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On the edge covering transversal edge Domination in Graphs

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Abstract

Let $G = (V, E)$ be any graph with n vertices and m edges. An edge dominating set which intersects every minimum edge covering set in a graph $G$is called an edge covering transversal edge dominating set of G. The minimum cardinality of an edge covering transversal edge dominating set is called an edge covering transversal edge domination number of $G$and is denoted by$\gamma_{eect}(G)$. Any edge covering transversal edge dominating set of cardinalities$\gamma_{eect}(G)$ is called a γ_{eect}-set of G. The edge covering transversal edge domination number of some standard graphs are determined. Some properties satisfied by this concept are studied.

Keywords: domination number, edge domination number, edge covering, edge covering number, edge covering transversal edge domination number.

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1. Introduction

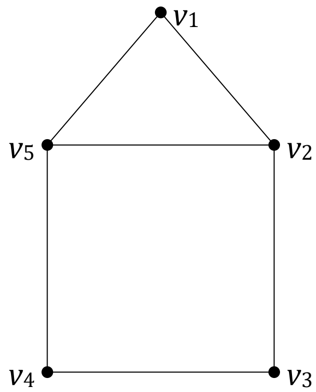
Let $G = (V, E)$ be any graph with $n$vertices and $m$edges. For any graph theoretic terminlogies not defined here, refer to the book of Bondy and Murthy [3]. One of the fastest growing areas in graph theory is the study of domination and related subset problems such as independence, covering and matching. By a graph G, we mean a non-trivial, finite, undirected graph with neither loops nor multiple edges. Two vertices $u$and $v$are said to be *adjacent* if $uv$is an edge of G. The *open neighbourhood of a vertex* $v$in a graph $G$is defined as the set $N_G(v) = \{v \in V(G) : uv \in E(G)\}$, while the *closed neighbourhood* of $v$in $G$is defined as $N_G[v] = N_G(v) \cup \{v\}$. For any vertex $v$in a graph G, the number of vertices adjacent to $v$is called the *degree* of v in G, denoted by $deg_G(v)$. If the degree of a vertex is 0, it is called an *isolated vertex*, while if the degree is 1, it is called an *end-vertex*. The *minimum degree* of vertices in $G$is defined by $\delta(G) = min\{deg(v)/v \in V(G)$. The *maximum degree* of vertices in $G$is defined by $\Delta(G) = max\{deg(v)/v \in V(G)$. A vertex $v$is called a *universal vertex* if $degG(v) = n - 1$.Two edges are said to *adjacent edges* if they have a common vertex. For any set $S$of vertices of G, the *induced subgraph* $< S >$is the maximal subgraph of G with vertex set.

A subset $S \subseteq V(G)$ is called a *dominating set* [3, 6, 7, 8]if every vertex $v \in V(G) \setminus S$is adjacent to a vertex $u \in S$. The *domination number*, $\gamma(G)$, of a graph Gdenotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph $G$is hence often called as a *γ-set* of G. A subset $S \subseteq E(G)$ is called an *edge dominating set* [1, 2] if every edge$f \in E(G) \setminus S$is adjacent to an edge$h \in S$. The *edge domination number*, $\gamma_e(G)$, of a graph Gdenotes the minimum cardinality of such edge dominating sets of G. A minimum edge dominating set of a graph $G$is hence often called as a γ_e*-set* of G. An *edge cove*r [5] of a graph is a set of edges such that every vertex of the graph is incident to atleast one edge of the set. A minimum edge covering is an edge covering of smallest possible size. The *edge covering number*$\rho(G)$ is the size of a minimum edge covering.Given a graph G and a collection of subsets of its vertices, a subset of $V(G)$ is called a transversal of G if it intersects each subset of the collection [4]. In this paper, we studied the concept of the edge covering transversal edge domination number of$G$.

