

Heterotic quantum cohomology

Jock McOrist^a and Eirik Eik Svanes^b

^a*Department of Mathematics, School of Science and Technology,
University of New England,
Armidale NSW 2350, Australia*

^b*Department of Mathematics and Physics, Faculty of Science and Technology,
University of Stavanger,
Kristine Bonnevis vei 22, 4021 Stavanger, Norway*

E-mail: jmcorist@une.edu.au, eirik.e.svanes@uis.no

ABSTRACT: It is believed, but not demonstrated, that the large radius massless spectrum of a heterotic string theory compactified to four-dimensional Minkowski space should obey equations that split into ‘F-terms’ and ‘D-terms’ in ways analogous to that of four-dimensional supersymmetric field theories. This is not easy to do directly as string theory is first quantised. Nonetheless, in this paper we demonstrate this splitting. We construct an operator $\bar{\mathcal{D}}$ whose kernel amounts to deformations solving ‘F-term’ type equations. In many previous works in this field, the spin connection is treated as an independent degree of freedom (and so is spurious or fake); here our results apply on the physical moduli space in which these fake degrees of freedom are eliminated. We utilise the moduli space metric, constructed in previous work, to define an adjoint operator $\bar{\mathcal{D}}^\dagger$. The kernel of $\bar{\mathcal{D}}^\dagger$ amounts to ‘D-term’ type equations. Put together, we show there is a $\bar{\mathcal{D}}$ -operator in which the massless spectrum are harmonic representatives of $\bar{\mathcal{D}}$. We conjecture that one could better study the moduli space of heterotic theories by studying the corresponding cohomology, a natural counterpart to studying the $\bar{\partial}$ -cohomology groups relevant to moduli of Calabi-Yau manifolds.

KEYWORDS: Flux Compactifications, Superstrings and Heterotic Strings, Supergravity Models, Superstring Vacua

ARXIV EPRINT: [2110.06549](https://arxiv.org/abs/2110.06549)

Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 1 |
| 2 | Review of results | 6 |
| 2.1 | α' -corrected supergravity | 6 |
| 2.2 | Holomorphic gauge and F-term equations | 8 |
| 2.3 | D-term equations | 9 |
| 2.4 | The Hodge decomposition | 10 |
| 3 | Warm-up: extension bundles for complex manifolds | 11 |
| 4 | Heterotic theories with spurious degrees of freedom | 13 |
| 4.1 | F-terms | 13 |
| 4.2 | The D-terms and the adjoint operator \overline{D}^\dagger | 16 |
| 5 | Eliminating the spurious degrees of freedom | 18 |
| 5.1 | Constructing the bundle Q | 18 |
| 5.2 | The D-terms and the adjoint operator \overline{D}^\dagger | 21 |
| 6 | Conclusion and outlook | 23 |
| A | Some useful relations | 24 |
| A.1 | Curvatures and Bianchi identity | 24 |
| A.2 | Hodge dual relations | 25 |
| B | Summary of notation of supergravity deformations | 26 |
| C | A supergravity no-go theorem | 28 |
| D | Constructing a family of \overline{D} operators | 32 |
| D.1 | Warm-up: complex manifolds and holomorphic bundles | 32 |
| D.2 | Hull-Strominger with spurious degrees of freedom | 33 |

1 Introduction

Once the famous paper [1] established the existence of a solution of ten-dimensional heterotic supergravity that realised a four-dimensional Minkowski space via the standard embedding, it was quickly realised that these vacua provide an easy route to describing viable phenomenological solutions of string theory. The standard embedding, roughly speaking, corresponds to a compactification on a Calabi-Yau 3-fold with vanishing torsion

$H = 0$, at least to order α'^3 in perturbation theory. More general solutions, in which H could be non-vanishing, corresponding to choices of holomorphic vector bundles with connection satisfying the hermitian Yang-Mills equation were realised in the beautiful paper of [2]. These solutions were shown to be unobstructed to all orders in α' perturbation theory. The conditions of supergravity were studied to first order in α' by [3, 4] which related the field equations to geometric conditions on the vector bundle and six-dimensional manifold, together with a non-trivial Bianchi identity. The equations were non-linear coupled PDEs and, although we know solutions exist, we do not yet have a good understanding of the moduli space of such solutions. Physically, this is important as quantum corrections are more easily studied than in the type II context, which involved non-perturbative objects like D-branes. One might hope that there is a nice geometric structure to the moduli space similar to that of the special geometry of Calabi-Yau manifolds, which exhibits a special Kähler structure where the space of deformations is described simply by the $\bar{\partial}$ -cohomology groups $H^{1,1}(X, \mathbb{C})$ and $H^{2,1}(X, \mathbb{C})$, where X is the Calabi-Yau manifold.

There has been a lot of recent progress in the past decade in trying to formulate both a cohomological description of the moduli space, and its differential geometric structure. In [5–7] the authors studied moduli of holomorphic bundles over complex manifolds in the heterotic context, and then later including the hermitian moduli and torsional solutions [8, 9], a good understanding of the infinitesimal moduli space of generic torsional compactifications emerged. At this time new mathematical tools had been developed, such as heterotic generalised geometry [10], putting the moduli problem on a more firm mathematical ground [11]. The massless spectrum in these works, however, often suffers from a problem in that they assume extra degrees of freedom that do not exist in string theory. The cohomology groups studied are not really the physical ones. In terms of the differential geometry, by studying the dimensional reduction and gauge symmetries a Kähler metric for the moduli space of heterotic string theory was proposed in [12]. Remarkably, it is formally very similar to what was studied in the special geometry of Calabi-Yau manifolds; this lends hope to the belief there is an underlying special geometry of heterotic theories. The equations describing the small deformations have been understood in terms of the superpotential [13], and the same superpotential correctly describes the Yukawa couplings [14]. There have also been developments in terms of holomorphic string algebroids [15, 16], higher order deformations [17], and a universal geometry picture of the moduli space [18, 19].¹

Of course, many open questions still remain. For example, the original studies of the moduli problem of the Hull-Strominger system [8, 9, 13] focused on the F-term equations. We call these F-terms because they can be shown to derive from a functional labelled the superpotential. In holomorphic gauge, the superpotential is a holomorphic section over the moduli space, just as it is for four-dimensional supersymmetric field theories. Interestingly however, one needs to impose a constraint which is the heterotic Bianchi identity is satisfied. One can phrase the F-term conditions as the existence of a holomorphic top-form, and a holomorphic vector bundle, and a relation between the contorsion of the metric on \mathcal{X} and

¹For brevity, we have omitted references to many important works on heterotic moduli, particularly from the world-sheet point of view. The papers [20–24] and references therein constitute a non-exhaustive list.

the three-form H . In [8, 9] it was shown that a deformation preserving these conditions can naturally be viewed as an element of the first cohomology of a holomorphic double extension bundle \mathcal{Q} . In [17] it was further shown that preserving the constraints coming from D-terms, that is the Yang-Mills condition on the bundle and a conformally balanced geometry, comes down to picking a particular representative of this cohomology class. This is similar to the moduli space of hermitian Yang-Mills connections A on a holomorphic vector bundle \mathcal{E} . Here the holomorphic condition imposes that the infinitesimal deformations of A define a class in $H^{(0,1)}(\text{End}\mathcal{E})$, while the Yang-Mills constraint imposes that we choose the *harmonic* representative. A similar observation can be made for the Calabi-Yau moduli problem, where it has been shown that *holomorphic gauge* equals *harmonic gauge*, see for example [19], where the moduli are given by harmonic forms. In this paper we explore this subtlety further for the heterotic moduli problem. We point out that as the moduli naturally take values in the holomorphic double-extension \mathcal{Q} , again in a holomorphic gauge as described in [12, 18, 19], the notion of harmonic must be considered in the context of a natural inner product on \mathcal{Q} . The correct inner product to consider derives from the Kähler metric on the moduli space [12], and using this, the D-term conditions now impose a harmonic gauge with respect to this metric.

The second question we address concerns a subtlety often ignored in previous work. Recall that part of the defining equations of the Hull-Strominger system includes the heterotic Bianchi identity

$$dH = -\frac{\alpha'}{4} (\text{Tr } F^2 - \text{Tr } R^2) ,$$

where H is the heterotic NS flux, F is the curvature of a connection A on a holomorphic vector bundle \mathcal{E} , while R is the curvature of a connection Θ on the tangent bundle \mathcal{T}_X . Which connection to use on \mathcal{T}_X is a long debated subject. In the usual field choice,² both the world-sheet [3] and supergravity analysis [29] point to natural connection, often referred to as the Hull-connection. Using this connection however presents a problem for mathematicians, in that the system is only accurate modulo $\mathcal{O}(\alpha'^2)$ corrections. Physically this is to be expected, but in order to make exact mathematical statements of the system, ala Yau’s Theorem for Calabi-Yau manifolds or the Donaldson-Uhlenbeck-Yau Theorem for Hermitian Yang-Mills connections, then an exact system of equations is required. A common “cheat” then is to promote the Hull connection to a hermitian Yang-Mills connection in its own right. This renders an exact system of equations, which is also more in line with the generalised geometry perspective [10, 30]. This is also the approach used in most previous studies of the moduli problem of the Hull-Strominger system.

However, introducing a new connection on the tangent bundle also introduces new nonphysical spurious modes in the moduli problem, corresponding to deformations of this connection. These modes can be interpreted as field redefinitions [28], and the true moduli space would then be given as a complicated sub-variety. This is a bit unsatisfactory for two reasons. Firstly, one would like to understand better exactly how the space of physical deformations fits within the larger moduli space including these spurious modes. Secondly,

²It has been pointed out many places that field redefinitions may be used to change the connection, see e.g. [25–28], but this is a rather subtle point as explained below.

and more crucially, once we inevitably come to quantise the classical system of equations in order to understand the quantum moduli space, fluctuating these nonphysical modes will again have an impact which we ignore at our peril. It may be possible to deal with these modes through Lagrange-multiplier techniques or by other means. Alternatively, one might go back to the beginning and redo the moduli story using the correct physical Hull-connection. This is the approach we have initiated in the current paper. To be more specific, we have derived the cohomology which correctly counts the physical spectrum of solutions, without extra spurious degrees of freedom. The answer turns out to be both interesting and subtle, prompting many future research questions to explore. See Conclusions and Outlook section 6.

Setting the stage. In this paper we study heterotic vacua that have a large radius limit. In this limit, the vacua are solutions of α' -corrected supergravity. This effective field theory is fixed up to and including α'^2 by supersymmetry and the result is also consistent with string scattering amplitudes. We refer to the resulting equations of motion as the Hull-Strominger system [3, 4]. The vacua we study have a smooth limit $\alpha' \rightarrow 0$ which forces $H \rightarrow 0$, and a ten-dimensional spacetime of the form

$$\mathcal{M}_{10} = \mathbb{R}^{3,1} \times \mathcal{X},$$

with \mathcal{X} a complex 3-fold with $c_1(\mathcal{X}) = 0$. With an appropriate gauge fixing the dilaton is constant up to order α'^3 [31], and the background is topologically a CY manifold with a holomorphic vector bundle \mathcal{E} admitting a connection A that satisfies the hermitian Yang-Mills equation. These are the only compact supergravity solutions with a valid $\alpha' \rightarrow 0$ limit, which guarantees the supergravity solutions are also solutions of string theory up to non-perturbative corrections. There are also solutions in which as $\alpha' \rightarrow 0$, the three-form H is non-trivial. However, one must study either non-compact manifolds, be non-perturbative in the dilaton via, e.g. non-geometric solutions, or consider solutions without a convergent α' expansion. We do not consider such solutions here because the higher order perturbative α' corrections to the Hull-Strominger equations (and likely worldsheet instantons) will play an important role modifying the higher order α' behaviour, and these α' corrections have not been completely determined yet. We expand upon our justification for restricting to solutions with a smooth $\alpha' \rightarrow 0$ limit in appendix C.

The moduli space is a finite dimensional complex Kähler manifold \mathcal{M} . See appendix B for an overview of the notation use when discussing moduli, following [12]. Each point $y \in \mathcal{M}$ corresponds to a heterotic vacuum and there is a Kuranishi map which relates tangent vectors δy on \mathcal{M} to deformations of fields.³ Small gauge transformations correspond to deforming the Kuranishi map and there is a choice in which the map is holomorphic [19]. We call this holomorphic gauge. In holomorphic gauge, there is a relation between deformations of the connection Θ on the tangent bundle $\mathcal{T}_{\mathcal{X}}$ and the moduli of \mathcal{X} [18]

$$\delta\Theta^{(0,1)} = \delta y^\alpha \mathfrak{D}_\alpha \Theta, \quad \text{where} \quad \mathfrak{D}_\alpha \Theta_{\bar{\mu}^\nu \sigma} = \nabla_\sigma \Delta_{\alpha\bar{\mu}^\nu} + i \nabla^\nu (\partial_\alpha \omega)_{\sigma\bar{\mu}}. \quad (1.1)$$

³See [6, 32] for previous work on Kuranishi maps and higher order deformations in the heterotic context.

Here we have introduced holomorphic and anti-holomorphic coordinates on \mathcal{M} as $y^a = (y^\alpha, y^{\bar{\beta}})$ using the complex structure inherited from the heterotic theory. On \mathcal{X} real coordinates are denoted x^m , complex coordinates $(x^\mu, x^{\bar{\nu}})$ and ∇_σ is the Levi-Civita connection of the Ricci-flat metric on \mathcal{X} . We denote J the complex structure on \mathcal{X} and (in holomorphic gauge) a holomorphic deformation is a $(0, 1)$ -form valued in the holomorphic tangent bundle. We write this in the notation of [12] viz. $\delta y^\alpha \Delta_\alpha \bar{\mu}^\nu$. Similarly, if ω is the hermitian form on \mathcal{X} then $(\partial_\alpha \omega)_{\sigma \bar{\mu}}$ is a holomorphic deformation of the hermitian form restricted to its $(1, 1)$ component. In holomorphic gauge this is equal (up to a factor of i) to a gauge invariant deformation of the B-field, denoted $\mathcal{B}_\alpha^{(1,1)}$ [19]. The analogue of complexified Kähler deformations for heterotic theories is the combination

$$\mathcal{Z}_\alpha = \mathcal{B}_\alpha + i\partial_\alpha \omega.$$

The key point is that $\delta\Theta^{(0,1)}$ is fixed in terms of Δ_α and $\mathcal{Z}_\alpha^{(1,1)}$.

In contrast, in [8, 9, 11] the deformation $\delta\Theta^{(0,1)}$ is treated as a degree of freedom independent of the other moduli with its own set of parameters. That is, $\delta\Theta^{(0,1)}$ corresponds to an arbitrary element of $H^1(\text{End } \mathcal{T}_\mathcal{X})$, and with these spurious degrees of freedom one finds a \bar{D} -operator on an extension bundle Q in which $\bar{D}^2 = 0$ and its first cohomology is related to the deformations of heterotic theories plus the spurious degrees of freedom. Specifically, one first considers the Atiyah extension, as described in section 3 and first applied to the heterotic context in [5, 6]

$$0 \longrightarrow \text{End } \mathcal{E} \xrightarrow{i_1} Q_1 \xrightarrow{\pi_1} \mathcal{T}_\mathcal{X}^{(1,0)} \longrightarrow 0.$$

This sequence governs the combined moduli space of complex manifolds with a holomorphic vector bundle. As described in section 4, one then extends this bundle by $\text{End}(\mathcal{T}_\mathcal{X})$ to account for the spurious deformations of the connection $\delta\Theta^{(0,1)}$

$$0 \longrightarrow \text{End}(\mathcal{T}_\mathcal{X}) \xrightarrow{i_2} Q_2 \xrightarrow{\pi_2} Q_1 \longrightarrow 0.$$

Finally, to account for the deformations of the hermitian structure, one further extends this bundle by the holomorphic cotangent bundle as

$$0 \longrightarrow \mathcal{T}_\mathcal{X}^{*(1,0)} \xrightarrow{i} Q \xrightarrow{\pi} Q_2 \longrightarrow 0.$$

The extension class is derived from the heterotic anomaly cancellation condition (or Bianchi identity), and gives rise to a natural \bar{D} -operator on Q whose first cohomology counts the spectrum including the spurious deformations. This cohomology can then be described in terms of more familiar cohomologies using homological algebra and long-exact sequences [8, 9].

A very natural question to ask is then what happens to the operator \bar{D} and bundle Q when we eliminate the spurious degrees of freedom? Our main goal is to answer this question. What we find is that in holomorphic gauge [19], the natural object to consider is now a further extension \mathcal{Q} of the Atiyah extension given by (5.9)

$$0 \longrightarrow \mathcal{T}_\mathcal{X}^{*(1,0)} \xrightarrow{i} \mathcal{Q} \xrightarrow{\pi} Q_1 \longrightarrow 0.$$

The bundle \mathcal{Q} governs the moduli problem without the spurious degrees of freedom, with a $\bar{\mathcal{D}}$ operator acting as a differential on this bundle. Its kernel amounts to ‘F-term’ type equations. The extension class is again derived from the heterotic anomaly cancellation condition. It should however be noted that in this case, the extension class is no longer tensorial due to the appearance of a holomorphic covariant derivative, see (5.1). As a result, the new differential operator $\bar{\mathcal{D}}$ is no longer a connection on \mathcal{Q} , as it fails to satisfy the Leibniz rule. However, it is still a perfectly well defined linear operator on \mathcal{Q} . Moreover, given an inner product on \mathcal{Q} one can define an adjoint of $\bar{\mathcal{D}}$, and use it to describe harmonic forms with respect to $\bar{\mathcal{D}}$.

Our next goal is therefore to use the moduli space metric constructed in [12, 18]⁴ to construct an adjoint operator $\bar{\mathcal{D}}^\dagger$ and show that the co-kernel of $\bar{\mathcal{D}}$ describe D-term type equations. D-term equations here refer to first order deformations of the Hermitian-Yang-Mills equation and balanced equations.⁵

In other words, the physical massless moduli of the heterotic supergravity at large radius are harmonic representatives of the cohomology of the $\bar{\mathcal{D}}$ -operator. This is the main result of the paper.

Hodge-theory further says that these harmonic forms are in one-to-one correspondence with the cohomology of $\bar{\mathcal{D}}$.⁶ Using long exact sequences in cohomology and kernels of extension maps, this cohomology can again be computed in terms of more familiar cohomologies such as the complex structure moduli $H^{(0,1)}(\mathcal{T}_X^{(1,0)})$, the bundle moduli $H^{(0,1)}(\text{End}\mathcal{E})$, and the hermitian moduli $H^{(0,1)}(\mathcal{T}_X^{*(1,0)})$.

In the next section we review some results setting up our notation. In section 3 we review how extension bundles describe deformations of complex manifolds with a holomorphic vector bundle. In section 4 we revisit and refine the calculation in [8, 9], including the spurious degrees of freedom. We find a family of operators $\bar{\mathcal{D}}$ on a double extension, and using the metric in [12] show its adjoint describes deformations of the Hermitian-Yang-Mills equation and balanced equation. In section 5, we construct an extension \mathcal{Q} and $\bar{\mathcal{D}}$ -operator whose cohomology describe the F-terms and the harmonic representatives the D-terms. We give a review of our main results in section 6, and discuss future directions.

2 Review of results

2.1 α' -corrected supergravity

The heterotic action is fixed by supersymmetry up to and including α'^2 corrections. There is nice basis of fields in which the action, equations of motion and supersymmetry variations are particularly compact. This was constructed in [34] to order α' by supersymmetrising the Lorentz-Chern-Simons terms in H and extended to α'^2 in [29]. The action and Bianchi

⁴This is the natural string theory moduli space metric constructed in α' -corrected supergravity, and so is valid for the solutions of Hull-Strominger with a smooth $\alpha' \rightarrow 0$ limit.

⁵We use F-term equations and D-terms equations loosely: without an off-shell definition of string theory, viz. string field theory, they don’t really make sense.

⁶In upcoming work we further analyse the operator $\bar{\mathcal{D}}$ [33], investigating its ellipticity properties, Hodge-theory etc.

identity for H is unique up to field redefinitions; there is no ambiguity in the theory. The action, in the field basis of [31], is given by

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10} X \sqrt{g_{10}} e^{-2\Phi} \left\{ \mathcal{R} - \frac{1}{2} |H|^2 + 4(\partial\Phi)^2 - \frac{\alpha'}{4} \left(\text{Tr} |F|^2 - \text{Tr} |R(\Theta^+)|^2 \right) \right\} + \mathcal{O}(\alpha'^3). \quad (2.1)$$

Our notation is such that μ, ν, \dots are holomorphic indices along \mathcal{X} with coordinates x ; m, n, \dots are real indices along \mathcal{X} . The 10D Newton constant is denoted by κ_{10} , $g_{10} = -\det(g_{MN})$, Φ is the 10D dilaton, \mathcal{R} is the Ricci scalar evaluated using the Levi-Civita connection and F is the Yang-Mills field strength with the trace taken in the adjoint of the gauge group.

