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Performance Measure of Hierarchical Structures for Multi-Agent Systems

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Abstract: This paper investigates the robustness of linear consensus networks which are designed under a hierarchical scheme based on Cartesian product. For robustness analysis, consensus networks are subjected to additive white Gaussian noise. To quantify the robustness of the network, we use $\mathcal{H}_2$-norm: the square root of the expected value of the steady state dispersion of network states. We compare several classes of undirected and directed graph topologies. We show that the hierarchical structures, designed under the Cartesian product-based hierarchy, outperforms the single-layer structures in robustness. We provide simulations to support the analytical results presented in this paper.

Keywords: Distributed Control, Hierarchical Structures, Multi-Agent Systems, Performance Measure

1. INTRODUCTION

Distributed consensus in Multi-agents Systems (MAS) has been a topic of great attention in control community due to many potential applications; such as sensor networks [1, 2], cooperative control of unmanned aerial vehicles [3, 4] and biological networks [5]. In general consensus refers to agree on a common state value. Since relative information is available only, thus distributed consensus is the only option for a large network. In realistic scenario, network systems mostly suffer from communication link failure, node failure, and often, noise appearing at every node. Almost every MAS suffers from additive white Gaussian noise (AWGN). Thus, the robustness analysis of MAS against AWGN is an important topic.

Robustness of consensus networks is widely discussed in [6–13]. Robustness against additive white Gaussian noise is quantified as $\mathcal{H}_2$-norm while dealing with consensus network [14–21]. An edge-based approach plays an important role for analyzing the $\mathcal{H}_2$ performance of the consensus networks. The addition of cycles and trees improves the performance and convergence rate of the networked system [17, 22]. A wide-ranging performance analysis of linear consensus networks with random graph topologies in the presence of external disturbances have been discussed in [20].

Lower and upper bound on the $\mathcal{H}_2$-norm is also found useful for the guaranteed performance of the network in the presence of external disturbances. Various lower and upper bounds for the first and second-order consensus networks are obtained in [14, 23, 24], while for the robustness of uncertain consensus networks, detailed analyses can be found in [21, 25–29]. By designing observer-based feedback decentralized control law, one can find limits and scaling laws for the stability analysis and performance measure of interconnected linear systems [30].

In [31], an optimal bipartite consensus control problem of networked systems with unknown dynamics using model free reinforcement learning method is proposed. In [32], an improved distributed dual consensus DC-ADMM (alternating direction method of multipliers) is proposed without dual gap to overcome the issue of resource allocation and its dual problem. Furthermore, for less computation of complicated objective functions, a distributed inexact dual consensus is proposed.

To quantify the robustness of consensus network using $\mathcal{H}_2$-norm, it is standard to assume the Laplacian matrix of a graph as *normal* [20, 21, 33] . With this standard assumption, we state the contribution of this paper in the sequel.

In this paper we investigate the performance measure of consensus networks for directed and undirected graphs designed under Cyclic Pursuit (CP) scheme, Cartesian Product-based Hierarchical (CPH) scheme and compare the $\mathcal{H}_2$-norm of the system. We analyze the robustness of Single Layer and Hierarchical control structures in the presence of external disturbances.

Our contributions in this paper are as follows:

- We analyse the robustness of the hierarchical structures of MAS, while considering the additive white

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Gaussian noise at each node independently and their comparison with the single layer scheme.

- We show that hierarchical structures are more efficient and robust than the single layer structures by considering those digraphs whose Laplacian matrices are normal. $\mathcal{H}_2$-norm is the measure of robustness.

In Section II preliminaries and mathematical background are discussed. Section III discusses the robustness of noisy undirected consensus networks. Section IV deals with robustness of directed consensus networks. In Section V simulation results are discussed. Finally, Section VI concludes the paper.

2. PRELIMINARIES AND BACKGROUND

2.1. Graph Theory

Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with a non-empty node set $\mathcal{V} = \{1, 2, ..., n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Let $|\mathcal{V}(\mathcal{G})| = n$ be the number of nodes and $|\mathcal{E}| = m$ be the number of edges in $\mathcal{G}$. Laplacian matrix associated with $\mathcal{G}$ is defined as:

$$\mathcal{L} = \mathcal{D} - \mathcal{A}, \tag{1}$$

where $\mathcal{A} = \mathcal{A}(\mathcal{G}) \in \mathbb{R}^{n \times n}$ is an Adjacency matrix with positive entries, of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined such that $[\mathcal{A}(\mathcal{G})]_{ij} = 1$ if $(i, j) \in \mathcal{E}$ and 0 otherwise. $\mathcal{D}$ is a diagonal matrix containing the out degree of each node

$$\mathcal{D} = [\mathcal{D}]_{ii} = \sum_j [\mathcal{D}]_{ij}.$$

We can sort eigenvalues of $\mathcal{L}$ for undirected graph in ascending order as, $\lambda_1(\mathcal{G}) = 0, \leq \lambda_2(\mathcal{G}), \leq \ldots, \lambda_n(\mathcal{G})$. Superscript $\top$, represents the transpose of a matrix. A centering matrix M_n is defined as $M_n := (I_n - \frac{1}{n}\mathbf{1}\mathbf{1}_n^\top)$. An incidence matrix for a graph is denoted by the matrix $E = E(\mathcal{G})$ contains $\{0, \pm 1\}$ and defined as:

$$[E]_{ij} = \begin{cases} -1 & \text{if edge } j \text{ terminates at node } i \\ 1 & \text{if edge } j \text{ begins at node } i \\ 0 & \text{otherwise} \end{cases}$$

So the graph Laplacian can also be written as:

$$\mathcal{L} = E\mathcal{W}E^\top. \tag{2}$$

where $\mathcal{W}$ is the weight function that maps an edge set to a scalar value such that $\mathcal{W} : \mathcal{E} \mapsto \mathbb{R}$.