2. On the edge covering transversal edge domination number of a graph

Definition 2.1. An edge dominating set which intersects every minimum edge covering set in a graph $G$is called an edge covering transversal edge dominating set of G.The minimum cardinality of an edge covering transversal edge dominating set is called an *edge covering transversal edge domination number* of $G$and is denoted by $\gamma_{eect}(G)$. Any edge covering transversal edge dominating set of cardinalities$\gamma_{eect}(G)$ is called a γ_{eect}-set of G.

Example 2.2. For the graph Ggiven in Fig. 2.1, $D_1 = \{v_2v_3, v_4v_5\}$ and $D_2 = \{v_2v_5, v_3v_4\}$ are the only two minimum edge dominating sets of G.
Also,$S_1 = \{v_1v_5, v_2v_3, v_3v_4\}$,$S_2 = \{v_1v_2, v_2v_3, v_4v_5\}$, $S_3 = \{v_1v_2, v_2v_5, v_3v_4\}$, $S_4 = \{v_1v_5, v_2v_3, v_4v_5\}$ are the only 4 minimum edge covering sets of G. Since $D_1 \cap S_3 = \phi$ and $D_2 \cap S_4 = \phi$, $D_1$and $D_2$are not an edge covering transversal edge dominating sets of G and so $\gamma_{eect}(G) = 3$.Now,$D_3 = \{v_1v_2, v_2v_3, v_4v_5\}$ is a γ_{eect}-set of G, so that $\gamma_{eect}(G) = 3$.



G

Figure 2.1

Theorem 2.3. A set $S$of edges of $G = K_{r,r}$ $(r \geq 2)$ is a minimum edge covering transversal edge domination of Gif and only if Sconsists of n-independent edges.

Proof. Let S be any set of r-independent edges of $G = K_{r,s}$ $(2 \leq r \leq s)$. Then S is a minimum edge covering set of G. Since $S \cap D = \phi$for any edge covering transversal edge dominating set $D$of G, $S$is an edge covering transversal edge dominating set of G. Hence, it follows that $\gamma_{eect}(G) \leq r$. If $\gamma_{eect}(G) \leq r$, then there exists an edge covering transversal edge dominating set *S'*, such that $|S'| < r$. Therefore, there exists atleast one vertex *v* of *G*, such that *v* is not incident with any edge of *S'* and so *S' is* not an edge covering transversal edge dominating set of G, which is a contradiction. Hence, *S* is a minimum edge covering transversal edge dominating set of G.

Conversely, let Sbe a minimum edge covering set of *G*. Let *S' be* any set of *n*independent set of edges of G.Then as in first part of this theorem, S' is a minimum edge covering set of G. Therefore, $|S'| = r$. Hence, $|S| = r$. If $S$is not independent, then there exists a vertex $v$of Gsuch that, $v$is not independent with any edge of S. Hence $S$is not an edge covering transversal edge dominating set of *G*, which is a contradiction. Thus, *S* consists of *r*-independent edges.

Theorem 2.4. A set S of edges of $G = K_{r,s}(2 \leq r \leq s)$ is a minimum edge covering transversal edge domination of G, if and only if Sconsists of $r - 1$ independent edges of $G$and $s - r + 1$ adjacent edges of$G$.

Proof. Let $X = \{u_1, u_2, \ldots\ldots, u_r\}$ and $Y = \{v_1, v_2, \ldots\ldots, v_s\}$ be a bipartition of G. Let Sbe any set of $r - 1$ independent edges of G and $s - r + 1$ adjacent edges of G.Since each vertex of $G$is incident with an edge of S, it follows that $\gamma_{eect}(G) < s$.

If $\gamma_{eect}(G) < s$, then there exists an edge covering transversal edge dominating set S' of Gsuch that $|S'| < s$. Therefore, there exists atleast one vertex $v$of Gsuch that $v$is not incident with any edge of S' and so S' is not an edge covering transversal edge dominating set of G, which is a contradiction. Hence, $S$is a minimum edge covering transversal edge dominating set of G.

