality (2), we get, with probability at least $\frac{9}{10}$

$$\hat{A} \leq C_{max}^*(\hat{I}) + h \cdot c \leq (1 + \delta)C_{max}^*(I_{big}) + \frac{15\delta n}{m} + 2h \cdot c.$$

If $m \leq \frac{n\epsilon}{20h \cdot c}$, with $\delta = \frac{\epsilon}{20}$, we get $h \cdot c \leq \frac{\delta n}{m}$. Thus, we get,

$$\begin{aligned} \hat{A} &\leq (1 + \delta)C_{max}^*(I_{big}) + \frac{15\delta n}{m} + 2h \cdot c \\ &\leq (1 + \delta)C_{max}^*(I_{big}) + \frac{17\delta n}{m} \\ &\leq (1 + \delta)C_{max}^*(I_{round}) + \frac{17\delta n}{m} && \text{by (9)} \\ &\leq (1 + \delta)^2 C_{max}^*(I) + \frac{17\delta n}{m} && \text{by (8)} \\ &\leq (1 + 20\delta)C_{max}^*(I) && \text{by } C_{max}^*(I) \geq \frac{n}{m} \\ &\leq (1 + \epsilon)C_{max}^*(I), && \text{by } \delta = \frac{\epsilon}{20} \end{aligned}$$

and

$$\begin{aligned}
\hat{A} &\geq C_{max}^*(\hat{I}) \\
&\geq (1 - \delta)(C_{max}^*(I_{big}) - h \cdot c) && \text{by (2)} \\
&\geq (1 - \delta)(C_{max}^*(I_{big}) - \frac{\delta n}{m}) && \text{by } h \cdot c \leq \frac{\delta n}{m} \\
&\geq (1 - \delta)(C_{max}^*(I_{round}) - \frac{9\delta n}{m}) && \text{by (9)} \\
&\geq (1 - \delta)(C_{max}^*(I) - \frac{9\delta n}{m}) && \text{by (8)} \\
&\geq (1 - \delta)(1 - 9\delta)C_{max}^*(I) && \text{by } C_{max}^*(I) \geq \frac{n}{m} \\
&\geq (1 - 10\delta)C_{max}^*(I) && \text{by } \delta = \frac{\epsilon}{20} \\
&\geq (1 - \epsilon)C_{max}^*(I)
\end{aligned}$$

■

Based on the above theorem, when $m = o(n^{1/2})$, Randomized-Algorithm is a sublinear time approximation scheme.

Corollary 23. *When $m = o(n^{1/2})$, Randomized-Algorithm1 is a randomized $(1 + \epsilon)$ -approximation scheme for $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ that runs in sublinear time.*

4.2. Sublinear Time Algorithm for $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$

In this section, we will generalize Randomized-Algorithm1 to solve the general problem $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$. The idea is basically similar except some pre-processing is needed because we do not know the processing time range of the top αn jobs. Specifically, our sublinear algorithm first samples some jobs to determine the upper bound of the largest job, then samples enough number of jobs to generate the estimated sketch of the input, and finally computes the approximation of the optimal value based on the estimated sketch of the input. The details are given in Randomized-Algorithm2.

Like the Randomized-Algorithm1, the time complexity of Randomized-Algorithm2 is dominated by the sample size n' . However, n' in this algorithm depends on n . Still, we will show in the lemma below that the running time of the algorithm is sublinear when $m = o(n^{1/2})$.

