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Optimal orientations of Vertex-multiplications of Trees with Diameter 4

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Abstract

Koh and Tay proved a fundamental classification of G vertex-multiplications into three classes $\mathscr{C}_0, \mathscr{C}_1$ and $\mathscr{C}_2$. They also showed that any vertex-multiplication of a tree with diameter at least 3 does not belong to the class $\mathscr{C}_2$. Of interest, G vertex-multiplications are extensions of complete n-partite graphs and Gutin characterised complete bipartite graphs with orientation number 3 (or 4 resp.) via an ingenious use of Sperner's theorem. In this paper, we investigate vertex-multiplications of trees with diameter 4 in $\mathscr{C}_0$ (or $\mathscr{C}_1$) and exhibit its intricate connections with problems in Sperner Theory, thereby extending Gutin's approach. Let s denote the vertex-multiplication of the central vertex. We almost completely characterise the case of even s and give a complete characterisation for the case of odd $s \geq 3$.

Keywords: optimal orientation, orientation number, vertex-multiplication, Sperner families, antichains, cross-intersecting.
MSC 2020: 05C12, 05C20, 05D05.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. In this paper, we consider only graphs with no loops or parallel edges. For any vertices $v, x \in V(G)$, the *distance* from v to x, $d_G(v,x)$, is defined as the length of a shortest path from v to x. For $v \in V(G)$, its *eccentricity* $e_G(v)$ is defined as $e_G(v) = \max\{d_G(v,x) \mid x \in V(G)\}$. The *diameter* of G, denoted by $d(G)$, is defined as $d(G) = \max\{e_G(v) \mid v \in V(G)\}$. The above notions are defined similarly for a digraph D; and we refer the reader to [1] for any undefined terminology. For a digraph D, a vertex x is said to be *reachable* from another vertex v if $d_D(v,x) < \infty$. The *outset* and *inset* of a vertex $v \in V(D)$ are defined to be $O_D(v) = \{x \in V(D) \mid v \to x\}$ and $I_D(v) = \{y \in V(D) \mid y \to v\}$ respectively. The *outdegree* $\deg^+_D(v)$ and *indegree* $\deg^-_D(v)$ of a vertex $v \in V(D)$ are defined by $\deg^+_D(v) = |O_D(v)|$ and $\deg^-_D(v) = |I_D(v)|$ respectively. If there is no ambiguity, we shall omit the subscript for the above notation.

An *orientation* D of a graph G is a digraph obtained from G by assigning a direction to every edge $e \in E(G)$. An orientation D of G is said to be *strong* if every two vertices in $V(D)$ are mutually reachable. An edge $e \in E(G)$ is a *bridge* if $G - e$ is disconnected. Robbins' One-way Street Theorem [19] states that a connected graph G has a strong orientation if and only if G is bridgeless.

Given a connected and bridgeless graph G, let $\mathscr{D}(G)$ be the family of strong orientations of G. The *orientation number* of G is defined as

$$\bar{d}(G) = \min\{d(D) \mid D \in \mathscr{D}(G)\}.$$

Any orientation D in $\mathscr{D}(G)$ with $d(D) = \bar{d}(G)$ is called an *optimal orientation* of G. The general problem of finding the orientation number of a connected and bridgeless graph is very difficult. Moreover, Chvátal and Thomassen [3] proved that it is NP-hard to determine whether a graph admits an orientation of diameter 2. Hence, it is natural to focus on special classes of graphs. The orientation number was evaluated for various classes of graphs, such as the complete graphs [2, 15, 18] and complete bipartite graphs [7, 20]. Of interest, Gutin ingeniously made use of a celebrated result in combinatorics, Sperner's theorem (see Theorem 2.1), to determine a characterisation of complete bipartite graphs with orientation number 3 (or 4 resp.).

Theorem 1.1 (Šoltés [20] and Gutin [7]). *For $q \geq p \geq 2$,*

$$\bar{d}(K(p,q)) = \begin{cases} 3, & \text{if } q \leq \binom{p}{\lfloor p/2 \rfloor}, \\ 4, & \text{if } q > \binom{p}{\lfloor p/2 \rfloor}. \end{cases}$$

In 2000, Koh and Tay [11] studied the orientation numbers of a family of graphs known as the G vertex-multiplications. They extended the results on complete n-partite graphs. Let G be a given connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For any sequence of n positive integers (s_i), a G *vertex-multiplication* (also known as an *extension* of G in [1]), denoted by $G(s_1, s_2, \ldots, s_n)$, is the graph with vertex set $V^* = \bigcup_{i=1}^{n} V_i$ and edge set E^*, where V_i's are pairwise disjoint sets with $|V_i| = s_i$, for $i = 1, 2, \ldots, n$; and for any $u, v \in V^*$, $uv \in E^*$ if and only if $u \in V_i$ and $v \in V_j$ for some $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ such that $v_i v_j \in E(G)$. For instance, if $G \cong K_n$, then the graph $G(s_1, s_2, \ldots, s_n)$ is a complete n-partite graph with partite sizes $s_1, s_2, \ldots, s_n$. Also, we say G is a *parent graph* of a graph H if $H \cong G(s_1, s_2, \ldots, s_n)$ for some sequence (s_i) of positive integers.

For $i = 1, 2, \ldots, n$, we denote the x-th vertex in V_i by (x, v_i), i.e., $V_i = \{(x, v_i) \mid x = 1, 2, \ldots, s_i\}$. Hence, two vertices (x, v_i) and (y, v_j) in V^* are adjacent in $G(s_1, s_2, \ldots, s_n)$ if and only if $i \neq j$ and $v_i v_j \in E(G)$. For convenience, we write $G^{(s)}$ in place of $G(s, s, \ldots, s)$ for any positive integer s, and it is understood that the number of s's is equal to the order of G, n. Thus, $G^{(1)}$ is simply the graph G itself.

The G vertex-multiplications are a natural generalisation of complete multipartite graphs. Optimal orientations minimising the diameter can also be used to solve a variant of the Gossip Problem on a graph G. The Gossip Problem attributed to Boyd by Hajnal et al. [9] is stated as follows:

"There are n ladies, and each one of them knows an item of scandal which is not known to any of the others. They communicate by telephone, and whenever two ladies make a call, they pass on to each other, as much scandal as they know at that time. How many calls are needed before all ladies know all the scandal?"

The Problem has been the source of many papers that have studied the spread of information by telephone calls, conference calls, letters and computer networks. One can imagine a network of people modelled by a G vertex-multiplication where the parent graph is G and persons within a partite set are not allowed to communicate directly with each other, for perhaps secrecy or disease containment reasons.

The following theorem by Koh and Tay [11] provides a fundamental classification on G vertex-multiplications.

Theorem 1.2 (Koh and Tay [11]). *Let G be a connected graph of order $n \geq 3$. If $s_i \geq 2$ for $i = 1, 2, \ldots, n$, then $d(G) \leq \bar{d}(G(s_1, s_2, \ldots, s_n)) \leq d(G) + 2$.*

In view of Theorem 1.2, all graphs of the form $G(s_1, s_2, \ldots, s_n)$, with $s_i \geq 2$ for all $i = 1, 2, \ldots, n$, can be classified into three classes $\mathscr{C}_j$, where

$$\mathscr{C}_j = \{G(s_1, s_2, \ldots, s_n) \mid \bar{d}(G(s_1, s_2, \ldots, s_n)) = d(G) + j\},$$

for $j = 0, 1, 2$. Henceforth, we assume $s_i \geq 2$ for $i = 1, 2, \ldots, n$. The following lemma was found useful in proving Theorem 1.2.

Lemma 1.3 (Koh and Tay [11]). *Let s_i, t_i be integers such that $s_i \leq t_i$ for $i = 1, 2, \ldots, n$. If the graph $G(s_1, s_2, \ldots, s_n)$ admits an orientation F in which every vertex v lies on a cycle of length not exceeding m, then $\bar{d}(G(t_1, t_2, \ldots, t_n)) \leq \max\{m, d(F)\}$.*

To discuss further, we need some notation. In this paper, let T_4 (or simply T unless stated otherwise) be a tree of diameter 4 with vertex set $V(T_4) = \{v_1, v_2, \ldots, v_n\}$. We further denote by c, the unique central vertex of T_4, i.e., $e_{T_4}(\mathsf{c}) = 2$, and the neighbours of c by $[i]$, i.e., $N_{T_4}(\mathsf{c}) = \{[i] \mid i = 1, 2, \ldots, \deg_{T_4}(\mathsf{c})\}$. For each $i = 1, 2, \ldots, \deg_{T_4}(\mathsf{c})$, we further denote the neighbours of $[i]$, excluding c, by $[\alpha, i]$, i.e., $N_{T_4}([i]) - \{\mathsf{c}\} = \{[\alpha, i] \mid \alpha = 1, 2, \ldots, \deg_{T_4}([i]) - 1\}$. Figure 1 illustrates the use of this notation.

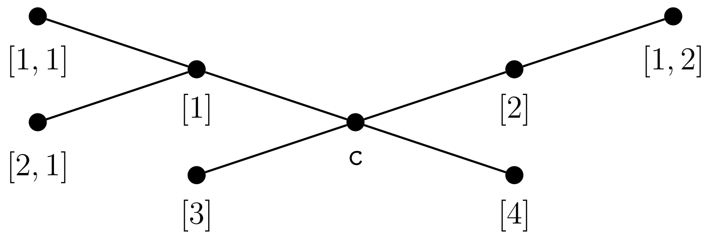

Figure 1: Labelling vertices in a T_4.

From here onwards, let $\mathcal{T} = T_4(s_1, s_2, \ldots, s_n)$ be a vertex-multiplication of a tree T_4. In $\mathcal{T}$, the integer s_i corresponds to the vertex v_i, $i \neq n$, while $s_n = s$ corresponds to c. We will loosely use the two denotations of a vertex, for example, if $v_i = [j]$, then $s_i = s_{[j]}$. Also, if $X \subseteq \mathbb{N}_k = \{1, 2, \ldots, k\}$, where $k \in \mathbb{Z}^+$, and $v \in V(T)$, then set $(X, v) = \{(i, v) \mid i \in X\}$. In particular, $(\mathbb{N}_s, \mathsf{c}) = \{(1, \mathsf{c}), (2, \mathsf{c}), \ldots, (s, \mathsf{c})\}$. For any set $\lambda \subseteq (\mathbb{N}_s, \mathsf{c})$, $\bar{\lambda} = (\mathbb{N}_s, \mathsf{c}) - \lambda$ denotes its complement set.

A vertex v in a graph G is a *leaf* if $deg_G(v) = 1$. For a given T_4, set $E = \{[i] \mid [i]$ is an leaf in $T_4\}$. For a given $\mathcal{T}$ of T_4, set $\mathcal{T}(A_j) = \{[i] \mid s_{[i]} = j,\ 1 \leq i \leq \deg_T(\mathsf{c}),\ [i] \notin E\}$, where j is a positive integer. If there is no ambiguity, we will use A_j instead of $\mathcal{T}(A_j)$. Similarly, $A_{\leq j}$ and $A_{\geq j}$ denote the corresponding sets, when the condition $s_{[i]} = j$ is replaced by $s_{[i]} \leq j$ and $s_{[i]} \geq j$ respectively. For example, if T_4 is as given in Figure 1, then $E = \{[3], [4]\}$; furthermore, if $s_i = 2$ for all $i = 1, 2, \ldots, n$, in $\mathcal{T}$, then $A_2 = \{[1], [2]\}$.

Theorem 1.2 was generalised to digraphs by Gutin et al. [8]. Ng and Koh [16] and Wong and Tay [26] investigated vertex-multiplications of cycles and Cartesian products of graphs respectively. Koh and Tay [12] studied vertex-multiplications of trees. Since trees with diameter at most 2 are parent graphs of complete bipartite graphs and are completely solved, Koh and Tay considered trees of diameter at least 3. They proved that vertex-multiplications of trees with diameter 3, 4 or 5 does not belong to the class $\mathscr{C}_2$ and those with diameter at least 6 belong to the class $\mathscr{C}_0$.

Theorem 1.4 (Koh and Tay [12]). *If T is a tree of order n and $d(T) = 3, 4$ or 5, then $T(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$.*

Theorem 1.5 (Koh and Tay [12]). *If T is a tree of order n and $d(T) \geq 6$, then $T(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.*

Wong and Tay [23] proved a characterisation for vertex-multiplications of trees with diameter 5 in $\mathscr{C}_0$ and $\mathscr{C}_1$.

Theorem 1.6 (Wong and Tay [23]). *Let T be a tree of diameter 5 with central vertices c_1 and c_2, and s_i corresponding to c_i for $i = 1, 2$. Furthermore for $i = 1, 2$, denote $E'_i = \{u \mid u \in N_T(\mathsf{c}_i) - \{\mathsf{c}_{3-i}\}, u$ is not an leaf in $T\}$ and $m_i = \min\{s_u \mid u \in E'_i\}$.*
(a) If $s_1 \geq 3$, or $s_2 \geq 3$, or $m_1, m_2 \geq 4$, then $T(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.
(b) Suppose $s_1 = s_2 = 2$ and $2 \leq m_1 \leq 3$ or $2 \leq m_2 \leq 3$. Then, $T(s_1, s_2 \ldots, s_n) \in \mathscr{C}_0$ if and only if $|E'_j| = 1$ for some $j = 1, 2$.

Koh and Tay [12] obtained some results regarding membership in $\mathscr{C}_0$ or $\mathscr{C}_1$ for vertex-multiplications of trees with diameter 4.

Theorem 1.7 (Koh and Tay [12])**.** *For a given* T_4*,*
(a) if $\deg_{T_4}(\mathsf{c}) = 2$*, then* $\mathcal{T} \in \mathscr{C}_0$*.*
(b) if $\deg_{T_4}(\mathsf{c}) \geq 3$*, then* $T_4^{(2)} \in \mathscr{C}_1$*.*

In this paper, we further investigate vertex-multiplications of trees with diameter 4 and almost completely classify them as $\mathscr{C}_0$ or $\mathscr{C}_1$. The techniques required here exhibit intricate connections with problems in Sperner Theory. In Section 2, we provide the main tools, which comprise well-known results from Sperner Theory and structural properties of optimal orientations of a $\mathcal{T}$. Section 3 focuses primarily on the case where s is even and the findings are summarised in Theorem 1.8. In Section 4, we prove a complete characterisation of vertex-multiplications of trees with odd $s \geq 3$, namely Theorem 1.9. We point out that Propositions 3.4 and 3.5 hold for all integers $s \geq 2$ and $s \geq 3$ respectively.

Theorem 1.8. *Let* T_4 *be a tree of diameter* 4 *with the only central vertex* c*. Suppose* s *is even for a* $\mathcal{T}$*. Then,*

(a) For $s = 2$:

$\lvert A_2 \cup A_3\rvert$	$\lvert A_{\geq 4}\rvert$	$\mathcal{T} \in \mathscr{C}_0 \iff \cdots$	**Proposition**
0	≥ 2	Always true.	3.4
≥ 1	≥ 0	$\deg_T(\mathsf{c}) = 2$.	3.2

Table 1: Summary for $T_4(s_1, s_2, \ldots, s_n)$, where $s = 2$.

(b) For $s \geq 4$:

$\lvert A_2\rvert$	$\lvert A_3\rvert$	$\lvert A_{\geq 4}\rvert$	$\mathcal{T} \in \mathscr{C}_0 \iff \cdots$	**Proposition**
0	0	≥ 2	Always true.	3.4
0	≥ 1	≥ 0	$\lvert A_3\rvert \leq \binom{s}{s/2} + \binom{s}{(s/2)+1} - 2$.	3.9
≥ 2	0	0	(i) $\lvert A_2\rvert \leq \binom{s}{\lceil s/2\rceil} - 1$, if $\lvert A_2\rvert < \deg_T(\mathsf{c})$, (ii) $\lvert A_2\rvert \leq \binom{s}{\lceil s/2\rceil}$, if $\lvert A_2\rvert = \deg_T(\mathsf{c})$.	3.5
≥ 1	0	≥ 1	(i) $\lvert A_2\rvert \leq \binom{s}{s/2} - 2$, if $\lvert A_{\geq 4}\rvert \geq 2$ or $\lvert A_{\geq 2}\rvert < \deg_T(\mathsf{c})$, (ii) $\lvert A_2\rvert \leq \binom{s}{s/2} - 1$, otherwise.	3.10
≥ 1	1	0	(i) $\lvert A_2\rvert \leq \binom{s}{s/2} - 2$ if $\lvert A_{\geq 2}\rvert < \deg_T(\mathsf{c})$, (ii) $\lvert A_2\rvert \leq \binom{s}{s/2} - 1$, if $\lvert A_{\geq 2}\rvert = \deg_T(\mathsf{c})$.	3.11

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The final case is incomplete and excludes the case $|A_2| \geq 1$, $|A_3| = 1$ and $|A_{\geq 4}| = 0$.