2.2. Cartesian Product Of The Graph

CPH scheme is an algebraic approach to design Complex networks [34]. Let $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ are two factor digraphs. Cartesian product of $\mathcal{G}_1$ and $\mathcal{G}_2$ is a graph $\mathcal{G} = \mathcal{G}_1 \Box \mathcal{G}_2$, having vertex set $\mathcal{V}_1 \times \mathcal{V}_2$ and there is an edge from vertex (i, p) to (j, q) in $\mathcal{V}$ if and only if either $i = j$ and $(p, q) \in \mathcal{E}_2$, or $p = q$ and $(i, j) \in \mathcal{E}_1$ [33]. An example of the Cartesian product is shown in Fig.1

It is worth mentioning that if two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected then $\mathcal{G} = \mathcal{G}_1 \Box \mathcal{G}_2$ is also connected [34].

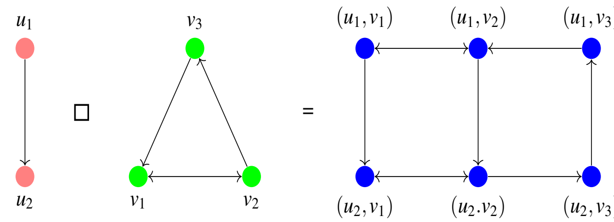

Fig. 1. Cartesian Product of the Graphs.

2.3. Robustness of Linear Time Invariant Systems

Consider a Linear Time Invariant (LTI) system with external stochastic noise as given below:

$$\dot{x}(t) = Ax(t) + \xi(t), \tag{3}$$

$$y(t) = Cx(t). \tag{4}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^m$ is the output of the system, $\xi(t) \in \mathbb{R}^n$ is the Additive White Gaussian Noise (AWGN) with zero mean and identity co-variance,

$$\mathbb{E}[\xi(t)\xi^\top(t)] = I_n\delta(t - \tau), \tag{5}$$

where $\delta(t)$ is a Dirac delta function. Dispersion of the system is defined by the square root of the following quantity: $||y(t)|| = (y^\top(t)y(t))$. Thus the robustness of a system can be measured as:

$$\mathcal{H}_2(A; \mathcal{Q}) := \lim_{t\to\infty} \mathbb{E}[||y(t)||] = \lim_{t\to\infty} \mathbb{E}[x^\top(t)\mathcal{Q}x(t)]^{\frac{1}{2}}, \tag{6}$$

where $\mathcal{Q} = C^\top C$ and $C = I_n$ with $I_n \in \mathbb{R}^{n\times n}$. $\mathcal{H}_2$-norm of the system from ξ to y is $[tr(CXC^\top)]^{\frac{1}{2}}$, where X is a solution of the Lyapunov equation $AX + XA^\top + BB^\top = 0$ and tr represents the trace of a matrix. $\mathcal{H}_2$-norm of the system is given by [14]:

$$\mathcal{H}_2(A; \mathcal{Q}) := \left(\sum_{i=1}^{n} \frac{1}{2} (\Re(\lambda_i(A)))^{-1} \right)^{\frac{1}{2}}, \tag{7}$$

If $A_s = (A^\top + A)/2$, then the bounds are [14]:

$$-\sum_{i=1}^{n} \frac{1}{2} \Re(\lambda_i(A))^{-1} \leq \mathcal{H}_2(A; \mathcal{Q}) \leq -\sum_{i=1}^{n} \frac{1}{2} (\lambda_i(A_s))^{-1} \tag{8}$$

It is assumed that the system given in (3)-(4) is stable with input noise co-variance (5) and the unitary matrix C. Furthermore, lower bound in (8) is achieved if and only if A is *normal* [14] and the symmetric part, denoted by, A_s is Hurwitz.

3. ROBUSTNESS OF UNDIRECTED CONSENSUS NETWORK

In this section, we will discuss the performance of the MAS designed under CP and CPH scheme in the presence of external disturbances for undirected graphs .

3.1. Single Layer Control Strategy

Cyclic Pursuit (CP) scheme is one of the single layer control strategy to solve the consensus problem. Consider a network of homogeneous agents model as a single-integrator:

$$\dot{x}_i(t) = u_i(t), \tag{9}$$

where $x_i(t)$ is the state of ith agent and $u_i(t)$ is the control input expressed as:

$$u_i(t) = \sum_{j=1}^{n} k_{ij}(x_j(t) - x_i(t)), \tag{10}$$

where $x_i(t)$ is the state of ith agent and k_{ij} represents the weight (gain) of an edge. For undirected graphs $k_{ij} = k_{ji} > 0$, if there is a link form i to j or vice versa.
The resulting autonomous system is

$$\dot{x}(t) = -\mathcal{L}(\mathcal{G})x(t), \tag{11}$$
$$y(t) = Cx(t). \tag{12}$$

It is assumed that the network is noise free. Consensus is said to be achieved in the sense that $\lim_{t\to\infty} ||x_i - x_j|| = 0, \forall i, j = 1, 2, ..., n$.