Algorithm Randomized-Algorithm2

Input: Parameters $m, c, h, \epsilon, \alpha$

Jobs: $(p_j, dp_j), 1 \leq j \leq n, 1 \leq dp_j \leq h$

Output: An approximation of the optimal makespan

- 1: determine the upper bound of the largest job:
 - 2: let $\delta = \frac{\epsilon}{20}, k = \lfloor \log_{1+\delta} \frac{cn}{\delta} \rfloor$, and $\gamma = \frac{1}{10hk}$
 - 3: let $n_0 = 1$ if $\alpha = 1$, and $n_0 = \left\lceil \frac{\ln \gamma}{\ln(1-\alpha)} \right\rceil$ if $\alpha < 1$
 - 4: sample n_0 jobs uniformly at random
 - 5: let w_0 be the largest processing time among all the n_0 sampled jobs
 - 6: determine the sample size n' :
 - 7: let $p = \frac{5\alpha\delta}{2c^2 h \cdot k \cdot m}$, and $\beta = \delta p$
 - 8: let $n' = \frac{2}{\alpha\beta^2} \cdot \ln \frac{2}{\gamma}$
 - 9: sample n' jobs uniformly at random
 - 10: remove those jobs whose processing time is at most $\frac{\delta w_0}{n}$ from the sampled jobs
 - 11: compute the sketch of the remaining sample jobs $SKJ'_\delta = \{(d, u, n'_{d,u})\}$
 - 12: compute the estimated sketch of all jobs \widehat{SKJ}_δ
 - 13: let $\tau(n, h, c) = n \cdot p$
 - 14: $\widehat{SKJ}_\delta = \emptyset$
 - 15: **for** each $(d, u, n'_{d,u}) \in SKJ'_\delta$ **do**
 - 16: let $\hat{e}_{d,u} = n \cdot \frac{n'_{d,u}}{n'}$
 - 17: if $\hat{e}_{d,u} > 2\tau(n, h, c)$
 - 18: $\widehat{SKJ}_\delta = \widehat{SKJ}_\delta \cup \{(d, u, \hat{e}_{d,u})\}$
 - 19: **end for**
 - 20: compute the estimated makespan
 - 21: let $u_- = \lfloor \log_{1+\delta} \frac{\delta w_0}{n} \rfloor, u_+ = \lfloor \log_{1+\delta} c w_0 \rfloor$
 - 22: let $rp_{u_+} = c w_0$
 - 23: for each $u_- \leq u < u_+$
 - 24: let $rp_u = (1 + \delta)^{u+1}$
 - 25: for each $d, 1 \leq d \leq h$
 - 26: let $\hat{A}_d = \frac{1}{m} \sum_{u=u_-}^{u_+} (\hat{e}_{d,u} \cdot rp_u)$, where $(d, u, \hat{e}_{d,u}) \in \widehat{SKJ}_\delta$
 - 27: $\hat{A} = \sum_{d=1}^h \left(\lfloor \hat{A}_d \rfloor + c w_0 \right)$
 - 28: return \hat{A}
-

Lemma 24. *Randomized-Algorithm2 runs in time $O(\frac{c^4 h^2}{\alpha^3 \epsilon^6} \cdot m^2 \log^2(\frac{cn}{\epsilon}) \log(\frac{h}{\epsilon} \log(\frac{cn}{\epsilon})))$.*

Proof: The running time is dominated by the sampling of $n_0 + n' = O(n')$ jobs. Thus its running time is

$$O(n') = O(\frac{1}{\alpha \beta^2} \cdot \ln \frac{2}{\gamma}) = O(\frac{1}{\alpha \delta^2} (\frac{c^2 h k m}{\alpha \delta})^2 \log h k) = O(\frac{c^4 h^2}{\alpha^3 \epsilon^6} \cdot m^2 \log^2(\frac{cn}{\epsilon}) \log(\frac{h}{\epsilon} \log(\frac{cn}{\epsilon}))).$$

■

The next lemma shows that by sampling n_0 jobs, we can get a good estimate of the largest processing time $p_{[n]}$.

Lemma 25. *With probability at least $1 - \gamma$, $p_{[n]} \leq cw_0$.*

Proof: Since we sample the jobs uniformly, the probability that a job from the top αn largest jobs is selected is α . The probability that no job from top αn largest jobs is sampled is at most $(1 - \alpha)^{n_0} \leq \gamma$, which implies that with probability of at least $1 - \gamma$, some jobs from the top αn largest jobs are sampled, which means $w_0 \geq p_{[1-\alpha]n}$ and $p_{[n]} \leq cw_0$.

