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Maclaurin-type inequalities for Riemann–Liouville fractional integrals

Abstract. In the present article, an equality is established by using the well-known Riemann–Liouville fractional integrals. With the aid of this equality, some Euler–Maclaurin-type inequalities are given in the case of differentiable convex functions. Moreover, we give an example using graphs in order to show that our main result is correct.

1. Introduction. The inequality theory is a famous subject in many mathematical areas and remains an interesting research field with a great deal of applications. In addition, convex functions have also a significant place in the theory of inequalities. Furthermore, fractional calculus has been the focus of attraction for mathematicians in mathematical sciences because of its fundamental properties and applications in real-life problems. In consequence of the importance of fractional calculus, mathematicians have studied several fractional integral inequalities. For new inequalities bounds can be proved by using not only Hermite–Hadamard-type inequalities but also Simpson, Newton, and Euler–Maclaurin-type inequalities.

Let us consider $f \in L_1[a,b]$. The *Riemann–Liouville integrals* $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

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and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\quad x<b,$$

respectively [11, 17]. Here, $\Gamma(\alpha)$ denotes the Gamma function and it is defined as

$$\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}du.$$

The fractional integral reduces to the classical integral in the case of $\alpha=1$.

Simpson's inequalities have the following Simpson's rules:

i. Simpson's quadrature formula (Simpson's 1/3 rule) is formulated as follows:

$$\int_a^b f(x)\,dx\approx\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]. \tag{1}$$

ii. Simpson's second formula or the Newton–Cotes quadrature formula (Simpson's 3/8 rule (cf. [3])) is formulated as follows:

$$\int_a^b f(x)\,dx\approx\frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]. \tag{2}$$

iii. The corresponding dual Simpson's 3/8 formula – the Maclaurin rule based on the Maclaurin formula (cf. [3]) is formulated as follows:

$$\int_a^b f(x)\,dx\approx\frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]. \tag{3}$$

Formulae (1), (2), and (3) hold for every function f with continuous 4th derivative on $[a,b]$.

The most popular Newton–Cotes quadrature containing three-point Simpson-type inequality is formulated as follows:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ denote a four times differentiable and continuous function on (a,b) and let $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

Dragomir [6] proved an estimation of the remainder for Simpson's quadrature formula for the case of bounded variation functions and gave its applications in theory of special means. Moreover, some fractional Simpson-type inequalities for the case of functions whose second derivatives in absolute value are convex are given in [12]. Budak et al. [1] established some variants of Simpson-type inequalities for the case of differentiable convex functions and generalized fractional integrals. For further information concerned

Simpson-type inequalities and some properties of Riemann–Liouville fractional integrals, the reader is referred to [2, 16] and the references therein.

The classical closed type quadrature rule is the Simpson 3/8 rule based on the Simpson 3/8 inequality formulated as follows:

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ denote a four times differentiable and continuous function on (a, b), and let $\left\| f^{(4)} \right\|_{\infty} = \sup_{x \in (a,b)} \left| f^{(4)}(x) \right| < \infty$. Then, one has the inequality*

$$\left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{6480} \left\| f^{(4)} \right\|_{\infty} (b-a)^4 .$$

Simpson's second rule has the rule of three-point Newton–Cotes quadrature, hence evaluations for three steps quadratic kernel are sometimes called Newton-type results in the literature. Newton-type inequalities have been investigated extensively by many researchers. For instance, Erden et al. [8] proved some new integral inequalities of Newton-type for the case of functions whose first derivative in absolute value at certain power are arithmetically-harmonically convex. By using the Riemann–Liouville fractional integrals, several Newton-type inequalities for the case of differentiable convex functions were proved and some fractional Newton-type inequalities for the case of bounded variation functions were presented in [13]. Moreover, Gao and Shi [10] proved new Newton-type inequalities based on convexity and some applications for special cases of real functions are also established. Furthermore, Sitthiwirattham et al. [20] presented some Newton-type inequalities by using Riemann–Liouville fractional integrals and several fractional Newton-type inequalities for the case of bounded variation functions were given. For further information concerning Newton-type inequalities including convex differentiable functions, the reader is referred to [14, 15, 18] and the references therein.

The corresponding dual Simpson's 3/8 formula – the Maclaurin rule based on the Maclaurin inequality is formulated as follows:

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ denote a four times differentiable and continuous function on (a, b), and let $\left\| f^{(4)} \right\|_{\infty} = \sup_{x \in (a,b)} \left| f^{(4)}(x) \right| < \infty$. Then, the following inequality holds:*

$$\left| \frac{1}{8} \left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{7}{51840} \left\| f^{(4)} \right\|_{\infty} (b-a)^4 .$$

Dedić et al. [4] established a set of inequalities by using the Euler–Maclaurin formulae and the results are applied to obtain some error estimates in the case of the Maclaurin quadrature rules. Furthermore, a set of inequalities is established by using the Euler–Simpson 3/8 formulae. The results are implemented to get some error estimates for the case of the Simpson 3/8 quadrature rules in [5]. The reader is referred to [9, 3, 19] and the references therein for further information about inequalities of this kind.