Now, consider the consensus dynamics of the network with noise

$$\dot{x}(t) = -\mathcal{L}x(t) + \zeta(t), \tag{13}$$
$$y(t) = Cx(t), \tag{14}$$

where $\zeta(t) \in \mathbb{R}^n$ is the Additive White Gaussian Noise (AWGN) with zero mean and identity covariance. It follows from [14] that $\mathcal{H}_2$-norm from ζ to y is bounded as below:

$$\sum_{i=2}^{n} \frac{1}{2\Re(\lambda_i(\mathcal{L}))} \leq \mathcal{H}_2(-\mathcal{L}; \mathcal{Q}) \leq \sum_{i=2}^{n} \frac{1}{\lambda_i(\mathcal{L} + \mathcal{L}^\top)} \tag{15}$$

where $\mathcal{Q} = M_n$ is the centering matrix for $\mathcal{H}_2$-norm. Since the graph is connected so only one marginally stable mode of the consensus network with dynamics given in (13) exits with the eigenvector $\mathbf{1_n}$, where $\mathbf{1_n}$ is a column vector with every entry equal to 1 and rest of the modes are stable. This marginally stable mode is not observable from output $y(t)$ because $M_n\mathbf{1_n} = \mathbf{0}$. Moreover, lower bound is achieved if and only if $\mathcal{L}$ is normal. Theorem 5 in [14] shows that the upper and lower bound given in (15) can be further tightened by considering $(13) - (14)$ as stable system with normal state matrix $\mathcal{L}$ as:

$$\sum_{i=2}^{n} \frac{\lambda_i(\mathcal{Q})}{2\Re(\lambda_i(\mathcal{L}))} \leq \mathcal{H}_2(-\mathcal{L}; \mathcal{Q}) \leq \sum_{i=2}^{n} \frac{\lambda_{n-i+2}(\mathcal{Q})}{2\Re(\lambda_i(\mathcal{L}))} \tag{16}$$

The lower and upper bounds are achieved if and only if $\mathcal{Q} = q(I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top), \forall q \geq 0$ [14].

3.2. Cartesian Product-Based Hierarchical Scheme

Hierarchical structures are useful to improve the convergence rate and performance measurement of the consensus protocol [7, 35, 36]. Cartesian product-based hierarchical structure (CPH) is a scheme in which complex systems are designed by taking the Cartesian product of graphs [34]. The Cartesian product of L number of graphs that makes an L-layer hierarchical network can be written as:

$$\mathcal{G}_L = \mathcal{G}_{n_1} \square \mathcal{G}_{n_2} \square ... \square \mathcal{G}_{n_L}, \tag{17}$$

where $\mathcal{G}_{n_1}, \mathcal{G}_{n_2}, ..., \mathcal{G}_{n_L}$ are balanced and strongly connected graphs, n_1 is the number of agents in a group and n_L is the number of groups in the L-layer hierarchy [34]. The underlying linear consensus protocol structure remain the same, but unlike single-layer MAS, in hierarchical structure of MAS, every agent has a neighboring agent in every layer. This facilitates convergence rate.

Exploiting the properties of the Kronecker product, Kronecker sum and Cartesian product of strongly connected balanced digraphs, the Laplacian associated with the digraph $\mathcal{G}_\mathcal{L}$ can be written as:

$$\mathcal{L}(\mathcal{G}_{n_1} \square \mathcal{G}_{n_2} \square \cdots \square \mathcal{G}_{n_L}) = \mathcal{L}_1(\mathcal{G}_{n_1}) \oplus \mathcal{L}_2(\mathcal{G}_{n_2}) \oplus ... \oplus \mathcal{L}_L(\mathcal{G}_{n_L}),$$

where $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_L$ represents the Laplacian matrices of $\mathcal{G}_{n_1}$,$\mathcal{G}_{n_2}$ and $\mathcal{G}_{n_L}$. In the above equation $\otimes$ and $\oplus$ represents the Kronecker product and Kronecker sum respectively. State and output equation for MAS in L-layer CPH strategy can be written as:

$$\dot{\hat{x}}_L = \hat{M}_L \hat{x}_L + \xi(t), \tag{18}$$
$$y(t) = C\hat{x}(t). \tag{19}$$

where

$$\hat{M}_L = -\mathcal{L}(\mathcal{G}_{n_1} \square \mathcal{G}_{n_2} \square \cdots \square \mathcal{G}_{n_L})$$

and

$$\hat{x}_L = (x_{1,1}, ..., x_{n_1,1}; x_{1,n_2}, ..., x_{n_1,n_2}; ...; x_{1,n_2}, ..., x_{n_1,n_L/n_1})$$

is state vector of $n_L = \prod_{m=1}^{L} n_m$ agents.
System Matrix for L-layer CPH scheme is given by:

$$\begin{aligned} \hat{M}_L(\mathcal{G}_{n_1} \square \cdots \square \mathcal{G}_{n_L}) = & -\mathcal{L}_1 \otimes I_{n_2} \otimes I_{n_3} \otimes \cdots \otimes I_{n_L} \\ & - I_{n_1} \otimes \mathcal{L}_2 \otimes I_{n_3} \otimes \cdots \otimes I_{n_L} - \cdots \\ & - I_{n_1} \otimes \cdots \otimes I_{n_{L-2}} \otimes \mathcal{L}_{L-1} \otimes I_{n_L} \\ & - I_{n_1} \otimes I_{n_2} \otimes \cdots \otimes I_{n_{L-1}} \otimes \mathcal{L}_L. \end{aligned}$$