■

The next lemma is similar to Lemma 19 which states that $\hat{e}_{d,u}$ is a good estimate of the number of corresponding jobs in the input instance, $n_{d,u}$. The only difference is that here we focus on the jobs whose processing time is at least $\frac{\delta w_0}{n}$. The proof is the same and we omit here.

Lemma 26. *For any d, u , let $\hat{e}_{d,u}$ be the value computed by Randomized-Algorithm2, then we have:*

(i) *If $n_{d,u} \geq \tau(n, h, c)$, $\Pr((1 - \delta)n_{d,u} \leq \hat{e}_{d,u} \leq (1 + \delta)n_{d,u}) \geq 1 - \gamma$, and*

(ii) *If $n_{d,u} < \tau(n, h, c)$, $\Pr(\hat{e}_{d,u} \leq 2\tau(n, h, c)) \geq 1 - \gamma$.*

Like Theorem 22, we can prove that Randomized-Algorithm2 is an approximation scheme.

Theorem 27. *For $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, when $m \leq \frac{n \cdot \alpha \cdot \epsilon}{20c^2 \cdot h}$, Randomized-Algorithm2 is a randomized $(1 + \epsilon)$ -approximation scheme that runs in time $O(\frac{c^4 h^2}{\alpha^3 \epsilon^6} \cdot m^2 \log^2(\frac{cn}{\epsilon}) \log(\frac{h}{\epsilon} \log(\frac{cn}{\epsilon})))$.*

Proof: The running time follows from Lemma 24. We consider the approximation ratio only. The proof is similar to that of Theorem 22. For the input instance I , let I_{round} be the instance

obtained from I by rounding up the processing times for the jobs with $p_j \geq \frac{\delta w_0}{n}$. Let $C_{max}^*(I)$ and $C_{max}^*(I_{round})$ respectively be the optimal makespan for I and I_{round} . Then we still have the same inequalities between $C_{max}^*(I_{round})$ and $C_{max}^*(I)$:

$$C_{max}^*(I) \leq C_{max}^*(I_{round}) \leq (1 + \delta)C_{max}^*(I) \quad (10)$$

Let I_{big} be the instance obtained from the instance I_{round} by removing not only the (d, u) -groups with $n_{d,u} < 3\tau(n, h, c)$ but also the groups of the jobs whose processing time is less than $\frac{\delta w_0}{n}$. The total processing time of the jobs with processing time less than $\frac{\delta w_0}{n}$ is at most $n \cdot \frac{\delta w_0}{n} \leq \delta w_0 \leq \delta C_{max}^*(I_{round})$. The other jobs removed belong to the groups with $n_{d,u} < 3\tau(n, h, c)$, and each of these jobs has processing time at least $\frac{\delta w_0}{n}$ and at most cw_0 . There are at most $h \cdot k$ such groups where $k = \log_{1+\delta} \frac{cn}{\delta}$ as defined in the algorithm. Thus the total processing time of these jobs is at most

$$3\tau(n, h, c) \cdot h \cdot k \cdot cw_0 = 3n \cdot \frac{5\alpha\delta}{2c^2 \cdot h \cdot k \cdot m} \cdot h \cdot k \cdot cw_0 \leq \frac{8\delta \cdot \alpha \cdot nw_0}{cm}.$$

Thus we have

$$C_{max}^*(I_{round}) - \delta C_{max}^*(I_{round}) - \frac{8\delta \cdot \alpha \cdot nw_0}{cm} \leq C_{max}^*(I_{big}) \leq C_{max}^*(I_{round}) \quad (11)$$