Conversely, let$S$be a minimum edge covering transversal edge dominating set of G. Let S' be any set of $r - 1$ independent edges of $G$and $r - s + 1$ adjacent edges of G. Then as in the first part of this theorem, $S$is a minimum edge covering transversal edge dominating set of G. Therefore $|S'| = s$. Hence $|S| = s$. Let us assume that $S = S_1 \cup S_2$, where S_1consists of independent edges and S_2consists of adjacent edges of G. If $|S_1| \leq s - 2$, then S_2must contain atmost edges$s - r$. Then there exists atleast one vertex $v$of X, such that $v$is not incident with any edge of $S$and so $S$is not an edge covering transversal edge dominating set of G, which is a contradiction. Therefore Sconsists of $s - 1$ independent edges of $G$and $s - r + 1$ adjacent edges of G.

Corollary 2.5. For the complete bipartite graph, $K_{r,s}(2 \leq r \leq s)$, $\gamma_{eect}(G) = s$.

Theorem 2.6. For the complete graph $G = K_n(n \geq 4)$ with $n$even, a set $S$of edges of $G$is a minimum edge covering transversal edge domination of Gif and only if Sconsists of $\frac{n}{2}$independent edges.

Proof. The proof is similar to the proof of the Theorem 2.4.

Theorem 2.7. For the complete graph $G = K_n(n \geq 5)G$ with$n$odd, a set $S$of edges of $G$is a minimum edge covering transversal edge dominating set of Gif and only if S consists of$\frac{n-3}{2}$independent edges and two adjacent edges of G.

Proof. The proof is similar to the proof of the Theorem 2.4.

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Corollary2.8. For the Complete graph

$$G = K_n(n \geq 4),\ \gamma_{eect}(G) = \begin{cases} \frac{n}{2} \text{ if} n \text{ is even} \\ \frac{n+1}{2} \text{ if } n \text{ is odd} \end{cases}$$

Theorem 2.9. For the Cycle $G = C_n(n \geq 4)$,

$$\gamma_{eect}(G) = \begin{cases} \frac{n}{2} \text{ if } n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil \text{ if } n \text{ is odd} \end{cases}$$

Proof. Let $C_n: \{v_1, v_2, v_3, \ldots., v_n, v_1\}$ be the cycle. We consider the following two cases

Case (i). $n$is even.

Let $n = 2k(k \geq 2)$, then $S_1 = \{v_1v_2, v_3v_4, v_5v_6, \ldots, v_{2k-1}v_{2k}\}$ and $S_2 = \{v_2v_3, v_4v_5, v_6v_7, \ldots, v_{2k-2}v_{2k-1}, v_{2k}v_1\}$are the only two minimum edge covering sets of G. Let D be an edge dominating sets of G. Then it is clear that $D \cap S_1 \neq \phi$and$D \cap$

$S_2 \neq \phi$. Therefore $S_1$and $S_2$are the only two minimum edge covering transversal edge dominating sets of G, so that$\gamma_{eect}(G) = k = \frac{n}{2}$

Case (ii). $\boldsymbol{n}$ is odd

Let. $n = 2k + 1 (k \geq 2)$, then$S_1 = \{v_1v_2, v_3v_4, v_5v_6, \ldots, v_{2k-1}v_{2k}, v_{2nk}v_{2k+1}\}$ is a minimum edge covering transversal edge dominating set of G, so that$\gamma_{eect}(G) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 2.10. For the path $G = P_n (n \geq 4)$, $\gamma_{eect}(G) = \begin{cases} \frac{n}{2} \text{ if } n \text{ is even} \\ \frac{n+1}{2} \text{ if } n \text{ is odd} \end{cases}$

Proof. The proof is similar to the proof of the Theorem 2.9.

Theorem 2.11. For the star graph, $G = K_{1,n-1}$, $\gamma_{eect}(G) = n - 1$.