The aim of this article is to derive Euler–Maclaurin-type inequalities for the case of differentiable convex functions by using Riemann–Liouville fractional integrals and we present an example using graphs to display that our main result is correct. The basic definition of fractional calculus and other studies in this discipline are given in Section 1. We will prove an integral equality in Section 2 that is critical in proving the primary results of the presented paper. Furthermore, using the Riemann–Liouville fractional integrals, we will prove some inequalities of the Euler–Maclaurin-type inequalities for differentiable convex functions. Moreover, we will provide a graphical example and demonstrate the accuracy of the newly established inequalities. In Section 2, we will present some opinions about Euler–Maclaurin-type inequalities for the further directions of research.

2. Main results.

Lemma 1. *If $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function on (a,b) such that $f'\in L_1([a,b])$, then the following equality*

$$\begin{aligned}\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]&\\ -\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]&\\ =\frac{(b-a)}{2}\sum_{i=1}^{4}I_i&\end{aligned}\tag{4}$$

is valid. Here,

$$\begin{cases} I_1=\displaystyle\int_0^{\frac{1}{6}} t^{\alpha}\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt,\\[2ex] I_2=\displaystyle\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t^{\alpha}-\frac{3}{8}\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt,\\[2ex] I_3=\displaystyle\int_{\frac{1}{2}}^{\frac{5}{6}}\left(t^{\alpha}-\frac{5}{8}\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt,\\[2ex] I_4=\displaystyle\int_{\frac{5}{6}}^{1}\left(t^{\alpha}-1\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt.\end{cases}$$

Proof. With the help of integration by parts, we get

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{6}} t^{\alpha} \left[f'(tb+(1-t)a) - f'(ta+(1-t)b) \right] dt \\
&= \frac{1}{b-a} t^{\alpha} f(tb+(1-t)a) \Big|_0^{\frac{1}{6}} \\
&\quad - \frac{\alpha}{b-a} \int_0^{\frac{1}{6}} t^{\alpha-1} \left[f(tb+(1-t)a) + f(ta+(1-t)b) \right] dt \\
&= \frac{1}{6^{\alpha}(b-a)} \left(f\left(\frac{5a+b}{6}\right) + f\left(\frac{a+5b}{6}\right) \right) \\
&\quad - \frac{\alpha}{b-a} \int_0^{\frac{1}{6}} t^{\alpha-1} \left[f(tb+(1-t)a) + f(ta+(1-t)b) \right] dt.
\end{aligned}
\tag{5}
$$

In a similar manner, we obtain

$$
\begin{aligned}
I_2 &= \frac{1}{b-a} \Bigg[2\left(\frac{1}{2^{\alpha}} - \frac{3}{8}\right) f\left(\frac{a+b}{2}\right) \\
&\quad - \left(\frac{1}{6^{\alpha}} - \frac{3}{8}\right)\left(f\left(\frac{5a+b}{6}\right) + f\left(\frac{a+5b}{6}\right)\right)\Bigg] \\
&\quad - \frac{\alpha}{b-a} \int_{\frac{1}{6}}^{\frac{1}{2}} t^{\alpha-1} \left[f(tb+(1-t)a) + f(ta+(1-t)b) \right] dt,
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
I_3 &= \frac{1}{b-a} \Bigg[\left(\left(\frac{5}{6}\right)^{\alpha} - \frac{5}{8}\right)\left(f\left(\frac{5a+b}{6}\right) + f\left(\frac{a+5b}{6}\right)\right) \\
&\quad - 2\left(\frac{1}{2^{\alpha}} - \frac{5}{8}\right) f\left(\frac{a+b}{2}\right)\Bigg] \\
&\quad - \frac{\alpha}{b-a} \int_{\frac{1}{2}}^{\frac{5}{6}} t^{\alpha-1} \left[f(tb+(1-t)a) + f(ta+(1-t)b) \right] dt,
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
I_4 &= -\frac{1}{b-a} \left(\left(\frac{5}{6}\right)^{\alpha} - 1\right)\left(f\left(\frac{5a+b}{6}\right) + f\left(\frac{a+5b}{6}\right)\right) \\
&\quad - \frac{\alpha}{b-a} \int_{\frac{1}{2}}^{\frac{5}{6}} t^{\alpha-1} \left[f(tb+(1-t)a) + f(ta+(1-t)b) \right] dt.
\end{aligned}
\tag{8}
$$