As before, let \hat{I} be the instance corresponding to the sketch \widehat{SKJ}_δ , which contains $\hat{e}_{d,u}$ number of jobs with the depth of d and the processing time rp_u where $\hat{e}_{d,u} > 2\tau(n, h, c)$. Then the optimal makespan of \hat{I} , $C_{max}^*(\hat{I})$, is at least $\hat{A}_d = \frac{1}{m} \sum_{u=u_-}^{u_+} (\hat{e}_{d,u} \cdot rp_u)$. Between the I_{big} and \hat{I} , we can use similar argument for (6) and (7) to show that with probability at least $\frac{9}{10}$, we have

$$(1 - \delta)(C_{max}^*(I_{big}) - h \cdot cw_0) < C_{max}^*(\hat{I}) \quad (12)$$

and

$$C_{max}^*(\hat{I}) \leq (1 + \delta)C_{max}^*(I_{big}) + \frac{15\delta\alpha nw_0}{cm} + h \cdot cw_0 \quad (13)$$

The returned value $\hat{A} = \sum_{d=1}^h (\lfloor \hat{A}_d \rfloor + cw_0)$ is at least $C_{max}^*(\hat{I})$ and

$$\hat{A} = \sum_{d=1}^h (\lfloor \hat{A}_d \rfloor + cw_0) \leq C_{max}^*(\hat{I}) + h \cdot cw_0 \leq (1 + \delta)C_{max}^*(I_{big}) + \frac{15\delta\alpha n w_0}{cm} + 2h \cdot cw_0.$$

Assuming $m \leq \frac{n \cdot \alpha \cdot \epsilon}{20c^2 \cdot h} = \frac{n \cdot \alpha \cdot \delta}{c^2 \cdot h}$, then $h \cdot cw_0 \leq \frac{\delta\alpha n \cdot w_0}{cm}$, and combining with the above inequalities, we get

$$\begin{aligned} \hat{A} &\leq (1 + \delta)C_{max}^*(I_{big}) + \frac{17\delta\alpha n w_0}{cm} \\ &\leq (1 + \delta)C_{max}^*(I_{round}) + \frac{17\delta\alpha n w_0}{cm} && \text{by (11)} \\ &\leq (1 + \delta)C_{max}^*(I_{round}) + 17\delta C_{max}^*(I) && \text{by } C_{max}^*(I) \geq \frac{\alpha n \cdot w_0}{cm} \\ &\leq (1 + \delta)^2 C_{max}^*(I) + 17\delta C_{max}^*(I) && \text{by (10)} \\ &\leq (1 + 20\delta)C_{max}^*(I) \\ &\leq (1 + \epsilon)C_{max}^*(I) && \text{by } \delta = \frac{\epsilon}{20} \end{aligned}$$

and

$$\begin{aligned} \hat{A} &> C_{max}^*(\hat{I}) \\ &> (1 - \delta)(C_{max}^*(I_{big}) - h \cdot cw_0) && \text{by (12)} \\ &> (1 - \delta)((1 - \delta)C_{max}^*(I_{round}) - \frac{8\delta\alpha n w_0}{cm} - h \cdot cw_0) && \text{by (11)} \\ &> (1 - \delta)((1 - \delta)C_{max}^*(I_{round}) - \frac{9\delta\alpha n w_0}{cm}) && \text{by } h \cdot cw_0 \leq \frac{\delta\alpha n \cdot w_0}{cm} \\ &> (1 - \delta)((1 - \delta)C_{max}^*(I) - \frac{9\delta\alpha n w_0}{cm}) && \text{by (10)} \\ &> (1 - \delta)((1 - \delta)C_{max}^*(I) - 9\delta C_{max}^*(I)) && \text{by } C_{max}^*(I) \geq \frac{\alpha n w_0}{cm} \\ &\geq (1 - \delta)(1 - 10\delta)C_{max}^*(I) \\ &= (1 - \epsilon)C_{max}^*(I) && \text{by } \delta = \frac{\epsilon}{20} \end{aligned}$$

■

From Theorem 27, we can easily get the following corollaries.