We take the p -form norm as $|T|^2 = \frac{1}{p!} T_{M_1 \dots M_p} T^{M_1 \dots M_p}$ and the curvature squared terms correspond to

$$\text{Tr} |F|^2 = \frac{1}{2} \text{Tr} F_{MN} F^{MN} \quad \text{and} \quad \text{Tr} |R(\Theta^+)|^2 = \frac{1}{2} R_{MN}{}^A{}_B(\Theta^+) R^{MN B}{}_A(\Theta^+),$$

where the Riemann curvature is evaluated using a twisted connection

$$\Theta_M^\pm{}^A{}_B = \Theta_M{}^A{}_B \pm \frac{1}{2} H_M{}^A{}_B,$$

with Θ_M is the Levi-Civita connection and A, B are the tangent space indices. The three-form H satisfies a Bianchi-identity

$$dH = -\frac{\alpha'}{4} \left(\text{Tr} F^2 - \text{Tr} R^2 \right), \quad (2.2)$$

while at the same time it is related to the hermitian form as $H = d^c \omega$. With respect to a fixed complex structure this corresponds to

$$H = i \left(\partial - \bar{\partial} \right) \omega,$$

and so the Bianchi identity in this complex structure is

$$2i\partial\bar{\partial}\omega = \frac{\alpha'}{4} \left(\text{Tr} F^2 - \text{Tr} R^2 \right). \quad (2.3)$$

The decomposition of the connections into type is

$$A = \mathcal{A} - \mathcal{A}^\dagger, \quad \Theta = \theta - \theta^\dagger.$$

where $\mathcal{A} = A^{(0,1)}$ and $\theta = \Theta^{(0,1)}$ and we are using conventions in which A and Θ are antihermitian.

All we care about is that the Bianchi identity (2.2), equations of motion and supersymmetry variations match calculations from string scattering amplitudes [35] which has been checked [36]. This data is fixed, up to the usual caveat of field redefinitions. Indeed, if one were to perform a field redefinition, so that for example the curvature tensor R in the Bianchi identity is evaluated with a different connection, then the field redefinitions will propagate through the supersymmetry variations and equations of motion, likely losing

their simple closed form. For example, if one evaluated R using the Chern connection, then this requires a field redefinition in which resulting in a metric \tilde{g} that is no longer a tensor — it is charged under gauge transformations like the B-field — and this field redefinition is likely modify the supersymmetry variations already at order α' .⁷ We choose to use the usual conventions in which g is a metric tensor and R is evaluated with Θ^+ .

From now on we work to first order in α' . For emphasis, we will sometimes include $+\mathcal{O}(\alpha'^2)$ in our equations, but this will mostly be suppressed.

2.2 Holomorphic gauge and F-term equations

In appendix B we give a brief summary of the notation of small deformations of heterotic supergravity used in this paper, following [12, 18]. We use that notation to briefly summarise some of the equations the field deformations must satisfy. Consider a first deformation described by differentiating the fields by a real parameter y^a . Taking the derivatives of the supergravity equations of motion, they obey the following equations to first order in α' :

$$\begin{aligned} \bar{\partial}\Delta_a^\mu &= 0, \\ \bar{\partial}_{\mathcal{A}}(\mathfrak{D}_a\mathcal{A}) &= \Delta_a^\mu F_\mu, \\ \partial Z_a^{(0,2)} + \bar{\partial} Z_a^{(1,1)} &= 2i \Delta_a^\mu (\partial\omega)_\mu + \frac{\alpha'}{2} \text{Tr}(\mathfrak{D}_a\mathcal{A}F) - \frac{\alpha'}{2} \text{Tr}(\mathfrak{D}_a\Theta R), \\ \bar{\partial} Z_a^{(0,2)} &= 0. \end{aligned} \tag{2.4}$$

We assume $h^{(0,2)} = 0$, and so the last line is $Z_a^{(0,2)} = \bar{\partial}\beta_a^{(0,1)}$ and this means left hand side of the third line is $\bar{\partial}$ -exact.

There are equations coming from the HYM equation $\omega^2 F = 0$ and the balanced equation $d(\omega^2) = 0$ which are analysed below in section 2.3. Both these equations and (2.4) are invariant under small gauge transformations [19]:

$$\begin{aligned} \Delta_a^\mu &\sim \Delta_a^\mu + \bar{\partial}\varepsilon_a^\mu, & \mathfrak{D}_a\mathcal{A} &\sim \mathfrak{D}_a\mathcal{A} + \varepsilon_a^\mu F_\mu + \bar{\partial}_{\mathcal{A}}\phi_a, \\ Z_a &\sim Z_a + \varepsilon_a^m (H + i d\omega)_m + \frac{\alpha'}{2} \text{Tr}(F\phi_a) + d(\mathfrak{b}_a + i\varepsilon_a^m \omega_m), \\ \bar{Z}_a &\sim \bar{Z}_a + \varepsilon_a^m (H - i d\omega)_m + \frac{\alpha'}{2} \text{Tr}(F\phi_a) + d(\mathfrak{b}_a - i\varepsilon_a^m \omega_m), \end{aligned} \tag{2.5}$$

where $Z_a = \mathcal{B}_a + i\partial_a\omega$ and $\bar{Z}_a = \mathcal{B}_a - i\partial_a\omega$. A convenient choice of gauge fixing is holomorphic gauge:

$$\begin{aligned} \Delta_{\bar{\alpha}}^\mu &= 0, & \delta\Omega^{(3,0)} &= \delta y^\alpha k_\alpha \Omega, & \mathfrak{D}_{\bar{\alpha}}\mathcal{A} &= 0, \\ Z_{\bar{\alpha}}^{(1,1)} &= 0, & Z_{\bar{\alpha}}^{(0,2)} &= \bar{Z}_{\bar{\alpha}}^{(0,2)} = \bar{Z}_{\bar{\alpha}}^{(2,0)} = 0. \end{aligned} \tag{2.6}$$

⁷This is nicely described in [27]. If one wishes to preserve manifest (0,2)-supersymmetry on the worldsheet, then R in the Bianchi identity should be evaluated with the Chern connection. If one wishes to preserve manifest (0,1)-supersymmetry with g a conventional metric tensor then one evaluates the Bianchi identity with Θ^+ .

In this gauge, we have, for example,

$$\bar{\partial}\Delta_\alpha^\mu = 0, \quad \bar{\partial}\mathcal{A}(\mathfrak{D}_\alpha\mathcal{A}) + F_\mu\Delta_\alpha^\mu = 0, \quad (2.7)$$

where Δ_α^μ is a $(0,1)$ -form valued in $\mathcal{T}_X^{(1,0)}$. The gauge simplifies other equations such as

$$\bar{\partial}Z_\alpha^{(1,1)} - 2i\Delta_\alpha^\mu(\partial\omega)_\mu + \frac{\alpha'}{2}\text{Tr}(R\mathfrak{D}_\alpha\Theta) - \frac{\alpha'}{2}\text{Tr}(F\mathfrak{D}_\alpha\mathcal{A}) = 0, \quad (2.8)$$

and determines

$$Z_\alpha^{(1,1)} = 2i\mathfrak{D}_\alpha\omega^{(1,1)} = 2\mathcal{B}_\alpha^{(1,1)}, \quad \bar{Z}_\alpha^{(0,2)} = 2\mathcal{B}_\alpha^{(0,2)} = -2i\mathfrak{D}_\alpha\omega^{(0,2)}.$$

It should be noted that while the field $Z_\alpha^{(0,2)}$ vanishes in this gauge fixing, the field $\bar{Z}_\alpha^{(0,2)}$ does not. This field can be thought of as the antisymmetric part of $\Delta_{\alpha\bar{\mu}\bar{\nu}} = \Delta_{\alpha\bar{\mu}}^\rho g_{\rho\bar{\nu}}$. In the standard embedding, where $H = 0$, this term vanishes; in the more generic heterotic case it does not. Hence, it is a physical degree of freedom whose role in a heterotic vacuum is not yet clear. The role of $Z_\alpha^{(1,1)}$ is analogous to the complexified Kähler modulus for a CY manifold.

We refer to (2.7) and (2.8) as F-term equations.

2.3 D-term equations

An analysis of deformations in holomorphic gauge of the balanced equation and hermitian Yang-Mills equation was presented in [19]. This gives a relation between deformations in terms of adjoint operators. In this section we revisit this calculation massaging the equations with a prescience of results to come.

The top-form Ω on X has a norm

$$\|\Omega\| = \frac{1}{3!}\Omega_{mnp}\bar{\Omega}^{mnp}.$$

It is related to the dilaton $d\log\|\Omega\| = -2d\phi$. We gauge fix as in [19, 31] so that the dilaton is a constant on X and so in this gauge $\|\Omega\|$ is also a constant. Furthermore, in holomorphic gauge (2.6), $(\partial_\alpha\Omega)^{(3,0)} = k_\alpha\Omega$, where k_α is a constant on X . Using this, a first order variation of the norm is

$$\partial_\alpha\|\Omega\|^2 = \|\Omega\|^2(k_\alpha - \partial_\alpha\log\sqrt{g}).$$

As $\|\Omega\|$ and k_α are constants over X , it follows $\omega^{\lambda\bar{\sigma}}\partial_\alpha\omega_{\lambda\bar{\sigma}} = \frac{1}{2i}\omega^{\lambda\bar{\sigma}}Z_{\alpha\lambda\bar{\sigma}}$ is a constant on X .

A first order holomorphic variation of the balanced equation is

$$\partial\left(\omega Z_\alpha^{(1,1)}\right) = 0, \quad (2.9)$$

$$-i\bar{\partial}\left(\omega Z_\alpha^{(1,1)}\right) + 2\partial\left(\omega\Delta_\alpha^\mu\omega_\mu\right) = 0. \quad (2.10)$$

Using the Hodge dual relation for a $(1,1)$ -form (A.5), together with $\omega^{\lambda\bar{\sigma}}Z_{\alpha\lambda\bar{\sigma}}$ being a constant and the balanced equation, the first equation amounts to

$$\bar{\partial}^\dagger Z_\alpha^{(1,1)} = 0. \quad (2.11)$$

The second equation is

$$\partial \left(\partial_\alpha \omega^{(0,2)} \right) \omega + \bar{\partial} \left(\partial_\alpha \omega^{(1,1)} \right) \omega + \left(\partial_\alpha \omega^{(0,2)} \right) \partial \omega + \left(\partial_\alpha \omega^{(1,1)} \right) \bar{\partial} \omega = 0. \quad (2.12)$$

The Hodge dual of a (2,3)-form in (A.7) can be used together with the gauge fixed F-term equation (2.8) and the constraint $\nabla_\mu^{\text{Ch/H}} \Delta_\alpha^\mu = 0$ [19], as well as the background equations of motion $H_{\mu\nu}{}^\nu = 0$, $\omega^2 F = 0$, $\partial_\alpha \omega_{\mu\bar{\nu}} = 2i \Delta_{\alpha[\mu\bar{\nu}]}$ to rewrite this equation in terms of contractions. This result is

$$\frac{i}{2} \left(\bar{\partial} \omega \right)^{\nu\rho\bar{\mu}} Z_{\alpha\rho\bar{\mu}} - \frac{\alpha'}{4} \left[\text{Tr} \left(F^{\nu\bar{\mu}} \mathfrak{D}_\alpha \mathcal{A}_{\bar{\mu}} \right) - \text{Tr} \left(R^{\nu\bar{\mu}} \mathfrak{D}_\alpha \Theta_{\bar{\mu}} \right) \right] + \nabla^{\text{Ch/H}\bar{\mu}} \Delta_{\alpha\bar{\mu}}{}^\nu + i \Delta_{\alpha\bar{\rho}\bar{\lambda}} (\partial \omega)^{\nu\bar{\rho}\bar{\lambda}} = 0, \quad (2.13)$$

where $\nabla^{\text{Ch/H}\bar{\mu}}$ is the Chern or Hull connection (see [19] for discussion on this ambiguity). Because $H = \mathcal{O}(\alpha')$ and $\Delta_{\alpha[\bar{\rho}\bar{\lambda}]} = \mathcal{O}(\alpha'^2)$ [12], the last term is $\mathcal{O}(\alpha'^2)$ and so is dropped from hereon. A good consistency check is to derive this equation by differentiating $g^{\mu\bar{\nu}} H_{\mu\bar{\nu}\bar{\rho}} = 0$.

A first order variation of the Hermitian Yang-Mills equation can be written as

$$\bar{\partial}_{\mathcal{A}}^\dagger (\mathfrak{D}_\alpha \mathcal{A}) + \frac{1}{2} F^{\mu\bar{\nu}} Z_{\alpha\mu\bar{\nu}} = 0. \quad (2.14)$$

A similar equation is satisfied for the connection Θ on the tangent bundle to this order in α' . That is

$$\bar{\partial}_\Theta^\dagger (\mathfrak{D}_\alpha \Theta) + \frac{1}{2} R^{\mu\bar{\nu}} Z_{\alpha\mu\bar{\nu}} = 0. \quad (2.15)$$

In holomorphic gauge, we declare ‘F-term’ type equations to be (2.7)–(2.8). We do not have a good definition of string field theory, so we use this terminology with care. The motivation for the nomenclature is that these are the equations, together with an appropriate definition of holomorphy, that derive from a superpotential type construction [13, 14]. The equations (2.13)–(2.15) deriving from the balanced and HYM equation we declare to be D-term equations. A main outcome of this paper is to show that there is a $\bar{\mathcal{D}}$ -operator acting on first order fluctuations whose kernel corresponds to F-term equations, and co-kernel to the D-term equations. This helps justify this terminology.

2.4 The Hodge decomposition

The balanced and HYM equations play a role in determining the exact and co-exact terms in the Hodge decomposition of fields. We present this to emphasise the fact that deformations of a heterotic theory are not simply the harmonic representatives in the usual sense. Taking into account the F-term equations, the Hodge decomposition is

$$\begin{aligned} Z_\alpha^{(1,1)} &= Z_\alpha^{(1,1)\text{harm}} + \bar{\partial}^\dagger \xi_\alpha^{(1,2)}, \\ \Delta_\alpha &= \Delta_\alpha^{\text{harm}} + \bar{\partial} \kappa_\alpha, \\ \mathfrak{D}_\alpha \mathcal{A} &= \mathfrak{D}_\alpha \mathcal{A}^{\text{harm}} + \bar{\partial}_{\mathcal{A}} \Phi_\alpha + \bar{\partial}_{\mathcal{A}}^\dagger \Psi_\alpha^{(0,2)}. \end{aligned} \quad (2.16)$$

The role of the cohomology we aim to compute is to tell us what linear combinations of harmonic terms in (2.16) correspond to unobstructed deformations. The exact and co-exact terms in the Hodge decomposition are physical and so are important and are determined

by substituting into the HYM and balanced equations and solving the corresponding Poisson equations:

$$\begin{aligned} \xi_\alpha^{(1,2)} = & \square_{\bar{\partial}}^{-1} \left(\Delta_\alpha^\mu (\partial\omega)_\mu - \frac{i\alpha'}{4} \text{Tr} (\mathfrak{D}_\alpha \mathcal{A} F) \right. \\ & \left. + \frac{i\alpha'}{2} \left(\nabla_\nu \Delta_\alpha^\mu + \frac{1}{2} \nabla^\mu Z_{\alpha\nu}^{(0,1)} \right) R^\nu{}_\mu \right) + \bar{\partial}^\dagger\text{-closed}, \end{aligned} \quad (2.17a)$$

$$\kappa_\alpha^\mu = \square_{\bar{\partial}}^{-1} \left(-\frac{1}{2} (\partial^\dagger Z_\alpha)^\mu + \frac{1}{4i} (Z_{\alpha\bar{\nu}\rho}) (\partial\omega)^{\bar{\nu}\rho\mu} \right), \quad (2.17b)$$

$$\Psi_\alpha^{(0,2)} = \square_{\bar{\partial}_A}^{-1} (\Delta_\alpha^\mu F_\mu) + \bar{\partial}_A^\dagger\text{-closed}, \quad \Phi_\alpha = -\frac{1}{2} \square_{\bar{\partial}_A}^{-1} \left((Z_\alpha^{\mu\bar{\nu}}) F_{\mu\bar{\nu}} \right). \quad (2.17c)$$

The point here is that deformations of fields are not only highly coupled but they are not harmonic representatives. A natural question to ask is: can we change the gauge to find a harmonic decomposition? The answer is not likely without significant sacrifice. For example, one easily loses the holomorphic dependence of deformations on parameters. This is unlike the study of CY manifolds in which one studies holomorphic deformations that are also harmonic, viz. $\bar{\partial}$ -harmonic (1,1) forms and (2,1)-forms [37]. This derives from $H = 0$, an additional constraint that we do not have in the more general situation.

One point of this paper is to point out that without sacrificing holomorphy, the complicated Hodge decomposition beautifully reorganises itself into harmonic representatives of a \bar{D} -operator on a certain extension bundle, provided we use the appropriate inner product to define the adjoint of this operator. This provides a natural framework where holomorphic gauge becomes harmonic gauge.

3 Warm-up: extension bundles for complex manifolds

One approach to understanding (2.4)–(2.6) is inspired from the work of Atiyah [38], who described the deformations of a holomorphic bundle on a complex manifold. This was also studied in the heterotic context in [5, 6], and its instructive to review this with an eye towards the full heterotic story.

Consider a holomorphic bundle on a complex manifold. Atiyah, in 1955, pointed out that not all complex structure deformations are allowed as some may introduce a non-trivial $F^{(0,2)}$ component destroying holomorphy of the bundle. Those that are allowed satisfy

$$\bar{\partial}_A(\mathfrak{D}_\alpha \mathcal{A}) + F_\mu \Delta_\alpha^\mu = 0, \quad \text{and} \quad \bar{\partial} \Delta_\alpha = 0.$$

Intuitively such complex structure deformations may introduce $F^{(0,2)}$, but can be compensated by a simultaneous deformation of the bundle. A way to realise these constraints is to define a vector bundle Q_1 as a short exact sequence

$$0 \longrightarrow \text{End} \mathcal{E} \xrightarrow{i_1} Q_1 \xrightarrow{\pi_1} \mathcal{I}_X^{(1,0)} \longrightarrow 0, \quad (3.1)$$

where $\pi_1 : Q_1 \rightarrow \mathcal{T}_X$ is the canonical projection and i_1 the inclusion map.⁸ One can then study $(0, p)$ -forms valued in Q . These are vectors, schematically

$$\begin{pmatrix} \delta_\alpha \mathcal{A}^{(p)} \\ \Delta_\alpha^{(p)} \end{pmatrix},$$

where $\delta_\alpha \mathcal{A}^{(p)}$ is a $(0, p)$ -form valued in $\text{End } E$ and $\Delta_\alpha^{(p)}$ is a $(0, p)$ -form valued in $\mathcal{T}_X^{(1,0)}$.

There is a holomorphic structure defined by an operator \bar{D}_1 given by

$$\bar{D}_1 = \begin{pmatrix} \bar{\partial}_{\mathcal{A}} & \mathcal{F} \\ 0 & \bar{\partial} \end{pmatrix}, \quad \mathcal{F} : \Omega^{(0,p)}(\mathcal{T}_X^{(1,0)}) \rightarrow \Omega^{(0,p+1)}(\text{End } \mathcal{E}), \quad \mathcal{F}(\Delta_\alpha^{(p)}) = F_\mu \Delta_\alpha^{(p)\mu},$$

where, for example, $\Omega^{(p,q)}(\text{End } \mathcal{E})$ denotes (p, q) -forms on X valued in $\text{End } \mathcal{E}$. The action of \bar{D}_1 on a typical fibre of Q_1 vanishes if the field deformations obey the requisite equations of motion:

$$\bar{D}_1 \begin{pmatrix} \delta_\alpha \mathcal{A}^{(p)} \\ \Delta_\alpha^{(p)} \end{pmatrix} = \begin{pmatrix} \bar{\partial}_{\mathcal{A}}(\delta_\alpha \mathcal{A}^{(p)}) + \mathcal{F}(\Delta_\alpha^{(p)}) \\ \bar{\partial} \Delta_\alpha^{(p)} \end{pmatrix} = 0.$$

Furthermore, $\bar{D}_1^2 = 0$ if and only if the $F^{(0,2)} = 0$ and its Bianchi identity holds $d_A F = 0$.