The eigenvalues of the system matrix $\hat{M}_L$ are the elements in the set:

$$O = \{-\lambda_{i_1}^{\mathcal{L}_1} - \lambda_{i_2}^{\mathcal{L}_2} - \cdots - \lambda_{i_L}^{\mathcal{L}_L} | 1 \leq i_m \leq n_m, m = 1, 2 \cdots, L\},$$

(20)

where $\lambda_{i_m}^{\mathcal{L}_m}$ is the mth eigenvalue of the Laplacian matrix $\mathcal{L}_m$. Assume that the system (18) – (19) is stable with AWGN, where $\hat{M}_L$ is the negative Laplacian matrix of a weighted digraph which is *normal*, strongly connected and balanced so the bound can be given as:

$$\sum_{i=2}^{n}\frac{1}{2\Re(\lambda_i(\hat{M}_L))}\leq\mathcal{H}_2(\hat{M}_L;\mathcal{Q})\leq\sum_{i=2}^{n}\frac{1}{\lambda_i(\hat{M}_L+\hat{M}_L^\top)} \quad (21)$$

In (21) if $\hat{M}_L$ is *normal*, then lower and upper bound of $\mathcal{H}_2$ is same.

4. ROBUSTNESS OF DIRECTED CONSENSUS NETWORK

4.1. Single Layer Control Strategy

In this section, we consider linear consensus network design using CP scheme. It is assumed that all the directed graphs considered in this paper are weighted, strongly connected, balanced and stable modes of networks are observable from the output. Consider the noisy consensus network given in (13) – (14), we have:

$$\sum_{i=2}^{n}\frac{1}{2\Re(\lambda_i(\mathcal{L}))}\leq\mathcal{H}_2(-\mathcal{L};\mathcal{Q})\leq\sum_{i=2}^{n}\frac{1}{\lambda_i(\mathcal{L}_s)} \quad (22)$$

where $\mathcal{L}_s=(\mathcal{L}+\mathcal{L}^\top)$, $\mathcal{Q}=M_n$ and lower bound is achieved if and only if $\mathcal{L}$ is *normal*.

4.2. Cartesian Product-based Hierarchy

Cartesian product based hierarchical strategy is an algebraic approach to solve the average-consensus problem [34]. In the following theorem we show that the Laplacian of $\mathcal{G}=\mathcal{G}_1\square\mathcal{G}_2$ is *normal if and only if* the Laplacian $\mathcal{G}_1$ and $\mathcal{G}_2$ both are *normal*.

Theorem 1: Let $\mathcal{G}=\mathcal{G}_1\square\mathcal{G}_2$ be a balanced directed graph with the Laplacian matrix $\mathcal{L}$. The matrix $\mathcal{L}$ is normal if and only if the Laplacian matrices $\mathcal{L}_1$ and $\mathcal{L}_2$ of directed graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are normal.

Proof: Let $\mathcal{L}_1$ be the Laplacian matrix associated with the graph $\mathcal{G}_1$ and $\mathcal{L}_2$ be the Laplacian matrix associated with the graph $\mathcal{G}_2$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be *normal* matrices, that is $\mathcal{L}_1^\top\mathcal{L}_1=\mathcal{L}_1\mathcal{L}_1^\top$ and $\mathcal{L}_2^\top\mathcal{L}_2=\mathcal{L}_2\mathcal{L}_2^\top$.

We know that

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_1\otimes I_{n_2}+I_{n_1}\otimes\mathcal{L}_2\\ \mathcal{L}^\top\mathcal{L} &= (\mathcal{L}_1\otimes I_{n_2}+I_{n_1}\otimes\mathcal{L}_2)^\top(\mathcal{L}_1\otimes I_{n_2}+I_{n_1}\otimes\mathcal{L}_2)\\ &= (\mathcal{L}_1^\top\mathcal{L}_1\otimes I_{n_2})+(\mathcal{L}_1\otimes\mathcal{L}_2^\top)+(\mathcal{L}_1^\top\otimes\mathcal{L}_2)\\ &\quad+(I_{n_1}\otimes\mathcal{L}_2\mathcal{L}_2^\top)\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}\mathcal{L}^\top &= (\mathcal{L}_1\otimes I_{n_2}+I_{n_1}\otimes\mathcal{L}_2)(\mathcal{L}_1\otimes I_{n_2}+I_{n_1}\otimes\mathcal{L}_2)^\top\\ &= (\mathcal{L}_1\mathcal{L}_1^\top\otimes I_{n_2})+(\mathcal{L}_1\otimes\mathcal{L}_2^\top)+(\mathcal{L}_1^\top\otimes\mathcal{L}_2)\\ &\quad+(I_{n_1}\otimes\mathcal{L}_2\mathcal{L}_2^\top).\end{aligned}$$

Since $\mathcal{L}_1^\top\mathcal{L}_1=\mathcal{L}_1\mathcal{L}_1^\top$ and $\mathcal{L}_2^\top\mathcal{L}_2=\mathcal{L}_2\mathcal{L}_2^\top$. Thus $\mathcal{L}^\top\mathcal{L}=\mathcal{L}\mathcal{L}^\top$.