Corollary 28. *When $m = o(n^{1/2})$, Randomized-Algorithm2 is a randomized $(1 + \epsilon)$ -approximation*

scheme for $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, that runs in sublinear time.

Corollary 29. For any $\alpha = n^{-\phi}$ where $\phi \in (0, 1/3)$, if $m \leq \frac{n \cdot \alpha \cdot \epsilon}{20c^2 \cdot h}$, there is a randomized $(1 + \epsilon)$ -approximation scheme for $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$ and the algorithm runs in sublinear time $O(\frac{c^4 h^2}{\alpha^3 \epsilon^6} \cdot m^2 \log^2(\frac{cn}{\epsilon}) \log(\frac{h}{\epsilon} \log(\frac{cn}{\epsilon})))$.

Clearly, the algorithm will also work if there is no precedence constraint, i.e. all jobs have the same depth 1. This becomes the traditional load balancing problem.

Corollary 30. For any $\alpha = n^{-\phi}$ where $\phi \in (0, 1/3)$, if $m \leq \frac{n \cdot \alpha \cdot \epsilon}{20c^2 \cdot h}$, there is a randomized $(1 + \epsilon)$ -approximation scheme for $P \mid p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$ and the algorithm runs in sublinear time $O(\frac{c^4}{\alpha^3 \epsilon^6} \cdot m^2 \log^2(\frac{cn}{\epsilon}) \log(\frac{1}{\epsilon} \log(\frac{cn}{\epsilon})))$.

4.3. The Estimated Sketch of the Schedule

In this subsection, we will show that as the streaming algorithms in Section 3, the two sublinear time algorithms in this section can compute an **estimated sketch of a schedule** $\widehat{SKS} = \{t_d : 1 \leq d \leq h\}$ which describes a schedule where all the jobs of depth d are scheduled during the interval $[t_{d-1}, t_d]$ for $1 \leq d \leq h$ where $t_0 = 0$. And we will show that based on \widehat{SKS} , the Algorithm SketchToSchedule in Section 3.3 can, with high probability, generate a feasible schedule with the makespan at most $(1 + 2\epsilon)$ times the optimal makespan.

For the problem $P \mid prec, dp_j \leq h, p_{min} \leq c \cdot p_{max} \mid C_{max}$, we let the estimate sketch of a schedule be $\widehat{SKS} = \{t_d : 1 \leq d \leq h\}$ where $t_d = \sum_{i=1}^d (\lfloor \frac{A_i}{1-\delta} \rfloor + c + 3\lceil \tau(n, h, c) \rceil \cdot k \cdot c)$ for all $1 \leq d \leq h$. We have the following theorem for the estimate sketch of a schedule:

Theorem 31. For $P \mid prec, dp_j \leq h, p_{min} \leq c \cdot p_{max} \mid C_{max}$, given any $0 < \epsilon < 1$, when $m \leq \frac{n\epsilon}{20h \cdot c}$, Randomized-Algorithm1 can compute an estimated sketch of the schedule \widehat{SKS} , and based on the sketch, with probability at least $\frac{9}{10}$, Algorithm SketchToSchedule can generate a feasible schedule with the makespan at most $(1 + 2\epsilon)$ times the optimal makespan.

Proof: It is easy to see that Randomized-Algorithm1 can compute the estimate sketch of the schedule $\widehat{SKS} = \{t_d : 1 \leq d \leq h\}$. Now we will show that with high probability all the jobs with depth d can be feasibly scheduled during the interval $[t_{d-1}, t_d]$ for $1 \leq d \leq h$ where $t_0 = 0$. It

suffices to prove that with high probability all the jobs from the input instance I with depth d can be scheduled within an interval of length $\lfloor \frac{\hat{A}_d}{1-\delta} \rfloor + c + \lfloor 3\tau(n, h, c) \rfloor \cdot k \cdot c$. Since the instance I_{round} is obtained from I by rounding up the processing times, all we need to prove is that the jobs from the instance I_{round} with depth d can be scheduled within an interval of length $\lfloor \frac{\hat{A}_d}{1-\delta} \rfloor + c + \lfloor 3\tau(n, h, c) \rfloor \cdot k \cdot c$.