Under a small gauge transformation, which includes a small diffeomorphisms for generality,

$$\begin{aligned} \delta_\alpha \mathcal{A}^{(p)} &\longrightarrow \delta_\alpha \mathcal{A}^{(p)} + \varepsilon_\alpha^\mu F_\mu + \bar{\partial}_{\mathcal{A}} \phi_\alpha, & \delta_\alpha \mathcal{A}^{(p)\dagger} &\longrightarrow \delta_\alpha \mathcal{A}^{(p)\dagger} + \varepsilon_\alpha^{\bar{\mu}} F_{\bar{\mu}} - \partial_{\mathcal{A}^\dagger} \phi_\alpha, \\ \Delta_\alpha^{(p)\mu} &\longrightarrow \Delta_\alpha^{(p)\mu} + \bar{\partial} \varepsilon_\alpha^\mu, & \Delta_\alpha^{(p)\bar{\mu}} &\longrightarrow \Delta_\alpha^{(p)\bar{\mu}} + \bar{\partial} \varepsilon_\alpha^{\bar{\mu}}, \end{aligned} \quad (3.2)$$

where ε_α^m is a vector valued $(0, p-1)$ -form parameterising the small diffeomorphism, and ϕ_α is a $(0, p-1)$ -form valued in $\text{End } \mathcal{E}$ parameterising the small gauge transformation.

The bundle Q and operator \bar{D}_1 are defined with respect to holomorphic deformations only. The operator \bar{D}_1 is equivariant with respect to gauge transformations:

$$\bar{D}_1 \begin{pmatrix} \delta_\alpha \mathcal{A}^{(p)} + \varepsilon_\alpha^\mu F_\mu + \bar{\partial}_{\mathcal{A}} \phi_\alpha \\ \Delta_\alpha^{(p)} + \bar{\partial} \varepsilon_\alpha^\mu \end{pmatrix} = \bar{D}_1 \begin{pmatrix} \delta_\alpha \mathcal{A}^{(p)} \\ \Delta_\alpha^{(p)} \end{pmatrix}.$$

For $p = 1$ we enter the case of interest, which are deformations of a complex manifold with holomorphic bundle. The notation for deformations $\delta_\alpha \mathcal{A}^{(1)} \cong \mathfrak{D}_\alpha \mathcal{A}$ and $\Delta_\alpha^{(1)} \cong \Delta_\alpha$. The space of deformations is the cohomology $H^{(0,1)}(Q_1)$. However, as can be seen from (3.2) together with the bundle being stable, so $\bar{\partial}_{\mathcal{A}} \phi_\alpha = 0$ has only trivial solutions, any gauge transformation will introduce a non-holomorphic dependence on parameters. i.e. $\mathfrak{D}_\alpha \mathcal{A}^\dagger \neq 0$. This gauge choice is already made in many physical calculations such as the moduli space metric [12] or results deriving from the superpotential, e.g. [17]. So the cohomology is to be

⁸The sequence is exact so $\ker \pi_1 = \text{im } i_1 = \text{End } \mathcal{E}$ and by rank-nullity the dimension of a typical fiber at Q_1 is $\dim(\text{End } \mathcal{E}) + \dim(\mathcal{T}_X)$.

treated with care: we are always gauge fixed, and so always fixed a representative. This becomes more important in the situation to come.⁹

Setting this issue aside one can use a long exact sequence in cohomologies inherited from the short exact sequence to show

$$H^{(0,1)}(Q_1) = H_{\bar{\partial}_{\mathcal{A}}}^{(0,1)}(\text{End}\mathcal{E}) \oplus \ker \mathcal{F}. \quad (3.3)$$

The Hodge decomposition of an element of $H_{\bar{\partial}_1}^1(Q_1)$ is

$$\Delta_\alpha = \Delta_\alpha^{\text{harm}} + \bar{\partial}\kappa_\alpha, \quad \mathfrak{D}_\alpha\mathcal{A} = \mathfrak{D}_\alpha\mathcal{A}^{\text{harm}} + \bar{\partial}_{\mathcal{A}}\Phi_\alpha + \bar{\partial}_{\mathcal{A}}^\dagger\Psi_\alpha^{(0,2)}.$$

Equation (3.3) implies the tangent space to the moduli space is a direct sum. So there is a choice of coordinates for the moduli space in which parameters decompose into two components $y^\alpha = (y^{\alpha_1}, y^{\alpha_2})$. The first component is one-to-one with $H^1(\text{End}\mathcal{E})$:

$$\Delta_{\alpha_1} = \bar{\partial}\kappa_{\alpha_1}, \quad \mathfrak{D}_{\alpha_1}\mathcal{A} = \mathfrak{D}_{\alpha_1}\mathcal{A}^{\text{harm}} + \bar{\partial}_{\mathcal{A}}\Phi_{\alpha_1}.$$

where we use $\bar{\partial}_{\mathcal{A}}(\mathfrak{D}_{\alpha_1}\mathcal{A}) = 0$ and so $\Delta_{\alpha_1}{}^\mu F_\mu = 0$. This implies $\kappa_{\alpha_1}{}^\mu F_\mu = \bar{\partial}_{\mathcal{A}}(\dots)$, which has a solution provided we can invert F_μ and solve for $\kappa_{\alpha_1}{}^\mu$. The terms κ_{α_1} and Φ_{α_1} are completely undetermined at this point and so there is an infinite dimensional space of solutions. In the full heterotic theory, there are additional equations of motion, and these fix κ_{α_1} and Φ_{α_1} .

The second component of solutions is one-to-one with $\ker \mathcal{F} \subset H^1(\mathcal{T}_X^{(1,0)})$:

$$\delta y^{\alpha_2}\Delta_{\alpha_2} = \delta y^{\alpha_2}\left(\Delta_{\alpha_2}^{\text{harm}} + \bar{\partial}\kappa_{\alpha_2}\right), \quad \delta y^{\alpha_2}\mathfrak{D}_{\alpha_2}\mathcal{A} = \delta y^{\alpha_2}\left(\bar{\partial}_{\mathcal{A}}\Phi_{\alpha_2} + \bar{\partial}_{\mathcal{A}}^\dagger\Psi_{\alpha_2}^{(0,2)}\right),$$

where δy^{α_2} is a vector in $\ker \mathcal{F}$. That is, the cohomology tells us what harmonic representative of $H^1(\mathcal{T}_X^{(1,0)})$ appear in this equation. The Atiyah equation implies $\Psi_{\alpha_2}^{(0,2)} = \square_{\bar{\partial}_{\mathcal{A}}}^{-1}(\Delta_{\alpha_2}{}^\mu F_\mu)$ while κ_{α_2} and Φ_{α_2} are undetermined.

The lesson here is that in writing $H^{(0,1)}(Q_1) = H_{\bar{\partial}_{\mathcal{A}}}^{(0,1)}(\text{End}\mathcal{E}) \oplus \ker \mathcal{F}$ we do not mean that the field deformations decompose into a direct product. Instead, it is telling us about what combinations of harmonic forms can appear and that they have a Hodge decomposition. In this example we have some undetermined exact terms; in the full heterotic theory these are all fixed by the equations of motion and gauge fixing.

4 Heterotic theories with spurious degrees of freedom

4.1 F-terms

The situation in the Hull-Strominger system is more complicated as we have hermitian deformations, the B-field and a set of equations in (2.4)–(2.6) together with a Bianchi identity.

⁹The Kaluza-Klein reduction of heterotic supergravity at the standard embedding is almost always performed in harmonic gauge. Deformations of the complexified Kähler form and deformations of complex structure are then zero modes of Laplacians. Quantities such as the moduli space metric [37] are calculated with this particular choice of representative. Interestingly, this moduli space metric is not manifestly gauge invariant. Also note that the standard embedding holomorphic gauge and harmonic gauge are the same. Outside the standard embedding, holomorphic gauge is preferred as it naturally connects the complex structure of X with the moduli space \mathcal{M} .

The approach of [8, 9] is to include deformations of the tangent bundle $\delta\Theta^{(0,1)} = \delta y^\alpha \mathfrak{D}_\alpha \theta$ as independent degrees of freedom. The tangent bundle \mathcal{T}_X is holomorphic with $R^{(0,2)} = 0$, and the curvature two-form obeys the HYM equation $\omega^2 R = 0$.¹⁰ Hence, $\mathfrak{D}_\alpha \theta$ obeys an Atiyah equation, of the same form as the second equation of (2.4). These we refer to as spurious degrees of freedom because in the physical theory the deformations $\mathfrak{D}_\alpha \theta$ are determined in terms of the other deformations Δ_α , $\mathfrak{D}_\alpha \mathcal{A}$ and $\mathfrak{D}_\alpha \omega$. Nonetheless, they lead to a nice mathematical result which we review and refine here. Define an extension Q_2 of Q_1 :

$$0 \longrightarrow \text{End}(\mathcal{T}_X) \xrightarrow{i_2} Q_2 \xrightarrow{\pi_2} Q_1 \longrightarrow 0 ,$$

with a differential operator

$$\bar{D}_2 = \begin{pmatrix} \bar{\partial}_\theta & 0 & \mathcal{R} \\ 0 & \bar{\partial}_\mathcal{A} & \mathcal{F} \\ 0 & 0 & \bar{\partial} \end{pmatrix} ,$$

which acts on forms valued in Q_2 . The operator \mathcal{R} is analogous to \mathcal{F} : $\mathcal{R}(\Delta_\alpha) = R_\mu \Delta_\alpha^\mu$ where R_μ is the curvature two-form for θ . The operator $\bar{D}_2^2 = 0$ provided $R^{(0,2)} = 0$ and its Bianchi identity holds.

To incorporate hermitian deformations define a bundle Q by the short exact sequence

$$0 \longrightarrow \mathcal{T}_X^{*(1,0)} \xrightarrow{i} Q \xrightarrow{\pi} Q_2 \longrightarrow 0 .$$

We consider $(0, p)$ -forms valued in Q denoted as

$$\mathcal{Y}_\alpha^{(p)} = \begin{pmatrix} Z_{\alpha\nu}^{(p)} \\ \delta_\alpha \theta^{(p)} \\ \delta_\alpha \mathcal{A}^{(p)} \\ \Delta_\alpha^{(p)} \end{pmatrix} .$$

The first row is a $(0, p)$ -form valued in $\mathcal{T}_X^{*(1,0)}$, the middle two rows are $(0, p)$ -forms valued in $\text{End}\mathcal{T}_X$ and $\text{End}\mathcal{E}$ respectively, while the last line is a $(0, p)$ -form valued in $\mathcal{T}_X^{(1,0)}$. The special case of $\mathcal{Y}_\alpha^{(1)}$ corresponds to the field deformations of interest, where $Z_{\alpha\nu}^{(1)} \cong Z_{\alpha\nu\bar{\mu}} dx^{\bar{\mu}}$, $\delta_\alpha \mathcal{A}^{(1)} \cong \mathfrak{D}_\alpha \mathcal{A}$ and $\delta_\alpha \theta^{(1)} \cong \mathfrak{D}_\alpha \theta$. We will need this more general construction in later sections and when we construct the adjoint operator.

In the appendix section D we find a family of operators that satisfy $\bar{D}^2 = 0$ off-shell and reproduce the F-term equations. The operator in [9] is an example of this family. In this paper we use a \bar{D} -operator in this family whose presentation is simpler:

$$\bar{D} = \begin{pmatrix} -\bar{\partial} & \mathcal{H} \\ 0 & \bar{D}_2 \end{pmatrix} ,$$

where $\mathcal{H} : \Omega^{(0,p)}(Q_2) \rightarrow \Omega^{(0,p+1)}(\mathcal{T}_X^{*(1,0)})$ is a linear operator

$$\mathcal{H}_\nu \left(\delta_\alpha \theta^{(p)}, \delta_\alpha \mathcal{A}^{(p)}, \Delta_\alpha^{(p)} \right) = 2i \Delta_\alpha^{(p)\mu} (\partial\omega)_{\mu\nu} + \frac{\alpha^\lambda}{2} \left(\text{Tr} \left(\delta_\alpha \mathcal{A}^{(p)} F_\nu \right) - \text{Tr} \left(\delta_\alpha \theta^{(p)} R_\nu \right) \right) ,$$

¹⁰If you assume the Bianchi identity and supersymmetry variations hold then equations of motion, to first order in α^λ , hold if and only if $\omega^2 R = 0$. This result is valid to first order in α^λ as originally derived in [3, 39] with an elegant summary of the result in the appendix of [40].

and we show in section D.2 that $\bar{D}^2 = 0$ off-shell. We see that

$$\bar{D}\mathcal{Y}_\alpha^{(1)} = 0,$$

corresponds to the F-term equations (2.7)–(2.8).

We note that

$$\mathcal{H}_\nu \left(\bar{D}_2 \left(\delta_\alpha \theta^{(p)}, \delta_\alpha \mathcal{A}^{(p)}, \Delta_\alpha^{(p)} \right) \right) = \bar{\partial} \mathcal{H}_\nu \left(\delta_\alpha \theta^{(p)}, \delta_\alpha \mathcal{A}^{(p)}, \Delta_\alpha^{(p)} \right),$$

while under a small gauge transformation (2.5)

$$\begin{aligned} \mathcal{H}_\nu \left(\delta_\alpha \theta^{(p)} + \bar{\partial}_\theta \psi_\alpha + \varepsilon_\alpha^\mu R_\mu, \delta_\alpha \mathcal{A}^{(p)} + \bar{\partial}_\mathcal{A} \phi_\alpha + \varepsilon_\alpha^\mu F_\mu, \Delta_\alpha^{(p)} + \bar{\partial} \varepsilon_\alpha \right) \\ = \mathcal{H}_\nu \left(\delta_\alpha \theta^{(p)}, \delta_\alpha \mathcal{A}^{(p)}, \Delta_\alpha \right) + \bar{\partial} \mathcal{H}_\nu (\psi_\alpha, \phi_\alpha, \varepsilon_\alpha^\mu). \end{aligned} \quad (4.1)$$

It is important that the gauge symmetry has an action on the complexified hermitian term

$$\mathcal{Z}_\alpha^{(1,1)} \sim \mathcal{Z}_\alpha^{(1,1)} + 2i\varepsilon_\alpha^\mu (\partial\omega)_\mu + \frac{\alpha^\lambda}{2} \text{Tr}(\phi_\alpha F) + \partial(\mathfrak{b}_\alpha^{(0,1)} + i\varepsilon_\alpha^\mu \omega_\mu) + \bar{\partial} \mathfrak{b}_\alpha^{(1,0)}.$$

Here \mathfrak{b}_α is a one-form associated to gauge transformations of the B-field. At first sight, we seem to have a problem: $\bar{D}\mathcal{Y}_\alpha^{(1)} = 0$ is no longer satisfied as the term $\partial(\mathfrak{b}_\alpha^{(0,1)} + i\varepsilon_\alpha^\mu \omega_\mu)$ spoils the equation. The resolution is that (2.8) is derived from (2.4) in holomorphic gauge. There are no residual gauge transformations that preserve holomorphic gauge, which means any small gauge transformation is going to violate the gauge fixing condition. Instead of (2.8) we need to use is the third line of (2.4) and this involves $\mathcal{Z}_\alpha^{(0,2)}$, which is no longer zero. Indeed, we need to pair the transformation of $\mathcal{Z}_\alpha^{(1,1)}$ with

$$\mathcal{Z}_\alpha^{(0,2)} \sim \mathcal{Z}_\alpha^{(0,2)} + \bar{\partial} \left(\mathfrak{b}_\alpha^{(0,1)} + i\varepsilon_\alpha^\mu \omega_\mu \right).$$

Then we find that under a small gauge transformation

$$\bar{\partial} \mathcal{Z}_\alpha^{(1,1)} + \partial \mathcal{Z}_\alpha^{(0,2)} \sim \bar{\partial} \mathcal{Z}_\alpha^{(1,1)} + \partial \mathcal{Z}_\alpha^{(0,2)} + \bar{\partial} \mathcal{H}(\psi_\alpha, \phi_\alpha, \varepsilon_\alpha^\mu,).$$

As $h^{(0,2)} = 0$, the equations of motion imply $\mathcal{Z}_\alpha^{(0,2)} = \bar{\partial} \beta_\alpha^{(0,1)}$:

$$\bar{\partial} \mathcal{Z}_\alpha^{(1,1)} + \partial \mathcal{Z}_\alpha^{(0,2)} = \bar{\partial} \left(\mathcal{Z}_\alpha^{(1,1)} - \partial \beta_\alpha^{(0,1)} \right) = \mathcal{H}(\mathfrak{D}_\alpha \theta, \mathfrak{D}_\alpha \mathcal{A}, \Delta_\alpha),$$

where β_α is a gauge dependent quantity. In holomorphic gauge it vanishes. A gauge invariant formulation likely involves this combination. It would be interesting to relate these observations to the complexified gauge transformations studied in [41], particularly in light of the D-term calculations we perform below. The lesson is that the cohomological property

$$\mathcal{H} : H_{\bar{D}_1}^{(0,p)}(X, Q_1) \longrightarrow H_{\bar{\partial}}^{(1,p+1)}(X),$$

is subtle, and a better understanding of complexified gauge symmetries and GIT would probably help.

Setting this issue aside, it is shown in [8, 9] that there is a long exact sequence in cohomology to describe first order deformations of \bar{D} :

$$H_{\bar{D}}^{(0,1)}(Q) = H^{(0,1)}(\mathcal{T}_X^*) \oplus \ker \mathcal{H}, \quad \ker \mathcal{H} \subseteq H^1(Q_2), \quad (4.2)$$

where

$$H^{(0,1)}(Q_2) = H^{(0,1)}(\text{End } \mathcal{T}_X) \oplus H^{(0,1)}(\text{End } \mathcal{E}) \oplus (\ker \mathcal{F} \cap \ker \mathcal{R}). \quad (4.3)$$

There are also the hermitian Yang-Mills and balanced equations to take into account. We show their role appears in the adjoint operator \bar{D}^\dagger .

4.2 The D-terms and the adjoint operator \overline{D}^\dagger

A metric on Q describes how to pair field deformations to produce a real number. In string theory, a natural metric to use is the moduli space metric. As we preserve $N = 1$ supersymmetry in spacetime, the metric is Kähler. With the spurious degrees of freedom, this was first derived in [12] using a dimensional reduction of the α' -corrected supergravity theory considered here. The result in our notation is

$$g_{\alpha\bar{\beta}}^\# = \frac{1}{V} \int_X \left(\frac{1}{4} Z_\alpha^{(1,1)} \star \overline{Z}_\beta^{(1,1)} + \frac{\alpha'}{4} \text{Tr} \left(\mathfrak{D}_\alpha \theta \star \mathfrak{D}_{\bar{\beta}} \theta \right) - \frac{\alpha'}{4} \text{Tr} \left(\mathfrak{D}_\alpha \mathcal{A} \star \mathfrak{D}_{\bar{\beta}} \mathcal{A}^\dagger \right) + \Delta_\alpha^\mu \star \Delta_{\bar{\beta}}^{\bar{\nu}} g_{\mu\bar{\nu}} \right), \quad (4.4)$$

We have written the antihermitian connections as $A = \mathcal{A} - \mathcal{A}^\dagger$, $\mathcal{A} = A^{(0,1)}$ and $\mathfrak{D}_{\bar{\beta}} A = -\mathfrak{D}_{\bar{\beta}} \mathcal{A}^\dagger$ with similar expressions for $\Theta = \theta - \theta^\dagger$.

We take the metric on a pair of sections $\mathcal{Y}_\alpha^{(p)}$ and $\mathcal{Y}_\beta^{(p)}$ to be the natural generalisation of the moduli space metric:

$$\langle \mathcal{Y}_\alpha^{(p)}, \mathcal{Y}_\beta^{(p)} \rangle = \frac{1}{V} \int_X \left(\frac{1}{4} Z_{\alpha\nu}^{(p)} \star \overline{Z}_{\beta\bar{\mu}}^{(p)} g^{\nu\bar{\mu}} + \frac{\alpha'}{4} \text{Tr} \left(\delta_\alpha \theta^{(p)} \star \delta_{\bar{\beta}} \theta^{(p)\dagger} \right) - \frac{\alpha'}{4} \text{Tr} \left(\delta_\alpha \mathcal{A}^{(p)} \star \delta_{\bar{\beta}} \mathcal{A}^{(p)\dagger} \right) + \Delta_\alpha^{(p)\mu} \star \Delta_{\bar{\beta}}^{(p)\bar{\nu}} g_{\mu\bar{\nu}} \right), \quad (4.5)$$

where we hermitian conjugate in the appropriate way so the metric is real.