If we collect equalities from (5) to (8), then we readily get

$$\begin{aligned}\sum_{i=1}^{4} I_i &= \frac{1}{4(b-a)}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{5a+b}{6}\right)\right] \\ &\quad -\frac{\alpha}{b-a}\left[\int_0^1 t^{\alpha-1}f(tb+(1-t)a)\,dt+\int_0^1 t^{\alpha-1}f(ta+(1-t)b)\,dt\right].\end{aligned} \tag{9}$$

If we use the change of the variable $x=tb+(1-t)a$ and $x=ta+(1-t)b$ for $t\in[0,1]$ respectively, then equality (9) will be rewritten as follows:

$$\begin{aligned}\sum_{i=1}^{4} I_i &= \frac{1}{4(b-a)}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{5a+b}{6}\right)\right] \\ &\quad -\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right].\end{aligned} \tag{10}$$

Multiplying (10) by $\frac{(b-a)}{2}$, we obtain equality (4). This completes the proof of Lemma 1. □

Theorem 4. *Suppose that conditions of Lemma 1 hold and the function* $|f'|$ *is convex on* $[a,b]$*. Then, we have*

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right. \\ &\qquad\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right| \\ &\leq \frac{(b-a)}{2}\left(\Omega_1(\alpha)+\Omega_2(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right],\end{aligned} \tag{11}$$

where

$$\Omega_1(\alpha)=\int_0^{\frac{1}{6}}|t^{\alpha}|\,dt=\frac{1}{(\alpha+1)6^{\alpha+1}},$$

$$\begin{aligned}\Omega_2(\alpha)&=\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{3}{8}\right|dt \\ &=\begin{cases}\frac{1}{\alpha+1}\left(\frac{1}{2^{\alpha+1}}-\frac{1}{6^{\alpha+1}}\right)-\frac{1}{8}, & 0<\alpha<\frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{6}\right)}, \\ \frac{2\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{1+\frac{1}{\alpha}}+\frac{1}{\alpha+1}\left(\frac{1}{6^{\alpha+1}}+\frac{1}{2^{\alpha+1}}\right)-\frac{1}{4}, & \frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{6}\right)}\leq\alpha\leq\frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{2}\right)}, \\ \frac{1}{8}+\frac{1}{\alpha+1}\left(\frac{1}{6^{\alpha+1}}-\frac{1}{2^{\alpha+1}}\right), & \frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{2}\right)}<\alpha,\end{cases}\end{aligned}$$

$$\Omega_3(\alpha) = \int_{\frac{1}{2}}^{\frac{5}{6}} \left| t^{\alpha} - \frac{5}{8} \right| dt$$

$$= \begin{cases} \frac{1}{\alpha+1}\left(\left(\frac{5}{6}\right)^{\alpha+1} - \frac{1}{2^{\alpha+1}}\right) - \frac{5}{24}, & 0 < \alpha < \frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{1}{2}\right)}, \\ \frac{2\alpha}{\alpha+1}\left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}} + \frac{1}{\alpha+1}\left(\frac{1}{2^{\alpha+1}} + \left(\frac{5}{6}\right)^{\alpha+1}\right) - \frac{5}{6}, & \frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{1}{2}\right)} \leq \alpha \leq \frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{5}{6}\right)}, \\ \frac{5}{24} + \frac{1}{\alpha+1}\left(\frac{1}{2^{\alpha+1}} - \left(\frac{5}{6}\right)^{\alpha+1}\right), & \frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{5}{6}\right)} < \alpha, \end{cases}$$

and

$$\Omega_4(\alpha) = \int_{\frac{5}{6}}^{1} |t^{\alpha} - 1|\, dt = \frac{1}{6} + \frac{1}{\alpha+1}\left(\left(\frac{5}{6}\right)^{\alpha+1} - 1\right).$$

Proof. Consider modulus in Lemma 1. Then, we readily get the following inequality:

$$\begin{aligned} &\left| \frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right] \right. \\ &\qquad \left. - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \right| \\ &\leq \frac{(b-a)}{2}\left[\int_0^{\frac{1}{6}} |t^{\alpha}| \left| f'(tb + (1-t)a) - f'(ta + (1-t)b) \right| dt \right. \\ &\quad + \int_{\frac{1}{6}}^{\frac{1}{2}} \left| t^{\alpha} - \frac{3}{8} \right| \left| f'(tb + (1-t)a) - f'(ta + (1-t)b) \right| dt \\ &\quad + \int_{\frac{1}{2}}^{\frac{5}{6}} \left| t^{\alpha} - \frac{5}{8} \right| \left| f'(tb + (1-t)a) - f'(ta + (1-t)b) \right| dt \\ &\quad \left. + \int_{\frac{5}{6}}^{1} |t^{\alpha} - 1| \left| f'(tb + (1-t)a) - f'(ta + (1-t)b) \right| dt \right]. \end{aligned} \tag{12}$$