As the proof of Lemma 20, we split the jobs in I_{round} into two parts: those (d, u) -groups that are in I_{big} , and those (d, u) -groups that are not in I_{big} . For the jobs from the former, with probability at least $\frac{9}{10}$, all (d, u) -group in I_{big} are also included in \hat{I} and for each (d, u) -group of this type, we have $n_{d,u} \geq 3\tau(n, h, c)$ and $n_{d,u} \leq \frac{\hat{e}_{d,u}}{1-\delta}$. Thus all these jobs at depth d can be feasibly scheduled during the interval of length

$$\left\lceil \frac{1}{m} \sum_{u=0}^k (n_{d,u} \cdot r p_u) \right\rceil + c \leq \left\lceil \frac{1}{m} \sum_{u=0}^k \frac{(\hat{e}_{d,u} \cdot r p_u)}{(1-\delta)} \right\rceil + c = \left\lfloor \frac{\hat{A}_d}{1-\delta} \right\rfloor + c.$$

For the jobs from the latter (d, u) -groups, we have $n_{d,u} < 3\tau(n, h, c)$, and thus they can be feasibly scheduled on a single machine during an interval of length $\lfloor 3\tau(n, h, c) \rfloor \cdot k \cdot c$. Therefore, combining both types of jobs, we have that with probability at least $\frac{9}{10}$ all the jobs with depth d from I_{round} can be scheduled within a time interval of length $\lfloor \frac{\hat{A}_d}{1-\delta} \rfloor + c + \lfloor 3\tau(n, h, c) \rfloor \cdot c \cdot k$. Since the jobs in I_{round} are rounded up from those in I , the jobs depth d from I can also be scheduled within the same interval length.

Finally, it is easy to see that the Algorithm SketchToSchedule generates a feasible schedule of all the jobs with depth d from I during the interval $[t_{d-1}, t_d)$. The makespan of the final schedule is at most

$$t_h = \sum_{d=1}^h \left(\left\lfloor \frac{\hat{A}_d}{1-\delta} \right\rfloor + c + \lfloor 3\tau(n, h, c) \rfloor \cdot k \cdot c \right) \leq \sum_{d=1}^h \left(\frac{\hat{A}_d}{1-\delta} \right) + h \cdot c + \lfloor 3\tau(n, h, c) \rfloor \cdot h \cdot k \cdot c$$

Note $3\tau(n, h, c) \cdot h \cdot k \cdot c \leq \frac{8\delta n}{m}$, and $h \cdot c \leq \frac{\delta n}{m}$ when $m \leq \frac{n\epsilon}{20h \cdot c}$. Thus,

$$\begin{aligned}
t_h &\leq \sum_{d=1}^h \left(\frac{\hat{A}_d}{1-\delta} \right) + \frac{\delta n}{m} + \frac{8\delta n}{m} \\
&\leq \frac{1}{1-\delta} \sum_{d=1}^h (\hat{A}_d) + \frac{9\delta n}{m} \\
&\leq \frac{1}{1-\delta} \sum_{d=1}^h (\hat{A}_d) + 9\delta C_{max}^*(I) && \text{by } C_{max}^*(I) \geq \frac{n}{m} \\
&\leq \frac{\hat{A}}{1-\delta} + 9\delta C_{max}^*(I) \\
&\leq \frac{(1+20\delta)}{1-\delta} C_{max}^*(I) + 9\delta C_{max}^*(I) && \text{by Theorem 22} \\
&\leq (1+25\delta) C_{max}^*(I) + 9\delta C_{max}^*(I) && \text{by } \delta = \frac{\epsilon}{20} < \frac{1}{20} \\
&\leq (1+2\epsilon) C_{max}^*(I).
\end{aligned}$$