The metric allows us to define an adjoint operator \overline{D}^\dagger .

$$\langle \overline{D} \mathcal{Y}_\alpha^{(p)}, \mathcal{Y}_\beta^{(p+1)} \rangle = \langle \mathcal{Y}_\alpha^{(p)}, \overline{D}^\dagger \mathcal{Y}_\beta^{(p+1)} \rangle. \quad (4.6)$$

It is instructive to explicitly compute:

$$\begin{aligned} & \langle \overline{D} \mathcal{Y}_\alpha^{(p)}, \mathcal{Y}_\beta^{(p+1)} \rangle \\ &= \frac{1}{4V} \int_X \left(-\bar{\partial} Z_{\alpha\nu}^{(p)} + 2i \Delta_\alpha^{(p)\mu} (\partial\omega)_{\mu\nu} - \frac{\alpha'}{2} \text{Tr} \left(\delta_\alpha \theta^{(p)} R_\nu \right) + \frac{\alpha'}{2} \text{Tr} \left(\delta_\alpha \mathcal{A}^{(p)} F_\nu \right) \right) \star \overline{Z}_{\beta\bar{\lambda}}^{(p+1)} g^{\nu\bar{\lambda}} \\ & \quad + \frac{\alpha'}{4V} \text{Tr} \int \left(\bar{\partial}_\theta \left(\delta_\alpha \theta^{(p)} \right) + R_\mu \Delta_\alpha^{(p)\mu} \right) \star \delta_{\bar{\beta}} \theta^{\dagger(p+1)} \\ & \quad - \frac{\alpha'}{4V} \text{Tr} \int \left(\bar{\partial}_\mathcal{A} \left(\delta_\alpha \mathcal{A}^{(p)} \right) + F_\mu \Delta_\alpha^{(p)\mu} \right) \star \delta_{\bar{\beta}} \mathcal{A}^{(p+1)\dagger} + \frac{1}{V} \int \bar{\partial} \Delta_\alpha^{(p)\mu} \star \Delta_{\bar{\beta}}^{(p+1)\bar{\nu}} g_{\mu\bar{\nu}} \\ &= \frac{1}{V} \int_X \left\{ \frac{1}{4} Z_\alpha^{(p)} \star \left(-\partial^\dagger \overline{Z}_\beta^{(p+1)} \right) + \frac{\alpha'}{4} \text{Tr} \left(\delta_\alpha \theta^{(p)} \star \left((-1)^{p+1} \frac{1}{2} R_{\mu\bar{\nu}} \overline{Z}_\beta^{\mu\bar{\nu}(p+1)} + \partial_{\theta^\dagger}^\dagger \delta_{\bar{\beta}} \theta^{\dagger(p+1)} \right) \right) \right. \\ & \quad - \frac{\alpha'}{4} \text{Tr} \left(\delta_\alpha \mathcal{A}^{(p)} \star \left((-1)^{p+1} \frac{1}{2} F_{\nu\bar{\sigma}} \overline{Z}_\beta^{\nu\bar{\sigma}(p+1)} + \partial_{\mathcal{A}^\dagger}^\dagger \delta_{\bar{\beta}} \mathcal{A}^{\dagger(p+1)} \right) \right) \\ & \quad + \Delta_\alpha^{(p)\mu} \star \left((-1)^p \frac{i}{2} (\partial\omega)_\mu{}^{\rho\bar{\nu}} \overline{Z}_{\beta\rho\bar{\nu}}^{(p+1)} + \frac{\alpha'}{4} \text{Tr} R_{\mu\bar{\nu}} \delta_{\bar{\beta}} \theta^{\dagger\bar{\nu}(p+1)} \right. \\ & \quad \left. \left. - \frac{\alpha'}{4} \text{Tr} F_{\mu\bar{\nu}} \delta_{\bar{\beta}} \mathcal{A}^{\dagger\bar{\nu}(p+1)} + \partial^\dagger \Delta_{\bar{\beta}}^{(p+1)\mu} \right) \right\} \\ &= \langle \mathcal{Y}_\alpha^{(p)}, \overline{D}^\dagger \mathcal{Y}_\beta^{(p+1)} \rangle, \end{aligned}$$

where our notation is, for example, $Z_\alpha^{\mu\bar{\nu}(p+1)} = \frac{1}{p!} Z_\alpha^{\mu\bar{\nu}} \bar{\lambda}_1 \dots \bar{\lambda}_p dx^{\bar{\lambda}_1 \dots \bar{\lambda}_p}$. Comparing with (4.5) we identify

$$\bar{D}^\dagger = \begin{pmatrix} -\bar{\partial}^\dagger & 0 & 0 & 0 \\ \tilde{R}^\dagger & \bar{\partial}_\theta^\dagger & 0 & 0 \\ \tilde{F}^\dagger & 0 & \bar{\partial}_A^\dagger & 0 \\ \mathcal{H}^\dagger & \hat{\mathcal{R}}^\dagger & \hat{\mathcal{F}}^\dagger & \bar{\partial}^\dagger \end{pmatrix}, \quad (4.7)$$

where

$$\begin{aligned} \tilde{\mathcal{R}}^\dagger : \Omega^{(0,p+1)}(\mathcal{T}_X^{*(1,0)}) &\longrightarrow \Omega^{(0,p)}(\text{End } \mathcal{T}_X), & \tilde{R}^\dagger(Z_{\alpha\nu}^{(p+1)}) &= (-1)^p \frac{1}{2} R^{\nu\bar{\rho}} Z_{\alpha\nu\bar{\rho}}^{(p)}, \\ \tilde{\mathcal{F}}^\dagger : \Omega^{(0,p+1)}(\mathcal{T}_X^{(0,1)}) &\longrightarrow \Omega^{(0,p)}(\text{End } \mathcal{T}_X), & \tilde{F}^\dagger(Z_\alpha^{(p+1)}) &= (-1)^p \frac{1}{2} F^{\rho\bar{\nu}} Z_{\alpha\rho\bar{\nu}}^{(p)}, \\ \hat{\mathcal{R}}^\dagger : \Omega^{(0,p+1)}(\text{End } \mathcal{T}_X) &\longrightarrow \Omega^{(0,p)}(\mathcal{T}_X^{(0,1)}), & \hat{\mathcal{R}}^\dagger(\delta_\alpha \theta^{(p+1)}) &= -\frac{\alpha^\lambda}{4} \text{Tr}(R^{\nu\bar{\rho}} \delta_\alpha \theta_{\bar{\rho}}^{(p)}), \\ \hat{\mathcal{F}}^\dagger : \Omega^{(0,p+1)}(\text{End } \mathcal{E}) &\longrightarrow \Omega^{(0,p)}(\mathcal{T}_X^{(0,1)}), & \hat{\mathcal{F}}^\dagger(\delta_\alpha \mathcal{A}^{(p+1)}) &= \frac{\alpha^\lambda}{4} \text{Tr}(F^{\nu\bar{\rho}} \delta_\alpha \mathcal{A}_{\bar{\rho}}^{(p)}), \\ \mathcal{H}^\dagger : \Omega^{(0,p+1)}(\mathcal{T}_X^{*(1,0)}) &\longrightarrow \Omega^{(0,p)}(\mathcal{T}_X^{(1,0)}), & \mathcal{H}^\dagger(Z_{\alpha\nu}^{(p+1)}) &= (-1)^{p+1} \frac{i}{2} (\bar{\partial}\omega)^{\nu\rho\bar{\lambda}} Z_{\alpha\rho\bar{\lambda}}^{(p+1)}. \end{aligned} \quad (4.8)$$

The action of \bar{D}^\dagger on $\mathcal{Y}_\alpha^{(1)}$ is

$$\bar{D}^\dagger \begin{pmatrix} Z_{\alpha\nu}^{(1)} \\ \mathfrak{D}_\alpha \theta \\ \mathfrak{D}_\alpha \mathcal{A} \\ \Delta_{\alpha^\mu} \end{pmatrix} = \begin{pmatrix} \nabla^{\bar{\mu}} Z_{\alpha\nu\bar{\mu}} \\ \bar{\partial}_\theta^\dagger \mathfrak{D}_\alpha \theta + \frac{1}{2} R^{\nu\bar{\mu}} Z_{\alpha\nu\bar{\mu}} \\ \bar{\partial}_A^\dagger \mathfrak{D}_\alpha \mathcal{A} + \frac{1}{2} F^{\nu\bar{\mu}} Z_{\alpha\nu\bar{\mu}} \\ -\frac{i}{2} (\bar{\partial}\omega)^{\mu\rho\bar{\lambda}} Z_{\alpha\rho\bar{\lambda}} - \frac{\alpha^\lambda}{4} \text{Tr}(R^{\mu\bar{\rho}} \mathfrak{D}_\alpha \theta_{\bar{\rho}}) + \frac{\alpha^\lambda}{4} \text{Tr}(F^{\mu\bar{\rho}} \mathfrak{D}_\alpha \mathcal{A}_{\bar{\rho}}) - \nabla^{\text{Ch/H}\bar{\mu}} \Delta_{\alpha\bar{\mu}}{}^\nu \end{pmatrix}, \quad (4.9)$$

where we use that R and F are antihermitian and in the last line, we used the balanced equation to derive an identity

$$\bar{\partial}^\dagger \Delta^\mu = -\star \frac{i\omega^2}{2} \left(\partial \Delta^\mu + g^{\mu\bar{\sigma}} \partial g_{\bar{\sigma}\rho} \Delta^\rho \right) = -\nabla^{\text{Ch/H}\bar{\mu}} \Delta_{\alpha\bar{\mu}}{}^\nu. \quad (4.10)$$

We then find that

$$\bar{D}^\dagger \mathcal{Y}_\alpha^{(1)} = 0,$$

is precisely the D-term equations (2.13)–(2.15).

Hence we have derived the remarkable fact, that in holomorphic gauge (with spurious degrees of freedom) we have

$$\bar{D} \mathcal{Y}_\alpha^{(1)} = 0 \iff \text{F-terms}, \quad \bar{D}^\dagger \mathcal{Y}_\alpha^{(1)} = 0 \iff \text{D-terms}.$$

This is despite the fact the individual deformations such as $\mathfrak{D}_\alpha \mathcal{A}$ or Δ_{α^μ} are not harmonic representatives in the usual notion of harmonic, as demonstrated explicitly in (2.16), (2.17a)–(2.17c). Instead, if we work with $(0,1)$ -forms valued in Q then all we need to do is study harmonic representatives of this \bar{D} -operator. However, we are not yet at the physical case yet: we still need to eliminate the spurious degrees of freedom.

5 Eliminating the spurious degrees of freedom

The space of deformations of \bar{D} on the bundle Q is not the moduli space of string theory. Indeed, string scattering amplitudes and/or supersymmetry tell us that $\mathfrak{D}_\alpha \Theta$ is not an independent degree of freedom, but determined in terms of the remaining fields. In [18, 31] this is calculated to lowest order in α^λ :

$$\mathfrak{D}_\alpha \Theta_{\bar{\mu}^\nu \sigma} = \nabla_\sigma \Delta_{\alpha \bar{\mu}^\nu} + \frac{1}{2} \nabla^\nu Z_{\alpha \sigma \bar{\mu}}, \quad (5.1)$$

where

$$\nabla_\sigma \Delta_{\alpha \bar{\mu}^\nu} = \partial_\sigma \Delta_{\alpha \bar{\mu}^\nu} + \Gamma_{\sigma^\nu \lambda} \Delta_{\alpha \bar{\mu}^\lambda}, \quad \text{and} \quad \nabla_{\bar{\mu}} Z_{\alpha \sigma \bar{\nu}} = \partial_{\bar{\mu}} Z_{\alpha \sigma \bar{\nu}} - \Gamma_{\bar{\mu}^\lambda \bar{\nu}} \bar{Z}_{\alpha \sigma \bar{\lambda}}.$$

The Γ are the Levi-Civita symbols, which because this result is derived for solutions which have a smooth limit $\alpha^\lambda \rightarrow 0$ with $H \rightarrow 0$, are the same as the Chern connection.¹¹ The dilaton is constant to α^λ by a choice of gauge fixing [31]. See also appendix C of [19] for the calculation in the notation of this paper.

5.1 Constructing the bundle \mathcal{Q}

We now turn to constructing an extension bundle \mathcal{Q} whose sections describe the physical degrees of freedom. Recall that to first order in α^λ , the first order deformations satisfy an Atiyah equation $\bar{\partial}_\theta(\mathfrak{D}_\alpha \theta) = \Delta_\alpha^\mu R_\mu$. We check (5.1) satisfies this explicitly. First, note

$$\begin{aligned} \nabla_{\bar{\rho}} \nabla_\nu \Delta_\alpha^\mu &= \nabla_{\bar{\rho}} \left(\partial_\nu \Delta_\alpha^\mu + \Gamma_{\nu^\mu \lambda} \Delta_\alpha^\lambda \right), \\ [\nabla_{\bar{\rho}}, \nabla_\nu] \Delta_{\alpha \bar{\sigma}^\mu} &= (\partial_{\bar{\rho}} \Gamma_{\nu^\mu \lambda}) \Delta_{\alpha \bar{\sigma}^\lambda} + \left(\partial_\nu \Gamma_{\bar{\rho}^\lambda \bar{\sigma}} \right) \Delta_{\alpha \bar{\lambda}^\mu} = R^\mu{}_{\lambda \bar{\rho} \nu} \Delta_{\alpha \bar{\sigma}^\lambda} - R^{\bar{\lambda}}{}_{\bar{\sigma} \bar{\rho} \nu} \Delta_{\alpha \bar{\lambda}^\mu}, \end{aligned}$$

where $R^\mu{}_{\lambda \bar{\rho} \nu}$ is the Riemann curvature tensor on a complex manifold. Second,

$$\begin{aligned} \bar{\partial}(\mathfrak{D}_\alpha \Theta)^\mu{}_\nu &= dx^{\bar{\rho}} \nabla_{\bar{\rho}} \left(\nabla_\nu \Delta_\alpha^\mu + i \nabla^\mu \mathfrak{D}_\alpha \omega_\nu^{(0,1)} \right) \\ &= dx^{\bar{\rho}} [\nabla_{\bar{\rho}}, \nabla_\nu] \Delta_\alpha^\mu + \frac{1}{2} dx^{\bar{\rho}} [\nabla_{\bar{\rho}}, \nabla^\mu] Z_{\alpha \nu}^{(0,1)} + \frac{1}{2} \nabla^\mu \left(\bar{\partial} Z_{\alpha \nu}^{(0,1)} \right) \\ &= dx^{\bar{\rho} \bar{\sigma}} R^\mu{}_{\lambda \bar{\rho} \nu} \Delta_{\alpha \bar{\sigma}^\lambda} + dx^{\bar{\rho} \bar{\sigma}} R^{\bar{\lambda}}{}_{\bar{\sigma} \bar{\rho} \nu} \Delta_{\alpha \bar{\lambda}^\mu} \\ &= \Delta_\alpha^\lambda R^\mu{}_{\nu \lambda}{}^{(0,1)}, \end{aligned} \quad (5.2)$$

where $R^{\bar{\lambda}}{}_{\bar{\sigma} \bar{\rho} \nu} = R^{\bar{\lambda}}{}_{\bar{\rho} \nu \bar{\sigma}}$ is used in the penultimate line. We use that $R^{(0,2)} = 0$ and $\bar{\partial} Z_\alpha = \mathcal{O}(\alpha^\lambda)$. In fact, because Θ is an instanton (to first order in α^λ), $\mathfrak{D}_\alpha \Theta$ obeys an Atiyah equation and so we conclude that this result actually holds up to and including first order in α^λ . Hence, if there are any α^λ corrections to (5.1) then we know that in (5.2) that they must cancel.

A corollary of (5.2) is that the expression (5.1) is a solution of $\bar{D} \mathcal{Y}_\alpha^{(1)} = 0$ and so bona-fide element of the cohomology $H_D^1(Q)$. This serves as another consistency check and is part of our intuition that the true moduli space is some subspace of $H_D^1(Q)$.

¹¹As the connection Θ only appears together with a factor of α^λ , modulo higher orders we can also take Γ symbols to be the symbols of the Levi-Civita connection of the Ricci-flat metric.

Using the symmetry $\mathfrak{D}_\alpha \Theta_{\bar{\mu}}^{\bar{\nu}\bar{\sigma}} = -g^{\bar{\nu}\lambda} \mathfrak{D}_\alpha \Theta_{\bar{\mu}}^{\rho\lambda} g_{\rho\bar{\sigma}}$ we can write

$$\frac{\alpha^\lambda}{2} \text{Tr} (\mathfrak{D}_\alpha \Theta R) = \alpha^\lambda \left(\nabla_\nu \Delta_\alpha^\mu + \frac{1}{2} \nabla^\mu Z_{\alpha\nu}^{(0,1)} \right) R^\nu{}_\mu, \quad (5.3)$$

where $R^\nu{}_\mu = R_{\rho\bar{\sigma}}{}^\nu{}_\mu dx^{\rho\bar{\sigma}}$ is the Riemann tensor and we have used $R_{\bar{\mu}\nu} = -R_{\nu\bar{\mu}}$.

Now, using (5.2) with the Bianchi identity for R , (A.1) we find

$$\bar{\partial} \left(\text{Tr} \mathfrak{D}_\alpha \Theta^{(0,1)} R \right) = \Delta_\alpha^\mu \text{Tr} R_\mu \wedge R + \mathcal{O}(\alpha'^2). \quad (5.4)$$

Using (5.3) we see that a consequence of (5.4) is that

$$\bar{\partial} \left(\left(\nabla^\mu Z_{\alpha\nu}^{(0,1)} \right) R^\nu{}_\mu \right) = 0,$$

and so it should be $\bar{\partial}$ -exact. Indeed, this is the case and we now derive an explicit expression. Start by recalling that $R^\nu{}_\mu$ obeys a hermitian Yang-Mills equation $\nabla^\mu R^\nu{}_{\mu\rho\bar{\sigma}} = 0$ to first order in α^λ and so

$$\left(\nabla^\mu Z_{\alpha\nu}^{(0,1)} \right) R^\nu{}_\mu = \nabla^\mu \left(Z_{\alpha\nu}^{(0,1)} R^\nu{}_\mu \right).$$

As the Levi-Civita symbols are pure to this order in α^λ we find the term in parenthesis can be written as

$$Z_{\alpha\nu}^{(0,1)} R^\nu{}_\mu = -dx^\rho \left[\nabla_\rho, \bar{\partial} \right] Z_{\alpha\mu}^{(0,1)},$$

so that

$$\left(\nabla^\mu Z_{\alpha\nu}^{(0,1)} \right) R^\nu{}_\mu = -dx^\rho \nabla^\mu \left(\left[\nabla_\rho, \bar{\partial} \right] Z_{\alpha\mu}^{(0,1)} \right).$$

Using that $\bar{\partial}(Z_{\alpha\mu}^{(0,1)}) = \mathcal{O}(\alpha^\lambda)$ and that $R^{(0,2)} = 0$ we find

$$\left(\nabla^\mu Z_{\alpha\nu}^{(0,1)} \right) R^\nu{}_\mu = -\bar{\partial} \left(dx^\rho \nabla^\mu \nabla_\rho Z_{\alpha\mu}^{(0,1)} \right).$$

Note that $\nabla_\rho Z_{\alpha\mu}^{(0,1)} = \partial_\rho Z_{\alpha\mu}^{(0,1)} - \Gamma_{\rho\lambda}{}^\lambda{}_\mu Z_{\alpha\lambda}^{(0,1)}$ and so does not become ∂ .

Using these results, (2.8) can be rewritten as:

$$\bar{\partial} \left(Z_\alpha^{(1,1)} - \frac{\alpha^\lambda}{2} dx^\rho \left(\nabla^\mu \nabla_\rho Z_{\alpha\mu}^{(0,1)} \right) \right) - 2i \Delta_\alpha^\mu (\partial\omega)_\mu + \alpha^\lambda R^\nu{}_\mu \nabla_\nu (\Delta_\alpha^\mu) - \frac{\alpha^\lambda}{2} \text{Tr} (\mathfrak{D}_\alpha \mathcal{A} F) = 0. \quad (5.5)$$

It is natural to perform a field redefinition

$$\tilde{Z}_\alpha^{(1,1)} = Z_\alpha^{(1,1)} - \frac{\alpha^\lambda}{2} dx^\rho \nabla^\mu \nabla_\rho Z_{\alpha\mu}^{(0,1)}, \quad (5.6)$$

which is well-defined in α^λ -perturbation theory, and in particular, invertible. It is not overly surprising that the complexified hermitian form needs to be corrected in α^λ . The field redefinition will propagate into other relevant equations, such as deformations of the HYM and balanced equations and the moduli space metric. But it is the natural thing to do to realise a holomorphic structure on the extension bundle.