$\boldsymbol{|A_2|}$	$\boldsymbol{|A_3|}$	$\boldsymbol{|A_{\geq 4}|}$	$\boldsymbol{\mathcal{T} \in \mathscr{C}_0}$	**Proposition**
≥ 1	≥ 1	≥ 0	(a) $\mathcal{T} \in \mathscr{C}_0 \Rightarrow 2|A_2| + |A_3| \leq \binom{s}{s/2} + \binom{s}{(s/2)+1}$ $-\kappa^*_{s,\frac{s}{2}}(k)$ for some $k \leq |A_2| + |A_3|$. (b) There exists some $|A_2| + 1 \leq k \leq \min\{|A_2| + |A_3|, \binom{s}{s/2} - 1\}$ such that $2|A_2| + |A_3| \leq \binom{s}{s/2} + \binom{s}{(s/2)+1} - \kappa_{s,\frac{s}{2}}(k) - 3$ $\Rightarrow \mathcal{T} \in \mathscr{C}_0$. Note: $\kappa_{s,\frac{s}{2}}(\cdot)$ and $\kappa^*_{s,\frac{s}{2}}(\cdot)$ will be defined later.	3.12

Table 2: Summary for $\mathcal{T} = T_4(s_1, s_2, \ldots, s_n)$, where $s \geq 4$ is even.

Theorem 1.9. *Let T_4 be a tree of diameter 4 with the only central vertex c. Suppose $s \geq 3$ is odd for a $\mathcal{T}$. Then,*

$\boldsymbol{|A_2|}$	$\boldsymbol{|A_3|}$	$\boldsymbol{|A_{\geq 4}|}$	$\boldsymbol{\mathcal{T} \in \mathscr{C}_0 \iff \cdots}$	**Proposition**
0	≥ 1	≥ 0	$|A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$.	4.1
0	0	≥ 2	Always true.	3.4
≥ 2	0	0	(i) $|A_2| \leq \binom{s}{\lceil s/2 \rceil} - 1$, if $|A_2| < \deg_T(\mathsf{c})$, (ii) $|A_2| \leq \binom{s}{\lceil s/2 \rceil}$, if $|A_2| = \deg_T(\mathsf{c})$.	3.5
≥ 1	0	≥ 1	$|A_2| \leq \binom{s}{\lceil s/2 \rceil} - 1$.	4.8
≥ 1	1	0	$|A_2| \leq \binom{s}{\lceil s/2 \rceil} - 1$.	4.3
≥ 1	≥ 2	0	(i) $2|A_2| + |A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$, or (ii) $2|A_2| + |A_3| = 2\binom{s}{\lceil s/2 \rceil} - 1$, $|A_2| \geq \lceil \frac{s}{2} \rceil \lfloor \frac{s}{2} \rfloor$ and $s \geq 5$.	
≥ 1	1	≥ 1	$|A_2| \leq \binom{s}{\lceil s/2 \rceil} - 2$.	4.10
≥ 1	≥ 2	≥ 1	$2|A_2| + |A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$.	

Table 3: Summary for $\mathcal{T} = T_4(s_1, s_2, \ldots, s_n)$, where $s \geq 3$ is odd.

As we shall see in the proofs of Theorems 1.8 and 1.9, it is a key insight to partition $N_T(\mathsf{c})$ into 4 types of vertices, A_2, A_3, $A_{\geq 4}$ and E. Their sizes will then determine the equivalent conditions of an optimal orientation (except possibly Proposition 3.12). We shall consider cases dependent on these 4 sets. The lack of conformity in the equivalent conditions across all cases gives a compelling indication that the case distinctions are required.

2. Preliminaries

Our overarching approach is to reduce the investigation of optimal orientations of tree vertex-multiplication graphs to variants of problems in Sperner Theory; particularly concerning cross-intersecting antichains. The change in perspective grants us leverage on the following useful results in Sperner Theory.

For any $n \in \mathbb{Z}^+$, let $\mathbb{N}_n = \{1, 2, \ldots, n\}$ and $2^{\mathbb{N}_n}$ denote the power set of $\mathbb{N}_n$. For any integer k, $0 \leq k \leq n$, $\binom{\mathbb{N}_n}{k}$ denotes the collection of all k-subsets (i.e., subsets of cardinality k) of $\mathbb{N}_n$. Two families $\mathscr{A}, \mathscr{B} \subseteq 2^{\mathbb{N}_n}$ are said to be *cross-intersecting* if $A \cap B \neq \emptyset$ for all $A \in \mathscr{A}$ and all $B \in \mathscr{B}$. Two subsets X and Y of $\mathbb{N}_n$ are said to be *independent* if $X \nsubseteq Y$ and $Y \nsubseteq X$. An *antichain* or *Sperner family* $\mathscr{A}$ on $\mathbb{N}_n$ is a collection of pairwise independent subsets of $\mathbb{N}_n$, i.e., for all $X, Y \in \mathscr{A}$, $X \nsubseteq Y$.

Theorem 2.1 (Sperner [21])**.** *For any $n \in \mathbb{Z}^+$, if $\mathscr{A}$ is an antichain on $\mathbb{N}_n$, then $|\mathscr{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$. Furthermore, equality holds if and only if all members in $\mathscr{A}$ have the same size, $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$.*

Lih's theorem [14] provides the maximum size of an antichain with each member intersecting a fixed set and Griggs [6] determined all such maximum-sized antichains.

Theorem 2.2 (Lih [14])**.** *Let $n \in \mathbb{Z}^+$ and $Y \subseteq \mathbb{N}_n$. If $\mathscr{A}$ is an antichain on $\mathbb{N}_n$ such that $A \cap Y \neq \emptyset$ for all $A \in \mathscr{A}$, then*

$$|\mathscr{A}| \leq \binom{n}{\lceil n/2 \rceil} - \binom{n - |Y|}{\lceil n/2 \rceil}.$$

Theorem 2.3 (Griggs [6])**.** *Let $n \in \mathbb{Z}^+$ and $Y \subseteq \mathbb{N}_n$. If $\mathscr{A}$ is an antichain on $\mathbb{N}_n$ such that $A \cap Y \neq \emptyset$ for all $A \in \mathscr{A}$ and $|\mathscr{A}| = \binom{n}{\lceil n/2 \rceil} - \binom{n-|Y|}{\lceil n/2 \rceil}$, then $\mathscr{A}$ consists of exactly one of the following:*
(i) $\lceil \frac{n}{2} \rceil$-sets, or
(ii) $\frac{n-1}{2}$-sets for odd n and $|Y| \geq \frac{n+3}{2}$, or
(iii) $\frac{n+2}{2}$-sets for even n and $|Y| = 1$.

Given two cross-intersecting antichains $\mathscr{A}$ and $\mathscr{B}$ on $\mathbb{N}_n$, Ou [17], Frankl and Wong [5] and Wong and Tay [25] independently derived an upper bound for $|\mathscr{A}| + |\mathscr{B}|$. Furthermore, Wong and Tay [22] determined all extremal and almost-extremal cross-intersecting antichains for $\mathscr{A}$ and $\mathscr{B}$.

Theorem 2.4 (Ou [17], Frankl and Wong [5] and Wong and Tay [25])**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be two cross-intersecting antichains on $\mathbb{N}_n$, where $n \in \mathbb{Z}^+$ and $n \geq 3$. Then,*

$$|\mathscr{A}| + |\mathscr{B}| \leq \binom{n}{\lfloor (n+1)/2 \rfloor} + \binom{n}{\lceil (n+1)/2 \rceil}$$

Furthermore, equality holds if and only if $\{\mathscr{A}, \mathscr{B}\} = \{\binom{\mathbb{N}_n}{\lfloor (n+1)/2 \rfloor}, \binom{\mathbb{N}_n}{\lceil (n+1)/2 \rceil}\}$.

Theorem 2.5 (Wong and Tay [22])**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be two cross-intersecting antichains on $\mathbb{N}_n$, where $n \geq 3$ is an odd integer and $|\mathscr{A}| \geq |\mathscr{B}|$. Then, $|\mathscr{A}| + |\mathscr{B}| = 2\binom{n}{\lceil n/2 \rceil} - 1$ if and only if $\mathscr{A} = \binom{\mathbb{N}_n}{\lceil n/2 \rceil}$, $\mathscr{B} \subset \binom{\mathbb{N}_n}{\lceil n/2 \rceil}$ and $|\mathscr{B}| = \binom{n}{\lceil n/2 \rceil} - 1$.*

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Theorem 2.6 (Wong and Tay [22]). *Let $\mathscr{A}$ and $\mathscr{B}$ be two cross-intersecting antichains on $\mathbb{N}_n$, where $n \geq 4$ is an even integer and $|\mathscr{A}| \geq |\mathscr{B}|$. Then, $|\mathscr{A}|+|\mathscr{B}| = \binom{n}{n/2}+\binom{n}{(n/2)+1}-1$ if and only if*
(i) $\mathscr{A} = \binom{\mathbb{N}_n}{n/2}$, $\mathscr{B} \subset \binom{\mathbb{N}_n}{(n/2)+1}$ and $|\mathscr{B}| = \binom{n}{(n/2)+1} - 1$, or
(ii) $\mathscr{A} \subset \binom{\mathbb{N}_n}{n/2}$, $|\mathscr{A}| = \binom{n}{n/2} - 1$, and $\mathscr{B} = \binom{\mathbb{N}_n}{(n/2)+1}$.

Kruskal-Katona Theorem (KKT) is closely related to the *squashed order* of the k-sets. The squash relations $\leq_s$ and $<_s$ are defined as follows. For $A, B \in \binom{\mathbb{N}_n}{k}$, $A \leq_s B$ if the largest element of the symmetric difference $(A-B)\cup(B-A)$ is in B. Furthermore, denote $A <_s B$ if $A \leq_s B$ and $A \neq B$. For example, the 3-subsets of $\mathbb{N}_5$ in squashed order are: $\mathbf{123} <_s \mathbf{124} <_s \mathbf{134} <_s \mathbf{234} <_s \mathbf{125} <_s \mathbf{135} <_s \mathbf{235} <_s \mathbf{145} <_s \mathbf{245} <_s \mathbf{345}$. Here, we omit the braces and write $\boldsymbol{abc}$ to represent the set $\{a, b, c\}$, if there is no ambiguity. We shall denote the collections of the first m and last m k-subsets of $\mathbb{N}_n$ in squashed order by $F_{n,k}(m)$ and $L_{n,k}(m)$ respectively.

For a family $\mathscr{A} \subseteq \binom{\mathbb{N}_n}{k}$, the *shadow* and *shade* of $\mathscr{A}$ are defined as

$$\Delta\mathscr{A} = \{X \subseteq \mathbb{N}_n \mid |X| = k-1, X \subset Y \text{ for some } Y \in \mathscr{A}\}, \text{ if } k > 0, \text{ and}$$
$$\nabla\mathscr{A} = \{X \subseteq \mathbb{N}_n \mid |X| = k+1, Y \subset X \text{ for some } Y \in \mathscr{A}\}, \text{ if } k < n$$

respectively.

Then, KKT says that the shadow of a family $\mathscr{A}$ of k-sets has size at least that of the shadow of the first $|\mathscr{A}|$ k-sets in squashed order.

Theorem 2.7 (Kruskal [13], Katona [10] and Clements and Lindström [4]). *Let $\mathscr{A}$ be a collection of k-sets of $\mathbb{N}_n$ and suppose the k-binomial representation of $|\mathscr{A}|$ is $|\mathscr{A}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t}$, where $a_k > a_{k-1} > \cdots > a_t \geq t \geq 1$. Then,*

$$|\Delta\mathscr{A}| \geq |\Delta F_{n,k}(|\mathscr{A}|)| = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_t}{t-1}.$$

By considering the complements of sets in $F_{n,k}(m)$, the next lemma can be proved.

Lemma 2.8. *For any integer $0 \leq m \leq \binom{n}{k}$, $|\Delta F_{n,k}(m)| = |\nabla L_{n,n-k}(m)|$.*

Definition 2.9 (Wong and Tay [25]). Let n, r and m be integers such that $0 \leq m \leq \binom{n}{r}$. Define

$$\kappa_{n,r}(m) = |\Delta F_{n,r}(m)| - m \text{ and } \kappa^*_{n,r}(m) = \min_{0 \leq j \leq m} \kappa_{n,r}(j).$$

We remark that $\kappa_{n,\frac{n}{2}}(m) = |\nabla L_{n,\frac{n}{2}}(m)| - m$ by Lemma 2.8. Using KKT, Wong and Tay [25] derived an upper bound for cross-intersecting antichains with at most k disjoint pairs.

Theorem 2.10 (Wong and Tay [25]). *Let $n \geq 4$ be an even integer and $\mathscr{A}$ and $\mathscr{B}$ be two antichains on $\mathbb{N}_n$. Suppose there exist orderings of the elements $A_1, A_2, \ldots, A_{|\mathscr{A}|}$ in $\mathscr{A}$, and $B_1, B_2, \ldots, B_{|\mathscr{B}|}$ in $\mathscr{B}$, and some integer $k \leq \min\{|\mathscr{A}|, |\mathscr{B}|\}$, such that $A_i \cap B_j = \emptyset$ only if $i = j \leq k$. Then,*

$$|\mathscr{A}| + |\mathscr{B}| \leq \binom{n}{n/2} + \binom{n}{(n/2)+1} - \kappa^*_{n,\frac{n}{2}}(k),$$

where $\kappa^*_{n,\frac{n}{2}}(k) = 0$ *if* $k < 1 + \sum_{i=1}^{n/2} \binom{2i-1}{i}$ *and* $\kappa^*_{n,\frac{n}{2}}(k) < 0$ *otherwise. Furthermore, equality holds if*

(i) $k < 1 + \sum_{i=1}^{n/2} \binom{2i-1}{i}$, $\mathscr{A} = \binom{\mathbb{N}_n}{n/2}$ *and* $\mathscr{B} = \binom{\mathbb{N}_n}{(n/2)+1}$, *or*

(ii) $k \geq 1 + \sum_{i=1}^{n/2} \binom{2i-1}{i}$, $\mathscr{A} = \binom{\mathbb{N}_n}{n/2}$ *and* $\mathscr{B} = L_{n,\frac{n}{2}}(m) \cup \binom{\mathbb{N}_n}{(n/2)+1} - \nabla L_{n,\frac{n}{2}}(m)$, *where* $0 < m \leq k$ *is an integer such that* $\kappa^*_{n,\frac{n}{2}}(k) = \kappa_{n,\frac{n}{2}}(m)$.

Lastly, we prove here some key properties of an optimal orientation of a vertex-multiplication graph $\mathcal{T}$ in $\mathscr{C}_0$. Let D be an orientation of $\mathcal{T}$. If v_p and v_q, $1 \leq p, q \leq n$ and $p \neq q$, are adjacent vertices in the parent graph G, then for each i, $1 \leq i \leq s_p$, we denote by $O^{v_q}_D((i, v_p)) = \{(j, v_q) \mid (i, v_p) \to (j, v_q), 1 \leq j \leq s_q\}$ and $I^{v_q}_D((i, v_p)) = \{(j, v_q) \mid (j, v_q) \to (i, v_p), 1 \leq j \leq s_q\}$. If there is no ambiguity, we shall omit the subscript D for the above notation.

The next lemma is important but easy to verify.

Lemma 2.11. ***(Duality)*** *Let D be an orientation of a graph G. Let $\tilde{D}$ be the orientation of G such that $uv \in A(\tilde{D})$ if and only if $vu \in A(D)$. Then, $d(\tilde{D}) = d(D)$.*

Lemma 2.12. *Let D be an orientation of a $\mathcal{T}$ where $d(D) = 4$. Then, $d_D((p, [\alpha, i]), (q, [j])) = d_D((q, [j]), (p, [\alpha, i])) = 3$ for all $1 \leq i, j \leq \deg_T(\mathsf{c})$, $i \neq j$, $1 \leq \alpha \leq \deg_T([i]) - 1$, $1 \leq p \leq s_{[\alpha,i]}$ and $1 \leq q \leq s_{[j]}$.*

Proof: Note that $3 = d_T([\alpha, i], [j]) \leq d_D((p, [\alpha, i]), (q, [j])) \leq d(D) = 4$. Since there is no $[\alpha, i] - [j]$ path of even length in T, there is no $(p, [\alpha, i]) - (q, [j])$ path of even length in $\mathcal{T}$, in particular, no path of length 4. Hence, $d_D((p, [\alpha, i]), (q, [j])) = 3$. Similarly, $d_D((q, [j]), (p, [\alpha, i])) = 3$ may be proved. □

Since we are going to use this fact repeatedly, we state the following obvious lemma.

Lemma 2.13. *Let D be an orientation of a $\mathcal{T}$ where $d(D) = 4$. For $1 \leq i \leq \deg_T(\mathsf{c})$,*
(a) if $s_{[i]} = 2$, then for $1 \leq p \leq s_{[\alpha,i]}$ and $1 \leq \alpha \leq \deg_T([i]) - 1$, either $(2, [i]) \to (p, [\alpha, i]) \to (1, [i])$ or $(1, [i]) \to (p, [\alpha, i]) \to (2, [i])$.
(b) if $s_{[i]} = 3$, then for $1 \leq p \leq s_{[\alpha,i]}$ and $1 \leq \alpha \leq \deg_T([i]) - 1$, either $|O((p, [\alpha, i]))| = 1$ or $|I((p, [\alpha, i]))| = 1$.