■

For $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, we let the estimate sketch of a schedule be $\widehat{SKS} = \{t_d : 1 \leq d \leq h\}$ where $t_d = \sum_{i=1}^d (\lfloor \frac{\hat{A}_i}{1-\delta} \rfloor + cw_0 + \lfloor 3\tau(n, h, c) \rfloor \cdot k \cdot cw_0 + \lfloor \delta w_0 \rfloor)$ for all $1 \leq d \leq h$. For this sketch of the schedule, we can get similar conclusion.

Theorem 32. *For the problem $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$, given any $0 < \epsilon < 1$, and $m \leq \frac{n \cdot \alpha \cdot \epsilon}{20c^2 \cdot h}$, Randomized-Algorithm2 can generate an estimated sketch of the schedule \widehat{SKS} , and with probability at least $\frac{9}{10}$, Algorithm SketchToSchedule can produce based on \widehat{SKS} a feasible schedule with the makespan at most $(1+2\epsilon)$ times the optimal makespan.*

Proof: It is easy to see that Randomized-Algorithm2 can generate the estimated sketch of a schedule \widehat{SKS} . Now we will show that with high probability all the jobs with depth d can be feasibly scheduled during the interval $[t_{d-1}, t_d)$ for $1 \leq d \leq h$ where $t_0 = 0$. As the proof of Theorem 31, it suffices to prove that with high probability the jobs from the instance I_{round} with depth d can be scheduled within an interval of length $\lfloor \frac{\hat{A}_d}{1-\delta} \rfloor + cw_0 + \lfloor 3\tau(n, h, c) \rfloor \cdot k \cdot cw_0 + \lfloor \delta w_0 \rfloor$.

For this problem, the jobs in I_{round} can be split into three types: (1) (d, u) -groups that are in I_{big} , (2) (d, u) -groups corresponding to $(d, u, n_{d,u})$ where $n_{d,u} < 3\tau(n, h, c)$ for all $1 \leq d \leq h$ and $1 \leq u \leq k$ and the processing times of all jobs are greater than $\frac{\delta w_0}{n}$, and (3) jobs whose processing

times are no more than $\frac{\delta w_0}{n}$. We will bound the interval length needed to schedule jobs from each type. For type (1) jobs, as the proof in Theorem 27, with probability at least $\frac{9}{10}$, all (d, u) -group in I_{big} are also included in \hat{I} and for each (d, u) -group of this type, we have $n_{d,u} \geq 3\tau(n, h, c)$ and $n_{d,u} \leq \frac{\hat{e}_{d,u}}{1-\delta}$. Thus all these jobs at depth d can be feasibly scheduled during the interval of length

$$\left\lceil \frac{1}{m} \sum_{u=0}^k (n_{d,u} \cdot r p_u) \right\rceil + c w_0 \leq \left\lceil \frac{1}{m} \sum_{u=0}^k \frac{(\hat{e}_{d,u} \cdot r p_u)}{(1-\delta)} \right\rceil + c w_0 = \left\lceil \frac{\hat{A}_d}{1-\delta} \right\rceil + c w_0.$$

For type (2) jobs, it is easy to see that all the jobs at depth d can be feasibly scheduled during an interval of length $\lceil 3\tau(n, h, c) \rceil \cdot c w_0 \cdot k$; For type (3) jobs, since the processing times are integer, the processing time of each job must be at most $\lfloor \frac{\delta w_0}{n} \rfloor$. There are at most n such jobs at each depth, thus they can be feasibly scheduled during an interval of length $n \lfloor \frac{\delta w_0}{n} \rfloor = \lfloor \delta w_0 \rfloor$. Adding all these together, with probability at least $\frac{9}{10}$ all jobs at depth d from I_{round} can be scheduled into an interval of length

$$\left\lceil \frac{\hat{A}_d}{1-\delta} \right\rceil + c w_0 + \lceil 3\tau(n, h, c) \rceil \cdot k \cdot c w_0 + \lfloor \delta w_0 \rfloor.$$