Since $|f'|$ is convex, we have

$$\begin{aligned} &\left| \frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right] \right. \\ &\qquad \left. - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \right| \\ &\leq \frac{(b-a)}{2}\left[\int_0^{\frac{1}{6}} |t^{\alpha}| \left[t\left|f'(b)\right| + (1-t)\left|f'(a)\right| + t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right] dt \right. \\ &\quad + \int_{\frac{1}{6}}^{\frac{1}{2}} \left| t^{\alpha} - \frac{3}{8} \right| \left[t\left|f'(b)\right| + (1-t)\left|f'(a)\right| + t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right] dt \end{aligned}$$

$$+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha}-\frac{5}{8}\right|\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|+t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt$$

$$\left.+\int_{\frac{5}{6}}^{1}\left|t^{\alpha}-1\right|\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|+t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\right]$$

$$=\frac{(b-a)}{2}\left[\int_{0}^{\frac{1}{6}}\left|t^{\alpha}\right|dt+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{3}{8}\right|dt+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha}-\frac{5}{8}\right|dt+\int_{\frac{5}{6}}^{1}\left|t^{\alpha}-1\right|dt\right]$$
$$\times\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

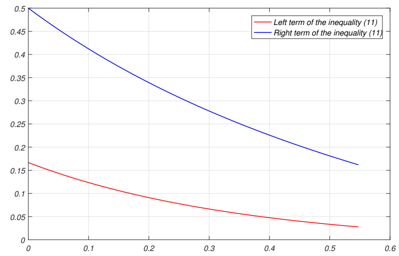
This is the end of the proof of Theorem 4. □

Example 1. Let us consider a function $f:[a,b]=[0,1]\to\mathbb{R}$ given by $f(x)=x^2$ in Theorem 4. Then, the left-hand side of (11) reduces to

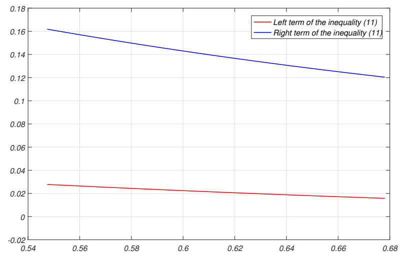
$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2}\left[J_{0+}^{\alpha}f(1)+J_{1-}^{\alpha}f(0)\right]\right|$$
$$=\left|\frac{1}{3}-\frac{\alpha}{2}\left[\int_{0}^{1}(1-t)^{\alpha-1}t^{2}dt+\int_{0}^{1}t^{\alpha-1}t^{2}dt\right]\right|$$
$$=\left|\frac{1}{3}-\frac{\alpha^{2}+\alpha+2}{2(\alpha+1)(\alpha+2)}\right|.$$

The right hand-side of (11) becomes

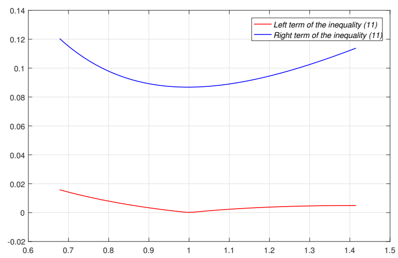
$$\begin{cases}\frac{1}{(\alpha+1)}\left[2\left(\frac{5}{6}\right)^{\alpha+1}-1\right]-\frac{1}{6}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{6}\right)},\\ \frac{2}{(\alpha+1)}\left[\frac{1}{6^{\alpha+1}}+\left(\frac{5}{6}\right)^{\alpha+1}+\alpha\left(\frac{3}{8}\right)^{1+\frac{1}{\alpha}}-\frac{1}{2}\right]-\frac{7}{24}, & \frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{6}\right)}<\alpha\leq\frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{1}{2}\right)},\\ \frac{2}{(\alpha+1)}\left[\frac{1}{6^{\alpha+1}}+\frac{1}{2^{\alpha+1}}+\left(\frac{5}{6}\right)^{\alpha+1}\right. & \\ \left.+\alpha\left(\frac{3}{8}\right)^{1+\frac{1}{\alpha}}+\alpha\left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}}-\frac{1}{2}\right]-\frac{11}{12}, & \frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{1}{2}\right)}<\alpha\leq\frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{2}\right)},\\ \frac{2}{(\alpha+1)}\left[\frac{1}{6^{\alpha+1}}+\left(\frac{5}{6}\right)^{\alpha+1}+\alpha\left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}}-\frac{1}{2}\right]-\frac{13}{24}, & \frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{2}\right)}<\alpha\leq\frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{5}{6}\right)},\\ \frac{1}{(\alpha+1)}\left[\frac{2}{6^{\alpha+1}}-1\right]+\frac{1}{2}, & \frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{5}{6}\right)}<\alpha.\end{cases}$$



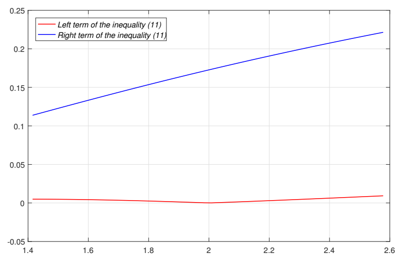
(A) Graph on the interval $0 < \alpha \le \frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{6}\right)}$.