Proof: Both statements follow from the fact that $O((p, [\alpha, i])) \neq \emptyset$ and $I((p, [\alpha, i])) \neq \emptyset$ for all $p = 1, 2, \ldots, s_{[\alpha,i]}$ so that D is a strong orientation. □

Example 2.14. To help the reader understand the following lemmas and the proof of Proposition 3.9, we use the orientation D shown in Figures 5 and 6 for this example. It will be shown later that $d(D) = 4$.
(a) Observe that $O((1, [1, i])) = \{(3, [i])\}$ for $i = 5, 6$, and $O^{\mathsf{c}}((3, [5])) = \{(1, \mathsf{c}), (2, \mathsf{c}), (3, \mathsf{c})\}$ and $O^{\mathsf{c}}((3, [6])) = \{(2, \mathsf{c}), (3, \mathsf{c}), (4, \mathsf{c})\}$ are independent.
(b) Note that $O((1, [1, 1])) = \{(1, [1]), (2, [1])\}$, $O^{\mathsf{c}}((1, [1])) = \{(1, \mathsf{c}), (2, \mathsf{c})\}$, $O^{\mathsf{c}}((2, [1])) = \{(3, \mathsf{c}), (4, \mathsf{c})\}$. It is easy to check that $O^{\mathsf{c}}((3, [5])) \not\subseteq O^{\mathsf{c}}((p, [1]))$ for $p = 1, 2$, and $O^{\mathsf{c}}((1, [1])) \cup O^{\mathsf{c}}((2, [1])) \not\subseteq O^{\mathsf{c}}((3, [5]))$.
In Lemmas 2.15 and 2.16, we prove that these observations hold generally.

Lemma 2.15. *Let D be an orientation of a $\mathcal{T}$ where $d(D) = 4$ and $1 \le i, j \le \deg_T(\mathsf{c})$, $i \ne j$, $1 \le \alpha \le \deg_T([i]) - 1$, $1 \le \beta \le \deg_T([j]) - 1$. Suppose $O^{\mathsf{c}}(u_i) = O^{\mathsf{c}}(v_i)$ for any $u_i, v_i \in O((1, [\alpha, i]))$ and $O^{\mathsf{c}}(u_j) = O^{\mathsf{c}}(v_j)$ for any $u_j, v_j \in O((1, [\beta, j]))$, then $O^{\mathsf{c}}(w_i)$ and $O^{\mathsf{c}}(w_j)$ are independent for any $w_i \in O((1, [\alpha, i]))$ and $w_j \in O((1, [\beta, j]))$.*

Proof: By Lemma 2.12, $d_D((1, [\alpha, i]), w_j) = 3$. Hence, it follows that $d_D(w_i, w_j) = 2$ and $O^{\mathsf{c}}(w_i) \not\subseteq O^{\mathsf{c}}(w_j)$. A similar argument shows $O^{\mathsf{c}}(w_j) \not\subseteq O^{\mathsf{c}}(w_i)$. □

Lemma 2.16. *Let D be an orientation of a $\mathcal{T}$ where $d(D) = 4$. Suppose $O((1, [\alpha, i])) = \{(1, [i])\}$ and $O((1, [\beta, j])) = \{(1, [j]), (2, [j])\}$ for $1 \le i, j \le \deg_T(\mathsf{c})$, $i \ne j$, $1 \le \alpha \le \deg_T([i]) - 1$, and $1 \le \beta \le \deg_T([j]) - 1$. Then, for each $p = 1, 2, \ldots, s_{[j]}$,*
(a) $O^{\mathsf{c}}((1, [i])) \not\subseteq O^{\mathsf{c}}((p, [j]))$,
(b) $O^{\mathsf{c}}((1, [j])) \cup O^{\mathsf{c}}((2, [j])) \not\subseteq O^{\mathsf{c}}((1, [i]))$.

Proof: (a) can be proved similarly to Lemma 2.15. By Lemma 2.12, $d_D((1, [\beta, j]), (1, [i])) = 3$, which implies $d_D((p, [j]), (1, [i])) = 2$ for some $p = 1, 2$. Hence, (b) follows. □

In view of the Duality Lemma, we remark that Lemmas 2.15 and 2.16 have their respective dual analogues in which the notion of 'out-sets' is replaced by 'in-sets'.

Lemma 2.17. *Let D be an orientation of a complete bipartite graph $K(p, q)$ with partite sets $V_1 = \{1_1, 1_2, \ldots, 1_p\}$ and $V_2 = \{2_1, 2_2, \ldots, 2_q\}$, $q \ge p \ge 3$. Suppose further for $1 \le i \le p$ that $\lambda_i \to 2_i \to \bar{\lambda}_i$, where $\lambda_i = \{1_i, 1_{i+1}, \ldots, 1_{i+\lceil \frac{p}{2} \rceil - 1}\}$. Then, $d_D(1_i, 1_j) = 2$ for any $1 \le i, j \le p$, $i \ne j$.*

Proof: Let $t \in \mathbb{N}_p$ such that $t \equiv i - \lceil \frac{p}{2} \rceil + 1 \pmod{p}$. Since $1_i \to \{2_i, 2_t\}$ and $V_1 - \{1_i\} \subseteq O(2_i) \cup O(2_t)$, it follows that $d_D(1_i, 1_j) = 2$ for $i \ne j$. □

3. Proof of Theorem 1.8

In proving the "only if" direction of the following propositions, we shall use a common setup forged with the following notions. For a $\mathcal{T}$, let D be an orientation of $\mathcal{T}$ with $d(D) = 4$. If $A_2 \ne \emptyset$, then by Lemma 2.13(a), we may assume without loss of generality in D that

$$(2, [i]) \to (1, [1, i]) \to (1, [i]) \text{ for any } [i] \in A_2. \tag{3.1}$$

Also, we let

$$B_2^O = \{O^{\mathsf{c}}((1, [i])) \mid [i] \in A_2\} \text{ and } B_2^I = \{I^{\mathsf{c}}((2, [i])) \mid [i] \in A_2\}. \tag{3.2}$$

Note that $(1, [i])$ ($(2, [i])$ resp.) is effectively the only 'outlet' ('inlet' resp.) for the vertex $(1, [1, i])$ if $[i] \in A_2$.

Analogously, if $A_3 \ne \emptyset$, then by Lemma 2.13(b), we can partition A_3 into A_3^O and A_3^I, where

$$\left.\begin{aligned} A_3^O &= \{[i] \mid \forall \alpha, 1 \le \alpha \le \deg_T([i]) - 1, \forall p, 1 \le p \le s_{[\alpha,i]}, |O((p, [\alpha, i]))| = 1\}, \\ A_3^I &= \{[i] \mid \exists \alpha, 1 \le \alpha \le \deg_T([i]) - 1, \exists p, 1 \le p \le s_{[\alpha,i]}, |O((p, [\alpha, i]))| = 2\}. \end{aligned}\right\} \tag{3.3}$$

Without loss of generality, we assume in D that

$$\left.\begin{aligned} &\{(1, [i]), (2, [i])\} \to (1, [1, i]) \to (3, [i]) \text{ if } [i] \in A_3^O, \\ \text{and } &(3, [i]) \to (1, [1, i]) \to \{(1, [i]), (2, [i])\} \text{ if } [i] \in A_3^I. \end{aligned}\right\} \tag{3.4}$$

We also let

$$B_3^O = \{O^{\mathsf{c}}((3,[i])) \mid [i] \in A_3^O\} \text{ and } B_3^I = \{I^{\mathsf{c}}((3,[i])) \mid [i] \in A_3^I\}. \quad (3.5)$$

Note that $(3,[i])$ is effectively the only 'outlet' ('inlet' resp.) for the vertex $(1,[1,i])$ if $[i] \in A_3^O$ (A_3^I resp.). Furthermore, both $B_2^O \cup B_3^O$ and $B_2^I \cup B_3^I$ are antichains on $(\mathbb{N}_s, \mathsf{c})$ by Lemma 2.15 and its dual respectively.

Example 3.1. Let D be the orientation shown in Figures 5 and 6. Then, $A_3^I = \{[1],[2],[3],[4]\}$, $A_3^O = \{[5],[6]\}$, $B_3^I = \{\{(2,\mathsf{c}),(3,\mathsf{c})\},\{(1,\mathsf{c}),(4,\mathsf{c})\},\{(1,\mathsf{c}),(3,\mathsf{c})\},\{(2,\mathsf{c}),(4,\mathsf{c})\}\}$, and $B_3^O = \{\{(1,\mathsf{c}),(2,\mathsf{c}),(3,\mathsf{c})\},\{(2,\mathsf{c}),(3,\mathsf{c}),(4,\mathsf{c})\}\}$.

As the problem differs for $s = 2$ from $s \geq 4$, we consider them separately.

Proposition 3.2. *Suppose $s = 2$ and $A_2 \cup A_3 \neq \emptyset$ for a $\mathcal{T}$. Then, $\mathcal{T} \in \mathscr{C}_0$ if and only if $\deg_T(\mathsf{c}) = 2$.*

Proof: ($\Rightarrow$) Since $\mathcal{T} \in \mathscr{C}_0$, there exists an orientation D of $\mathcal{T}$, where $d(D) = 4$. As $A_2 \cup A_3 \neq \emptyset$, we assume (3.1)-(3.5) here. From $d(T) = 4$, it follows that $|A_{\geq 2}| \geq 2$. We shall consider two cases to show $|A_{\geq 2}| = 2$ and $E = \emptyset$.

Case 1. $|A_2 \cup A_3^O| > 0$.

Let $[i^*] \in A_2 \cup A_3^O$ and $\delta = 1$ if $[i^*] \in A_2$, and $\delta = 3$ if $[i^*] \in A_3^O$. For all $[i] \in N_T(\mathsf{c}) - \{[i^*]\}$ and all $p = 1, 2, \ldots, s_{[i]}$, since $d_D((1,[1,i^*]),(p,[i])) = 3 = d_D((1,[1,i]),(\delta,[i^*]))$ by Lemma 2.12, we may assume without loss of generality that $(2,\mathsf{c}) \to (\delta,[i^*]) \to (1,\mathsf{c})$, which implies $(1,\mathsf{c}) \to (p,[i]) \to (2,\mathsf{c})$. Now, if $[i],[j] \in A_{\geq 2} - \{[i^*]\}$, then $d_D((1,[1,i]),(1,[j])) > 3$, a contradiction to Lemma 2.12. Hence, $|A_{\geq 2} - \{[i^*]\}| \leq 1$ and thus, $|A_{\geq 2}| = 2$. If $E \neq \emptyset$, then a similar argument follows for $[i] \in A_{\geq 2} - \{[i^*]\}$ and $[j] \in E$.

Case 2. $|A_2 \cup A_3^O| = 0$ and $|A_3^I| > 0$.

Then, A_3^I behaves like A_3^O in $\tilde{D}$. The result follows from Case 1 by the Duality Lemma.

Remark 3.3. We note the difference in the definition (3.3) of A_3^O and A_3^I respectively. For the argument, we actually needed only a partition A_3^O and A_3^I of A_3 satisfying

$$\begin{aligned}
[i] \in A_3^O &\implies \exists \alpha, 1 \leq \alpha \leq \deg_T([i]) - 1, \exists p, 1 \leq p \leq s_{[\alpha,i]}, |O((p,[\alpha,i]))| = 1,\\
\text{and } [i] \in A_3^I &\implies \exists \alpha, 1 \leq \alpha \leq \deg_T([i]) - 1, \exists p, 1 \leq p \leq s_{[\alpha,i]}, |I((p,[\alpha,i]))| = 1.
\end{aligned}$$

If A_3^O and A_3^I were each defined using existential quantifiers instead, their intersection may be nonempty. We may arbitrarily include these elements in A_3^O or A_3^I (but not both) to get a partition. However, for the sake of a well-defined partition, we used (3.3). We emphasize that this does not affect the duality effect in the argument and shall repeatedly apply this.

($\Leftarrow$) This follows from Theorem 1.7(a). □

For the following, note that Proposition 3.4 holds for all integers $s \geq 2$ while Proposition 3.5 and Corollary 3.8 hold for all integers $s \geq 3$.

Proposition 3.4. *If $s \geq 2$ and $A_2 = A_3 = \emptyset$ for a $\mathcal{T}$, then $\mathcal{T} \in \mathscr{C}_0$.*

Proof: Let $\mathcal{H} = T(t_1, t_2, \ldots, t_n)$ be the subgraph of $\mathcal{T}$, where $t_{[i]} = 4$ for all $[i] \in \mathcal{T}(A_{\geq 4})$ and $t_v = 2$ otherwise. We will use A_j for $\mathcal{H}(A_j)$ for the remainder of this proof. Note that $A_j \neq \emptyset$ if and only if $j = 4$. Define an orientation D for $\mathcal{H}$ as follows.

$$\{(2,[i]),(3,[i])\} \to (1,[\alpha,i]) \to \{(1,[i]),(4,[i])\} \to (2,[\alpha,i]) \to \{(2,[i]),(3,[i])\},$$
$$\text{and } \{(1,[i]),(2,[i])\} \to (1,\mathsf{c}) \to \{(3,[i]),(4,[i])\} \to (2,\mathsf{c}) \to \{(1,[i]),(2,[i])\}$$

for all $[i] \in A_4$ and $1 \leq \alpha \leq \deg_T([i]) - 1$.

$$(2,\mathsf{c}) \to \{(1,[j]),(2,[j])\} \to (1,\mathsf{c})$$

for all $[j] \in E$. (See Figure 2 for D when $s = 2$.)

It can be verified that $d(D) = 4$; this part of the proof is omitted for brevity and we refer the interested reader to [24] for details. Since every vertex lies in a directed C_4 for D and $d(D) = 4$, $\bar{d}(\mathcal{T}) \leq \max\{4, d(D)\}$ by Lemma 1.3. With $\bar{d}(\mathcal{T}) \geq d(T) = 4$, it follows that $\bar{d}(\mathcal{T}) = 4$. □

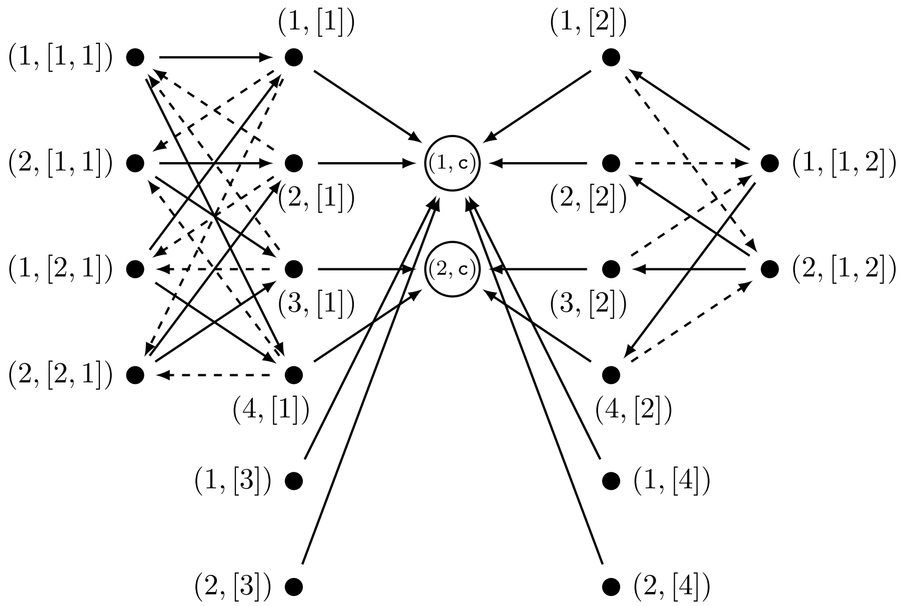

Figure 2: Orientation D, where $A_4 = \{[1], [2]\}$, $E = \{[3], [4]\}$.
Note that the parent graph is the tree in Figure 1.

Note: For clarity, the arcs directed from (p, c) to $(q, [i])$ are omitted, while the arcs directed from $(q, [i])$ to $(r, [\alpha, i])$ are represented by dashed (- - -) lines. The same simplification is used for Figures 3 to 10.

Proposition 3.5. *Suppose $s \geq 3$ and $A_{\geq 3} = \emptyset$ for a $\mathcal{T}$. Then,*

$$\mathcal{T} \in \mathscr{C}_0 \iff \begin{cases} |A_2| \leq \binom{s}{\lceil s/2 \rceil} - 1, & \text{if } |A_2| < \deg_T(\mathsf{c}), \\ |A_2| \leq \binom{s}{\lceil s/2 \rceil}, & \text{if } |A_2| = \deg_T(\mathsf{c}). \end{cases}$$

Proof: ($\Rightarrow$) Since $\mathcal{T} \in \mathscr{C}_0$, there exists an orientation D of $\mathcal{T}$, where $d(D) = 4$. As $A_2 \neq \emptyset$, we assume (3.1)-(3.2) here. By Sperner's theorem, $|A_2| = |B_2^O| \leq \binom{s}{\lceil s/2 \rceil}$. So, we are done if $|A_2| = \deg_T(\mathsf{c})$.