There is a cohomology that captures these field deformations. To describe it, we need to generalise some of the above results above to $(0, p)$ -forms. First, we ansatz that (5.1) generalises to

$$\delta_\alpha \theta^{(p)\nu}{}_\sigma = \nabla_\sigma \Delta_\alpha^{(p)\nu} + \frac{1}{2} \nabla^\nu Z_{\alpha\sigma}^{(p)}. \quad (5.7)$$

Second we need the generalisation of the \bar{D} -operator for the spurious case, written in components in (D.10). The only non-trivial part is to evaluate $\text{Tr } \delta_\alpha \theta^{(p)} R_\nu$ using (5.7) and substitute into the first line of (D.10) giving

$$\widehat{Z}_\nu^{(p+1)} = -\bar{\partial} \widetilde{Z}_{\alpha\nu}^{(p)} + 2i \Delta_\alpha^{(p)\mu} (\partial\omega)_{\mu\nu} + \alpha^\lambda \nabla_\sigma (\Delta_\alpha^{(p)\tau}) R^\sigma{}_{\tau\nu\bar{\lambda}} dx^{\bar{\lambda}} + \frac{\alpha^\lambda}{2} \text{Tr} (\delta_\alpha \mathcal{A}^{(p)} F_\nu) .$$

A consistency check is that the relation (5.7) satisfies the second line of (D.10). We also have the field redefinition

$$\widetilde{Z}_{\alpha\nu}^{(p)} = Z_{\alpha\nu}^{(p)} + \frac{\alpha^\lambda}{2} (-1)^p \nabla^\mu \nabla_\nu Z_{\alpha\mu}^{(p)} . \tag{5.8}$$

Now construct the extension bundle. The extension Q_1 defined in (3.1) is unchanged. Q_2 is now not necessary. Consider a short exact sequence

$$0 \longrightarrow \mathcal{T}_X^{*(1,0)} \xrightarrow{i} \mathcal{Q} \xrightarrow{\pi} Q_1 \longrightarrow 0 , \tag{5.9}$$

where $\mathcal{T}_X^{*(1,0)}$ are holomorphic co-vectors (forms). p -forms valued in \mathcal{Q} are denoted as in the previous subsection

$$\mathcal{Y}_\alpha^{(p)} = \begin{pmatrix} \widetilde{Z}_{\alpha\nu}^{(p)} \\ \delta_\alpha \mathcal{A}^{(p)} \\ \Delta_\alpha^{(p)} \end{pmatrix} .$$

We have abused notation in using the same letter $\mathcal{Y}_\alpha^{(p)}$ as for the spurious case, but it should be clear from context what we mean.

Define the operator \bar{D} on sections valued in this bundle as

$$\bar{D} = \begin{pmatrix} -\bar{\partial} & \mathcal{H}^{\text{new}} \\ 0 & \bar{D}_1 \end{pmatrix} ,$$

where the map, or extension class, $\mathcal{H}^{\text{new}} : \Omega^{(0,p)}(Q_1) \rightarrow \Omega^{(0,p+1)}(\mathcal{T}_X^{*(1,0)})$ is now given by

$$\mathcal{H}_\nu^{\text{new}} (\delta_\alpha \mathcal{A}^{(p)}, \Delta_\alpha^{(p)}) = 2i \Delta_\alpha^{(p)\mu} (\partial\omega)_{\mu\nu} - \alpha^\lambda \nabla_\rho (\Delta_\alpha^{(p)\lambda}) R^\rho{}_{\lambda\nu} + \frac{\alpha^\lambda}{2} \text{Tr} (\delta_\alpha \mathcal{A}^{(p)} F_\nu) ,$$

and where $R^\rho{}_{\lambda\nu} = R^\rho{}_{\lambda\nu\bar{\sigma}} dx^{\bar{\sigma}}$. The action of \bar{D} is

$$\bar{D} \begin{pmatrix} \widetilde{Z}_{\alpha\nu}^{(p)} \\ \delta_\alpha \mathcal{A}^{(p)} \\ \Delta_\alpha^{(p)} \end{pmatrix} = \begin{pmatrix} -\bar{\partial} \widetilde{Z}_{\alpha\nu}^{(p)} + 2i \Delta_\alpha^{(p)\mu} (\partial\omega)_{\mu\nu} - \alpha^\lambda (\nabla_\rho \Delta_\alpha^{(p)\lambda}) R^\rho{}_{\lambda\nu} + \frac{\alpha^\lambda}{2} \text{Tr} (\delta_\alpha \mathcal{A}^{(p)} F_\nu) \\ \bar{\partial}_{\mathcal{A}} (\delta_\alpha \mathcal{A}^{(p)}) + F_\mu \Delta_\alpha^{(p)\mu} \\ \bar{\partial} \Delta_\alpha^{(p)} \end{pmatrix} . \tag{5.10}$$

This operator satisfies $\bar{D}^2 = 0$ after using the Bianchi identity (2.3).¹² Furthermore, $\bar{D} \mathcal{Y}_\alpha^{(1)} = 0$ corresponds to precisely the F-term equations (2.7)–(2.8) together with (5.1).

¹²Note that the operator \bar{D} squares to zero on the nose if one modifies the Hull-Strominger system to use the curvature of the zeroth order Ricci-flat Calabi-Yau metric in the $\text{Tr } R^2$ term in the Bianchi identity. Otherwise, this only holds modulo α'^2 corrections.

Finally, it should be noted that in contrast to the spurious case of section 4, this operator is not a connection in that for a function f and section $\mathcal{Y}_\alpha^{(p)}$ of \mathcal{Q} the Leibniz rule is not satisfied:

$$\overline{\mathcal{D}}(f \mathcal{Y}_\alpha^{(p)}) \neq \overline{\partial}f \wedge \mathcal{Y}_\alpha^{(p)} + f \overline{\mathcal{D}}\mathcal{Y}_\alpha^{(p)}. \quad (5.11)$$

This is due to the appearance of a holomorphic derivative in the new extension map $\mathcal{H}_\nu^{\text{new}}$.¹³ Nonetheless, $\overline{\mathcal{D}}$ is still a well defined linear operator on \mathcal{Q} whose cohomology can be computed using long exact sequences like before. Moreover, given an inner product on \mathcal{Q} , one can define the adjoint of $\overline{\mathcal{D}}$ and study its harmonic modes.¹⁴

5.2 The D-terms and the adjoint operator $\overline{\mathcal{D}}^\dagger$

The moduli space metric, after including (5.1), is [12, 18]

$$g_{\alpha\beta}^\sharp = \frac{1}{V} \int_X \left(\Delta_\alpha^\mu \star \Delta_{\overline{\beta}}^{\overline{\nu}} g_{\mu\overline{\nu}} + \frac{1}{4} Z_\alpha^{(1,1)} \star \overline{Z}_{\overline{\beta}}^{(1,1)} + \frac{\alpha^\lambda}{4} \text{Tr} \left(\mathfrak{D}_\alpha A \star \mathfrak{D}_{\overline{\beta}} A \right) \right) \\ + \frac{\alpha^\lambda}{2V} \int_X d^6x \sqrt{g} \left(\Delta_{\alpha\overline{\mu\nu}} \Delta_{\overline{\beta}\rho\sigma} + \frac{1}{4} Z_{\alpha\rho\overline{\mu}} \overline{Z}_{\overline{\beta}\sigma\overline{\nu}} \right) R^{\rho\overline{\mu}\sigma\overline{\nu}} + \mathcal{O}(\alpha^2), \quad (5.12)$$

where we have used holomorphic gauge (2.6) in writing $Z_\alpha^{(1,1)} = 2i\mathfrak{D}_\alpha\omega^{(1,1)}$. We need to write down $g_{\alpha\beta}^\sharp$ in the new variables $\widetilde{Z}_\alpha^{(1,1)}$ via the field redefinition (5.6). First note that

$$\int_X Z_\alpha^{(1,1)} \star \overline{Z}_{\overline{\beta}}^{(1,1)} = \\ \int_X \left\{ \widetilde{Z}_\alpha^{(1,1)} \star \overline{Z}_{\overline{\beta}}^{(1,1)} - \frac{\alpha^\lambda}{2} \left(\nabla^\mu \nabla_\rho \widetilde{Z}_{\alpha\mu}^{(0,1)} \right) dx^\rho \star \overline{Z}_{\overline{\beta}}^{(1,1)} - \frac{\alpha^\lambda}{2} \widetilde{Z}_\alpha^{(1,1)} \star \left(\nabla^{\overline{\mu}} \nabla_{\overline{\rho}} \overline{Z}_{\overline{\beta}\overline{\mu}}^{(1,0)} dx^{\overline{\rho}} \right) \right\}.$$

The second term is

$$\int_X \left(\nabla^\mu \nabla_\rho \widetilde{Z}_{\alpha\mu}^{(0,1)} dx^\rho \right) \star \overline{Z}_{\overline{\beta}}^{(1,1)} = - \int_X d^6x \sqrt{g} \left(\nabla^\mu \nabla_\rho \widetilde{Z}_{\alpha\mu\overline{\lambda}} \right) \overline{Z}_{\overline{\beta}}^{\rho\overline{\lambda}} \\ = \int_X \left(\nabla^\mu \widetilde{Z}_{\alpha\mu}^{(0,1)} \right) \star \left(\nabla^{\overline{\tau}} \overline{Z}_{\overline{\beta}\sigma}^{(1,0)} \right) - \int_X d^6x \sqrt{g} \left([\nabla^\mu, \nabla_\rho] \widetilde{Z}_{\alpha\mu\overline{\lambda}} \right) \overline{Z}_{\overline{\beta}}^{\rho\overline{\lambda}}.$$

The commutator involves both the Riemann curvature and Ricci tensor $\text{Ric}^{\delta\overline{\tau}} = -R^\delta{}_\mu{}^{\mu\overline{\tau}}$:

$$\left[\nabla^\mu, \nabla^{\overline{\tau}} \right] \widetilde{Z}_{\alpha\mu\overline{\lambda}} = -R^\delta{}_\mu{}^{\mu\overline{\tau}} \widetilde{Z}_{\alpha\delta\overline{\lambda}} - R^{\overline{\delta}}{}_\lambda{}^{\mu\overline{\tau}} \widetilde{Z}_{\alpha\mu\overline{\delta}} = \text{Ric}^{\delta\overline{\tau}} \widetilde{Z}_{\alpha\delta\overline{\lambda}} - R^{\overline{\delta}}{}_\lambda{}^{\mu\overline{\tau}} \widetilde{Z}_{\alpha\mu\overline{\delta}}.$$

The Ricci tensor vanishes to this order in α^λ . So

$$\int_X \left(\nabla^\mu \nabla_\rho \widetilde{Z}_{\alpha\mu}^{(0,1)} dx^\rho \right) \star \overline{Z}_{\overline{\beta}}^{(1,1)} = \int_X \left(\nabla^\mu \widetilde{Z}_{\alpha\mu}^{(0,1)} \right) \star \left(\nabla^{\overline{\tau}} \overline{Z}_{\overline{\beta}\overline{\tau}}^{(1,0)} \right) + \int_X d^6x \sqrt{g} R^{\mu\overline{\delta}\sigma\overline{\tau}} \widetilde{Z}_{\alpha\mu\overline{\delta}} \overline{Z}_{\overline{\beta}\sigma\overline{\tau}}.$$

The third term follows in the same way. Hence, the moduli space metric in these variables is

$$g_{\alpha\beta}^\sharp = \frac{1}{V} \int_X \left(\Delta_\alpha^\mu \star \Delta_{\overline{\beta}}^{\overline{\nu}} g_{\mu\overline{\nu}} + \frac{1}{4} \widetilde{Z}_\alpha^{(1,1)} \star \overline{Z}_{\overline{\beta}}^{(1,1)} + \frac{\alpha^\lambda}{4} \text{Tr} \left(\mathfrak{D}_\alpha A \star \mathfrak{D}_{\overline{\beta}} A \right) \right) \\ - \frac{\alpha^\lambda}{4V} \int_X \left(\nabla^\mu \widetilde{Z}_{\alpha\mu}^{(0,1)} \right) \star \left(\nabla^{\overline{\tau}} \overline{Z}_{\overline{\beta}\overline{\tau}}^{(1,0)} \right) - \frac{\alpha^\lambda}{8V} \int_X d^6x \sqrt{g} R^{\mu\overline{\delta}\sigma\overline{\tau}} \widetilde{Z}_{\alpha\mu\overline{\delta}} \overline{Z}_{\overline{\beta}\sigma\overline{\tau}} \\ + \frac{\alpha^\lambda}{2V} \int_X d^6x \sqrt{g} \left(\Delta_{\alpha\overline{\mu\nu}} \Delta_{\overline{\beta}\rho\sigma} \right) R^{\rho\overline{\mu}\sigma\overline{\nu}}. \quad (5.13)$$

¹³We are grateful to Mario Garcia Fernandez for pointing this out.

¹⁴The Hodge-theory, elliptic properties and other relations of $\overline{\mathcal{D}}$ to the Hull-Strominger system will be studied in an upcoming paper [33].

It is straightforward to generalise this calculation to a metric on $(0, p)$ -forms:

$$\begin{aligned}
 \langle \mathcal{Y}_\alpha^{(p)}, \mathcal{Y}_\beta^{(p)} \rangle &= \frac{1}{V} \int_X \left\{ \frac{1}{4} \tilde{Z}_{\alpha\nu}^{(p)} \star \tilde{Z}_\beta^{(p)\nu} - \frac{\alpha'}{4} \left(\nabla^\mu \tilde{Z}_{\alpha\mu}^{(p)} \right) \star \left(\nabla^{\bar{\tau}} \tilde{Z}_{\beta\bar{\tau}}^{(p)} \right) \right. \\
 &\quad + \frac{\alpha'}{8} (-1)^p R^{\mu\bar{\delta}\sigma\bar{\tau}} \tilde{Z}_{\alpha\mu\bar{\delta}}^{(p)} \star \tilde{Z}_{\beta\sigma\bar{\tau}}^{(p)} - \frac{\alpha'}{4} \text{Tr} \left(\delta_\alpha \mathcal{A}^{(p)} \star \delta_{\bar{\beta}} \mathcal{A}^{(p)\dagger} \right) \\
 &\quad \left. + \Delta_\alpha^{(p)\mu} \star \Delta_\beta^{(p)\bar{\nu}} g_{\mu\bar{\nu}} + \frac{\alpha'}{2} (-1)^{p+1} R^{\bar{\mu}\rho}{}_{\nu\bar{\sigma}} \Delta_{\alpha\bar{\mu}}^{(p)\nu} \star \Delta_{\bar{\beta}\rho}^{(p)\bar{\sigma}} \right\}, \quad (5.14)
 \end{aligned}$$

where $\Delta_{\alpha\bar{\mu}}^{(p)\bar{\nu}} = \frac{1}{(p-1)!} \Delta_{\alpha\bar{\mu}\bar{\lambda}_1 \dots \bar{\lambda}_{p-1}}^{(p)} \bar{\nu} dx^{\bar{\lambda}_1 \dots \bar{\lambda}_{p-1}}$. For $p = 0$, the third and last terms are not present. We have also used

$$\begin{aligned}
 &\frac{\alpha'}{4} \text{Tr} \left(\delta_\alpha \theta^{(p)} \star \delta_{\bar{\beta}} \theta^{\dagger(p)} \right) = \\
 &\frac{\alpha'}{2} (-1)^{p+1} \left\{ \frac{1}{4} \left(Z_\alpha^{(p)} \right)_{\mu\bar{\delta}} \star \left(\bar{Z}_\beta^{(p)} \right)_{\sigma\bar{\tau}} R^{\mu\bar{\delta}\sigma\bar{\tau}} + \left(\Delta_\alpha^{(p)} \right)_{\bar{\mu}}{}^\nu \star \left(\Delta_\beta^{(p)} \right)_\sigma{}^{\bar{\tau}} R^{\bar{\mu}\sigma}{}_{\nu\bar{\tau}} \right\}.
 \end{aligned}$$

Using the derivative operator in (5.10) and the metric above, we compute the adjoint following the logic of section 4.2. Many of the terms follow through identically. For ease of notation, we now drop the tilde on $Z_\alpha^{(p)}$. The adjoint is

$$\begin{aligned}
 &\langle \bar{D} \mathcal{Y}_\alpha^{(p)}, \mathcal{Y}_\beta^{(p+1)} \rangle \\
 &= \frac{1}{V} \int_X \left\{ \frac{1}{4} \left[-\bar{\partial} Z_{\alpha\nu}^{(p)} + \frac{\alpha'}{2} \text{Tr} \left(\delta_\alpha \mathcal{A}^{(p)} F_\nu \right) + 2i \Delta_\alpha^{(p)\mu} (\partial\omega)_{\mu\nu} - \alpha' \nabla_\sigma \Delta_\alpha^{(p)\tau} R_{\nu\bar{\lambda}}{}^\sigma{}_\tau dx^{\bar{\lambda}} \right] \star \bar{Z}_\beta^{(p+1)\nu} \right. \\
 &\quad + \frac{\alpha'}{4} \bar{\partial} \left(\nabla^\sigma Z_{\alpha\sigma}^{(p)} \right) \star \left(\nabla^{\bar{\tau}} \bar{Z}_{\beta\bar{\tau}}^{(p+1)} \right) + \frac{\alpha'}{8} (-1)^p R^{\sigma\bar{\lambda}\tau\bar{\nu}} \left(-\bar{\partial} Z_\alpha^{(p)} \right)_{\sigma\bar{\lambda}} \star \bar{Z}_{\beta\tau\bar{\nu}}^{(p)} \\
 &\quad - \frac{\alpha'}{4} \text{Tr} \left[\left(\bar{\partial}_\mathcal{A} \delta_\alpha \mathcal{A}^{(p)} + F_\mu \Delta_\alpha^{(p)\mu} \right) \star \delta_{\bar{\beta}} \mathcal{A}^{\dagger(p+1)} \right] \\
 &\quad \left. + \left(\bar{\partial} \Delta_\alpha^{(p)} \right)^\mu \star \Delta_\beta^{(p+1)\bar{\nu}} g_{\mu\bar{\nu}} + \frac{\alpha'}{2} (-1)^{p+1} R^{\bar{\mu}\rho}{}_{\nu\bar{\sigma}} \left(\bar{\partial} \Delta_\alpha^{(p)} \right)_{\bar{\mu}}{}^\nu \star \left(\Delta_\beta^{(p+1)} \right)_\rho{}^{\bar{\sigma}} \right\} \\
 &= \frac{1}{V} \int_X \left\{ \frac{1}{4} Z_{\alpha\nu}^{(p)} \star \left[-\partial^\dagger \bar{Z}_\beta^{(p+1)\nu} + \frac{\alpha'}{2} (-1)^p R^{\tau\bar{\sigma}\nu\bar{\lambda}} \nabla_{\bar{\lambda}} \bar{Z}_{\beta\tau\bar{\sigma}}^{(p+1)} \right] \right. \\
 &\quad - \frac{\alpha'}{4} \nabla^\mu Z_{\alpha\mu}^{(p)} \star \nabla^{\bar{\tau}} \left(-\partial^\dagger \bar{Z}_\beta^{(p+1)} \right)_{\bar{\tau}} + \frac{\alpha'}{8} (-1)^p R^{\tau\bar{\nu}\sigma\bar{\delta}} Z_{\alpha\sigma\bar{\delta}}^{(p)} \star \left(-\partial^\dagger Z_{\beta\tau\bar{\nu}}^{(p+1)} \right) \\
 &\quad - \frac{\alpha'}{4} \text{Tr} \left(\delta_\alpha \mathcal{A}^{(p)} \star \left(\partial_{\mathcal{A}^\dagger}^\dagger \delta_{\bar{\beta}} \mathcal{A}^{\dagger(p+1)} + \frac{1}{2} (-1)^{p+1} F_{\mu\bar{\nu}} Z_\beta^{\mu\bar{\nu}(p+1)} \right) \right) \\
 &\quad + \Delta_\alpha^{(p)\mu} \star \left[(-1)^p \frac{i}{2} (\partial\omega)^{\rho\bar{\nu}\bar{\lambda}} \bar{Z}_{\beta\rho\bar{\nu}}^{(p+1)} - \frac{\alpha'}{4} \text{Tr} F^{\bar{\lambda}\nu} \delta_{\bar{\beta}} \mathcal{A}_\nu^{\dagger(p+1)} + \partial^\dagger \Delta_\beta^{(p+1)\bar{\lambda}} \right. \\
 &\quad - \frac{\alpha'}{4} (-1)^p R^{\bar{\lambda}\nu\sigma\bar{\tau}} \nabla_\nu \bar{Z}_{\beta\sigma\bar{\tau}}^{(p+1)} + \frac{\alpha'}{2} (-1)^p R^{\bar{\lambda}\rho}{}_{\mu\bar{\sigma}} \nabla_{\bar{\lambda}} \Delta_{\beta\rho}^{(p+1)\bar{\sigma}} \left. \right] g_{\mu\bar{\lambda}} \\
 &\quad \left. + \frac{\alpha'}{2} (-1)^{p+1} \Delta_{\alpha\bar{\nu}}^{(p)} \star \left(\partial^\dagger \Delta_\beta^{(p+1)} \right)_\rho{}^{\bar{\sigma}} R_{\mu\bar{\sigma}}{}^{\rho\bar{\nu}} \right\} \\
 &= \langle \mathcal{Y}_\alpha^{(p)}, \bar{D}^\dagger \mathcal{Y}_\beta^{(p+1)} \rangle.
 \end{aligned}$$

We compare with the metric (5.14) and work to first order in α^λ to deduce the action of the adjoint operator on sections

$$\overline{\mathcal{D}}^\dagger \mathcal{Y}_\alpha^{(p+1)} = \begin{pmatrix} \widehat{\mathcal{Z}}_\alpha^{(p)} \\ \widehat{\delta}_\alpha \mathcal{A}^{(p)} \\ \widehat{\Delta}_\alpha^{(p)\mu} \end{pmatrix}, \tag{5.15}$$

where

$$\begin{aligned} \widehat{\mathcal{Z}}_\alpha^{(p)} &= -\overline{\partial}^\dagger \mathcal{Z}_\alpha^{(p+1)\bar{\nu}} - \frac{\alpha^\lambda}{2} (-1)^p R^{\tau\bar{\sigma}\lambda\bar{\nu}} \nabla_\lambda \mathcal{Z}_{\alpha\tau\bar{\sigma}}^{(p+1)}, \\ \widehat{\delta}_\alpha \mathcal{A}^{(p)} &= \overline{\partial}_\mathcal{A}^\dagger \delta_\alpha \mathcal{A}^{(p+1)} + \frac{1}{2} (-1)^p F^{\rho\bar{\nu}} \mathcal{Z}_{\alpha\rho\bar{\nu}}^{(p+1)}, \\ \widehat{\Delta}_\alpha^{(p)\mu} &= (-1)^{p+1} \frac{i}{2} (\overline{\partial}\omega)^{\mu\rho\bar{\lambda}} \mathcal{Z}_{\alpha\rho\bar{\lambda}}^{(p+1)} + \frac{\alpha^\lambda}{4} \text{Tr} \left(F^{\mu\bar{\rho}} \delta_\alpha \mathcal{A}_\rho^{(p+1)} \right) - \nabla^{\text{Ch/H}\bar{\nu}} \Delta_{\alpha\bar{\nu}}^{(p)\mu} \\ &\quad + \frac{\alpha^\lambda}{2} (-1)^p R^\mu{}_\rho{}^{\sigma\bar{\tau}} \nabla_\sigma \Delta_{\alpha\bar{\tau}}^{(p)\rho} - \frac{\alpha^\lambda}{4} (-1)^p R^{\mu\bar{\rho}\sigma\bar{\tau}} \nabla_{\bar{\rho}} \mathcal{Z}_\alpha^{(p)}{}_{\sigma\bar{\tau}}. \end{aligned} \tag{5.16}$$

That is, the first line is a component of the balanced equation (2.11), after the redefinition (5.6); the second line is the vanishing of the HYM equation (2.14); the third line is the other component of the balanced equation (2.13), where we use (1.1).