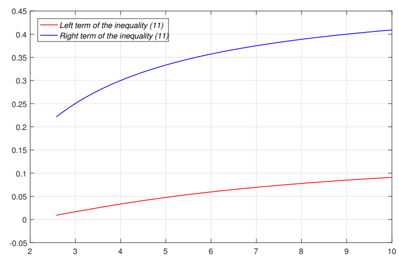
(B) Graph on the interval $\frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{6}\right)} < \alpha \le \frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{1}{2}\right)}$.



(C) Graph on the interval $\frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{1}{2}\right)} < \alpha \le \frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{2}\right)}$.



(D) Graph on the interval $\frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{2}\right)} < \alpha \le \frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{5}{6}\right)}$.



(E) Graph on the interval $\frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{5}{6}\right)} < \alpha \le 10$.

FIGURE 1. Graph of both sides of (11) in Example 1, depending on α, computed and plotted with MATLAB.

As one can see from Figure (1A) to Figure (1E), the left-hand side of (11) in Example 1 is always below the right-hand side of this equation, for all values of $\alpha \in (0, 10]$.

Corollary 1. *If we assign $\alpha = 1$ in Theorem 4, then we obtain the Euler–Maclaurin-type inequality*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{25(b-a)}{576}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Theorem 5. *Suppose that the assumptions of Lemma 1 are satisfied and the function $|f'|^q$, $q>1$ is convex on $[a,b]$. Then*

$$\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.\\
&\qquad\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\\
&\leq \frac{(b-a)}{2}\left[\left(\varphi_1(\alpha,p)+\varphi_4(\alpha,p)\right)\left[\left(\frac{11|f'(a)|^q+|f'(b)|^q}{72}\right)^{\frac{1}{q}}\right.\right.\\
&\qquad\left.+\left(\frac{|f'(a)|^q+11|f'(b)|^q}{72}\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\varphi_2(\alpha,p)+\varphi_3(\alpha,p)\right)\left[\left(\frac{2|f'(a)|^q+|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right.\\
&\qquad\left.\left.+\left(\frac{|f'(a)|^q+2|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right]\right].
\end{aligned} \tag{13}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ and

$$\begin{cases}
\varphi_1(\alpha,p)=\left(\int_0^{\frac{1}{6}}|t^{\alpha}|^p\,dt\right)^{\frac{1}{p}}=\left(\frac{1}{(\alpha p+1)6^{\alpha p+1}}\right)^{\frac{1}{p}},\\
\varphi_2(\alpha,p)=\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{3}{8}\right|^p dt\right)^{\frac{1}{p}},\\
\varphi_3(\alpha,p)=\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha}-\frac{5}{8}\right|^p dt\right)^{\frac{1}{p}},\\
\varphi_4(\alpha,p)=\left(\int_{\frac{5}{6}}^{1}(1-t^{\alpha})^p\,dt\right)^{\frac{1}{p}}.
\end{cases}$$

Proof. If we apply Hölder inequality in (12), then we readily obtain

$$\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.\\
&\qquad\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\\
&\leq\frac{(b-a)}{2}\left[\left(\int_0^{\frac{1}{6}}|t^{\alpha}|^p\,dt\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{6}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.\right.\\
&\qquad\left.+\left(\int_0^{\frac{1}{6}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{3}{8}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha}-\frac{5}{8}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\int_{\frac{5}{6}}^{1}|t^{\alpha}-1|^p\,dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{5}{6}}^{1}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\qquad\left.\left.+\left(\int_{\frac{5}{6}}^{1}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right]\right].
\end{aligned}$$