Now, assume $|A_2| < \deg_T(\mathsf{c})$ and let $[i^*] \in E$. If $|O^{\mathsf{c}}((1,[i^*]))| \geq \lceil \frac{s}{2} \rceil$, then $d_D((1,[1,i]),(1,[i^*])) = 3$ implies $O^{\mathsf{c}}((1,[i])) \cap I^{\mathsf{c}}((1,[i^*])) \neq \emptyset$ for all $[i] \in A_2$. It follows from Lih's theorem that $|A_2| = |B_2^O| \leq \binom{s}{\lceil s/2 \rceil} - \binom{s-|I^{\mathsf{c}}((1,[i^*]))|}{\lceil s/2 \rceil} \leq \binom{s}{\lceil s/2 \rceil} - \binom{\lceil s/2 \rceil}{\lceil s/2 \rceil} = \binom{s}{\lceil s/2 \rceil} - 1$. If $|O^{\mathsf{c}}((1,[i^*]))| \leq \lfloor \frac{s}{2} \rfloor$, then $d_D((1,[i^*]),(1,[1,i])) = 3$ implies $I^{\mathsf{c}}((2,[i])) \cap O^{\mathsf{c}}((1,[i^*])) \neq \emptyset$ for all $[i] \in A_2$. It follows from Lih's theorem that $|A_2| = |B_2^I| \leq \binom{s}{\lceil s/2 \rceil} - \binom{s-|O^{\mathsf{c}}((1,[i^*]))|}{\lceil s/2 \rceil} \leq \binom{s}{\lceil s/2 \rceil} - \binom{\lceil s/2 \rceil}{\lceil s/2 \rceil} = \binom{s}{\lceil s/2 \rceil} - 1$.

Remark 3.6. On account of the above part, it is intuitive to let $O^{\mathsf{c}}((1,[i])) = O^{\mathsf{c}}((2,[i]))$ and $|O^{\mathsf{c}}((1,[i]))| = \lfloor \frac{s}{2} \rfloor$ in constructing an optimal orientation D of $\mathcal{T}$. Indeed, this is our plan if $|A_2|$ is big enough (i.e., $|A_2| \geq s$). However, there are some potential drawbacks of this approach if $|A_2|$ is small (i.e., $|A_2| < s$). For instance, consider $s = 5$ and $\deg_T(\mathsf{c}) = |A_2| = 2$. If we assigned $O^{\mathsf{c}}((p,[1])) = \{(1,\mathsf{c}),(2,\mathsf{c})\}$ and $O^{\mathsf{c}}((p,[2])) = \{(1,\mathsf{c}),(3,\mathsf{c})\}$ for $p = 1,2$, then $\deg^+((1,\mathsf{c})) = 0$ and $\deg^-((j,\mathsf{c})) = 0$ for $j = 4,5$. Consequently, D will not be a strong orientation. Hence, we consider cases dependent on $|A_2|$ to circumvent this problem; namely, they are Cases 1 and 2 for small $|A_2|$, and Cases 3 and 4 for large $|A_2|$.

($\Leftarrow$) Without loss of generality, assume $A_2 = \{[i] \mid i \in \mathbb{N}_{|A_2|}\}$. Thus, it is taken that $E = \{[i] \mid i \in \mathbb{N}_{\deg_T(\mathsf{c})} - \mathbb{N}_{|A_2|}\}$ if $|A_2| < \deg_T(\mathsf{c})$. Let $\mathcal{H} = T(t_1,t_2,\ldots,t_n)$ be the subgraph of $\mathcal{T}$, where $t_{\mathsf{c}} = s$ and $t_v = 2$ for all $v \neq \mathsf{c}$. We will use A_j for $\mathcal{H}(A_j)$ for the remainder of this proof.

Case 1. $|A_2| = \deg_T(\mathsf{c})$ (i.e., $E = \emptyset$) and $|A_2| \leq s$.

Define an orientation D_1 for $\mathcal{H}$ as follows.

$$(2,[i]) \to (1,[\alpha,i]) \to (1,[i]) \to (2,[\alpha,i]) \to (2,[i]) \tag{3.6}$$

for all $[i] \in A_2$ and $1 \leq \alpha \leq \deg_T([i]) - 1$.

$$(\mathbb{N}_s,\mathsf{c}) - \{(i,\mathsf{c})\} \to \{(1,[i]),(2,[i])\} \to (i,\mathsf{c}) \tag{3.7}$$

for all $1 \leq i \leq |A_2| - 1$.

$$\begin{aligned}(\mathbb{N}_s,\mathsf{c}) - \{(k,\mathsf{c}) \mid |A_2| \leq k \leq s\} &\to \{(1,[|A_2|]),(2,[|A_2|])\} \\ &\to \{(k,\mathsf{c}) \mid |A_2| \leq k \leq s\}.\end{aligned} \tag{3.8}$$

Claim 1: For all $v,w \in V(D_1)$, $d_{D_1}(v,w) \leq 4$.

Case 1.1. $v,w \in \{(1,[\alpha,i]),(2,[\alpha,i]),(1,[i]),(2,[i])\}$ for each $[i] \in A_2$ and $1 \leq \alpha \leq \deg_T([i]) - 1$.

This is clear since (3.6) is a directed C_4.

Case 1.2. For each $[i],[j] \in A_2$, $i \neq j$, each $1 \leq \alpha \leq \deg_T([i]) - 1$, and each $1 \leq \beta \leq \deg_T([j]) - 1$,
(i) $v = (p,[\alpha,i]), w = (q,[\beta,j])$ for $p,q = 1,2$, or
(ii) $v = (p,[\alpha,i]), w = (q,[i])$ for $p,q = 1,2$, or
(iii) $v = (p,[i]), w = (q,[\beta,j])$ for $p,q = 1,2$.

If $i \neq j$, then, by (3.6)-(3.8), $(p,[\alpha,i]) \to (p,[i]) \to (i,\mathsf{c}) \to (1,[j]) \to (2,[\beta,j])$ and $(p,[\alpha,i]) \to (p,[i]) \to (i,\mathsf{c}) \to (2,[j]) \to (1,[\beta,j])$.

Case 1.3. $v = (x_1, \mathsf{c})$ and $w = (x_2, \mathsf{c})$ for $x_1 \neq x_2$ and $1 \leq x_1, x_2 \leq s$.

If $x_2 < |A_2|$, then $(x_1, \mathsf{c}) \to (1, [x_2]) \to (x_2, \mathsf{c})$ by (3.7). If $x_1 < |A_2| \leq x_2 \leq s$, then $(x_1, \mathsf{c}) \to (1, [|A_2|]) \to (x_2, \mathsf{c})$ by (3.7)-(3.8). If $|A_2| \leq x_1, x_2 \leq s$, then $(x_1, \mathsf{c}) \to (1, [1]) \to (1, \mathsf{c}) \to (1, [|A_2|]) \to (x_2, \mathsf{c})$ by (3.7)-(3.8).

Case 1.4. $v \in \{(1, [i]), (2, [i]), (1, [\alpha, i]), (2, [\alpha, i])\}$ for each $[i] \in A_2$, $1 \leq \alpha \leq \deg_T([i]) - 1$, and $w = (j, \mathsf{c})$ for $1 \leq j \leq s$.

If $j = i$, or $i = |A_2| \leq j \leq s$, then $(p, [\alpha, i]) \to (p, [i]) \to (j, \mathsf{c})$ for $p = 1, 2$, by (3.6)-(3.8). If $j \neq i$ and $j < |A_2|$, then $(p, [\alpha, i]) \to (p, [i]) \to (i, \mathsf{c}) \to (1, [j]) \to (j, \mathsf{c})$ for $p = 1, 2$, by (3.6)-(3.7). If $i < |A_2| \leq j \leq s$, then $(p, [\alpha, i]) \to (p, [i]) \to (i, \mathsf{c}) \to (1, [|A_2|]) \to (j, \mathsf{c})$ for $p = 1, 2$, by (3.6)-(3.8).

Case 1.5. $v = (j, \mathsf{c})$ for each $1 \leq j \leq s$, and $w \in \{(1, [i]), (2, [i]), (1, [\alpha, i]), (2, [\alpha, i])\}$ for each $[i] \in A_2$ and $1 \leq \alpha \leq \deg_T([i]) - 1$.

If $j < |A_2|$ and $j \neq i$, or $i < |A_2| \leq j \leq s$, then $(j, \mathsf{c}) \to (p, [i]) \to (3 - p, [\alpha, i])$ for $p = 1, 2$, by (3.6)-(3.8). If $i = j < |A_2|$, then $(j, \mathsf{c}) \to (1, [|A_2|]) \to (|A_2|, \mathsf{c}) \to (p, [i]) \to (3 - p, [\alpha, i])$ for $p = 1, 2$, by (3.6)-(3.8). If $i = |A_2| \leq j \leq s$, then $(j, \mathsf{c}) \to (1, [1]) \to (1, \mathsf{c}) \to (p, [|A_2|]) \to (3 - p, [\alpha, |A_2|])$, for $p = 1, 2$, by (3.6)-(3.8).

Case 1.6. $v = (p, [i])$ and $w = (q, [j])$, where $1 \leq p, q \leq 2$, $i \neq j$, and $[i], [j] \in A_2$.

This follows from the fact that $|O^{\mathsf{c}}((p, [i]))| > 0$, $|I^{\mathsf{c}}((q, [j]))| > 0$, and $d_{D_1}((r_1, \mathsf{c}), (r_2, \mathsf{c})) = 2$ for any $r_1 \neq r_2$ and $1 \leq r_1, r_2 \leq |A_2|$ by Case 1.3.

Case 2. $|A_2| < \deg_T(\mathsf{c})$ (i.e., $E \neq \emptyset$) and $|A_2| < s$.

Define an orientation D_2 for $\mathcal{H}$ as follows.

$$(2, [i]) \to (1, [\alpha, i]) \to (1, [i]) \to (2, [\alpha, i]) \to (2, [i]) \tag{3.9}$$

for all $[i] \in A_2$ and $1 \leq \alpha \leq \deg_T([i]) - 1$.

$$(\mathbb{N}_s, \mathsf{c}) - \{(i, \mathsf{c})\} \to \{(1, [i]), (2, [i])\} \to (i, \mathsf{c}) \tag{3.10}$$

for all $1 \leq i \leq |A_2|$.

$$\begin{aligned} &(\mathbb{N}_{|A_2|}, \mathsf{c}) \to \{(p, [i]) \mid p = 1, 2; [i] \in E\} \\ &\to \{(k, \mathsf{c}) \mid |A_2| < k \leq s\} \to \{(q, [j]) \mid q = 1, 2; [j] \in A_2\}. \end{aligned} \tag{3.11}$$

(See Figure 3 for D_2 when $s = 5$.)

Claim 2: For all $v, w \in V(D_2)$, $d_{D_2}(v, w) \leq 4$.

In view of the similarity between D_1 and D_2, it suffices to check the following.

Case 2.1. For each $[i] \in A_2$, each $[j] \in E$, and each $1 \leq \alpha \leq \deg_T([i]) - 1$,
(i) $v = (p, [\alpha, i]), w = (q, [j])$ for $p, q = 1, 2$, or
(ii) $v = (p, [i]), w = (q, [j])$ for $p, q = 1, 2$, or
(iii) $v = (q, [j]), w = (p, [\alpha, i])$ for $p, q = 1, 2$.

By (3.9)-(3.11), (i) and (ii) follow from $(p, [\alpha, i]) \to (p, [i]) \to (i, \mathsf{c}) \to \{(1, [j]), (2, [j])\}$. Similarly for (iii), $\{(1, [j]), (2, [j])\} \to (s, \mathsf{c}) \to (3 - p, [i]) \to (p, [\alpha, i])$.

Case 2.2. $v = (x_1, \mathsf{c})$ and $w = (x_2, \mathsf{c})$ for $x_1 \neq x_2$ and $1 \leq x_1, x_2 \leq s$.

If $x_2 \leq |A_2|$, then $(x_1, \mathsf{c}) \to (1, [x_2]) \to (x_2, \mathsf{c})$ by (3.10). If $x_1 \leq |A_2|$ and $|A_2| + 1 \leq x_2 \leq s$, then $(x_1, \mathsf{c}) \to (1, [j]) \to (x_2, \mathsf{c})$ for any $[j] \in E$ by (3.11). If $|A_2| + 1 \leq x_1, x_2 \leq s$, then $(x_1, \mathsf{c}) \to (1, [1]) \to (1, \mathsf{c}) \to (1, [j]) \to (x_2, \mathsf{c})$ for any $[j] \in E$ by (3.10) and (3.11).

Case 2.3. $v \in \{(1, [i]), (2, [i])\}$ for each $[i] \in E$, and $w = (j, \mathsf{c})$ for $1 \leq j \leq s$.

For $1 \leq j \leq |A_2|$, $\{(1, [i]), (2, [i])\} \to (s, \mathsf{c}) \to (1, [j]) \to (j, \mathsf{c})$ by (3.10) and (3.11). For $|A_2| + 1 \leq j \leq s$, $\{(1, [i]), (2, [i])\} \to (j, \mathsf{c})$ by (3.11).

Case 2.4. $v = (j, \mathsf{c})$ for each $1 \leq j \leq s$, and $w \in \{(1, [i]), (2, [i])\}$ for each $[i] \in E$.

By (3.11), for any $1 \leq j \leq |A_2|$, $(j, \mathsf{c}) \to \{(1, [i]), (2, [i])\}$. For $|A_2| + 1 \leq j \leq s$, $(j, \mathsf{c}) \to (1, [1]) \to (1, \mathsf{c}) \to \{(1, [i]), (2, [i])\}$ by (3.10) and (3.11).

Case 2.5. $v = (p, [i])$ and $w = (q, [j])$, where $1 \leq p, q \leq 2$, and $[i], [j] \in E$.

Here, it is possible that $i = j$. Note that $\{(1, [i]), (2, [i])\} \to (s, \mathsf{c}) \to (1, [1]) \to (1, \mathsf{c}) \to \{(1, [j]), (2, [j])\}$ by (3.10) and (3.11).

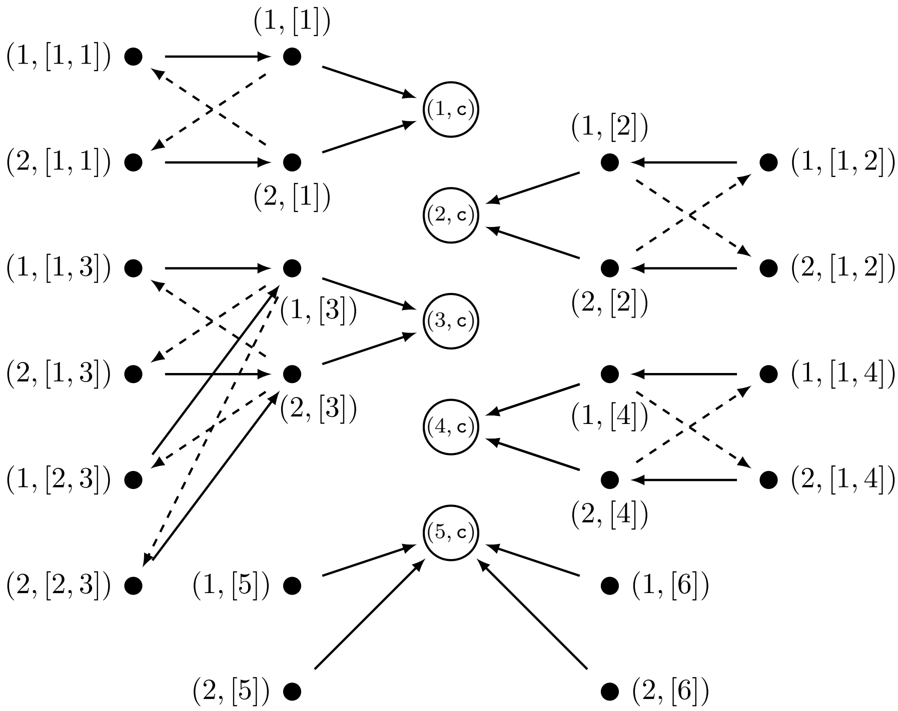

Figure 3: Orientation D_2 for $\mathcal{H}$, Case 2. $s = 5$, $\deg_T(\mathsf{c}) = 6$, $A_2 = \{[1], [2], [3], [4]\}$, $E = \{[5], [6]\}$.

To settle Cases 3 and 4 (and forthcoming propositions), we require the following notation.

Definition 3.7. Set $\{\lambda_1, \lambda_2, \ldots, \lambda_{\binom{s}{\lceil s/2 \rceil}}\} = \binom{(\mathbb{N}_s, \mathsf{c})}{\lceil s/2 \rceil}$, i.e., the set containing all $\lceil \frac{s}{2} \rceil$-subsets of $(\mathbb{N}_s, \mathsf{c})$. In particular, for $1 \leq i \leq s$, let $\lambda_i = \{(i, \mathsf{c}), (i+1, \mathsf{c}), \ldots, (i + \lceil \frac{s}{2} \rceil - 1, \mathsf{c})\}$, the sets containing $\lceil \frac{s}{2} \rceil$ vertices in consecutive (cyclic) order starting from (i, c). For example, $\lambda_2 = \{(2, \mathsf{c}), (3, \mathsf{c}), \ldots, (\lceil \frac{s}{2} \rceil + 1, \mathsf{c})\}$. The denotation of the remaining λ_i's can be arbitrary.

Case 3. $|A_2| = \deg_T(\mathsf{c})$ (i.e., $E = \emptyset$) and $s < |A_2| \le \binom{s}{\lceil s/2 \rceil}$. (If $s = 3$, this case does not apply, and we refer to Case 1 instead.)

Define an orientation D_3 for $\mathcal{H}$ as follows.

$$(2,[i]) \to (1,[\alpha,i]) \to (1,[i]) \to (2,[\alpha,i]) \to (2,[i]), \tag{3.12}$$
$$\text{and } \lambda_i \to \{(1,[i]),(2,[i])\} \to \bar{\lambda}_i \tag{3.13}$$

for all $[i] \in A_2$ and $1 \le \alpha \le \deg_T([i]) - 1$. We point out that the $\lceil \frac{s}{2} \rceil$-sets $\lambda_1, \lambda_2, \ldots, \lambda_{|A_2|}$ ($\lfloor \frac{s}{2} \rfloor$-sets $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{|A_2|}$ resp.) are used as 'in-sets' ('out-sets' resp.) to construct B_2^I (B_2^O resp.).