Conversely, let $\mathcal{L}\mathcal{L}^\top=\mathcal{L}^\top\mathcal{L}$, which can be written as:

$$(\mathcal{L}_1\mathcal{L}_1^\top-\mathcal{L}_1^\top\mathcal{L}_1)\otimes I_{n_2}+I_{n_1}\otimes(\mathcal{L}_2\mathcal{L}_2^\top-\mathcal{L}_2^\top\mathcal{L}_2)=0 \quad (23)$$

Now pre-multiply (23) with $(I_{n_1}\otimes\mathbf{1}_{n_2}^\top)$, we get

$$\begin{aligned}&(I_{n_1}\otimes\mathbf{1}_{n_2}^\top)\left((\mathcal{L}_1\mathcal{L}_1^\top-\mathcal{L}_1^\top\mathcal{L}_1)\otimes I_{n_2}\right)+\\ &\qquad(I_{n_1}\otimes\mathbf{1}_{n_2}^\top)\left(I_{n_1}\otimes(\mathcal{L}_2\mathcal{L}_2^\top-\mathcal{L}_2^\top\mathcal{L}_2)\right)=0\end{aligned} \quad (24)$$

As $(\mathbf{1}_{n_2}^\top\mathcal{L}_2^\top\mathcal{L}_2)=(\mathcal{L}_2\mathbf{1}_{n_2})^\top\mathcal{L}_2=0$

Since $\mathcal{G}=\mathcal{G}_1\square\mathcal{G}_2$ is balanced directed graph. By *Theorem 1* in [34], $\mathcal{G}_1$ and $\mathcal{G}_2$ are also balanced directed graphs. Consequently, we have $\mathcal{L}_1\mathbf{1}_{n_1}=0$, $\mathcal{L}_2\mathbf{1}_{n_2}=0$, $\mathbf{1}_{n_1}^\top\mathcal{L}_1=0$, and $\mathbf{1}_{n_2}^\top\mathcal{L}_2=0$ Thus, (24) can be written as:

$$(\mathcal{L}_1\mathcal{L}_1^\top-\mathcal{L}_1^\top\mathcal{L}_1)\otimes\mathbf{1}_{n_2}^\top=0 \quad (25)$$

Post-multiply (25) by $\mathbf{1}_{n_2}$, we have

$$(\mathcal{L}_1\mathcal{L}_1^\top-\mathcal{L}_1^\top\mathcal{L}_1)=0$$

$$\mathcal{L}_1\mathcal{L}_1^\top=\mathcal{L}_1^\top\mathcal{L}_1.$$

Similarly pre-multiply (23) with $(I_{n_1}^\top\otimes\mathbf{1}_{n_2})$, we get

$$I_{n_1}^\top\otimes(\mathcal{L}_2\mathcal{L}_2^\top-\mathcal{L}_2^\top\mathcal{L}_2)=0$$

and the result follows. □

Corollary 1: The Laplacian $\mathcal{L}$ of the balanced directed graph $\mathcal{G}=\mathcal{G}_1\square\mathcal{G}_2\square\mathcal{G}_3...\square\mathcal{G}_n$ is *normal if and only if* the Laplacian matrices $\mathcal{L}_1,\mathcal{L}_2,\cdots,\mathcal{L}_n$ of the graphs $\mathcal{G}_1,\mathcal{G}_2,\cdots,\mathcal{G}_n$ are *normal*, respectively.

Consider the linear consensus system (18)-(19) with $\hat{M}_L=-\mathcal{L}$, which is a system matrix of a strongly connected graph. If we consider $\mathcal{L}$ is normal and $\mathcal{Q}=C^\top C$ with $\mathcal{Q}\mathbf{1}=\mathbf{0}$, then

$$\sum_{i=2}^{n}\frac{\lambda_i(\mathcal{Q})}{2\Re\{\lambda_i(\hat{M}_L)\}}\leq\mathcal{H}_2(\hat{M}_L;\mathcal{Q})\leq\sum_{i=2}^{n}\frac{\lambda_{n-i+2}(\mathcal{Q})}{2\Re\{\lambda_i(\hat{M}_L)\}} \quad (26)$$

In above equation both bounds are achieved if and only if $\mathcal{Q}=q(I_n-\frac{1}{n}\mathbf{1}\mathbf{1}_n^\top),\forall q>0$.

Proposition 1: The *normal* Laplacian matrix $\hat{M}_L$ associated with the graph $\mathcal{G}_L$ is stable.