We observe that the case $\overline{\mathcal{D}}^\dagger \mathcal{Y}_\alpha^{(1)} = 0$ amounts to precisely the D-terms!

6 Conclusion and outlook

So we have shown that the physical field deformations of the Hull-Strominger system, in holomorphic gauge, are given by harmonic representatives of the $\overline{\mathcal{D}}$ operator on the new \mathcal{Q} bundle, with the spurious degrees of freedom eliminated. These are harmonic with respect to the inner product on \mathcal{Q} derived from the moduli space metric in [19]. We see

$$\overline{\mathcal{D}} \mathcal{Y}_\alpha^{(1)} = 0 \iff \text{F-terms}, \quad \overline{\mathcal{D}}^\dagger \mathcal{Y}_\alpha^{(1)} = 0 \iff \text{D-terms}.$$

Along the way we also demonstrated that the non-physical calculation, in which one has additional spurious degrees of freedom, are captured by a $\overline{\mathcal{D}}$ -operator with an analogous relation between the F-terms and D-terms and $\overline{\mathcal{D}}$ -closure and $\overline{\mathcal{D}}$ -coclosure. That means that with respect to the $\overline{\mathcal{D}}$ -operator we could equally well call holomorphic gauge, harmonic gauge.

It would be interesting to reconcile these results with the complexified gauge transformations and moment maps in a GIT-type analysis of these solutions studied in [16, 41]. There are also examples of mathematical interest, such as that related to the Hopf surface which do not have an obvious valid α^λ -expansion, yet appear to be governed by a (0, 2) superconformal algebra [42]. Nonetheless, one could still try to construct the adjoint operator $\overline{\mathcal{D}}^\dagger$ and compare with the space of deformations obtained. It would also be interesting to see to what extent the structure discovered in this paper extends to higher orders in α^λ . We expect the generalised geometry approach of [11, 15, 16, 41, 43] to be relevant for these scenarios.

Another avenue of interest is to investigate further the properties of this new found operator $\bar{\mathcal{D}}$. As mentioned above, $\bar{\mathcal{D}}$ is not a connection on \mathcal{Q} as it fails to satisfy the Leibniz rule. However, one can still investigate its properties like ellipticity, Hodge theory, etc. [33]. One could also ask about higher orders in deformation theory. Presumably at second and third order in deformation theory there are relations with the L_3 algebra described in [17]. As noted there and elsewhere, with the spurious modes included there is a close relation with generalised geometry, as the connection $\bar{\mathcal{D}}$ forms a natural part of a generalised holomorphic Courant algebroid. This connection is however lost for the operator $\bar{\mathcal{D}}$.¹⁵ In spite of this, we do believe it is important to understand the moduli problem without the spurious modes included. In particular, when we inevitably come to analysing the quantum moduli problem. For example, the action constructed in [17] hints towards a (quasi-)topological theory for heterotic geometries, in close analogy with holomorphic Chern-Simons theory. One may hope to quantise this theory and define a notion of topological invariants and enumerative geometry for heterotic geometries, which would generalise the well-known Donaldson-Thomas invariants [44]. However, if we ignore the subtleties of fluctuating the spurious modes, we run the risk of introducing unwanted anomalies, effective couplings, etc. Moreover, as pointed out in appendix C, the spurious modes come with a negative definite kinetic term in the supergravity action, and ignoring this can lead to all sorts of craziness in the quantum theory. It may be possible to handle the extra modes using some sophisticated Lagrange-multiplier technique or Palatini formalism. However, string theory has told us what the correct degrees of freedom are, and so it might be wise to stick to them.

Finally, it is known in the study of CY manifolds that special geometry involves relations between the moduli and matter sectors. It would be interesting to extend these results to the charged matter sector utilising the matter metric [14]. The results here are reminiscent of the study of the tt^* equations of the special geometry of CY manifolds [45], in which one studies a fiber bundle whose base manifold is the moduli space and fibres are harmonic elements of the cohomologies $H^{(2,1)} \oplus H^{(1,1)}$. Until now, we haven't been able to demonstrate a cohomology with harmonic representative capturing the deformations of the full heterotic theory. This might be a new door worth opening.

Acknowledgments

We would like to thank L. Álvarez Cónsul, A. Ashmore, A. Coimbra, X. de la Ossa, M. Garcia Fernandez, R. González Molina, M. Larfors, M. Magill, R. Sisca, G. Smith, C. Strickland-Constable, G. Tartaglino-Mazzucchelli, R. Tellez, D. Tennyson, and D. Waldram for useful discussions.

A Some useful relations

A.1 Curvatures and Bianchi identity

The Bianchi identity for a curvature R evaluated with a connection Θ is

$$dR^m{}_n + \Theta^m{}_p R^p{}_n - R^m{}_p \Theta^p{}_n = 0, \quad \Leftrightarrow \quad d_\Theta R = 0. \tag{A.1}$$

¹⁵We thank the JHEP referee for making this remark.

For now, we take R to be the Riemann curvature tensor. It has symmetry properties

$$R_{mnpq} = -R_{mnqp} = -R_{nmpq} = R_{pqmn}.$$

We have need for commutators of derivatives to lowest order in α' e.g.

$$[\nabla^\mu, \nabla^{\bar{\tau}}]Z_{\alpha\mu\bar{\lambda}} = -R^\delta{}_\mu{}^{\mu\bar{\tau}}Z_{\alpha\delta\bar{\lambda}} - R^{\bar{\delta}}{}_{\bar{\lambda}}{}^{\mu\bar{\tau}}Z_{\alpha\mu\bar{\delta}} = \text{Ric}^{\delta\bar{\tau}}Z_{\alpha\delta\bar{\lambda}} - R^{\bar{\delta}}{}_{\bar{\lambda}}{}^{\mu\bar{\tau}}Z_{\alpha\mu\bar{\delta}}.$$

It involves both the Riemann curvature and Ricci tensor $\text{Ric}^{\delta\bar{\tau}} = -R^\delta{}_\mu{}^{\mu\bar{\tau}}$: the Ricci tensor vanishes to this order in α' .

It is useful to note that

$$2i\bar{\partial}(\bar{\partial}\omega)_{\mu\nu} = 2i(\bar{\partial}\partial\omega)_{\mu\nu} = -\frac{\alpha'}{4}\text{Tr}(F^2)_{\mu\nu} = \frac{\alpha'}{2}\text{Tr}F_\mu F_\nu,$$

where we omit the $\text{Tr}R^2$ term as it follows in the obvious way with a minus sign.

A.2 Hodge dual relations

It is useful to recount from the appendix of [19] adjoint differential operators and some Hodge dual relations for forms on X .

The Hodge \star operator acts on type

$$\star : \Omega^{(p,q)}(X) \rightarrow \Omega^{(N-q,N-p)}(X).$$

Given k -forms η, ξ and a metric $ds^2 = g_{mn}dx^m \otimes dx^n$ on X , the Hodge dual defines an inner product

$$(\cdot, \cdot) : \Omega^k(X) \times \Omega^k(X) \rightarrow \mathbb{R},$$

with

$$(\eta, \xi) = \frac{1}{V} \frac{1}{k!} \int_X d^6x \sqrt{g} \eta^{m_1 \dots m_k} \xi_{m_1 \dots m_k}.$$

Now consider two forms η_k and ξ_l , where the subscript denotes their degree and $k \leq l$. Contraction is

$$\lrcorner : \Omega^k(X) \times \Omega^l(X) \rightarrow \Omega^{l-k}(X),$$

and acts as follows

$$\eta_k \lrcorner \xi_l = \frac{1}{k!(l-k)!} \eta^{m_1 \dots m_k} \xi_{m_1 \dots m_k n_1 \dots n_{l-k}} dx^{n_1 \dots n_{l-k}} = \frac{1}{k!} \eta^{m_1 \dots m_k} \xi_{m_1 \dots m_k}.$$

An interesting feature of this operator is that it is the adjoint of the wedge product

$$(\sigma_{l-k} \lrcorner \xi_l, \eta_k) = (\xi_l, \sigma_{l-k} \wedge \eta_k). \tag{A.2}$$

Recall, the de Rahm operator d can be written

$$d = dx^m \nabla_m^{\text{LC}}.$$

Using (A.2) and integration by parts

$$(d\eta, \xi) = (dx^m \nabla_m^{\text{LC}} \eta, \xi) = (\nabla_m^{\text{LC}} \eta, \xi^m) = (\eta, -\nabla_m^{\text{LC}} \xi^m).$$

It follows that

$$d^\dagger \xi_k = -\nabla_m^{\text{LC}} \xi^m = -\frac{1}{(k-1)!} \nabla_n^{\text{LC}} \xi^n_{m_1 \dots m_{k-1}} dx^{m_1 \dots m_{k-1}}.$$

The de Rham differential splits into the sum of Dolbeault operators $d = \partial + \bar{\partial}$. Analogously, the codifferential also splits $d^\dagger = \partial^\dagger + \bar{\partial}^\dagger$ where

$$\begin{aligned} \partial^\dagger : \Omega^{(p,q)}(X) &\rightarrow \Omega^{(p-1,q)}(X), & \partial^\dagger &= -\star \bar{\partial} \star, \\ \bar{\partial}^\dagger : \Omega^{(p,q)}(X) &\rightarrow \Omega^{(p,q-1)}(X), & \bar{\partial}^\dagger &= -\star \partial \star. \end{aligned} \tag{A.3}$$

One-forms, type (1, 0):

$$\star \eta^{(1,0)} = -i \eta^{(1,0)} \frac{\omega^2}{2}. \tag{A.4}$$

Two-forms, types (2, 0) and (1, 1):

$$\begin{aligned} \star \eta^{(2,0)} &= \eta^{(2,0)} \omega, \\ \star \eta^{(1,1)} &= -i \eta_\mu^\mu \frac{\omega^2}{2} - \eta^{(1,1)} \omega = (\omega \lrcorner \eta^{(1,1)}) \frac{\omega^2}{2} - \eta^{(1,1)} \omega, \end{aligned} \tag{A.5}$$

Three-forms, types (3, 0) and (2, 1):

$$\begin{aligned} \star \eta^{(3,0)} &= -i \eta^{(3,0)}, \\ \star \eta^{(2,1)} &= i \eta^{(2,1)} - \eta_\mu^\mu \eta^{(1,0)} \omega = i \eta^{(2,1)} - i (\omega \lrcorner \eta^{(2,1)}) \omega, \end{aligned} \tag{A.6}$$

Type (2, 3):

$$\star \eta^{(2,3)} = \frac{i}{2} \eta_{\mu\nu}^{\mu\nu} \eta^{(0,1)} = \frac{i}{2} \omega \lrcorner (\omega \lrcorner \eta^{(2,3)}). \tag{A.7}$$

B Summary of notation of supergravity deformations

We give a brief summary of some notation in describing deformations of heterotic supergravity to first order in α' .

We consider heterotic supergravity, correct to first order in α' , on $\mathbb{R}^{3,1} \times X$, where X is a complex manifold. There is moduli space \mathcal{M} with real coordinates denoted y^a . The moduli space is complex, in fact it is Kähler, and so there is a choice of holomorphic coordinates which we denote $y^a = (y^\alpha, y^{\bar{\beta}})$. For each $y \in \mathcal{M}$ there is a solution to the quantum corrected supergravity equations of motion which we denote $[\mathcal{X}, \mathcal{E}, H]$ and refer to this as a heterotic structure. If we pick a $y \in \mathcal{M}$ and consider small deformations in the neighbourhood $y \rightarrow y + \delta y$ then the corresponding data in the heterotic solution is modified, for example the complex structure is deformed $J \rightarrow J + \delta J$. We need to relate the field deformations, e.g. δJ , to the deformations of the parameters δy^a and this is done by considering derivatives. The details of this can be found in the references [12, 14, 18], but we very briefly state some results here in order to be complete.

First, consider the integrable complex structure J and with corresponding complex coordinates $x^m = (x^\mu, x^{\bar{\nu}})$. Deformations can be expressed in terms of the undeformed complex structure as

$$\delta J = \delta J_{\bar{\nu}}^\mu dx^{\bar{\nu}} \otimes \partial_\mu + \delta J_{\nu}^{\bar{\mu}} dx^\nu \otimes \partial_{\bar{\mu}}.$$

It is convention to express this in terms of a tensor and to relate the field deformation to small changes in parameters y^a :

$$\delta J_{\bar{\nu}}^\mu = \delta y^a (2i \Delta_a \bar{\nu}^\mu).$$

We often think of deformations of fields as appropriate derivatives with respect to parameters. For example, in this case, this would mean $\partial_a J_{\mu}^{\bar{\nu}} = 2i \Delta_a \bar{\nu}^\mu$. Demanding the Nijenhuis tensor be preserved implies $\bar{\partial} \Delta_a^\mu = 0$, in which we understand $\Delta_a^\mu \cong \Delta_a \bar{\nu}^\mu dx^{\bar{\nu}}$. The parameters, collectively denoted y^a , are also coordinates for a manifold M , the moduli space.

There is a compatible hermitian form ω . A deformation of ω can be written as

$$\delta \omega^{(2,0)} = \delta y^a \Delta_a^{\bar{\mu}} \omega_{\bar{\mu}}, \quad \delta \omega^{(1,1)} = \delta y^a (\partial_a \omega)^{(1,1)}, \quad \delta \omega^{(0,2)} = \delta y^a \Delta_a^\mu \omega_\mu, \quad \omega_m = \omega_{mnd} dx^n.$$

The gauge field A transforms under gauge symmetries

$$A \rightarrow {}^\Phi A = \Phi A \Phi^{-1} - (d\Phi) \Phi^{-1}, \quad (\text{B.1})$$

and under the background gauge principle its deformation transforms homogenously $\delta \mathcal{A} \rightarrow \Phi \delta \mathcal{A} \Phi^{-1}$. To relate this deformation and its gauge property to the coordinates of the moduli space one introduces covariant derivatives with respect to parameters

$$\delta \mathcal{A} = \delta y^a \mathfrak{D}_a \mathcal{A},$$

where

$$\mathfrak{D}_a A = \partial_a A - d_A \Lambda_a, \quad \text{and where } d_A \Lambda_a = d\Lambda_a + [A, \Lambda_a]. \quad (\text{B.2})$$

Here $\Lambda_a dy^a$ is a connection on the moduli space and transforms appropriately under gauge transformations

$$\Lambda_a \rightarrow {}^\Phi \Lambda_a = \Phi \Lambda_a \Phi^{-1} - (\partial_a \Phi) \Phi^{-1}. \quad (\text{B.3})$$

When it does so, the covariant derivative transforms homogeneously: ${}^\Phi \mathfrak{D}_a \mathcal{A} = \Phi \mathfrak{D}_a \mathcal{A} \Phi^{-1}$.

Consider the three-form H

$$H = dB - \frac{\alpha'}{4} (\text{CS}[A] - \text{CS}[\Theta]), \quad \text{CS}[A] = \text{Tr} \left(\text{Ad}A + \frac{2}{3} A^3 \right), \quad (\text{B.4})$$

defined so that it satisfies the Bianchi identity. Here Θ is the gauge potential for frame transformations.¹⁶ Under background gauge transformations

$${}^\Phi B = B - \frac{\alpha'}{4} (\text{Tr}(AY) - \text{Tr}(\Theta Z) - U), \quad \frac{1}{3} \text{Tr}(Y^3) = dU, \quad (\text{B.5})$$

¹⁶Note that Θ is a field dependent object, but we leave it arbitrary for this appendix.

where $Y = \Phi^{-1}d\Phi$ and $Z = \Psi^{-1}d\Psi$, where Φ is a gauge transformation and Ψ a frame transformation of \mathcal{T}_X . A variation of H is

$$\delta H = \delta y^a \left(d\mathcal{B}_a - \frac{\alpha'}{2} \text{Tr}(\mathfrak{D}_a A F) + \frac{\alpha'}{2} \text{Tr}(\mathfrak{D}_a \Theta R) \right), \quad (\text{B.6})$$

where \mathcal{B}_a is well-defined two-form defined as

$$\mathcal{B}_a = \mathfrak{D}_a B + \frac{\alpha'}{4} \text{Tr}(A \mathfrak{D}_a A) - \frac{\alpha'}{4} \text{Tr}(\Theta \mathfrak{D}_a \Theta) - d\mathbb{B}_a, \quad (\text{B.7})$$

$$\mathfrak{D}_a B = \partial_a B - \frac{\alpha'}{4} \text{Tr}(\Lambda_a dA) + \frac{\alpha'}{4} \text{Tr}(\Pi_a dA). \quad (\text{B.8})$$

Here Π is the connection on the moduli space responsible for frame transformations, and the field \mathbb{B}_a is a 1-form on X .

It is convenient to form the complexified combinations

$$\mathcal{Z}_a = \mathcal{B}_a + i \partial_a \omega, \quad \bar{\mathcal{Z}}_a = \mathcal{B}_a - i \partial_a \omega,$$

where we denote $\mathfrak{D}_a \omega^{(p,q)} = (\partial_a \omega)^{(p,q)}$ while on a real form $\mathfrak{D}_a \omega = \partial_a \omega$. \mathcal{Z}_a is the heterotic analogue of a complexified Kähler deformation in the theory of Calabi-Yau manifolds.

C A supergravity no-go theorem

No-go theorems are well-known in the context of type II and M-theory flux vacua. Classical supergravity equations of motion, when integrated over a compact manifolds, force all non-trivial fluxes to vanish. The evasion of these theorems comes in the form of α' -corrections to supergravity, interpreted as the contribution of orientifold planes, objects with negative stress-energy tensor. Here we briefly recount an analogous result in heterotic string theory; this demonstrates that if one wants solutions of supergravity then $H = \mathcal{O}(\alpha')$.

In this section only we work to α'^3 , as it is not complicated to include the α'^2 correction. The equations of motion are given by

$$\begin{aligned} \mathcal{R} - 4(\nabla\Phi)^2 + 4\nabla^2\Phi - \frac{1}{2}|H|^2 - \frac{\alpha'}{4} \left(\text{Tr}|F|^2 - \text{Tr}|R(\Theta^+)|^2 \right) + \mathcal{O}(\alpha'^3) &= 0, \\ \mathcal{R}_{MN} + 2\nabla_M\nabla_N\Phi - \frac{1}{4}H_{MAB}H_N{}^{AB} - \frac{\alpha'}{4} \left(\text{Tr}F_{MP}F_N{}^P \right. \\ &\quad \left. - R_{MPAB}[\Theta^+]R_N{}^{PAB}[\Theta^+] \right) + \mathcal{O}(\alpha'^3) &= 0, \\ \nabla^M \left(e^{-2\Phi} H_{MNP} \right) + \mathcal{O}(\alpha'^3) &= 0, \\ \mathcal{D}^{-M} \left(e^{-2\Phi} F_{MN} \right) + \mathcal{O}(\alpha'^3) &= 0, \end{aligned} \quad (\text{C.1})$$

here $M, N = 0, \dots, 9$, $\mathcal{D}^- = \nabla^- + [A, \cdot]$, with ∇^- computed with respect to the Θ^- connection, and \mathcal{R}_{MN} is the Ricci tensor. We restrict to the internal manifold X directions and by manipulating the equations above we can write

$$\nabla^2 \left(e^{-2\Phi} \right) = e^{-2\Phi} |H|^2 + \frac{\alpha'}{4} e^{-2\Phi} \left(\text{Tr}|F|^2 - \text{Tr}|R(\Theta^+)|^2 \right) + \mathcal{O}(\alpha'^3). \quad (\text{C.2})$$

This is actually the form of the equation of motion derived in [46] by computing the two-loop beta function.