Claim 3: For all $v, w \in V(D_3)$, $d_{D_3}(v,w) \le 4$.

Case 3.1. $v, w \in \{(1,[\alpha,i]),(2,[\alpha,i]),(1,[i]),(2,[i])\}$ for each $[i] \in A_2$ and $1 \le \alpha \le \deg_T([i]) - 1$.

This is clear since (3.12) is a directed C_4.

Case 3.2. For each $[i],[j] \in A_2$, $i \ne j$, each $1 \le \alpha \le \deg_T([i]) - 1$, and each $1 \le \beta \le \deg_T([j]) - 1$,
(i) $v = (p,[\alpha,i]), w = (q,[\beta,j])$ for $p,q = 1,2$, or
(ii) $v = (p,[\alpha,i]), w = (q,[j])$ for $p,q = 1,2$, or
(iii) $v = (p,[i]), w = (q,[\beta,j])$ for $p,q = 1,2$.

By (3.12)-(3.13), since $O^{\mathsf{c}}((p,[i])) = \bar{\lambda}_i \not\subseteq \bar{\lambda}_j = O^{\mathsf{c}}((q,[j]))$, there exists a vertex $(x,\mathsf{c}) \in \bar{\lambda}_i \cap \lambda_j$ such that $(p,[\alpha,i]) \to (p,[i]) \to (x,\mathsf{c}) \to (3-q,[j]) \to (q,[\beta,j])$.

Case 3.3. $v = (r_1,\mathsf{c})$ and $w = (r_2,\mathsf{c})$ for $r_1 \ne r_2$ and $1 \le r_1, r_2 \le s$.

Here, we want to prove a stronger claim, $d_{D_3}((r_1,\mathsf{c}),(r_2,\mathsf{c})) = 2$. For $1 \le k \le s$, let $x_k = (1,[k])$ and observe from (3.13) that $\lambda_k \to x_k \to \bar{\lambda}_k$. The subgraph induced by $V_1 = (\mathbb{N}_s,\mathsf{c})$ and $V_2 = \{x_k \mid 1 \le k \le s\}$ is a complete bipartite graph $K(V_1,V_2)$. By Lemma 2.17, $d_{D_3}((r_1,\mathsf{c}),(r_2,\mathsf{c})) = 2$.

Case 3.4. $v \in \{(1,[i]),(2,[i]),(1,[\alpha,i]),(2,[\alpha,i])\}$ for each $[i] \in A_2$ and $1 \le \alpha \le \deg_T([i]) - 1$, and $w = (r,\mathsf{c})$ for $1 \le r \le s$.

Note that there exists some $1 \le k \le s$ such that $d_{D_3}(v,(k,\mathsf{c})) \le 2$ by (3.12)-(3.13). If $k = r$, we are done. If $k \ne r$, then $d_{D_3}((k,\mathsf{c}),(r,\mathsf{c})) = 2$ by Case 3.3. Hence, it follows that $d_{D_3}(v,w) \le d_{D_3}(v,(k,\mathsf{c})) + d_{D_3}((k,\mathsf{c}),w) = 4$.

Case 3.5. $v = (r,\mathsf{c})$ for $1 \le r \le s$ and $w \in \{(1,[i]),(2,[i]),(1,[\alpha,i]),(2,[\alpha,i])\}$ for each $[i] \in A_2$ and $1 \le \alpha \le \deg_T([i]) - 1$.

Note that there exists some $1 \le k \le s$ such that $d_{D_3}((k,\mathsf{c}),w) \le 2$ by (3.12)-(3.13). If $k = r$, we are done. If $k \ne r$, then $d_{D_3}((r,\mathsf{c}),(k,\mathsf{c})) = 2$ by Case 3.3. Hence, it follows that $d_{D_3}(v,w) \le d_{D_3}(v,(k,\mathsf{c})) + d_{D_3}((k,\mathsf{c}),w) = 4$.

Case 3.6. $v = (p,[i])$ and $w = (q,[j])$, where $1 \le p,q \le 2$ and $[i],[j] \in A_2$.

This follows from the fact that $|O^{\mathsf{c}}((p,[i]))| > 0$, $|I^{\mathsf{c}}((q,[j]))| > 0$, and $d_{D_3}((r_1,\mathsf{c}),(r_2,\mathsf{c}))$ $= 2$ for any $r_1 \ne r_2$ and $1 \le r_1, r_2 \le s$ by Case 3.3.

Case 4. $|A_2| < \deg_T(\mathsf{c})$ (i.e., $E \neq \emptyset$) and $s \leq |A_2| \leq \binom{s}{\lceil s/2 \rceil} - 1$. (If $s = 3$, this case does not apply and we refer to Case 2 instead.)

We define an orientation D_4 for $\mathcal{H}$ by making a slight enchancement to D_3. Noting that $|A_2| \leq \binom{s}{\lceil s/2 \rceil} - 1$, we include in D_4 these extra arcs:

$$\lambda_{\binom{s}{\lceil s/2 \rceil}} \to \{(1,[j]),(2,[j])\} \to \bar{\lambda}_{\binom{s}{\lceil s/2 \rceil}}$$

for all $[j] \in E$. (See Figure 4 for D_4 when $s = 5$.)

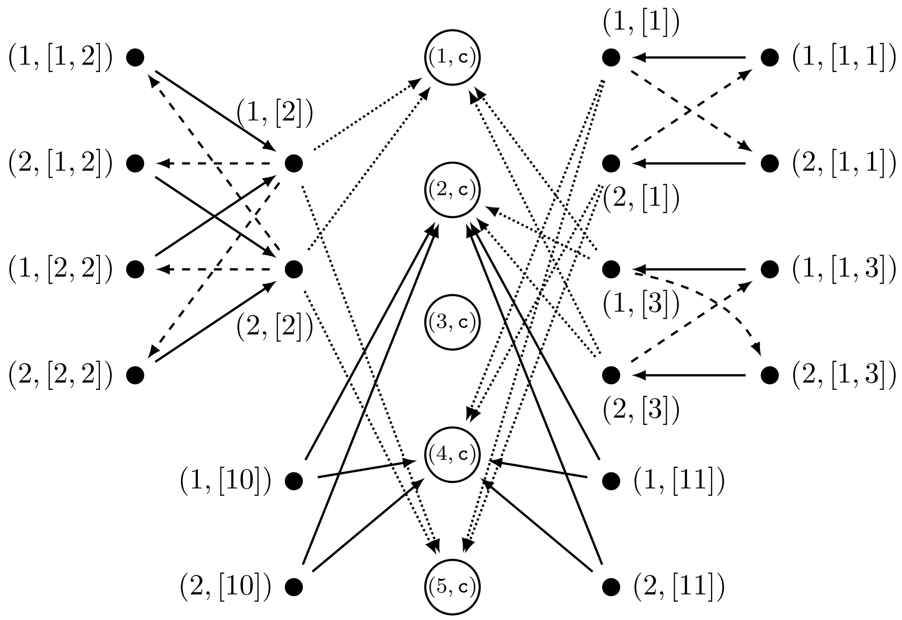

Figure 4: Orientation D_4 for $\mathcal{H}$, Case 4. $s = 5$, $\deg_T(\mathsf{c}) = 11$, $A_2 = \{[1],[2],\ldots,[9]\}$, $E = \{[10],[11]\}$. Here, we assume $\lambda_{\binom{s}{\lceil s/2 \rceil}} = \{(1,\mathsf{c}),(3,\mathsf{c}),(5,\mathsf{c})\}$. For clarity, we only show the vertices $[\alpha, i]$ and $[i]$ for $i = 1, 2, 3, 10, 11$.

Note: In addition to the simplification noted in Figure 2, in Figures 4 to 10, we use densely dotted (⋯⋯) (densely dashdotdotted (-⋯-⋯-) resp.) lines to elucidate the 'out-sets' B_2^O and B_3^O (complements of the 'in-sets' B_2^I and B_3^I resp.); and in cases where both coincide, the densely dotted lines take precedent.

Claim 4: For all $v, w \in V(D_4)$, $d_{D_4}(v,w) \leq 4$.

In view of the similarity between D_3 and D_4, it suffices to check the following.

Case 4.1. $v \in \{(1,[i]),(2,[i])\}$ for each $[i] \in E$ and $1 \leq \alpha \leq \deg_T([i]) - 1$, and $w = (r, \mathsf{c})$ for $1 \leq r \leq s$.

This follows from the fact that $|O^{\mathsf{c}}((p,[i]))| > 0$ for $p = 1, 2$, and $d_{D_4}((r_1,\mathsf{c}),(r_2,\mathsf{c})) = 2$ for any $r_1 \neq r_2$ and $1 \leq r_1, r_2 \leq s$ by Case 3.3.

Case 4.2. $v = (r, \mathsf{c})$ for $1 \leq r \leq s$ and $w \in \{(1,[i]),(2,[i])\}$ for each $[i] \in E$ and $1 \leq \alpha \leq \deg_T([i]) - 1$.

This follows from the fact that $|I^{\mathsf{c}}((p,[i]))| > 0$ for $p = 1, 2$, and $d_{D_4}((r_1,\mathsf{c}),(r_2,\mathsf{c})) = 2$ for any $r_1 \neq r_2$ and $1 \leq r_1, r_2 \leq s$ by Case 3.3.

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DOI: 10.20429/tag.2023.10106

Case 4.3. $v = (p, [i])$ and $w = (q, [j])$, where $1 \leq p, q \leq 2$ and $[i], [j] \in E$.

This follows from the fact that $|O^{\mathsf{c}}((p, [i]))| > 0$, $|I^{\mathsf{c}}((q, [j]))| > 0$, and $d_{D_4}((r_1, \mathsf{c}), (r_2, \mathsf{c})) = 2$ for any $r_1 \neq r_2$ and $1 \leq r_1, r_2 \leq s$ by Case 3.3.

Hence, $d(D_i) = 4$ for $i = 1, 2, 3, 4$. Since every vertex lies in a directed C_4 for D_i and $d(D_i) = 4$, $\bar{d}(\mathcal{T}) \leq \max\{4, d(D_i)\}$ by Lemma 1.3, and thus $\bar{d}(\mathcal{T}) = 4$. □

Corollary 3.8. *Suppose $s \geq 3$ for a $\mathcal{T}$. If*
(i) $|A_{\geq 2}| \leq \binom{s}{\lceil s/2 \rceil} - 1$, or
(ii) $|A_{\geq 2}| \leq \binom{s}{\lceil s/2 \rceil}$ and $|A_{\geq 2}| = \deg_T(\mathsf{c})$,
then $\mathcal{T} \in \mathscr{C}_0$.

Proof: Note in the proof of Proposition 3.5 that every vertex lies in a directed C_4 for each orientation D_i and $d(D_i) \leq 4$, for $i = 1, 2, 3, 4$. By Lemma 1.3, $\bar{d}(\mathcal{T}) \leq \max\{4, d(D_i)\}$ for $i = 1, 2, 3, 4$, and thus $\bar{d}(\mathcal{T}) = 4$. □

For the remaining propositions of this section, we consider even $s \geq 4$. The proof of Proposition 3.9 is centered on a reduction to cross-intersecting antichains and Theorems 2.4 and 2.6.

Proposition 3.9. *Suppose $s \geq 4$ is even, $A_2 = \emptyset$ and $A_3 \neq \emptyset$ for a $\mathcal{T}$. Then, $\mathcal{T} \in \mathscr{C}_0$ if and only if $|A_3| \leq \binom{s}{s/2} + \binom{s}{(s/2)+1} - 2$.*

Proof: ($\Rightarrow$) Since $\mathcal{T} \in \mathscr{C}_0$, there exists an orientation D of $\mathcal{T}$, where $d(D) = 4$. As $A_3 \neq \emptyset$, we assume (3.3)-(3.5) here. By Sperner's theorem, $|B_3^O| \leq \binom{s}{s/2}$ and $|B_3^I| \leq \binom{s}{s/2}$. If $|B_3^O| = 0$ or $|B_3^I| = 0$, then $|A_3| = |B_3^O| + |B_3^I| \leq \binom{s}{s/2}$. Therefore, we assume $|B_3^O| > 0$ and $|B_3^I| > 0$.

Observe also that for each $[i] \in A_3^O$ and each $[j] \in A_3^I$, $d_D((1, [i]), (1, [j])) = 4$ implies $X \cap Y \neq \emptyset$ for all $X \in B_3^O$ and all $Y \in B_3^I$. By Theorem 2.4, $|A_3| = |B_3^O| + |B_3^I| \leq \binom{s}{s/2} + \binom{s}{(s/2)+1}$. Suppose $|A_3| > \binom{s}{s/2} + \binom{s}{(s/2)+1} - 2$ for a contradiction. It follows from Theorems 2.4 and 2.6 that $\{B_3^O, B_3^I\} = \{\mathscr{A}, \mathscr{B}\}$, where
(1) $\mathscr{A} = \binom{(\mathbb{N}_s, \mathsf{c})}{s/2}$, $\mathscr{B} = \binom{(\mathbb{N}_s, \mathsf{c})}{(s/2)+1}$, or
(2) $\mathscr{A} = \binom{(\mathbb{N}_s, \mathsf{c})}{s/2}$, $\mathscr{B} \subset \binom{(\mathbb{N}_s, \mathsf{c})}{(s/2)+1}$ and $|\mathscr{B}| = \binom{s}{(s/2)+1} - 1$, or
(3) $\mathscr{A} \subset \binom{(\mathbb{N}_s, \mathsf{c})}{s/2}$, $|\mathscr{A}| = \binom{s}{s/2} - 1$, and $\mathscr{B} = \binom{(\mathbb{N}_s, \mathsf{c})}{(s/2)+1}$.

Case 1. $B_3^O = \binom{(\mathbb{N}_s, \mathsf{c})}{s/2}$.

Let $[i] \in A_3^I$. For all $[j] \in A_3^O$ and $p = 1, 2$, $d_D((1, [1, j]), (p, [i])) = 3$ implies $X \cap I^{\mathsf{c}}((p, [i])) \neq \emptyset$ for all $X \in B_3^O$. It follows that $|I^{\mathsf{c}}((p, [i]))| \geq \frac{s}{2} + 1$ for all $p = 1, 2$. As a result, $O^{\mathsf{c}}((1, [i]))$ and $O^{\mathsf{c}}((2, [i]))$ are independent. Otherwise, $O^{\mathsf{c}}((1, [i])) \cup O^{\mathsf{c}}((2, [i])) \subset X$ for some $X \in B_3^O$, which contradicts Lemma 2.16(b).

Subcase 1.1. $B_3^I = \binom{(\mathbb{N}_s, \mathsf{c})}{(s/2)+1}$.

Let $[i^*] \in A_3^I$. For all $[i] \in A_3^I - \{[i^*]\}$ and $p = 1, 2$, $d_D((p, [i^*]), (1, [1, i])) = 3$ implies $X \cap O^{\mathsf{c}}((p, [i^*])) \neq \emptyset$ for all $X \in B_3^I - \{I^{\mathsf{c}}((3, [i^*]))\}$. Consequently, we have either $|O^{\mathsf{c}}((p, [i^*]))| \geq \frac{s}{2}$, or $O^{\mathsf{c}}((p, [i^*])) = O^{\mathsf{c}}((3, [i^*]))$ for each $p = 1, 2$. Since $|I^{\mathsf{c}}((p, [i^*]))| \geq \frac{s}{2} + 1$, $|O^{\mathsf{c}}((p, [i^*]))| < \frac{s}{2}$. Hence, $O^{\mathsf{c}}((1, [i^*])) = O^{\mathsf{c}}((3, [i^*])) = O^{\mathsf{c}}((2, [i^*]))$, a contradiction to $O^{\mathsf{c}}((1, [i^*]))$ and $O^{\mathsf{c}}((2, [i^*]))$ being independent.

Subcase 1.2. $B_3^I \subset \binom{(\mathbb{N}_s,\mathsf{c})}{(s/2)+1}$ and $|B_3^I| = \binom{s}{(s/2)+1} - 1$.

Let $\binom{(\mathbb{N}_s,\mathsf{c})}{(s/2)+1} - B_3^I = \{\psi\}$. If $|O^{\mathsf{c}}((p,[i]))| \leq \frac{s}{2} - 2$ for some $[i] \in A_3^I$ and some $p = 1, 2$, then there are $\binom{|I^{\mathsf{c}}((p,[i]))|}{(s/2)+1} \geq \binom{(s/2)+2}{(s/2)+1} = \frac{s}{2} + 2 \geq 4 > 2$ $(\frac{s}{2}+1)$-subsets of $I^{\mathsf{c}}((p,[i]))$, i.e., $X \subset I^{\mathsf{c}}((p,[i]))$ for some $X \in B_3^I - \{I^{\mathsf{c}}((3,[i]))\}$, a contradiction to the dual of Lemma 2.16(a). So, for each $[i] \in A_3^I$ and each $p = 1, 2$, we have either $|O^{\mathsf{c}}((p,[i]))| \geq \frac{s}{2}$, or $O^{\mathsf{c}}((p,[i])) = O^{\mathsf{c}}((3,[i]))$, or $O^{\mathsf{c}}((p,[i])) = \bar{\psi}$. Since $|I^{\mathsf{c}}((p,[i]))| \geq \frac{s}{2} + 1$ and $O^{\mathsf{c}}((1,[i]))$ and $O^{\mathsf{c}}((2,[i]))$ are independent, we may assume without loss of generality that $O^{\mathsf{c}}((1,[i])) = \bar{\psi}$ and $O^{\mathsf{c}}((2,[i])) = O^{\mathsf{c}}((3,[i]))$ for each $[i] \in A_3^I$.