Proof: The *normal* Laplacian matrix $\hat{M}_L$ can be writen as:

$$\hat{M}_L=-\mathcal{L}(\mathcal{G}_{n_1}\square\mathcal{G}_{n_2}\square\cdots\square\mathcal{G}_{n_L})$$

The eigenvalues of the system matrix $\hat{M}_L$ are the elements in the following set O: [34]

$$O=\{-\lambda_{i_1}^{\mathcal{L}_1}-\lambda_{i_2}^{\mathcal{L}_2}-\cdots-\lambda_{i_L}^{\mathcal{L}_L}|1\leq i_m\leq n_m,m=1,2\cdots,L\}.$$

Since all the eigenvalues of the $\hat{M}_L$ are negative except a simple zero eigenvalue. Hence the system matrix $\hat{M}_L$ is stable. □

In the following theorem, we will quantify the robustness of the CP and CPH scheme in terms of performance measure of the MAS.

Theorem 2: Let n be the number of agents in CPH strategy such that $n = n_1 n_2$. Let the Laplacian matrix of the network designed under CPH strategy be normal. Let e_{ss}^{cp} be the dispersion of consensus network under the CP strategy with the same number of agents as in CPH strategy. In addition, the Laplacian matrix of CP strategy is normal and $e_{ss}^{cp} \geq d_1$. Let e_{ss}^{cph} be the dispersion of consensus network under CPH strategy and $e_{ss}^{cph} \leq d_2$, where d_1 is the lower bound of CP strategy and d_2 is the upper bound of CPH strategy, then $d_2 < d_1$.

Proof: The dispersion of the system designed under the CP strategy is given by

$$\sum_{i=2}^{n} \frac{1}{2\Re\{\lambda_i(\mathcal{L})\}} \leq e_{ss}^{cp} = \frac{n^2-1}{12}.$$

Now, consider the lower bound of e_{ss}^{cp} given by:

$$d_1 = \frac{\lambda_2(\mathcal{Q})}{2\Re\{\lambda_2(\mathcal{L})\}} + \frac{\lambda_3(\mathcal{Q})}{2\Re\{\lambda_3(\mathcal{L})\}} + \cdots + \frac{\lambda_n(\mathcal{Q})}{2\Re\{\lambda_n(\mathcal{L})\}} \quad (27)$$

In CP strategy real part of eigenvalue of matrix $\mathcal{L}$ can be written as:

$$\Re\{\lambda_i(\mathcal{L})\} = cos\left(\frac{2\pi(i-1)}{n}\right) - 1. \quad (28)$$

for $i = 1, 2, \cdots, n$. As n increases, $\Re\{\lambda_i(\mathcal{L})\}$ decreases.

Now consider the CPH strategy with the same number of agents as in the CP strategy for L-layers given by, $n = \prod_{m=1}^{L} n_m$. Let $\hat{M}_L$ be the negative Laplacian associated with the Cartesian product of the graph, then the dispersion from the consensus value is upper bounded by:

$$e_{ss}^{cph} \leq \sum_{i=2}^{n} \frac{\lambda_{n-i+2}(\mathcal{Q})}{2\Re\{\lambda_i(\hat{M}_L)\}},$$

Let

$$d_2 = \sum_{i=2}^{n} \frac{\lambda_{n-i+2}(\mathcal{Q})}{2\Re\{\lambda_i(\hat{M}_L)\}},$$

for $i = 2, 3, \cdots n$, where d_2 is the upper bound of e_{ss}^{cph}.

$$d_2 = \frac{\lambda_n(\mathcal{Q})}{2\Re\{\lambda_1(\hat{M}_L)\}} + \frac{\lambda_{n-1}(\mathcal{Q})}{2\Re\{\lambda_2(\hat{M}_L)\}} + \cdots + \frac{\lambda_2(\mathcal{Q})}{2\Re\{\lambda_n(\hat{M}_L)\}} \quad (29)$$

In this case, the close form of the real parts of eigenvalues of the system matrix $\hat{M}_L$ of the CPH strategy is given by:

$$\Re\{\lambda_{i_m}^{\mathcal{L}_k}\} = cos\left(\frac{2\pi(i_m-1)}{n_k}\right) - 1 \quad (30)$$

for $k = 1, 2, ..., L$, and n_k is number of groups in L-layer hierarchy. [34]. Since $\mathcal{Q} = (I_n - \frac{1}{n}\mathbf{1}\mathbf{1}_n^\top)$ is a centering matrix with $\mathcal{Q}\mathbf{1} = \mathbf{0}$ and it has n identical eigenvalues so, it is clear from (27), (29) and the set $O = \{-\lambda_{i_1}^{\mathcal{L}_1} - \lambda_{i_2}^{\mathcal{L}_2} - \cdots - \lambda_{i_L}^{\mathcal{L}_L} | 1 \leq i_m \leq n_m, m = 1, 2 \cdots, L\}$, that values of the real parts of eigenvalues of $\hat{M}_L$ will be greater than the values of the real parts of eigenvalues of $\mathcal{L}$. Thus the results. □

The above theorem quantifies the performance measure of MAS designed under CP and CPH scheme. Moreover, CPH scheme outperforms the CP scheme in terms of *robustness*. Note that the robustness is shown by introducing AWGN in the nodes of system. The CPH scheme provides an efficient way to design and analyze the complex networks.