Integrating both sides over X , using that X is compact, we see

$$\int d^6x \sqrt{g} |e^{-\Phi} H|^2 + \frac{\alpha'}{4} \int_X d^6x \sqrt{g} \left(\text{Tr} |F|^2 - \text{Tr} |R(\Theta^+)|^2 \right) = 0.$$

As $|H|^2$ is positive definite, we immediately see that $H = \mathcal{O}(\alpha')$. So if we want to study solutions of string theory in which the Hull-Strominger system is a well-behaved description, then $H = \mathcal{O}(\alpha')$. From this, we can use the gauge fixing to set the dilaton to be a constant and we are in the regime studied in this paper.

If we didn't have the contribution of the last term there would be no solution at all — this is the classical supergravity no-go theorem. We evade this because of the Green-Schwarz anomaly cancellation condition between the classical supergravity contributions and the α' -correction involving the curvature squared terms. In certain backgrounds heterotic theories are dual to type IIB theories, and this contribution can be understood as coming from the contribution of $O7$ -planes in type IIB, which carry negative energy density. This explains physically why the contribution from the $\text{Tr} R^2$ term in the moduli space metric appears with the opposite sign from all the other terms. That is [12],

$$g_{\alpha\bar{\beta}}^\# = \frac{1}{V} \int_X \left(\frac{1}{4} Z_\alpha^{(1,1)} \star \bar{Z}_\beta^{(1,1)} + \frac{\alpha'}{4} \text{Tr} \left(\mathfrak{D}_\alpha A \star \mathfrak{D}_{\bar{\beta}} \mathcal{A} \right) + \Delta_{\alpha^\mu} \star \Delta_{\bar{\beta}^\nu} g_{\mu\nu} - \frac{\alpha'}{4} \text{Tr} \left(\mathfrak{D}_\alpha \Theta \star \mathfrak{D}_{\bar{\beta}} \Theta \right) \right).$$

This does not mean the moduli space metric has indefinite signature: it is constructed in supergravity and so necessarily $\alpha' \ll 1$ meaning the last term, no matter its value, cannot change the signature of the metric. Even after substituting for the spurious degrees of freedom, the same logic will apply.

If one wanted to study solutions where α' is large, then if we want it to be relevant at all to string theory (along with all its beautiful results such as mirror symmetry, special geometry and sigma models) then we need to include the higher order α' -corrections. It is already known that there is an α'^2 correction to the hermitian sector of the metric [31]. As pointed out in that paper, there are going to be α'^3 corrections, such as the α'^3 correction to the moduli space metric calculated using mirror symmetry [47] at the standard embedding

$$\begin{aligned} K_{\alpha^3} &= -\log \left(\frac{4}{3} \int_X \omega^3 + 8\alpha'^3 \zeta(3) \chi(X) \right) \\ &\approx -\log \left(\frac{4}{3} \int_X \omega^3 \right) + 8\alpha'^3 \frac{\zeta(3) \chi(X)}{V} + \dots, \end{aligned} \tag{C.3}$$

where V is the volume of X , ζ is the Riemann zeta function and χ the Euler number. Presumably, if we understood $(0,2)$ -mirror symmetry, we could determine all the α' -corrections to the moduli space metric.

With all deformations turned on, the Kähler potential for the moduli space metric $g_{\alpha\bar{\beta}}^\#$ was first shown to be [12]

$$K = -\log \left(\frac{4}{3} \int_X \omega^3 \right) - \log \left(i \int_X \Omega \bar{\Omega} \right). \tag{C.4}$$

It is remarkable that this also encodes the deformations of the vector bundle despite involving quantities which naively belong to X .¹⁷ In the gauge in which the dilaton is constant [31], the norm $\|\Omega\|$ is constant over X and so the compatibility relation between Ω and ω is

$$i \int_X \Omega \bar{\Omega} = \frac{1}{3!} \int_X \|\Omega\|^2 \omega^3.$$

Substituting into (C.4) we see that

$$K = -\log \left(\int_X \|\Omega\| \omega^3 \right), \tag{C.5}$$

where we have dropped irrelevant numerical constants.

We now show this is equivalent to the four-dimensional dilaton as written in Einstein frame. Indeed, in Einstein frame of the dimensional reduction of heterotic supergravity to $\mathbb{R}^{3,1}$, the ten-dimensional dilaton is written¹⁸

$$\Phi = \Phi_0 + \varphi(X^e) + \alpha'^3 \Phi_3(X, x) + \dots,$$

where $X^e = (X^0, \dots, X^3)$ are the $\mathbb{R}^{3,1}$ coordinates and x^m are the coordinates of X . The first term is the zero mode and so a constant. There is a purely $d = 4$ fluctuation $\varphi(X)$ and the next non-zero term is Φ_3 which is cubic order in α' . The canonical definition of the four-dimensional dilaton, even for non-Kähler manifolds is (see for example [48]):

$$e^{-2\phi_4} = e^{-2\varphi(X^e)} \frac{1}{g_s^2 V_0} \int_X \frac{1}{3!} \omega^3 + \mathcal{O}(\alpha'^3),$$

where $g_s = e^{\Phi_0}$ is the zero-mode of the dilaton and is the string coupling constant. For the Hull-Strominger system to be a good approximation to string theory, we require $g_s \rightarrow 0$. V_0 is a reference volume of X , measured at any point in moduli space, and appears in order for the dilaton to be dimensionless.

A consequence of supersymmetry is that

$$d \log \|\Omega\| = -2d\Phi + \mathcal{O}(\alpha'^2), \tag{C.6}$$

where d is the ten-dimensional exterior derivative. We therefore get

$$e^{-2\phi_4} = \frac{1}{g_s^2 V_0} \int_X \frac{1}{3!} \|\Omega\| \omega^3 + \mathcal{O}(\alpha'^2).$$

Supersymmetry here is important. The four-dimensional dilaton appears in a combination with the universal axion — the B -field whose legs span $\mathbb{R}^{3,1}$: $S = b + ie^{-2\phi_4}$. In the language of $N = 1$, $d = 4$ supersymmetry S is a component of a linear multiplet.

We see that, up to irrelevant constants,

$$2\phi_4 = -\log \int_X \|\Omega\| \omega^3 - \log \left(\frac{i}{2} (\bar{S} - S) \right) + \mathcal{O}(\alpha'^2). \tag{C.7}$$

¹⁷There is additional term coming from the universal axio-dilaton which in the tradition of special geometry we do not write as it is completely decoupled from all the other parameters. We discuss it below.

¹⁸We again use that modulo an appropriate gauge fixing the dilaton is constant up to order α'^3 [31].

The second term we often do not include as the universal axio-dilaton is not coupled to any other fields at this order in α' and g_s and so its contribution to the moduli space metric is somewhat trivial [12].

As promised, we have shown that $K = 2\phi_4$ and the four-dimensional dilaton is the Kähler potential (C.4) first written in [12]. All we demanded here were the physically reasonable assumptions that supergravity is a good approximation, viz. $\alpha', g_s \rightarrow 0$.

Supersymmetry does not afford the dilaton functional any protection from α' corrections. The linear multiplet is a fully-fledged D-term and so will receive corrections at all orders in α' .¹⁹ That being so, we expect the first correction already at α'^2 . Many issues arise. It is not even clear, for example, if the four-dimensional dilaton and the Kähler potential for the moduli space metric at α'^2 are the same thing. Given the supergravity action and supersymmetry variations are known at this order, determining this correction is an interesting future direction. It would be very interesting to determine the α'^3 corrections as well, though the supersymmetry variations and equations of motion are not completely known at this order.

Recently, [16] took (C.7) as the Kähler potential for the moduli space metric to deformations of the first order α' Hull-Strominger system assuming the complex structure is fixed. They then related this to the Aeppli and Bott-Chern cohomology. In the limit $\alpha' \rightarrow 0$ with $H \rightarrow 0$, in which the Hull-Strominger system accurately describes string backgrounds, it reproduces [12] as we demonstrated above. Their metric also applies to formal solutions of the Hull-Strominger system one allow for α' to be large and for non-trivial dilaton with torsion $H = \mathcal{O}(1)$. However, as discussed above, such solutions are not obviously string solutions because of the non-trivial dilaton behaviour and because the $\alpha'^2, \alpha'^3, \dots$ corrections are not included. They do not find, for example, the α'^3 correction to the metric predicted from mirror symmetry in the special case of the standard embedding cf. (C.3). This term will dominate of the lower order contributions in such solutions. Furthermore, interesting pathologies can also arise in the large α' limit, such as the moduli space metric being degenerate and so not Kähler as required by $N = 1$ $d = 4$ supersymmetry. This reinforces the importance of including perturbative $\alpha'^2, \alpha'^3, \dots$ -corrections to the Hull-Strominger system — they will dominate over the lower order Hull-Strominger equations at string scale volumes. No doubt α' -corrections in the form of worldsheet instantons will play an even more important role as well — it is these that encode geometric invariants of interest in pure mathematics.

So if we want to apply the formalism of the Hull-Strominger system to examples where $H = \mathcal{O}(1)$ then we must necessarily include the α' -corrections and g_s -corrections to the supersymmetry variations, equations of motion and in turn the moduli space metric. Even if the dilaton is constant, determining how the α' -corrections modify the Kähler potential beyond the first order result in [12, 18] for heterotic theories remains an open question. These higher order terms will dominate and qualitatively change the behaviour of the metric. For example, such terms should presumably force the metric to remain positive definite even though we have a contribution at order α' with a negative sign. We know this must be

¹⁹We thank Gabriele Tartaglino-Mazzucchelli for explaining this to us.

the case as there are strong arguments from [2, 49] that at least in α' -perturbation theory these geometries define sigma models that flow to unitary conformal field theories with positive definite kinetic terms for the moduli fields, and so, positive definite metric.

The role of the dilaton is even more mysterious: if there are string solutions on compact manifolds, then they are not described by conventional sigma models. As pointed out in [50], the $(0, 2)$ sigma model for the Hopf surface — a simple geometric example in which the dilaton is non-trivial at zeroth order in α' — is not unitary and so the sigma model does not describe a physical string theory. In fact, the dilaton scalar is not even globally well-defined on the Hopf surface, so it is not clear how it solves the supergravity equations of motion (C.1) globally. Finding a suitable modification of this worldsheet theory and Hopf surface geometry so that we have a unitary theory flowing to a unitary $(0, 2)$ conformal field theory would be very interesting. Recently, [42] conjectured some of the algebraic structures such a theory should have — it would be very interesting to try to turn this into a description of a unitary conformal field theory.

D Constructing a family of $\bar{\mathcal{D}}$ operators

We construct a family of $\bar{\mathcal{D}}$ -operators on the Q bundle of $(0, p)$ -forms that square to zero and $\bar{\mathcal{D}}\mathcal{Y}_\alpha^{(1)} = 0$ corresponds to the F-terms; $\bar{\mathcal{D}}^\dagger\mathcal{Y}_\alpha^{(1)} = 0$ corresponds to the D-terms.

D.1 Warm-up: complex manifolds and holomorphic bundles

As a warm-up we start by setting $\mathcal{Z}_\alpha^{(p)} = 0$ and $\delta_\alpha\Theta^{(p)} = 0$. The F-term equations (2.7) restrict to

$$\bar{\partial}_{\mathcal{A}}(\mathfrak{D}_\alpha\mathcal{A}) + F_\mu\Delta_\alpha^\mu = 0, \quad \bar{\partial}\Delta_\alpha^\mu = 0, \tag{D.1}$$

while the D-term equations are (2.13)–(2.14).

$$0 = \bar{\partial}_{\mathcal{A}}^\dagger(\mathfrak{D}_\alpha\mathcal{A}), \quad 0 = -\frac{\alpha'}{4}\text{Tr}(F^{\nu\bar{\mu}}\mathfrak{D}_\alpha\mathcal{A}_{\bar{\mu}}) + \nabla^{\bar{\mu}}\Delta_{\alpha\bar{\mu}}{}^\nu. \tag{D.2}$$

We define \bar{D}_1 to be

$$\bar{D}_1 = \begin{pmatrix} \bar{\partial}_{\mathcal{A}} & \mathcal{F} \\ 0 & \tilde{\mu}_p\bar{\partial} \end{pmatrix}, \quad \mathcal{F}(\Delta_\alpha^p) = \tilde{\lambda}_p F_\mu\Delta_\alpha^{p\mu}.$$

with $\tilde{\lambda}_p, \tilde{\mu}_p$ arbitrary coefficients. We see that $\bar{D}_1^2 = 0$ amounts to

$$\bar{\partial}_{\mathcal{A}}\tilde{\lambda}_p F_\mu\Delta_\alpha^{p\mu} + \tilde{\lambda}_{p+1}\tilde{\mu}_p F_\mu\bar{\partial}\Delta_\alpha^{p\mu},$$

and so

$$-\tilde{\lambda}_p + \tilde{\lambda}_{p+1}\tilde{\mu}_p = 0. \tag{D.3}$$

We use the appropriate restriction of moduli space metric (4.5)

$$\langle \mathcal{Y}_\alpha^{(p)}, \mathcal{Y}_\beta^{(p)} \rangle = \frac{1}{V} \int_X \left(-\frac{\alpha'}{4} \text{Tr} \left(\delta_\alpha \mathcal{A}^{(p)} \star \delta_{\bar{\beta}} \mathcal{A}^{(p)\dagger} \right) + \Delta_\alpha^{(p)\mu} \star \Delta_{\bar{\beta}}^{(p)\bar{\nu}} g_{\mu\bar{\nu}} \right).$$

to compute the adjoint operator

$$\begin{aligned} \langle \bar{D}_1 \mathcal{Y}_\alpha^{(p)}, \mathcal{Y}_\beta^{(p+1)} \rangle &= \frac{1}{V} \int_{\mathcal{X}} \left\{ -\frac{\alpha'}{4} \text{Tr} \left(\delta_\alpha \mathcal{A}^{(p)} \star \left(\partial_{\mathcal{A}^\dagger}^\dagger \delta_{\bar{\beta}} \mathcal{A}^{\dagger(p+1)} \right) \right) \right. \\ &\quad \left. + \Delta_\alpha^{(p)\mu} \star \left(-\tilde{\lambda}_p \frac{\alpha'}{4} \text{Tr} F^{\bar{\lambda}\nu} \delta_{\bar{\beta}} \mathcal{A}_\nu^{\dagger(p+1)} + \tilde{\mu}_p \partial^\dagger \Delta_{\bar{\beta}}^{(p+1)\bar{\lambda}} \right) g_{\mu\bar{\lambda}} \right\} \\ &= \langle \mathcal{Y}_\alpha^{(p)}, \bar{D}_1^\dagger \mathcal{Y}_\beta^{(p+1)} \rangle, \end{aligned}$$

From this we read off

$$\bar{D}_1^\dagger \mathcal{Y}_\alpha^{(p+1)} = \begin{pmatrix} \bar{\partial}_{\mathcal{A}}^\dagger \delta_\alpha \mathcal{A}^{(p+1)} \\ \tilde{\lambda}_p \frac{\alpha'}{4} \text{Tr} F^{\mu\bar{\nu}} \delta_\alpha \mathcal{A}_{\bar{\nu}}^{(p+1)} - \tilde{\mu}_p \nabla^{\bar{\nu}} \Delta_{\alpha\bar{\nu}}^{(p+1)\mu} \end{pmatrix}.$$

We check that $(\bar{D}_1^\dagger)^2 = 0$. Indeed, the only non-trivial part is

$$(\bar{D}_1^\dagger)^2 = \left(-\tilde{\lambda}_{p-1} + \tilde{\lambda}_p \tilde{\mu}_{p-1} \right) \frac{\alpha'}{4} \text{Tr} F^{\mu\bar{\tau}} \nabla^{\bar{\lambda}} \delta_\alpha \mathcal{A}_{\bar{\tau}\bar{\lambda}}^{(p+1)} = 0,$$

and as expected we find exactly (D.3), the same condition on the coefficients as for $\bar{D}_1^2 = 0$. This is a good consistency check.

We now compare with the F-terms and D-terms. The F-terms amount to $\bar{D}_1 \mathcal{Y}_\alpha^{(1)} = 0$ and so this fixes $\tilde{\lambda}_1 = 1$. The D-term equation (D.2) corresponds to $\bar{D}_1^\dagger \mathcal{Y}_\alpha^{(1)} = 0$ and we see

$$\tilde{\lambda}_0 = \tilde{\mu}_0.$$

One simple solution is to pick $\tilde{\mu}_p = \tilde{\lambda}_p = 1$. We generalise this calculation to include the spin connection and hermitian terms.

D.2 Hull-Strominger with spurious degrees of freedom

We now do this for the entire Hull-Strominger system with the spurious degrees of freedom. We introduce a \bar{D} operator with p -graded coefficients in the same way. Its action in components is

$$\begin{aligned} \widehat{Z}_{\alpha\nu}^{(p+1)} &= \mu_p \bar{\partial} Z_{\alpha\nu}^{(p)} + 2i\varepsilon_p \Delta_\alpha^{(p)\mu} (\partial\omega)_{\mu\nu} + \frac{\alpha'}{2} \lambda_p \left(\text{Tr} (\delta_\alpha \mathcal{A}^{(p)} F_\nu) - \text{Tr} (\delta_\alpha \theta^{(p)} R_\nu) \right), \\ \widehat{\delta}_\alpha \mathcal{A}^{(p+1)} &= \bar{\partial}_{\mathcal{A}} \delta_\alpha \mathcal{A}^{(p)} + \tilde{\lambda}_p F_\mu \Delta_\alpha^{(p)\mu}, \\ \widehat{\delta}_\alpha \theta^{(p+1)} &= \bar{\partial}_\theta \delta_\alpha \theta^{(p)} + \tilde{\lambda}_p R_\mu \Delta_\alpha^{(p)\mu}, \\ \widehat{\Delta}_\alpha^{(p+1)} &= \tilde{\mu}_p \bar{\partial} \Delta_\alpha^{(p)}. \end{aligned} \tag{D.4}$$

As a matrix we write this as

$$\bar{D} = \begin{pmatrix} \mu_p \bar{\partial} & \mathcal{H}_p \\ 0 & \bar{D}_2 \end{pmatrix}, \tag{D.5}$$

where $\mathcal{H}_p(\Delta_\alpha^{(p)}, \delta_\alpha \mathcal{A}^{(p)}) = 2i\varepsilon_p \Delta_\alpha^{(p)\mu} (\partial\omega)_{\mu\nu} + \frac{\alpha^\lambda}{2} \lambda_p \left(\text{Tr}(\delta_\alpha \mathcal{A}^{(p)} F_\nu) - \text{Tr}(\delta_\alpha \theta^{(p)} R_\nu) \right)$ and \bar{D}_2 is the natural generalisation of \bar{D}_1 taken to act on both $\delta \mathcal{A}^{(p)}$ and $\delta \theta^{(p)}$.

$$\bar{D} \begin{pmatrix} Z_{\alpha\nu}^{(p)} \\ \delta_\alpha \theta^{(p)} \\ \delta_\alpha \mathcal{A}^{(p)} \\ \Delta_\alpha^{(p)} \end{pmatrix} = \begin{pmatrix} -\bar{\partial} Z_{\alpha\nu}^{(p)} + 2i\Delta_\alpha^{(p)\mu} (\partial\omega)_{\mu\nu} + \frac{\alpha^\lambda}{2} \left(\text{Tr}(\delta_\alpha \mathcal{A}^{(p)} F_\nu) - \text{Tr}(\delta_\alpha \theta^{(p)} R_\nu) \right) \\ \bar{\partial}_\theta (\mathfrak{D}_\alpha \theta) + \mathcal{R}(\Delta_\alpha) \\ \bar{\partial}_{\mathcal{A}} (\mathfrak{D}_\alpha \mathcal{A}) + \mathcal{F}(\Delta_\alpha) \\ \bar{\partial} \Delta_\alpha \end{pmatrix}. \quad (\text{D.6})$$

We now check that $\bar{D}^2 = 0$. From the previous subsection we see that $\bar{D}_2^2 = 0$ amounts to $-\tilde{\lambda}_p + \tilde{\lambda}_{p+1} \tilde{\mu}_p = 0$. The rest of the calculation amounts to checking

$$\widehat{Z}_{\alpha\nu}^{(p+2)} = \mu_{p+1} \bar{\partial} \widehat{Z}_{\alpha\nu}^{(p+1)} + 2i\varepsilon_{p+1} \widehat{\Delta}_\alpha^{(p+1)\mu} (\partial\omega)_{\mu\nu} + \frac{\alpha^\lambda}{2} \lambda_{p+1} \left(\text{Tr}(\widehat{\delta}_\alpha \mathcal{A}^{(p+1)} F_\nu) - \text{Tr}(\widehat{\delta}_\alpha \theta^{(p+1)} R_\nu) \right) = 0,$$

We find this is the case if

$$\begin{aligned} \lambda_{p+1} + \lambda_p \mu_{p+1} &= 0, & \tilde{\lambda}_p - \tilde{\lambda}_{p+1} \tilde{\mu}_p &= 0, \\ \varepsilon_p \mu_{p+1} + \varepsilon_{p+1} \tilde{\mu}_p &= 0, & \mu_{p+1} \varepsilon_p + \lambda_{p+1} \tilde{\lambda}_p &= 0, \end{aligned} \quad (\text{D.7})$$

together with the Bianchi identity

$$2i\partial\bar{\partial}\omega = \frac{\alpha^\lambda}{4} \text{Tr} F^2 - \frac{\alpha^\lambda}{4} \text{Tr} R^2.$$

and we have included the condition for $\bar{D}_2^2 = 0$. Just as for $\bar{D}_1^2 = 0$ in the previous section, this calculation is offshell — we do not need to use the equations of motion.