Now, we claim that there exists some $[j] \in A_3^I$ such that $|\bar{\psi} \cup O^{\mathsf{c}}((3,[j]))| = \frac{s}{2}$. Note that $|\bar{\psi} \cup O^{\mathsf{c}}((3,[j]))| = \frac{s}{2}$ if and only if $|\bar{\psi} \cap O^{\mathsf{c}}((3,[j]))| = \frac{s}{2} - 2$ if and only if $|\psi \cap O^{\mathsf{c}}((3,[j]))| = 1$. Since $\binom{|\bar{\psi}|}{(s/2)-2}\binom{|\psi|}{1} = \binom{(s/2)-1}{(s/2)-2}\binom{(s/2)+1}{1} = \frac{s^2}{4} - 1 \geq 3$ and $|B_3^I| = \binom{s}{(s/2)+1} - 1$, the claim follows. Hence, $O^{\mathsf{c}}((1,[j])) \cup O^{\mathsf{c}}((2,[j])) = O^{\mathsf{c}}((1,[j])) \cup O^{\mathsf{c}}((3,[j])) = \bar{\psi} \cup O^{\mathsf{c}}((3,[j])) = O^{\mathsf{c}}((3,[k]))$ for some $[k] \in A_3^O$. This contradicts Lemma 2.16(b).

Case 2. $B_3^I = \binom{(\mathbb{N}_s,\mathsf{c})}{s/2}$.

If $B_3^O = \binom{(\mathbb{N}_s,\mathsf{c})}{(s/2)+1}$ ($B_3^O \subset \binom{(\mathbb{N}_s,\mathsf{c})}{(s/2)+1}$ and $|B_3^O| = \binom{s}{(s/2)+1} - 1$ resp.), then the result follows from Subcase 1.1 (Subcase 1.2 resp.) by the Duality Lemma.

Case 3. $B_3^O = \binom{(\mathbb{N}_s,\mathsf{c})}{(s/2)+1}$, $B_3^I \subset \binom{(\mathbb{N}_s,\mathsf{c})}{s/2}$ and $|B_3^I| = \binom{s}{s/2} - 1$

Let $[i] \in A_3^I$. For all $[j] \in A_3^O$ and $p = 1, 2$, $d_D((1,[1,j]),(p,[i])) = 3$ implies $X \cap I^{\mathsf{c}}((p,[i])) \neq \emptyset$ for all $X \in B_3^O$. It follows that $|I^{\mathsf{c}}((p,[i]))| \geq \frac{s}{2}$ for all $p = 1, 2$. Furthermore, $O^{\mathsf{c}}((1,[i]))$ and $O^{\mathsf{c}}((2,[i]))$ are independent. Otherwise, $O^{\mathsf{c}}((1,[i])) \cup O^{\mathsf{c}}((2,[i])) \subset X$ for some $X \in B_3^O$, which contradicts Lemma 2.16(b).

Let $\binom{(\mathbb{N}_s,\mathsf{c})}{s/2} - B_3^I = \{\lambda\}$. If $|O^{\mathsf{c}}((p,[i]))| \leq \frac{s}{2} - 1$ for some $p = 1, 2$, and some $[i] \in A_3^I$, then there are $\binom{|I^{\mathsf{c}}((p,[i]))|}{s/2} \geq \binom{(s/2)+1}{s/2} = \frac{s}{2} + 1 \geq 3$ $\frac{s}{2}$-subsets of $I^{\mathsf{c}}((p,[i]))$, i.e., $X \subset I^{\mathsf{c}}((p,[i]))$ for some $X \in B_3^I - \{I^{\mathsf{c}}((3,[i]))\}$, a contradiction to the dual of Lemma 2.16(a). Consequently, we have for each $p = 1, 2$, $|O^{\mathsf{c}}((p,[i]))| \geq \frac{s}{2} + 1$, or $O^{\mathsf{c}}((p,[i])) = O^{\mathsf{c}}((3,[i]))$, or $O^{\mathsf{c}}((p,[i])) = \bar{\lambda}$. Since $|I^{\mathsf{c}}((p,[i]))| \geq \frac{s}{2}$ and $O^{\mathsf{c}}((1,[i]))$ and $O^{\mathsf{c}}((2,[i]))$ are independent, we may assume without loss of generality that $O^{\mathsf{c}}((1,[i])) = \bar{\lambda}$ and $O^{\mathsf{c}}((2,[i])) = O^{\mathsf{c}}((3,[i]))$ for each $[i] \in A_3^I$.

Now, we claim that there exists some $[j] \in A_3^I$ such that $|\bar{\lambda} \cup O^{\mathsf{c}}((3,[j]))| = \frac{s}{2} + 1$. Note that $|\bar{\lambda} \cup O^{\mathsf{c}}((3,[j]))| = \frac{s}{2} + 1$ if and only if $|\bar{\lambda} \cap O^{\mathsf{c}}((3,[j]))| = \frac{s}{2} - 1$ if and only if $|\lambda \cap O^{\mathsf{c}}((3,[j]))| = 1$. Since $\binom{|\bar{\lambda}|}{(s/2)-1}\binom{|\lambda|}{1} = \binom{s/2}{(s/2)-1}\binom{s/2}{1} = \frac{s^2}{4} \geq 4$ and $|B_3^I| = \binom{s}{(s/2)+1} - 1$, the claim follows. Hence, $O^{\mathsf{c}}((1,[j])) \cup O^{\mathsf{c}}((2,[j])) = \bar{\lambda} \cup O^{\mathsf{c}}((3,[j])) = O^{\mathsf{c}}((3,[k]))$ for some $[k] \in A_3^O$. This contradicts Lemma 2.16(b).

Case 4. $B_3^I = \binom{(\mathbb{N}_s,\mathsf{c})}{(s/2)+1}$, $B_3^O \subset \binom{(\mathbb{N}_s,\mathsf{c})}{s/2}$ and $|B_3^O| = \binom{s}{s/2} - 1$.

This follows from Case 3 by the Duality Lemma.

($\Leftarrow$) If $|A_{\geq 3}| \leq \binom{s}{s/2} - 1$, then by Corollary 3.8(i), $\mathcal{T} \in \mathscr{C}_0$. Hence, we assume $|A_{\geq 3}| \geq \binom{s}{s/2}$ hereafter, on top of the hypothesis that $|A_3| \leq \binom{s}{s/2} + \binom{s}{(s/2)+1} - 2$. If $|A_3| \geq \binom{s}{s/2} - 2$, define $A_3^\diamond = A_3$. Otherwise, let $A_3^\diamond = A_3 \cup A^*$, where A^* is an arbitrary subset of $A_{\geq 4}$ such that $|A_3^\diamond| = \binom{s}{s/2} - 2$. Then, let $A_4^\diamond = A_{\geq 4} - A_3^\diamond$. Furthermore, assume without loss of generality that $A_3^\diamond = \{[i] \mid i \in \mathbb{N}_{|A_3^\diamond|}\}$ and $A_4^\diamond = \{[i] \mid i \in \mathbb{N}_{|A_3^\diamond|+|A_4^\diamond|} - \mathbb{N}_{|A_3^\diamond|}\}$.

Let $\mathcal{H} = T(t_1, t_2, \ldots, t_n)$ be the subgraph of $\mathcal{T}$, where $t_{\mathsf{c}} = s$, $t_{[i]} = 3$ for all $[i] \in \mathcal{T}(A_3^\diamond)$, $t_{[j]} = 4$ for all $[j] \in \mathcal{T}(A_4^\diamond)$ and $t_v = 2$ otherwise. We will use A_j for $\mathcal{H}(A_j)$ for the remainder of this proof. Let $\binom{(\mathbb{N}_s, \mathsf{c})}{(s/2)+1} = \{\psi_i \mid i = 1, 2, \ldots, \binom{s}{(s/2)+1}\}$ and recall λ_i from Definition 3.7. Define an orientation D of $\mathcal{H}$ as follows.

$$\begin{aligned}&(3,[i]) \to \{(1,[\alpha,i]),(2,[\alpha,i])\} \to \{(1,[i]),(2,[i])\}, \text{ and}\\ &\bar{\lambda}_1 = \lambda_{\frac{s}{2}+1} \to (1,[i]) \to \lambda_1 \to (2,[i]) \to \lambda_{\frac{s}{2}+1}\end{aligned}$$

for all $1 \leq i \leq \binom{s}{s/2} - 2$ and all $1 \leq \alpha \leq \deg_T([i]) - 1$.

$$\lambda_{i+1} \to (3,[i]) \to \bar{\lambda}_{i+1}$$

for all $1 \leq i \leq \frac{s}{2} - 1$.

$$\lambda_{i+2} \to (3,[i]) \to \bar{\lambda}_{i+2}$$

for all $\frac{s}{2} \leq i \leq \binom{s}{s/2} - 2$, i.e., excluding λ_1 and $\lambda_{\frac{s}{2}+1}$, the $\frac{s}{2}$-sets λ_i's are used as 'in-sets' to construct B_3^I.

$$\begin{aligned}&\{(1,[j]),(2,[j])\} \to \{(1,[\beta,j]),(2,[\beta,j])\} \to (3,[j]),\\ &\lambda_1 \to (1,[j]) \to \lambda_{\frac{s}{2}+1} \to (2,[j]) \to \lambda_1, \text{ and}\\ &\bar{\psi}_{j+2-\binom{s}{s/2}} \to (3,[j]) \to \psi_{j+2-\binom{s}{s/2}}\end{aligned}$$

for all $\binom{s}{s/2} - 1 \leq j \leq |A_3|$ and all $1 \leq \beta \leq \deg_T([j]) - 1$, i.e., the $(\frac{s}{2}+1)$-sets $\psi_1, \psi_2, \ldots, \psi_{|A_3|+2-\binom{s}{s/2}}$ are used as 'out-sets' to construct B_3^O.

$$\begin{aligned}&(2,[\gamma,k]) \to \{(2,[k]),(4,[k])\} \to (1,[\gamma,k]) \to \{(1,[k]),(3,[k])\} \to (2,[\gamma,k]),\\ &\text{and } \lambda_{\frac{s}{2}+1} \to \{(1,[k]),(4,[k])\} \to \lambda_1 \to \{(2,[k]),(3,[k])\} \to \lambda_{\frac{s}{2}+1}\end{aligned}$$

for all $[k] \in A_4$ and all $1 \leq \gamma \leq \deg_T([k]) - 1$.

$$\lambda_1 \to \{(1,[l]),(2,[l])\} \to \lambda_{\frac{s}{2}+1}$$

for any $[l] \in E$. (See Figures 5 and 6 when $s = 4$.)

It can be verified that $d(D) = 4$ and every vertex lies in a directed C_4; this part of the proof is omitted for brevity and we refer the interested reader to [24] for details. Hence, $\bar{d}(\mathcal{T}) \leq \max\{4, d(D)\}$ by Lemma 1.3 and we have $\bar{d}(\mathcal{T}) = 4$. □

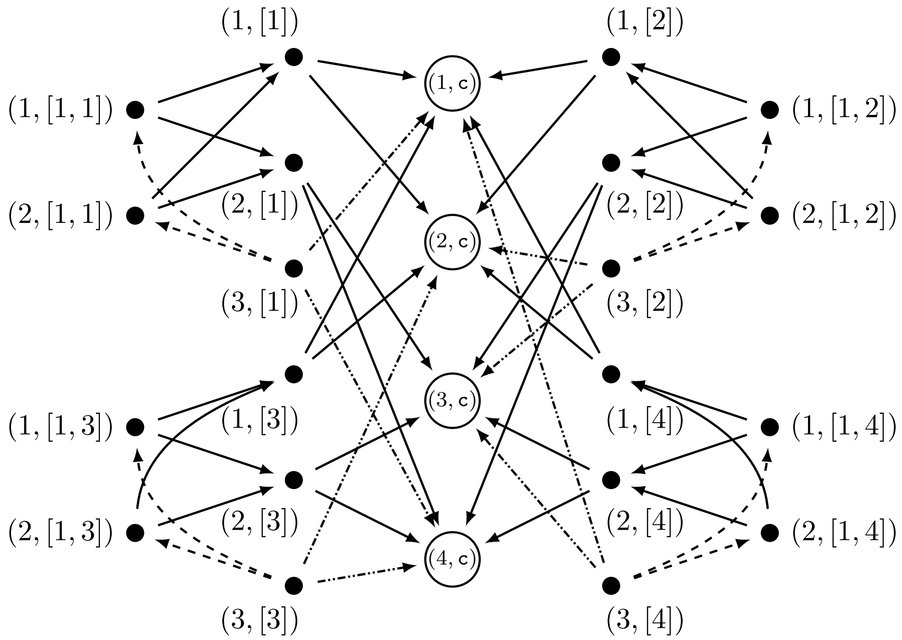

Figure 5: Partial orientation D for $\mathcal{H}$ for $s = 4$,
$A_3 = \{[1], [2], \ldots, [6]\}$, $A_4 = \{[7], [8]\}$, $E = \{[9], [10]\}$; showing $[i]$ for $1 \leq i \leq \binom{s}{s/2} - 2$.

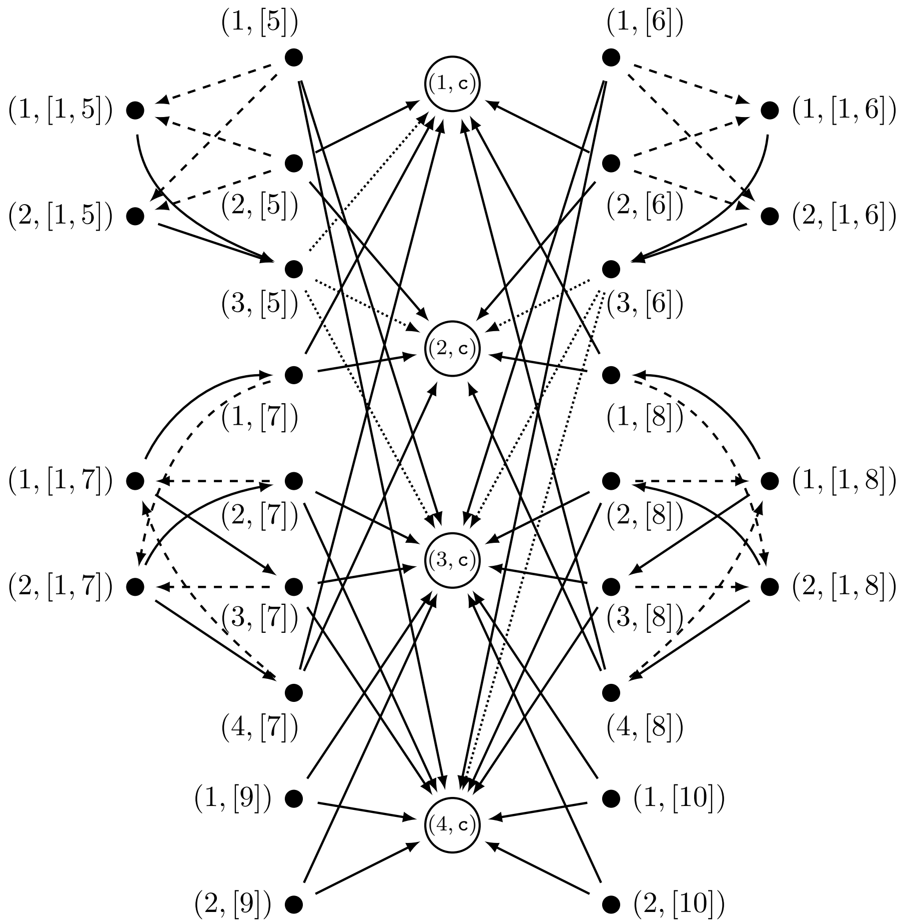

Figure 6: Partial orientation D for $\mathcal{H}$ for $s = 4$;
showing $[i]$ for $\binom{s}{s/2} - 1 \leq i \leq |A_3|$ or $[i] \in A_4$ or $[i] \in E$.

Using similar ideas to Propositions 3.5 and 3.9, and Lih's and Griggs' theorems, we can prove Propositions 3.10 and 3.11. Hence, their proofs are omitted for brevity and can be found in [24].