Theorem 2.12. For the wheel graph $G = K_1 + C_{n-1} (n \geq 5)$,

$$\gamma_{eect}(G) = \begin{cases} \dfrac{n-1}{2} \text{ if } n-1 \text{ } is \text{ even} \\ \dfrac{n+1}{2} \text{ if } n-1 \text{ is odd} \end{cases}$$

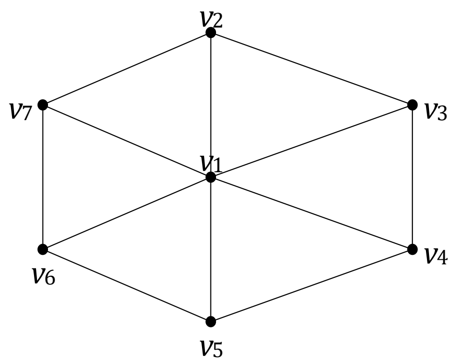

Figure 2.2

Proof. Let$V(K_1) = \{x\}$ and $C_{n-1} be\ v_1, v_2, v_3, \ldots, v_{n-1}, v_1$.

We consider the following cases.

Case (i).$\boldsymbol{n} - \mathbf{1}$ is even.

Let $n - 1 = 2k (k \geq 3)$. Then $S_1 = \{v_1v_2, v_3v_4, v_5v_6, \ldots, v_{2k-1}v_{2k}\}$ and $S_2 = \{v_2v_3, v_4v_5, v_6v_7, \ldots, v_{2k-2}v_{2k-1}, v_{2k}v_1\}$are the only two minimum edge covering sets of G. Then it is clear that $D \cap S_1 \neq \phi$and that $D \cap S_2 \neq \phi$. Therefore $S_1$and $S_2$are the only two minimum edge covering transversal edge dominating sets of G, so that$\gamma_{eect}(G) = k = \frac{n-1}{2}$.

Case (ii). $\boldsymbol{n-1}$ is odd.
Let $n-1 = 2k+1 (k \geq 2)$. Then $S_1 = \{v_1v_2, v_3v_4, v_5v_6, \dots, v_{2k-1}v_{2k}, v_{2k}v_{2k+1}\}$ is a minimum edge covering transversal edge dominating set of G, so that $\gamma_{eect}(G) = \left\lceil \frac{n-1}{2} \right\rceil$.

Theorem 2.13. Let G be a connected graph with m edges ($m \geq 2$), then $1 \leq \gamma_{eect}(G) \leq m$.

Theorem 2.14. For any graph $G, \gamma_e(G) \leq \gamma_{eect}(G) \leq \gamma_e(G) + \Delta(G)$.
Proof. Let D be a γ_{eectt}-set of G. Then D itself is an edge dominating set. Therefore $\gamma_e(G) \leq |D| = \gamma_{eect}(G)$. Now, let S be a γ_e-set in G and let v be a vertex of maximum degree. Therefore $deg(v) = \Delta(G)$. Then every minimum edge covering transversal edge dominating set of G contains an edge of $< E[N(v)] >$ so $S \cup < E[N(v)] >$ is an edge covering transversal edge dominating set. Also, since S intersects $< E[N(v)] >$ $|S \cup < E[N(v)] >| \leq \gamma_e(G) + \Delta(G)$. Then $\gamma_e(G) \leq \gamma_{eect}(G) \leq \gamma_e(G) + \Delta(G)$.

Remark 2.15. The bound in the Theorem 2.14 is sharp.

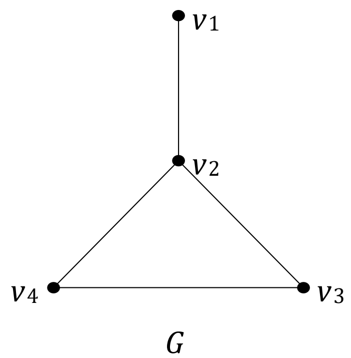

Figure 2.3

For the graph given in Figure 2.3, $S_1 = \{v_1v_2, v_3v_4\}$ is a γ_e-set of G and γ_{eect}-set of G so that $\gamma_e(G) = \gamma_{eect}(G)$.

3. Conclusions

In this paper, we obtain some results related to edge covering transversal domination in graphs and this work gives the scope for an extensive study of various domination parameters of these graphs.

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