Similar as before, we can use Algorithm SketchToSchedule to generate a feasible schedule with the makespan at most

$$\begin{aligned} t_h &= \sum_{d=1}^h \left(\left\lceil \frac{\hat{A}_d}{1-\delta} \right\rceil + c w_0 + \lceil 3\tau(n, h, c) \rceil \cdot k \cdot c w_0 + \lfloor \delta w_0 \rfloor \right) \\ &\leq \left(\sum_{d=1}^h \frac{\hat{A}_d}{1-\delta} \right) + h \cdot (c w_0 + 3\tau(n, h, c) \cdot k \cdot c w_0 + \delta w_0) \\ &\leq \left(\sum_{d=1}^h \frac{\hat{A}_d}{1-\delta} \right) + (1 + \delta) h \cdot c w_0 + 3\tau(n, h, c) \cdot h \cdot k \cdot c w_0 \end{aligned}$$

By line 13 of Randomized-Algorithm2, $3\tau(n, h, c) \cdot h \cdot k \cdot c w_0 \leq \frac{8\delta\alpha n \cdot w_0}{cm}$ which implies $h \cdot c w_0 \leq \frac{\delta\alpha n \cdot w_0}{cm}$.

With $m \leq \frac{n \cdot \alpha \cdot \epsilon}{20c^2 \cdot h}$, we get $t_h \leq \left(\sum_{d=1}^h \frac{\hat{A}_d}{1-\delta} \right) + (1 + \delta) \frac{\delta\alpha n \cdot w_0}{cm} + \frac{8\delta\alpha n \cdot w_0}{cm}$. Since $C_{max}^*(I) \geq \frac{\alpha n \cdot w_0}{cm}$, we have $t_h \leq \left(\sum_{d=1}^h \frac{\hat{A}_d}{1-\delta} \right) + 10\delta C_{max}^*(I) \leq \frac{\hat{A}}{1-\delta} + 10\delta C_{max}^*(I)$. By Theorem 22, and $\delta = \frac{\epsilon}{20}$, we have

$$t_h \leq \frac{(1 + 20\delta)}{1 - \delta} C_{max}^*(I) + 10\delta C_{max}^*(I) \leq (1 + 2\epsilon) C_{max}^*.$$

This completes the proof. ▀

5. Conclusions

In this work, we studied the parallel machine precedence constrained scheduling problems $P \mid prec, dp_j \leq h, p_{max} \leq c \cdot p_{min} \mid C_{max}$ and $P \mid prec, dp_j \leq h, p_{[n]} \leq c \cdot p_{[(1-\alpha)n]} \mid C_{max}$. We focused on two types of computing paradigms, sublinear space algorithms and sublinear time algorithms, which are inspired by the boost of multitude of data in manufacturing and service industry. It is worth mentioning that in spite of the inapproximability result that there does not exist a polynomial time approximation algorithm with approximation ratio better than $\frac{4}{3}$ unless $P=NP$, our algorithms imply that both problems admit approximation schemes if m satisfies certain condition. Moreover, our algorithms for precedence constrained problems also imply the sublinear approximation algorithms for the popular load balancing problem where jobs are independent.

Our work not only provides an algorithmic solutions to the studied problem under big data model, but also provide a methodological framework for designing sublinear approximation algorithms that can be used for solving other scheduling problems. In particular, besides outputting the approximate value of the optimal makespan, we introduced the concept of “the sketch of a schedule” to cater the need of generating a concrete schedule which approximates the optimal schedule. For our studied problems, it is also interesting to design sublinear approximation algorithms for other various precedence constraints and other performance criteria including total completion time, maximum tardiness, etc.

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