Proposition 3.10. *Suppose $s \geq 4$ is even, $A_2 \neq \emptyset$, $A_3 = \emptyset$ and $A_{\geq 4} \neq \emptyset$ for a $\mathcal{T}$. Then,*

$$\mathcal{T} \in \mathscr{C}_0 \iff \begin{cases} |A_2| \leq \binom{s}{s/2} - 2, & \text{if } |A_{\geq 4}| \geq 2 \text{ or } |A_{\geq 2}| < \deg_T(\mathsf{c}), \\ |A_2| \leq \binom{s}{s/2} - 1, & \text{otherwise.} \end{cases}$$

Proposition 3.11. *Suppose $s \geq 4$ is even, $A_2 \neq \emptyset$, $|A_3| = 1$, and $A_{\geq 4} = \emptyset$ for a $\mathcal{T}$. Then,*

$$\mathcal{T} \in \mathscr{C}_0 \iff \begin{cases} |A_2| \leq \binom{s}{s/2} - 2, & \text{if } |A_{\geq 2}| < \deg_T(\mathsf{c}), \\ |A_2| \leq \binom{s}{s/2} - 1, & \text{if } |A_{\geq 2}| = \deg_T(\mathsf{c}). \end{cases}$$

Proposition 3.12. *Suppose $s \geq 4$ is even, $A_2 \neq \emptyset$ and either $|A_3| \geq 2$ or $|A_3| = 1$ and $A_{\geq 4} \neq \emptyset$ for a $\mathcal{T}$.*
(a) If $\mathcal{T} \in \mathscr{C}_0$, then $2|A_2| + |A_3| \leq \binom{s}{s/2} + \binom{s}{(s/2)+1} - \kappa^*_{s,\frac{s}{2}}(k)$ for some $k \leq |A_2| + |A_3|$.
(b) If there exists some $|A_2|+1 \leq k \leq \min\{|A_2|+|A_3|, \binom{s}{s/2} - 1\}$ such that $2|A_2|+|A_3| \leq \binom{s}{s/2} + \binom{s}{(s/2)+1} - \kappa_{s,\frac{s}{2}}(k) - 3$, then $\mathcal{T} \in \mathscr{C}_0$.

Proof: (a) Since $\mathcal{T} \in \mathscr{C}_0$, there exists an orientation D of $\mathcal{T}$, where $d(D) = 4$. As $A_2 \neq \emptyset$ and $A_3 \neq \emptyset$, we assume (3.1)-(3.5) here. Partition A_3^O (A_3^I resp.) into $A_3^{O(D)}$ and $A_3^{O(S)}$ ($A_3^{I(D)}$ and $A_3^{I(S)}$ resp.), where

$$\left.\begin{aligned} A_3^{O(D)} &= \{[i] \in A_3^O \mid O^{\mathsf{c}}((1,[i])) \neq O^{\mathsf{c}}((2,[i]))\}, \\ A_3^{O(S)} &= \{[i] \in A_3^O \mid O^{\mathsf{c}}((1,[i])) = O^{\mathsf{c}}((2,[i]))\}, \\ A_3^{I(D)} &= \{[i] \in A_3^I \mid O^{\mathsf{c}}((1,[i])) \neq O^{\mathsf{c}}((2,[i]))\}, \\ \text{and} \quad A_3^{I(S)} &= \{[i] \in A_3^I \mid O^{\mathsf{c}}((1,[i])) = O^{\mathsf{c}}((2,[i]))\}. \end{aligned}\right\} \tag{3.14}$$

Note that $O^{\mathsf{c}}((1,[i]))$ and $O^{\mathsf{c}}((2,[i]))$ are *Different* (*Same* resp.) for $A_3^{O(D)}$ and $A_3^{I(D)}$ ($A_3^{O(S)}$ and $A_3^{I(S)}$ resp.).

Both $B_2^O \cup B_3^O \cup \{O^{\mathsf{c}}((1,[l])) \mid [l] \in A_3^{I(S)}\}$ and $B_2^I \cup B_3^I \cup \{I^{\mathsf{c}}((2,[l])) \mid [l] \in A_3^{O(S)}\}$ are antichains by Lemma 2.15 and its dual respectively. Furthermore, $d_D((1,[1,i]),(1,[1,j])) = 4$ for each $[i] \in A_2 \cup A_3^O \cup A_3^{I(S)}$ and $[j] \in A_2 \cup A_3^I \cup A_3^{O(S)}$, $i \neq j$, implies $O^{\mathsf{c}}((\delta_1,[i])) \cap I^{\mathsf{c}}((\delta_2,[j])) \neq \emptyset$ where

$$\delta_1 = \begin{cases} 1, & \text{if } [i] \in A_2 \cup A_3^{I(S)}, \\ 3, & \text{if } [i] \in A_3^O, \end{cases} \text{ and } \delta_2 = \begin{cases} 2, & \text{if } [j] \in A_2 \cup A_3^{O(S)}, \\ 3, & \text{if } [j] \in A_3^I. \end{cases}$$

Equivalently, for any $O^{\mathsf{c}}((\delta_1,[i])) \in B_2^O \cup B_3^O \cup \{O^{\mathsf{c}}((1,[l])) \mid [l] \in A_3^{I(S)}\}$ and $I^{\mathsf{c}}((\delta_2,[j])) \in B_2^I \cup B_3^I \cup \{I^{\mathsf{c}}((2,[l])) \mid [l] \in A_3^{O(S)}\}$, $O^{\mathsf{c}}((\delta_1,[i])) \cap I^{\mathsf{c}}((\delta_2,[j])) = \emptyset$ only if $[i] = [j] \in A_2 \cup A_3^{O(S)} \cup A_3^{I(S)}$. Then,

$$\begin{aligned} & 2|A_2| + |A_3| \\ &= 2|A_2| + |A_3^O| + |A_3^I| \\ &\leq (|A_2| + |A_3^O| + |A_3^{I(S)}|) + (|A_2| + |A_3^I| + |A_3^{O(S)}|) \\ &= |B_2^O \cup B_3^O \cup \{O^{\mathsf{c}}((1,[l])) \mid [l] \in A_3^{I(S)}\}| + |B_2^I \cup B_3^I \cup \{I^{\mathsf{c}}((2,[l])) \mid [l] \in A_3^{O(S)}\}| \\ &\leq \binom{s}{s/2} + \binom{s}{(s/2)+1} - \kappa^*_{s,\frac{s}{2}}(|A_2| + |A_3^{O(S)}| + |A_3^{I(S)}|) \text{ by Theorem 2.10,} \end{aligned}$$

where $k = |A_2| + |A_3^{O(S)}| + |A_3^{I(S)}| \le |A_2| + |A_3|$ is as required.

(b) Let the $\frac{s}{2}$-subsets of $\mathbb{N}_s$ in squashed order be $X_1, X_2, \ldots, X_{\binom{s}{s/2}}$. Note that $\bar{X}_i = X_{\binom{s}{s/2}-i}$ for $i = 1, 2, \ldots, \binom{s}{s/2}$ and $L_{s,\frac{s}{2}}(k) = \{X_{\binom{s}{s/2}}, X_{\binom{s}{s/2}-1}, \ldots, X_{\binom{s}{s/2}-k+1}\}$. Furthermore, let $\mu_i = (X_i, \mathsf{c})$ for $i = 1, 2, \ldots, \binom{s}{s/2}$. We also use the previous notation, $\binom{(\mathbb{N}_s, \mathsf{c})}{(s/2)+1} = \{\psi_i \mid i = 1, 2, \ldots, \binom{s}{(s/2)+1}\}$, and further assume $\{\psi_i \mid i = 1, 2, \ldots, \binom{s}{(s/2)+1} - |\nabla L_{s,\frac{s}{2}}(k)|\} = \{(Y, \mathsf{c}) \mid Y \in \binom{\mathbb{N}_s}{(s/2)+1} - \nabla L_{s,\frac{s}{2}}(k)\}$.

If $|A_{\ge 2}| \le \binom{s}{s/2} - 1$, then by Corollary 3.8(i), $\mathcal{T} \in \mathscr{C}_0$. Hence, we assume $|A_{\ge 2}| \ge \binom{s}{s/2}$. Let $A_2^\diamond = A_2 \cup A^*$, where A^* is an arbitrary subset of A_3 such that $|A_2^\diamond| = k - 1$; $A^* = \emptyset$ if $|A_2| = k - 1$. Then, let $A_3^\diamond = A_3 - A^*$. Furthermore, assume without loss of generality that $A_2^\diamond = \{[i] \mid i \in \mathbb{N}_{|A_2^\diamond|}\}$, $A_3^\diamond = \{[i] \mid i \in \mathbb{N}_{|A_2^\diamond|+|A_3^\diamond|} - \mathbb{N}_{|A_2^\diamond|}\}$ and $A_4^\diamond = \{[i] \mid i \in \mathbb{N}_{|A_2^\diamond|+|A_3^\diamond|+|A_4^\diamond|} - \mathbb{N}_{|A_2^\diamond|+|A_3^\diamond|}\}$.

Let $\mathcal{H} = T(t_1, t_2, \ldots, t_n)$ be the subgraph of $\mathcal{T}$, where $t_{\mathsf{c}} = s$, $t_{[i]} = 3$ for all $[i] \in \mathcal{T}(A_3^\diamond)$, $t_{[j]} = 4$ for all $[j] \in \mathcal{T}(A_4^\diamond)$ and $t_v = 2$ otherwise. We will use A_j for $\mathcal{H}(A_j)$ for the remainder of this proof. Define an orientation D of $\mathcal{H}$ as follows.

$$\begin{aligned}&(2,[i]) \to (1,[\alpha,i]) \to (1,[i]) \to (2,[\alpha,i]) \to (2,[i]), \text{ and}\\ &\mu_{i+1} \to \{(1,[i]),(2,[i])\} \to \bar{\mu}_{i+1}\end{aligned}$$

for all $1 \le i \le |A_2| = k-1$, and $1 \le \alpha \le \deg_T([i]) - 1$, i.e., the $\frac{s}{2}$-sets $\mu_2, \mu_3, \ldots, \mu_k$ ($\bar{\mu}_2, \bar{\mu}_3, \ldots, \bar{\mu}_k$ resp.) are used as 'in-sets' ('out-sets' resp.) to construct B_2^I (B_2^O resp.).

$$\begin{aligned}&(3,[j]) \to \{(1,[\beta,j]),(2,[\beta,j])\} \to \{(1,[j]),(2,[j])\},\\ &\bar{\mu}_1 = \mu_{\binom{s}{s/2}} \to (1,[j]) \to \mu_1 \to (2,[j]) \to \mu_{\binom{s}{s/2}}, \text{ and}\\ &\mu_{j+1} \to (3,[j]) \to \bar{\mu}_{j+1}\end{aligned}$$

for all $|A_2| + 1 \le j \le \binom{s}{s/2} - 2$ and all $1 \le \alpha \le \deg_T([j]) - 1$, i.e., the $\frac{s}{2}$-sets $\mu_{k+1}, \mu_{k+2}, \ldots, \mu_{\binom{s}{s/2}-1}$ are used as 'in-sets' to construct B_3^I.

$$\begin{aligned}&\{(1,[l]),(2,[l])\} \to \{(1,[\gamma,l]),(2,[\gamma,l])\} \to (3,[l]),\\ &\mu_1 \to (1,[l]) \to \mu_{\binom{s}{s/2}} \to (2,[l]) \to \mu_1, \text{ and}\\ &\bar{\psi}_{l+2-\binom{s}{s/2}} \to (3,[l]) \to \psi_{l+2-\binom{s}{s/2}}\end{aligned}$$

for all $\binom{s}{s/2} - 1 \le l \le |A_2| + |A_3|$ and all $1 \le \gamma \le \deg_T([l]) - 1$, i.e., the $(\frac{s}{2}+1)$-sets $\psi_1, \psi_2, \ldots, \psi_{\binom{s}{(s/2)+1} - |\nabla L_{s,\frac{s}{2}}(k)|}$ are used as 'out-sets' to construct B_3^O.

$$\begin{aligned}&(2,[\tau,x]) \to \{(2,[x]),(4,[x])\} \to (1,[\tau,x]) \to \{(1,[x]),(3,[x])\} \to (2,[\tau,x]),\\ &\text{and } \mu_{\binom{s}{s/2}} \to \{(1,[x]),(4,[x])\} \to \mu_1 \to \{(2,[x]),(3,[x])\} \to \mu_{\binom{s}{s/2}}\end{aligned}$$

for all $[x] \in A_4$ and all $1 \le \tau \le \deg_T([x]) - 1$.

$$\mu_1 \to \{(1,[y]),(2,[y])\} \to \mu_{\binom{s}{s/2}}$$

for any $[y] \in E$. (See Figures 7 and 8 when $s = 6$, $k = 13$.)

It can be verified that $d(D) = 4$ and every vertex lies in a directed C_4; this part of the proof is omitted for brevity and we refer the interested reader to [24] for details. Hence, $\bar{d}(\mathcal{T}) \le \max\{4, d(D)\}$ by Lemma 1.3 and thus $\bar{d}(\mathcal{T}) = 4$. □

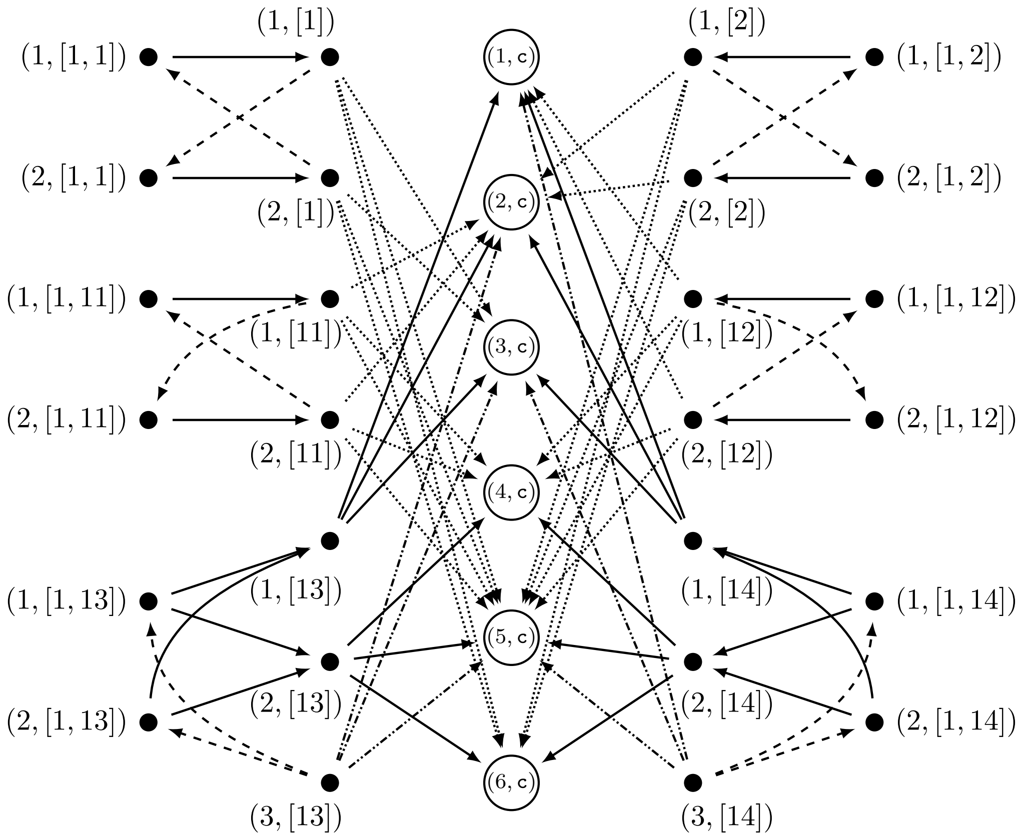

Figure 7: Partial orientation D for $\mathcal{H}$ for $s = 6$, $k = 13$;
$A_2 = \{[1], [2], \ldots, [12]\}$, $A_3 = \{[13], [14], \ldots, [20]\}$, $A_4 = \{[21], [22]\}$, $E = \{[23], [24]\}$.

Note that $\mathbf{123} <_s \mathbf{124} <_s \mathbf{134} <_s \mathbf{234} <_s \mathbf{125} <_s \mathbf{135} <_s \mathbf{235} <_s \mathbf{145} <_s \mathbf{245} <_s \mathbf{345} <_s \mathbf{126} <_s \mathbf{136} <_s \mathbf{236} <_s \mathbf{146} <_s \mathbf{246} <_s \mathbf{346} <_s \mathbf{156} <_s \mathbf{256} <_s \mathbf{356} <_s \mathbf{456}$. So, $L_{6,3}(13) = \{\mathbf{145}, \mathbf{245}, \ldots, \mathbf{456}\}$, $\binom{\mathbb{N}_6}{4} - \nabla L_{6,3}(13) = \{\mathbf{1234}, \mathbf{1235}\}$ and $\kappa_{6,3}(13) = 0$.