Since it is known that $|f'|^q$ is convex, we get

$$\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.\\
&\qquad\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\\
&\leq\frac{b-a}{2}\left[\left(\frac{1}{(\alpha p+1)\,6^{\alpha p+1}}\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{6}}t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q dt\right)^{\frac{1}{q}}\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left(\int_0^{\frac{1}{6}} t \left| f'(a) \right|^q + (1-t) \left| f'(b) \right|^q dt \right)^{\frac{1}{q}} \Bigg] \\
&+ \left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left| t^\alpha - \frac{3}{8} \right|^p dt \right)^{\frac{1}{p}} \left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}} t \left| f'(b) \right|^q + (1-t) \left| f'(a) \right|^q dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{1}{6}}^{\frac{1}{2}} t \left| f'(a) \right|^q + (1-t) \left| f'(b) \right|^q dt \right)^{\frac{1}{q}} \right] \\
&+ \left(\int_{\frac{1}{2}}^{\frac{5}{6}} \left| t^\alpha - \frac{5}{8} \right|^p dt \right)^{\frac{1}{p}} \left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}} t \left| f'(b) \right|^q + (1-t) \left| f'(a) \right|^q dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^{\frac{5}{6}} t \left| f'(a) \right|^q + (1-t) \left| f'(b) \right|^q dt \right)^{\frac{1}{q}} \right] \\
&+ \left(\int_{\frac{5}{6}}^{1} \left| t^\alpha - 1 \right|^p dt \right)^{\frac{1}{p}} \left[\left(\int_{\frac{5}{6}}^{1} t \left| f'(b) \right|^q + (1-t) \left| f'(a) \right|^q dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{5}{6}}^{1} t \left| f'(a) \right|^q + (1-t) \left| f'(b) \right|^q dt \right)^{\frac{1}{q}} \right] \Bigg] \\
&= \frac{(b-a)}{2} \left[\left(\left(\frac{1}{(\alpha p+1) 6^{\alpha p+1}} \right)^{\frac{1}{p}} + \left(\int_{\frac{5}{6}}^{1} (1-t^\alpha)^p dt \right)^{\frac{1}{p}} \right) \right. \\
&\quad \times \left[\left(\frac{11 |f'(a)|^q + |f'(b)|^q}{72} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 11 |f'(b)|^q}{72} \right)^{\frac{1}{q}} \right] \\
&\quad + \left(\left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left| t^\alpha - \frac{3}{8} \right|^p dt \right)^{\frac{1}{p}} + \left(\int_{\frac{1}{2}}^{\frac{5}{6}} \left| t^\alpha - \frac{5}{8} \right|^p dt \right)^{\frac{1}{p}} \right) \\
&\quad \left. \times \left[\left(\frac{2 |f'(a)|^q + |f'(b)|^q}{9} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 2 |f'(b)|^q}{9} \right)^{\frac{1}{q}} \right] \right].
\end{aligned}
$$

Thus, the proof of Theorem 5 is completed. □

Example 2. Let us consider a function $f:[a,b]=[0,1]\to\mathbb{R}$ given by $f(x)=x^2$ and $q=2$ in Theorem 5. Then, the left-hand side of (13) becomes

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2}\left[J_{0+}^{\alpha}f(1)+J_{1-}^{\alpha}f(0)\right]\right|$$
$$=\left|\frac{1}{3}-\frac{\alpha}{2}\left[\int_0^1(1-t)^{\alpha-1}t^2dt+\int_0^1 t^{\alpha-1}t^2dt\right]\right|=\left|\frac{1}{3}-\frac{\alpha^2+\alpha+2}{2(\alpha+1)(\alpha+2)}\right|.$$

The right hand-side of (13) reduces to

$$\frac{1}{2}\left[(\varphi_1(\alpha,2)+\varphi_4(\alpha,2))\frac{(1+\sqrt{11})}{3\sqrt{2}}+(\varphi_2(\alpha,2)+\varphi_3(\alpha,2))\frac{2(1+\sqrt{2})}{3}\right],$$

where

$$\begin{cases}\varphi_1(\alpha,2)=\left(\frac{1}{(2\alpha+1)6^{2\alpha+1}}\right)^{\frac{1}{2}},\\ \varphi_2(\alpha,2)=\left(\frac{1}{(2\alpha+1)}\left[\frac{1}{2^{2\alpha+1}}-\frac{1}{6^{2\alpha+1}}\right]-\frac{3}{4(\alpha+1)}\left[\frac{1}{2^{\alpha+1}}-\frac{1}{6^{\alpha+1}}\right]+\frac{3}{64}\right)^{\frac{1}{2}},\\ \varphi_3(\alpha,2)=\left(\frac{1}{(2\alpha+1)}\left[\left(\frac{5}{6}\right)^{2\alpha+1}-\frac{1}{2^{2\alpha+1}}\right]-\frac{5}{4(\alpha+1)}\left[\left(\frac{5}{6}\right)^{\alpha+1}-\frac{1}{2^{\alpha+1}}\right]+\frac{25}{192}\right)^{\frac{1}{2}},\\ \varphi_4(\alpha,2)=\left(\frac{1}{(2\alpha+1)}\left[1-\left(\frac{5}{6}\right)^{2\alpha+1}\right]-\frac{2}{(\alpha+1)}\left[1-\left(\frac{5}{6}\right)^{\alpha+1}\right]+\frac{1}{6}\right)^{\frac{1}{2}}.\end{cases}$$