Remark 1: As the number of agents in CP scheme increase, the real parts of the eigenvalues decrease, according to (22), $\mathcal{H}_2$-norm (performance measure) of CP scheme for directed consensus network deteriorates by $\mathcal{O}(n)$ [9].

Network designed under CPH scheme by considering the graphs having *normal* Laplacian matrix is more robust than the other networks design with cyclic pursuit graphs, both with the same number of nodes. The following example illustrates this fact.

Example 1: Consider a normal Laplacian matrix $\mathcal{L}_1$ obtain from two layer CPH scheme as

$$\mathcal{L}_1 = \begin{bmatrix} 2 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

and a cyclic pursuit graph with normal Laplacian matrix $\mathcal{L}_2 = circ\,[2 \;\; -1 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$, where circ denotes the circulant matrix of $\mathcal{L}_2$ [35]. The dimension of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the same and CPH scheme is more robust than the CP scheme because $\mathcal{H}_2$-norm is 1.3138 and 1.3994 respectively. This decrease in $\mathcal{H}_2$-norm for $\mathcal{L}_1$ and $\mathcal{L}_2$ comes at expense of adding two extra communication links in CPH scheme having normal Laplacian matrix $\mathcal{L}_1$.

Remark 2: CPH scheme is a methodical way of increasing the communication links in the network, improving the convergence rate and robustness. In contrast if we randomly increase the communication links in CP scheme it is not necessary to achieve average consensus with better convergence rate and robustness. Example 1 illustrates this fact. Also, from Table 1, Table 2 and Theorem 2 it is stated that the CPH scheme for undirected graphs is more robust than the directed graphs.

5. SIMULATION RESULTS

In this section, we present simulation results to support the theoretical results by considering several examples to give a better understanding about the convergence rate and robustness of MAS.

In the CP scheme, it is noticed that the convergence rate decreases and the robustness deteriorates as the size of network system increases. We design MAS for single, two, three and four layers by considering different number of agents under the CP and CPH scheme for both undirected and directed graphs. We consider balanced graphs with *normal* Laplacian matrices. For the sake of simplicity, we choose all the agents with homogeneous edge weights (gains) having value one. For the CP scheme while considering the undirected graph by adding AWGN in the system, $\mathcal{H}_2$-norm is 0.7906, 1.6202, 2.4410, 3.2596, 4.0774, 4.8947, 6.5288, 13.062 and 26.127 respectively for different number of agents.

When we consider the CPH scheme for two, three and four layers hierarchy the value of $\mathcal{H}_2$-norm decreases. These decreasing values of $\mathcal{H}_2$-norm shows that MAS designed under the CPH scheme has more robustness towards bearing the noise in the system. Since the $\mathcal{H}_2$- norm of the CPH scheme is less in comparison with the CP scheme, which concludes that the CPH scheme is more robust than the CP scheme. $\mathcal{H}_2$-norm for CP and the CPH scheme for undirected graphs is shown in Fig. 2.

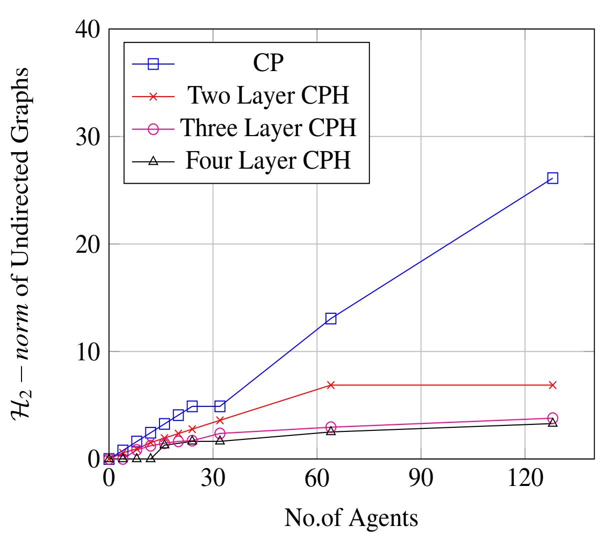

Fig. 2. Robustness of CP and CPH scheme for undirected graphs.

Now we consider directed graphs for checking the robustness of MAS designed under CP and the CPH scheme. For the CP scheme while considering the directed graphs, $\mathcal{H}_2$-norm is 1.118, 2.2913, 3.4521, 4.6098, 5.7663, 6.9222, 9.2331, 18.473 and 36.949 respectively. These values of $\mathcal{H}_2$-norm for directed graphs in comparison with undirected case for the CP scheme show that the CP scheme for undirected graphs is more robust than the directed graphs.When we consider the CPH scheme for two, three and four layers hierarchy for directed graphs the value of $\mathcal{H}_2$-norm decreases. Since the $\mathcal{H}_2$-norm is less for the CPH scheme than the CP scheme, which concludes that the CPH scheme is more robust than the CP scheme. Fig. 3 illustrates the $\mathcal{H}_2$-norm for CP and the CPH scheme for directed graphs.