We now compute the adjoint \bar{D}^\dagger using the metric (4.5).

$$\begin{aligned} &\langle \bar{D} \mathcal{Y}_\alpha, \mathcal{Y}_\beta \rangle \\ &= \frac{1}{4V} \int_X \left\{ \mu_p \bar{\partial} Z_{\alpha\nu}^{(p)} + 2i\varepsilon_p \Delta_\alpha^{(p)\mu} (\partial\omega)_{\mu\nu} + \frac{\alpha^\lambda}{2} \lambda_p \left(\text{Tr}(\delta_\alpha \mathcal{A}^{(p)} F_\nu) - \text{Tr}(\delta_\alpha \theta^{(p)} R_\nu) \right) \right\} \star \bar{Z}_\beta^\nu \\ &\quad - \frac{\alpha^\lambda}{4V} \int_X \left(\bar{\partial}_{\mathcal{A}} \delta_\alpha \mathcal{A}^{(p)} + \tilde{\lambda}_p F_\mu \Delta_\alpha^{(p)\mu} \right) \star \delta_{\bar{\beta}} \mathcal{A}^{\dagger(p+1)} + \frac{\alpha^\lambda}{4V} \int_X \left(\bar{\partial}_\theta \delta_\alpha \theta^{(p)} + \tilde{\lambda}_p R_\mu \Delta_\alpha^{(p)\mu} \right) \star \delta_{\bar{\beta}} \theta^{\dagger(p+1)} \\ &\quad + \frac{1}{V} \int \tilde{\mu}_p \bar{\partial} \Delta_\alpha^\mu \star \Delta_{\bar{\beta}}^{\dagger(p+1)}{}_\mu \\ &= \frac{1}{4V} \int_X Z_{\alpha\nu}^{(p)} \star \mu_p \partial^\dagger \bar{Z}_\beta^\nu - \frac{\alpha^\lambda}{4V} \int_X \delta_\alpha \mathcal{A}^{(p)} \star \left(\partial_{\mathcal{A}}^\dagger \delta_{\bar{\beta}} \mathcal{A}^{\dagger(p+1)} + (-1)^{p+1} \frac{\lambda_p}{2} F_{\nu\bar{\tau}} \bar{Z}_{\bar{\beta}}^{\nu\bar{\tau}} \right) \\ &\quad + \frac{\alpha^\lambda}{4V} \int_X \delta_\alpha \theta^{(p)} \star \left(\partial_\theta^\dagger \delta_{\bar{\beta}} \theta^{\dagger(p+1)} + (-1)^{p+1} \frac{\lambda_p}{2} R_{\nu\bar{\tau}} \bar{Z}_{\bar{\beta}}^{\nu\bar{\tau}} \right) \\ &\quad + \frac{1}{V} \int_X \Delta_\alpha^\mu \star \left\{ \frac{i}{2} \varepsilon_p (-1)^p (\partial\omega)_{\mu\nu\bar{\tau}} Z_{\bar{\beta}}^{\nu\bar{\tau}} - \tilde{\lambda}_p \frac{\alpha^\lambda}{4} \text{Tr} F_{\mu\bar{\tau}} \delta_{\bar{\beta}} \mathcal{A}^{\dagger\bar{\tau}} + \tilde{\lambda}_p \frac{\alpha^\lambda}{4} \text{Tr} R_{\mu\bar{\tau}} \delta_{\bar{\beta}} \theta^{\dagger\bar{\tau}} + \tilde{\mu}_p \partial^\dagger \Delta_{\bar{\beta}}^{\dagger(p+1)}{}_\mu \right\}. \end{aligned} \quad (\text{D.8})$$

So we find

$$\begin{aligned}
 & \overline{D}^\dagger \begin{pmatrix} Z_{\alpha\nu}^{(p+1)} \\ \delta_\alpha \theta^{(p+1)} \\ \delta_\alpha \mathcal{A}^{(p+1)} \\ \Delta_\alpha^{(p+1)\mu} \end{pmatrix} \\
 &= \begin{pmatrix} \mu_p \overline{\partial}^\dagger Z_{\alpha\nu}^{(p+1)} \\ \overline{\partial}_\theta^\dagger \delta_\alpha \theta^{(p+1)} + \frac{1}{2} (-1)^p \lambda_p R^{\rho\bar{\nu}} Z_{\alpha\rho\bar{\nu}}^{(p+1)} \\ \overline{\partial}_{\mathcal{A}}^\dagger \delta_\alpha \mathcal{A}^{(p+1)} + \frac{1}{2} (-1)^p \lambda_p F^{\rho\bar{\nu}} Z_{\alpha\rho\bar{\nu}}^{(p+1)} \\ \frac{i}{2} (-1)^{p+1} \varepsilon_p (\overline{\partial}\omega)^{\mu\rho\bar{\lambda}} Z_{\alpha\rho\bar{\lambda}} + \frac{\alpha'}{4} \tilde{\lambda}_p \left(\text{Tr} \left(F^{\mu\bar{\rho}} \delta_\alpha \mathcal{A}_\rho^{(p+1)} \right) - \text{Tr} \left(R^{\mu\bar{\rho}} \delta_\alpha \theta_\rho^{(p+1)} \right) \right) + \tilde{\mu}_p \overline{\partial}^\dagger \Delta_\alpha^{(p+1)\nu} \end{pmatrix}. \tag{D.9}
 \end{aligned}$$

A non-trivial check is that $(\overline{D}^\dagger)^2 = 0$ if and only if $\overline{D}^2 = 0$.

We now compare with the F-terms and the D-terms. The F-terms equations (2.7)–(2.8) follow if $\overline{D}\mathcal{Y}_\alpha^{(1)} = 0$ and $\mu_1 = -\varepsilon_1 = -\lambda_1$ and $\tilde{\lambda}_1 = 1$. The D-term equations are the variations of the HYM and balanced equations, written down in (2.11), (2.13), (2.14) and (2.15).²⁰ These correspond to $\overline{D}^\dagger\mathcal{Y}^{(1)} = 0$ with $\lambda_0 = 1$, $\varepsilon_0 = \tilde{\lambda}_0 = \tilde{\mu}_0$. One can check that these satisfy the consistency equations (D.7). Taken together these consistency checks are non-trivial.

We list two convenient solutions. The first is $\mu_p = \tilde{\lambda}_p = \tilde{\mu}_p = 1$ and $\varepsilon_p = \lambda_p = (-1)^p$. This corresponds to the \overline{D} -operator [9] in our conventions. A second choice is $\mu_p = -1$ and $\tilde{\lambda}_p = \tilde{\mu}_p = \lambda_p = \varepsilon_p = 1$, which avoids any explicit $(-1)^p$ factors. In components this second choice is

$$\begin{aligned}
 \widehat{Z}_{\alpha\nu}^{(p+1)} &= -\overline{\partial} Z_{\alpha\nu}^{(p)} + 2i\Delta_\alpha^{(p)\mu} (\partial\omega)_{\mu\nu} + \frac{\alpha'}{2} \left(\text{Tr} \left(\delta_\alpha \mathcal{A}^{(p)} F_\nu \right) - \text{Tr} \left(\delta_\alpha \theta^{(p)} R_\nu \right) \right), \\
 \widehat{\delta}_\alpha \mathcal{A}^{(p+1)} &= \overline{\partial}_{\mathcal{A}} \delta_\alpha \mathcal{A}^{(p)} + F_\mu \Delta_\alpha^{(p)\mu}, \\
 \widehat{\delta}_\alpha \theta^{(p+1)} &= \overline{\partial}_\theta \delta_\alpha \theta^{(p)} + R_\mu \Delta_\alpha^{(p)\mu}, \\
 \widehat{\Delta}_\alpha^{(p+1)} &= \overline{\partial} \Delta_\alpha^{(p)}. \tag{D.10}
 \end{aligned}$$

So in summary we have found a family of \overline{D} operators and its adjoint via the moduli space metric. The equations of motion correspond to $\overline{D}\mathcal{Y}^{(1)} = 0$ and $\overline{D}^\dagger\mathcal{Y}^{(1)} = 0$; in other words the first order deformations in holomorphic gauge correspond to the harmonic representatives of \overline{D} .

The non-spurious case follows in an identical manner to this case after making the appropriate substitution for $\delta_\alpha \theta^{(p)}$ and redefinition in Z_α discussed in the main text.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP³ supports the goals of the International Year of Basic Sciences for Sustainable Development.

²⁰Its worth noting the last term of (2.13) does not appear as it would come from the inner product of a pair of (0, 2)-deformations, $Z_\alpha^{(0,2)}$, which is $\mathcal{O}(\alpha'^2)$ and so dropped.

References

- [1] P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, *Vacuum Configurations for Superstrings*, *Nucl. Phys. B* **258** (1985) 46 [[INSPIRE](#)].
- [2] E. Witten, *New Issues in Manifolds of SU(3) Holonomy*, *Nucl. Phys. B* **268** (1986) 79 [[INSPIRE](#)].
- [3] C.M. Hull, *Compactifications of the Heterotic Superstring*, *Phys. Lett. B* **178** (1986) 357 [[INSPIRE](#)].
- [4] A. Strominger, *Superstrings with Torsion*, *Nucl. Phys. B* **274** (1986) 253 [[INSPIRE](#)].
- [5] L.B. Anderson, J. Gray, A. Lukas and B. Ovrut, *Stabilizing the Complex Structure in Heterotic Calabi-Yau Vacua*, *JHEP* **02** (2011) 088 [[arXiv:1010.0255](#)] [[INSPIRE](#)].
- [6] L.B. Anderson, J. Gray, A. Lukas and B. Ovrut, *The Atiyah Class and Complex Structure Stabilization in Heterotic Calabi-Yau Compactifications*, *JHEP* **10** (2011) 032 [[arXiv:1107.5076](#)] [[INSPIRE](#)].
- [7] L.B. Anderson, J. Gray, A. Lukas and B. Ovrut, *Stabilizing All Geometric Moduli in Heterotic Calabi-Yau Vacua*, *Phys. Rev. D* **83** (2011) 106011 [[arXiv:1102.0011](#)] [[INSPIRE](#)].
- [8] L.B. Anderson, J. Gray and E. Sharpe, *Algebroids, Heterotic Moduli Spaces and the Strominger System*, *JHEP* **07** (2014) 037 [[arXiv:1402.1532](#)] [[INSPIRE](#)].
- [9] X. de la Ossa and E.E. Svanes, *Holomorphic Bundles and the Moduli Space of $N = 1$ Supersymmetric Heterotic Compactifications*, *JHEP* **10** (2014) 123 [[arXiv:1402.1725](#)] [[INSPIRE](#)].
- [10] M. Garcia-Fernandez, *Torsion-free generalized connections and Heterotic Supergravity*, *Commun. Math. Phys.* **332** (2014) 89 [[arXiv:1304.4294](#)] [[INSPIRE](#)].
- [11] M. Garcia-Fernandez, R. Rubio and C. Tipler, *Infinitesimal moduli for the Strominger system and Killing spinors in generalized geometry*, *Math. Ann.* **369** (2017) 539 [[arXiv:1503.07562](#)] [[INSPIRE](#)].
- [12] P. Candelas, X. de la Ossa and J. McOrist, *A Metric for Heterotic Moduli*, *Commun. Math. Phys.* **356** (2017) 567 [[arXiv:1605.05256](#)] [[INSPIRE](#)].
- [13] X. de la Ossa, E. Hardy and E.E. Svanes, *The Heterotic Superpotential and Moduli*, *JHEP* **01** (2016) 049 [[arXiv:1509.08724](#)] [[INSPIRE](#)].
- [14] J. McOrist, *On the Effective Field Theory of Heterotic Vacua*, *Lett. Math. Phys.* **108** (2018) 1031 [[arXiv:1606.05221](#)] [[INSPIRE](#)].
- [15] M. Garcia-Fernandez, R. Rubio and C. Tipler, *Holomorphic string algebroids*, *Trans. Am. Math. Soc.* **373** (2020) 7347 [[arXiv:1807.10329](#)] [[INSPIRE](#)].
- [16] M. Garcia-Fernandez, R. Rubio and C. Tipler, *Gauge theory for string algebroids*, [arXiv:2004.11399](#) [[INSPIRE](#)].
- [17] A. Ashmore, X. De La Ossa, R. Minasian, C. Strickland-Constable and E.E. Svanes, *Finite deformations from a heterotic superpotential: holomorphic Chern-Simons and an L_∞ algebra*, *JHEP* **10** (2018) 179 [[arXiv:1806.08367](#)] [[INSPIRE](#)].
- [18] P. Candelas, X. De La Ossa, J. McOrist and R. Sisca, *The Universal Geometry of Heterotic Vacua*, *JHEP* **02** (2019) 038 [[arXiv:1810.00879](#)] [[INSPIRE](#)].

- [19] J. McOrist and R. Sisca, *Small gauge transformations and universal geometry in heterotic theories*, *SIGMA* **16** (2020) 126 [[arXiv:1904.07578](#)] [[INSPIRE](#)].
- [20] E. Silverstein and E. Witten, *Criteria for conformal invariance of (0, 2) models*, *Nucl. Phys. B* **444** (1995) 161 [[hep-th/9503212](#)] [[INSPIRE](#)].
- [21] M. Becker, L.-S. Tseng and S.-T. Yau, *Moduli Space of Torsional Manifolds*, *Nucl. Phys. B* **786** (2007) 119 [[hep-th/0612290](#)] [[INSPIRE](#)].
- [22] J. McOrist, *The Revival of (0, 2) Linear Sigma Models*, *Int. J. Mod. Phys. A* **26** (2011) 1 [[arXiv:1010.4667](#)] [[INSPIRE](#)].
- [23] I.V. Melnikov and E. Sharpe, *On marginal deformations of (0, 2) non-linear sigma models*, *Phys. Lett. B* **705** (2011) 529 [[arXiv:1110.1886](#)] [[INSPIRE](#)].
- [24] I.V. Melnikov, *An Introduction to Two-Dimensional Quantum Field Theory with (0, 2) Supersymmetry*, Lecture Notes in Physics **951**, Springer (2019) [[DOI](#)] [[INSPIRE](#)].
- [25] C.M. Hull, *Anomalies, Ambiguities and Superstrings*, *Phys. Lett. B* **167** (1986) 51 [[INSPIRE](#)].
- [26] A. Sen, *(2, 0) Supersymmetry and Space-Time Supersymmetry in the Heterotic String Theory*, *Nucl. Phys. B* **278** (1986) 289 [[INSPIRE](#)].
- [27] I.V. Melnikov, R. Minasian and S. Sethi, *Heterotic fluxes and supersymmetry*, *JHEP* **06** (2014) 174 [[arXiv:1403.4298](#)] [[INSPIRE](#)].
- [28] X. de la Ossa and E.E. Svanes, *Connections, Field Redefinitions and Heterotic Supergravity*, *JHEP* **12** (2014) 008 [[arXiv:1409.3347](#)] [[INSPIRE](#)].
- [29] E.A. Bergshoeff and M. de Roo, *The Quartic Effective Action of the Heterotic String and Supersymmetry*, *Nucl. Phys. B* **328** (1989) 439 [[INSPIRE](#)].
- [30] A. Coimbra, R. Minasian, H. Triendl and D. Waldram, *Generalised geometry for string corrections*, *JHEP* **11** (2014) 160 [[arXiv:1407.7542](#)] [[INSPIRE](#)].
- [31] L. Anguelova, C. Quigley and S. Sethi, *The Leading Quantum Corrections to Stringy Kähler Potentials*, *JHEP* **10** (2010) 065 [[arXiv:1007.4793](#)] [[INSPIRE](#)].
- [32] P. Berglund, P. Candelas, X. de la Ossa, E. Derrick, J. Distler and T. Hubsch, *On the instanton contributions to the masses and couplings of E_6 singlets*, *Nucl. Phys. B* **454** (1995) 127 [[hep-th/9505164](#)] [[INSPIRE](#)].
- [33] J. McOrist, S. Picard and E.E. Svanes, *New Differentials of the Hull-Strominger System, and Mathematical Implications*, to appear (2022).
- [34] E. Bergshoeff and M. de Roo, *Supersymmetric Chern-Simons Terms in Ten-dimensions*, *Phys. Lett. B* **218** (1989) 210 [[INSPIRE](#)].
- [35] R.R. Metsaev and A.A. Tseytlin, *Order alpha-prime (Two Loop) Equivalence of the String Equations of Motion and the Sigma Model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor*, *Nucl. Phys. B* **293** (1987) 385 [[INSPIRE](#)].
- [36] W.A. Chemissany, M. de Roo and S. Panda, *α' -Corrections to Heterotic Superstring Effective Action Revisited*, *JHEP* **08** (2007) 037 [[arXiv:0706.3636](#)] [[INSPIRE](#)].
- [37] P. Candelas and X.C. de la Ossa, *Moduli space of Calabi-Yau manifolds*, in *XIII International School of Theoretical Physics: The Standard Model and Beyond*, Szczyrk Poland, September 19 1989.

- [38] M.F. Atiyah, *Complex analytic connections in fibre bundles*, *Trans. Amer. Math. Soc.* **85** (1957) 181.
- [39] S. Ivanov, *Heterotic supersymmetry, anomaly cancellation and equations of motion*, *Phys. Lett. B* **685** (2010) 190 [[arXiv:0908.2927](#)] [[INSPIRE](#)].
- [40] D. Martelli and J. Sparks, *Non-Kähler heterotic rotations*, *Adv. Theor. Math. Phys.* **15** (2011) 131 [[arXiv:1010.4031](#)] [[INSPIRE](#)].
- [41] A. Ashmore, C. Strickland-Constable, D. Tennyson and D. Waldram, *Heterotic backgrounds via generalised geometry: moment maps and moduli*, *JHEP* **11** (2020) 071 [[arXiv:1912.09981](#)] [[INSPIRE](#)].
- [42] L. Álvarez-Cónsul, A.D.A. de La Hera and M. Garcia-Fernandez, *(0, 2) Mirror Symmetry on homogeneous Hopf surfaces*, [arXiv:2012.01851](#) [[INSPIRE](#)].
- [43] M. Garcia-Fernandez, R. Rubio, C. Shahbazi and C. Tipler, *Canonical metrics on holomorphic Courant algebroids*, [arXiv:1803.01873](#) [[INSPIRE](#)].
- [44] S.K. Donaldson and R.P. Thomas, *Gauge theory in higher dimensions*, in *The Geometric Universe*, Huggett et al. eds., Oxford University Press (1998) [ISBN: 9780198500599].
- [45] S. Cecotti and C. Vafa, *Topological antitopological fusion*, *Nucl. Phys. B* **367** (1991) 359 [[INSPIRE](#)].
- [46] C.M. Hull and P.K. Townsend, *The Two Loop β -function for σ Models With Torsion*, *Phys. Lett. B* **191** (1987) 115 [[INSPIRE](#)].
- [47] P. Candelas, X.C. De La Ossa, P.S. Green and L. Parkes, *A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, *Nucl. Phys. B* **359** (1991) 21 [[INSPIRE](#)].
- [48] D. Cassani and A. Bilal, *Effective actions and $N = 1$ vacuum conditions from $SU(3) \times SU(3)$ compactifications*, *JHEP* **09** (2007) 076 [[arXiv:0707.3125](#)] [[INSPIRE](#)].
- [49] L. Witten and E. Witten, *Large Radius Expansion of Superstring Compactifications*, *Nucl. Phys. B* **281** (1987) 109 [[INSPIRE](#)].
- [50] Edward Witten, *Two-dimensional models with (0, 2) supersymmetry: Perturbative aspects*, *Adv. Theor. Math. Phys.* **11** (2007) 1 [[hep-th/0504078](#)].