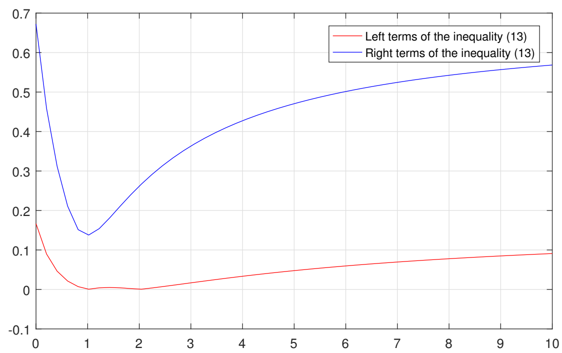


FIGURE 2. Graph of both sides of (13) in Example 2, depending on α, computed and plotted with MATLAB.

As one can see in Figure 2, the left-hand side of (13) in Example 2 is always below the right-hand side of this equation for all values of $\alpha \in (0, 10]$.

Corollary 2. *If we let $\alpha = 1$ in Theorem 5, then we obtain the following Euler–Maclaurin-type inequality:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq (b-a)\left[\left(\frac{1}{(p+1)\,6^{p+1}}\right)^{\frac{1}{p}}\left[\left(\frac{11\,|f'(a)|^q+|f'(b)|^q}{72}\right)^{\frac{1}{q}}\right.\right. \\
&\left.+\left(\frac{|f'(a)|^q+11\,|f'(b)|^q}{72}\right)^{\frac{1}{q}}\right] \\
&\quad+\left(\frac{1}{(p+1)}\left(\left(\frac{5}{24}\right)^{p+1}+\left(\frac{1}{8}\right)^{p+1}\right)\right)^{\frac{1}{p}}\left[\left(\frac{2\,|f'(a)|^q+|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right. \\
&\left.\left.+\left(\frac{|f'(a)|^q+2\,|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right]\right].
\end{aligned}$$

Theorem 6. *Let us suppose that the assumptions of Lemma 1 hold and the function $|f'|^q$, $q \geq 1$ is convex on $[a, b]$. Then, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right. \\
&\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right| \\
&\leq \frac{(b-a)}{2}\left[(\Omega_1(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_5(\alpha)\left|f'(b)\right|^q+(\Omega_1(\alpha)-\Omega_5(\alpha))\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.\right. \\
&\quad\left.+\left(\Omega_5(\alpha)\left|f'(a)\right|^q+(\Omega_1(\alpha)-\Omega_5(\alpha))\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right] \\
&\quad+(\Omega_2(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_6(\alpha)\left|f'(b)\right|^q+(\Omega_2(\alpha)-\Omega_6(\alpha))\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\Omega_6(\alpha)\left|f'(a)\right|^q+(\Omega_2(\alpha)-\Omega_6(\alpha))\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right] \\
&\quad+(\Omega_3(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_7(\alpha)\left|f'(b)\right|^q+(\Omega_3(\alpha)-\Omega_7(\alpha))\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\Omega_7(\alpha)\left|f'(a)\right|^q+(\Omega_3(\alpha)-\Omega_7(\alpha))\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right] \\
&\quad+(\Omega_4(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_8(\alpha)\left|f'(b)\right|^q+(\Omega_4(\alpha)-\Omega_8(\alpha))\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.\left.+\left(\Omega_8(\alpha)\left|f'(a)\right|^q+(\Omega_4(\alpha)-\Omega_8(\alpha))\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\right].
\end{aligned}$$

Here, $\Omega_1(\alpha)$, $\Omega_2(\alpha)$, $\Omega_3(\alpha)$ *and* $\Omega_4(\alpha)$ *are given in Theorem 4 and*

$$\Omega_5(\alpha) = \int_0^{\frac{1}{6}} |t^\alpha| \, tdt = \frac{1}{(\alpha+2)\,6^{\alpha+2}},$$

$$\Omega_6(\alpha) = \int_{\frac{1}{6}}^{\frac{1}{2}} \left| t^\alpha - \frac{3}{8} \right| tdt$$

$$= \begin{cases} \frac{1}{\alpha+2}\left(\frac{1}{2^{\alpha+2}} - \frac{1}{6^{\alpha+2}}\right) - \frac{1}{24}, & 0 < \alpha < \frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{6}\right)}, \\ \frac{\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{1+\frac{2}{\alpha}} + \frac{1}{\alpha+2}\left(\frac{1}{6^{\alpha+2}} + \frac{1}{2^{\alpha+2}}\right) - \frac{5}{96}, & \frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{6}\right)} \le \alpha \le \frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{2}\right)}, \\ \frac{1}{24} + \frac{1}{\alpha+2}\left(\frac{1}{6^{\alpha+2}} - \frac{1}{2^{\alpha+2}}\right), & \frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{1}{2}\right)} < \alpha, \end{cases}$$