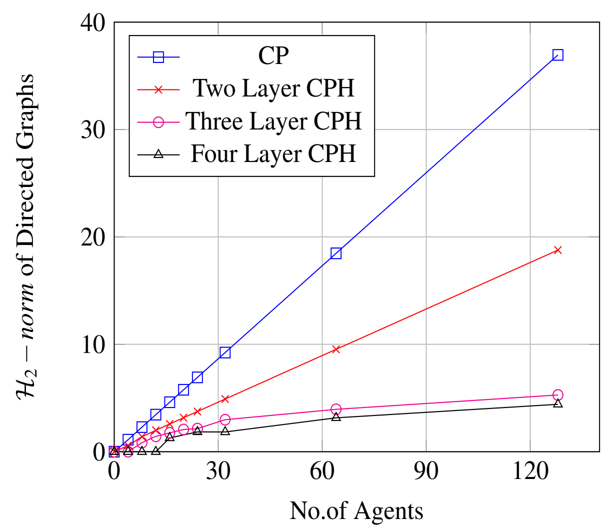

Fig. 3. Robustness of CP and CPH scheme for undirected graphs.

Moreover, Table 1 and Table 2 shows the changing trends in the values of $\mathcal{H}_2$-norm for undirected and directed graphs. It is worth mentioning that MAS designed under CP and the CPH scheme for undirected graphs are more robust than directed graphs. Also the performance of network designed under the CP scheme deteriorates badly with the growing size of network.

6. CONCLUSION

We analyzed the robustness of MAS having *normal* Laplacian matrices, while considering both undirected and directed graphs deigned under the CP and CPH scheme, perturbed by external AWGN. Robustness was measured by the $\mathcal{H}_2$-norm of the system and related to the properties of underlying graph topology. A comparison of the CP and CPH scheme was done by considering the different number of agents. For the CPH scheme we considered graphs with *normal* Laplacian matrices and performance measurement was observed. We proved that the CPH scheme is more robust than single layer strategy.

In future we aim to extend fuzzy-based control methods [37, 38] for multi-agent system to improve the robustness.

Table 1. $\mathcal{H}_2$-norm of the CP and CPH scheme with communication links for undirected graphs.

No.of Agents		**Layers**		**$\mathcal{H}_2$-norm**	
CP	**CPH**	**CP**	**CPH**	**CP**	**CPH**
1×4	2×2	1	2	0.7906	0.5774
1×8	2×4	1	2	1.6202	0.9335
1×8	$2\times2\times2$	1	3	1.6202	0.8416
1×12	2×6	1	2	2.4410	1.5248
1×12	$2\times2\times3$	1	3	2.4410	1.2252
1×16	2×8	1	2	3.2596	1.9442
1×16	$2\times2\times4$	1	3	3.2596	1.4649
1×16	$2\times2\times2\times2$	1	4	3.2596	1.2829
1×20	2×10	1	2	4.0774	2.3597
1×20	$2\times2\times5$	1	3	4.0774	1.6375
1×24	2×12	1	2	4.8947	2.7732
1×24	$2\times3\times4$	1	3	4.8947	1.6998
1×24	$2\times2\times2\times3$	1	4	4.8947	1.6375
1×32	2×16	1	2	6.5288	3.5964
1×32	$2\times2\times8$	1	3	6.5288	2.375
1×32	$2\times2\times2\times4$	1	4	6.5288	1.6419
1×64	2×32	1	2	13.062	6.873
1×64	$2\times4\times8$	1	3	13.062	2.954
1×64	$2\times2\times4\times4$	1	4	13.062	2.514
1×128	2×64	1	2	26.127	13.411
1×128	$4\times4\times8$	1	3	26.127	3.795
1×128	$2\times4\times4\times4$	1	4	26.127	3.294

Table 2. $\mathcal{H}_2$-norm of the CP and CPH scheme with communication links for directed graphs.

No.of Agents		**Layers**		**$\mathcal{H}_2$-norm**	
CP	**CPH**	**CP**	**CPH**	**CP**	**CPH**
1×4	2×2	1	2	1.1180	0.577
1×8	2×4	1	2	2.2913	1.3994
1×8	$2\times2\times2$	1	3	2.2913	0.8416
1×12	2×6	1	2	3.4521	1.9943
1×12	$2\times2\times3$	1	3	3.4521	1.4300
1×16	2×8	1	2	4.6098	2.5815
1×16	$2\times2\times4$	1	3	4.6098	1.7536
1×16	$2\times2\times2\times2$	1	4	4.6098	1.2829
1×20	2×10	1	2	5.7663	3.1651
1×20	$2\times2\times5$	1	3	5.7663	2.0685
1×24	2×12	1	2	6.9222	3.7467
1×24	$2\times\times4$	1	3	6.9222	2.1500
1×24	$2\times2\times2\times3$	1	4	6.9222	1.8469
1×32	2×16	1	2	9.2331	4.9070
1×32	$2\times2\times8$	1	3	9.2331	2.9824
1×32	$2\times2\times2\times4$	1	4	9.2331	1.8408
1×64	2×32	1	2	18.473	9.5345
1×64	$2\times4\times8$	1	3	18.473	3.947
1×64	$2\times2\times4\times4$	1	4	18.473	3.156
1×128	2×64	1	2	36.949	18.776
1×128	$4\times4\times8$	1	3	36.949	5.284
1×128	$2\times4\times4\times4$	1	4	36.949	4.416

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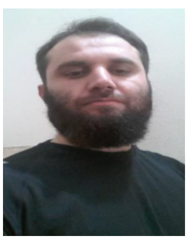

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