$$\Omega_7(\alpha) = \int_{\frac{1}{2}}^{\frac{5}{6}} \left| t^\alpha - \frac{5}{8} \right| tdt$$

$$= \begin{cases} \frac{1}{\alpha+2}\left(\left(\frac{5}{6}\right)^{\alpha+2} - \frac{1}{2^{\alpha+2}}\right) - \frac{5}{36}, & 0 < \alpha < \frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{1}{2}\right)}, \\ \frac{\alpha}{\alpha+2}\left(\frac{5}{8}\right)^{1+\frac{2}{\alpha}} + \frac{1}{\alpha+2}\left(\frac{1}{2^{\alpha+2}} + \left(\frac{5}{6}\right)^{\alpha+2}\right) - \frac{85}{288}, & \frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{1}{2}\right)} \le \alpha \le \frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{5}{6}\right)}, \\ \frac{5}{36} + \frac{1}{\alpha+2}\left(\frac{1}{2^{\alpha+2}} - \left(\frac{5}{6}\right)^{\alpha+2}\right), & \frac{\ln\left(\frac{5}{8}\right)}{\ln\left(\frac{5}{6}\right)} < \alpha, \end{cases}$$

$$\Omega_8(\alpha) = \int_{\frac{5}{6}}^{1} |t^\alpha - 1| \, tdt = \frac{11}{72} + \frac{1}{\alpha+2}\left(\left(\frac{5}{6}\right)^{\alpha+2} - 1\right).$$

Proof. If we use power-mean inequality in (12), then we have

$$\begin{aligned} &\left| \frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right) \right] \right. \\ &\left. - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\ &\le \frac{(b-a)}{2}\left[\left(\int_0^{\frac{1}{6}} |t^\alpha| \, dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{6}} |t^\alpha| \left| f'(tb+(1-t)a) \right|^q dt \right)^{\frac{1}{q}} \right. \\ &\quad + \left(\int_0^{\frac{1}{6}} |t^\alpha| \, dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{6}} |t^\alpha| \left| f'(ta+(1-t)b) \right|^q dt \right)^{\frac{1}{q}} \\ &\quad + \left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left| t^\alpha - \frac{3}{8} \right| dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left| t^\alpha - \frac{3}{8} \right| \left| f'(tb+(1-t)a) \right|^q dt \right)^{\frac{1}{q}} \end{aligned}$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{3}{8}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{3}{8}\right|\left|f'\left(ta+(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha}-\frac{5}{8}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha}-\frac{5}{8}\right|\left|f'\left(tb+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha}-\frac{5}{8}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha}-\frac{5}{8}\right|\left|f'\left(ta+(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{5}{6}}^{1}\left|t^{\alpha}-1\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{5}{6}}^{1}\left|t^{\alpha}-1\right|\left|f'\left(tb+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{5}{6}}^{1}\left|t^{\alpha}-1\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{5}{6}}^{1}\left|t^{\alpha}-1\right|\left|f'\left(ta+(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].$$

Since $|f'|^q$ is convex, we have

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{(b-a)}{2}\left[\left(\int_{0}^{\frac{1}{6}}\left|t^{\alpha}\right|dt\right)^{1-\frac{1}{q}}\left[\left(\int_{0}^{\frac{1}{6}}\left|t^{\alpha}\right|\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\int_{0}^{\frac{1}{6}}\left|t^{\alpha}\right|\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]$$
$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{3}{8}\right|dt\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{3}{8}\right|\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{3}{8}\right|\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]$$
$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha}-\frac{5}{8}\right|dt\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha}-\frac{5}{8}\right|\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha}-\frac{5}{8}\right|\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{5}{6}}^{1}|t^{\alpha}-1|\,dt\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{5}{6}}^{1}|t^{\alpha}-1|\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\int_{\frac{5}{6}}^{1}|t^{\alpha}-1|\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]\right].$$

Finally, this finishes the proof of Theorem 6. □

Corollary 3. *Consider $\alpha = 1$ in Theorem 6. Then, the following Euler–Maclaurin-type inequality holds:*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|$$
$$\leq\frac{(b-a)}{72}\left[\left(\frac{|f'(b)|^{q}+8|f'(a)|^{q}}{9}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+8|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left[\left(\frac{361|f'(b)|^{q}+863|f'(a)|^{q}}{576}\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\frac{361|f'(a)|^{q}+863|f'(b)|^{q}}{576}\right)^{\frac{1}{q}}\right]\right].$$

Conclusion. Some Euler–Maclaurin-type inequalities are presented for the case of differentiable convex functions by using Riemann–Liouville fractional integrals. In addition, we give an example using graphs in order to indicate that our main result is correct. Our results can be extended by mathematicians in future studies by applying different variations of convex function classes.

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