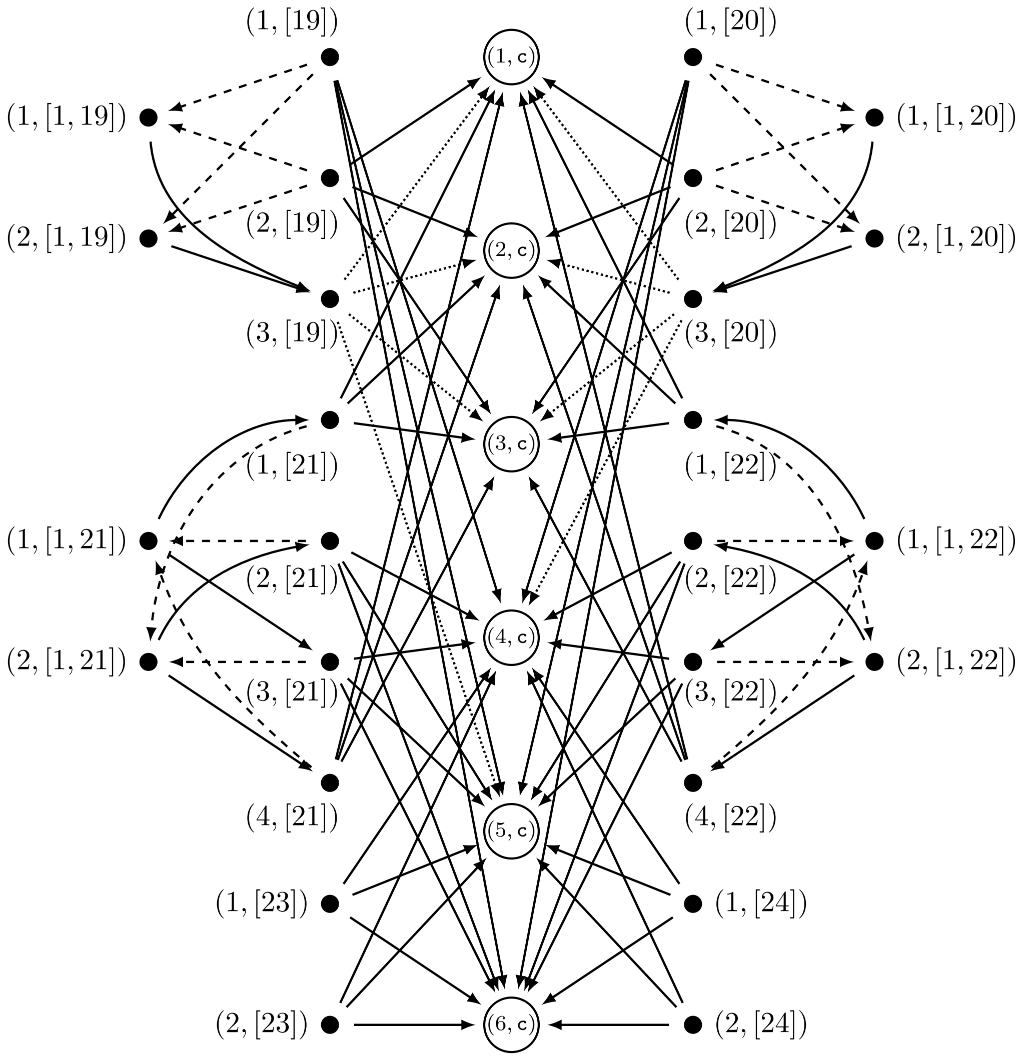

Figure 8: Partial orientation D for $\mathcal{H}$ for $s = 6$, $k = 13$;
$A_2 = \{[1], [2], \ldots, [12]\}$, $A_3 = \{[13], [14], \ldots, [20]\}$, $A_4 = \{[21], [22]\}$, $E = \{[23], [24]\}$.

This concludes the proof of Theorem 1.8. Unfortunately, we were not able to give a complete characterisation for Proposition 3.12. Our core idea is the consideration of cross-intersecting antichains with at most k disjoint pairs, thus, invoking Theorem 2.10. We believe the gap between the necessary and sufficient conditions ($\kappa^*_{s,\frac{s}{2}}(\cdot)$ and $\kappa_{s,\frac{s}{2}}(\cdot)$ resp.) may be further tightened if there is an analogue of Theorem 2.10 on *exactly* k disjoint pairs; more discussion may be found in the last section of [25].

4. Proof of Theorem 1.9

Similar to the previous section, we prove Theorem 1.9 by collating several propositions.

Proposition 4.1. *Suppose $s \geq 3$ is odd, $A_2 = \emptyset$ and $A_3 \neq \emptyset$ for a $\mathcal{T}$. Then, $\mathcal{T} \in \mathscr{C}_0$ if and only if $|A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$.*

Proof: ($\Rightarrow$) Since $\mathcal{T} \in \mathscr{C}_0$, there exists an orientation D of $\mathcal{T}$, where $d(D) = 4$. As $A_3 \neq \emptyset$, we assume (3.3)-(3.5) here. By Sperner's theorem, $|B_3^O| \leq \binom{s}{\lfloor s/2 \rfloor}$ and $|B_3^I| \leq \binom{s}{\lfloor s/2 \rfloor}$. So, if $|B_3^O| = 0$ or $|B_3^I| = 0$, then $|A_3| = |B_3^O| + |B_3^I| \leq \binom{s}{\lfloor s/2 \rfloor} \leq 2\binom{s}{\lceil s/2 \rceil} - 2$. Therefore, we assume $|B_3^O| > 0$ and $|B_3^I| > 0$.

In what follows, we first show that for any $[i] \in A_3$ and any $p = 1, 2, 3$, if $|O^c((p, [i]))|$ is too big ($> \lceil \frac{s}{2} \rceil$) or too small ($< \lfloor \frac{s}{2} \rfloor$), then $|A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$.

Case 1. There exists some $[i] \in A_3$ such that $|O^c((p, [i]))| > \lceil \frac{s}{2} \rceil$ for some $p = 1, 2, 3$.

For any $[j] \in A_3^O - \{[i]\}$, $d_D((1, [1, j]), (p, [i])) = 3$ implies $O^c((3, [j])) \cap I^c((p, [i])) \neq \emptyset$. By Lih's theorem, $|B_3^O| - 1 \leq |B_3^O - \{O^c((3, [i]))\}| \leq \binom{s}{\lceil s/2 \rceil} - \binom{s - |I^c((p,[i]))|}{\lceil s/2 \rceil} \leq \binom{s}{\lceil s/2 \rceil} - \binom{\lceil s/2 \rceil + 1}{\lceil s/2 \rceil} = \binom{s}{\lceil s/2 \rceil} - (\lceil \frac{s}{2} \rceil + 1) \leq \binom{s}{\lceil s/2 \rceil} - 3$. It follows that $|B_3^O| \leq \binom{s}{\lceil s/2 \rceil} - 2$ and $|A_3| = |B_3^O| + |B_3^I| \leq [\binom{s}{\lceil s/2 \rceil} - 2] + \binom{s}{\lfloor s/2 \rfloor} = 2\binom{s}{\lceil s/2 \rceil} - 2$.

Case 2. There exists some $[i] \in A_3$ such that $|O^c((p, [i]))| < \lfloor \frac{s}{2} \rfloor$ for some $p = 1, 2, 3$.

In other words, $|I^c((p, [i]))| > \lceil \frac{s}{2} \rceil$. Hence, this case follows from Case 1 by the Duality Lemma.

Case 3. For all $[i] \in A_3$ and all $p = 1, 2, 3$, $\lfloor \frac{s}{2} \rfloor \leq |O^c((p, [i]))| \leq \lceil \frac{s}{2} \rceil$.

Note that for all $[i] \in A_3^O$ and $[j] \in A_3^I$, $d_D((1, [1, i]), (1, [1, j])) = 4$ implies $X \cap Y \neq \emptyset$ for all $X \in B_3^O$ and $Y \in B_3^I$. Now, it suffices to consider the case where $B_3^O \cup B_3^I \subseteq \binom{(\mathbb{N}_s, \mathsf{c})}{\lceil s/2 \rceil}$. For otherwise, $|A_3| = |B_3^O| + |B_3^I| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$ by Theorems 2.4 and 2.5. Partition A_3^O (A_3^I resp.) into $A_3^{O(D)}$ and $A_3^{O(S)}$ ($A_3^{I(D)}$ and $A_3^{I(S)}$ resp.) as in (3.14).

Remark 4.2. Now, we shall make a series of assumptions on the structure of D, on which we will derive $|A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$ if any one fails to hold. We will then show that under all these assumptions, we still arrive at the same required conclusion.

Assumption 1: $|A_3^{O(D)}| \geq 2$ and $|A_3^{I(D)}| \geq 2$.

Suppose $|A_3^{O(D)}| \leq 1$. By the dual of Lemma 2.15, $\{I^c((1, [j])) \mid [j] \in A_3^{O(S)}\} \cup B_3^I$ is an antichain. By Sperner's theorem, $|A_3| = |B_3^O| + |B_3^I| = |A_3^{O(D)}| + |\{I^c((1, [j])) \mid [j] \in A_3^{O(S)}\}| + |B_3^I| \leq 1 + \binom{s}{\lfloor s/2 \rfloor} \leq 2\binom{s}{\lceil s/2 \rceil} - 2$. A similar argument follows if $|A_3^{I(D)}| \leq 1$.

Subcase 3.1. $|O^c((1, [i]))| = |O^c((2, [i]))| = \lfloor \frac{s}{2} \rfloor$ for some $[i] \in A_3^{O(D)}$.

For any $[j] \in A_3^I$ and $p = 1, 2$, $d_D((p, [i]), (1, [1, j])) = 3$ implies $O^c((p, [i])) \cap I^c((3, [j])) \neq \emptyset$, i.e., $I^c((3, [j])) \neq I^c((p, [i]))$. It follows that $|B_3^I| \leq \binom{s}{\lceil s/2 \rceil} - 2$. Hence, $|A_3| = |B_3^O| + |B_3^I| \leq \binom{s}{\lceil s/2 \rceil} + [\binom{s}{\lceil s/2 \rceil} - 2] = 2\binom{s}{\lceil s/2 \rceil} - 2$.

Subcase 3.2. $|O^c((1, [i]))| = |O^c((2, [i]))| = \lceil \frac{s}{2} \rceil$ for some $[i] \in A_3^{I(D)}$.

This follows from Subcase 3.1 by the Duality Lemma.

Subcase 3.3. $|O^c((1, [i^*]))| = \lfloor \frac{s}{2} \rfloor$ and $|O^c((2, [i^*]))| = \lceil \frac{s}{2} \rceil$ for some $[i^*] \in A_3^{O(D)}$.

($\star$) For any $[j] \in A_3^I$ and $p = 1, 2$, $d_D((1, [i^*]), (1, [1, j])) = 3$ implies $I^c((3, [j])) \neq I^c((1, [i^*]))$. It follows that $|B_3^I| \leq \binom{s}{\lceil s/2 \rceil} - 1$. Now, we are going to establish some assumptions regarding $A_3^{O(D)}$ and $A_3^{I(D)}$, and provide justifications accordingly.

Assumption 2A: $|O^c((1, [i]))| = \lfloor \frac{s}{2} \rfloor$ and $|O^c((2, [i]))| = \lceil \frac{s}{2} \rceil$ for all $[i] \in A_3^{O(D)}$.

Suppose there exists some $[i] \in A_3^{O(D)} - \{[i^*]\}$ such that $|O^c((1, [i]))| = |O^c((2, [i]))| = \lceil \frac{s}{2} \rceil$. Note by definition of $A_3^{O(D)}$ that $O^c((3, [i]))$ is equal to at most one of $O^c((1, [i]))$ and $O^c((2, [i]))$, say $O^c((3, [i])) \neq O^c((1, [i]))$. Also, for any $[j] \in A_3^O - \{[i]\}$, $d_D((1, [1, j]),$

$(1, [i])) = 3$ implies $O^c((3, [j])) \neq O^c((1, [i]))$. It follows that $|B_3^O| \leq \binom{s}{\lceil s/2 \rceil} - 1$. Hence, $|A_3| = |B_3^O| + |B_3^I| \leq 2[\binom{s}{\lceil s/2 \rceil} - 1] = 2\binom{s}{\lceil s/2 \rceil} - 2$. Therefore, and in view of Subcase 3.1, we may now assume $|O^c((1, [i]))| = \lfloor \frac{s}{2} \rfloor$ and $|O^c((2, [i]))| = \lceil \frac{s}{2} \rceil$ for all $[i] \in A_3^{O(D)}$.

Assumption 3A: $O^c((1, [i])) = O^c((1, [i^*]))$ for all $[i] \in A_3^{O(D)}$.

Suppose there exists some $[i] \in A_3^{O(D)} - \{[i^*]\}$ such that $O^c((1, [i])) \neq O^c((1, [i^*]))$. For any $[j] \in A_3^I$ and $x = i, i^*$, $d_D((1, [x]), (1, [1, j])) = 3$ implies $I^c((3, [j])) \neq I^c((1, [x]))$. It follows that $|B_3^I| \leq \binom{s}{\lceil s/2 \rceil} - 2$. Hence, $|A_3| = |B_3^O| + |B_3^I| \leq \binom{s}{\lceil s/2 \rceil} + [\binom{s}{\lceil s/2 \rceil} - 2] = 2\binom{s}{\lceil s/2 \rceil} - 2$. Thus, we may assume $O^c((1, [i])) = O^c((1, [i^*]))$ for all $[i] \in A_3^{O(D)}$.

Assumption 4A: $O^c((2, [i])) = O^c((3, [i]))$ for all $[i] \in A_3^{O(D)}$.

Suppose there exists some $[i] \in A_3^{O(D)}$ such that $O^c((2, [i])) \neq O^c((3, [i]))$. Also, for any $[j] \in A_3^O - \{[i]\}$, $d_D((1, [1, j]), (2, [i])) = 3$ implies $O^c((3, [j])) \neq O^c((2, [i]))$. It follows that $|B_3^O| \leq \binom{s}{\lceil s/2 \rceil} - 1$, and $|A_3| = |B_3^O| + |B_3^I| \leq 2[\binom{s}{\lceil s/2 \rceil} - 1] = 2\binom{s}{\lceil s/2 \rceil} - 2$. Therefore, the assumption follows.

Assumption 5A: $|O^c((1, [i]))| = |O^c((2, [i]))| = \lfloor \frac{s}{2} \rfloor$ for all $[i] \in A_3^{I(D)}$.

Suppose there exists some $[i] \in A_3^{I(D)}$ such that $|O^c((1, [i]))| = \lfloor \frac{s}{2} \rfloor$ and $|O^c((2, [i]))| = \lceil \frac{s}{2} \rceil$. For any $[j] \in A_3^O$, $d_D((1, [1, j]), (2, [i])) = 3$ implies $O^c((3, [j])) \neq O^c((2, [i]))$. So, $|B_3^O| \leq \binom{s}{\lceil s/2 \rceil} - 1$. Hence, $|A_3| = |B_3^O| + |B_3^I| \leq 2[\binom{s}{\lceil s/2 \rceil} - 1] = 2\binom{s}{\lceil s/2 \rceil} - 2$. Therefore, and in view of Subcase 3.2, we may now assume $|O^c((1, [i]))| = |O^c((2, [i]))| = \lfloor \frac{s}{2} \rfloor$ for all $[i] \in A_3^{I(D)}$.

Assumption 6A: $O^c((1, [i])) = O^c((1, [i^*]))$ and $O^c((2, [i])) = O^c((3, [i]))$ for all $[i] \in A_3^{I(D)}$.

Suppose there exists some $[i] \in A_3^{I(D)}$ and some $p = 1, 2$, such that $O^c((p, [i])) \neq O^c((1, [i^*]))$ and $O^c((p, [i])) \neq O^c((3, [i]))$. Also, for any $[j] \in A_3^I - \{[i]\}$, $d_D((p, [i]), (1, [1, j])) = 3$ implies $I^c((3, [j])) \neq I^c((p, [i]))$. Therefore, for all $X \in B_3^I$, $X \neq I^c((p, [i]))$ and recall from $(\star)$ that $X \neq I^c((1, [i^*]))$. It follows that $|B_3^I| \leq \binom{s}{\lceil s/2 \rceil} - 2$ and $|A_3| = |B_3^O| + |B_3^I| \leq \binom{s}{\lceil s/2 \rceil} + [\binom{s}{\lceil s/2 \rceil} - 2] = 2\binom{s}{\lceil s/2 \rceil} - 2$.

Therefore, for each $[i] \in A_3^{I(D)}$ and each $p = 1, 2$, either $O^c((p, [i])) = O^c((3, [i]))$ or $O^c((p, [i])) = O^c((1, [i^*]))$. By the definition of $A_3^{I(D)}$, we may assume without loss of generality that $O^c((1, [i])) = O^c((1, [i^*]))$ and $O^c((2, [i])) = O^c((3, [i]))$ for all $[i] \in A_3^{I(D)}$.

Now, with Assumptions 1, 2A-6A in place, consider $[j] \in A_3^{I(D)}$. For any $[k] \in A_3^{O(D)}$, $O^c((1, [k])) = O^c((1, [i^*])) = O^c((1, [j]))$ and $d_D((1, [j]), (1, [1, k])) = 3$ imply $O^c((1, [j])) \cap I^c((2, [k])) \neq \emptyset$. Equivalently, $O^c((1, [i^*])) = O^c((1, [j])) \not\subseteq O^c((2, [k]))$. Note also that there are $\lceil \frac{s}{2} \rceil$ number of $\lceil \frac{s}{2} \rceil$-supersets of $O^c((1, [i^*]))$. Recall that $O^c((2, [k])) = O^c((3, [k]))$, so that $\{O^c((2, [k])) \mid [k] \in A_3^{O(D)}\} = \{O^c((3, [k])) \mid [k] \in A_3^{O(D)}\} \subseteq \binom{(\mathbb{N}_s, \mathsf{c})}{\lceil s/2 \rceil}$. So, $|A_3^{O(D)}| \leq \binom{s}{\lceil s/2 \rceil} - \lceil \frac{s}{2} \rceil \leq \binom{s}{\lceil s/2 \rceil} - 2$. Since $\{I^c((1, [i])) \mid [i] \in A_3^{O(S)}\} \cup B_3^I$ is an antichain by the dual of Lemma 2.15, $|A_3^{O(S)}| + |A_3^I| = |\{I^c((1, [i])) \mid [i] \in A_3^{O(S)}\} \cup B_3^I| \leq \binom{s}{\lfloor s/2 \rfloor}$ by Sperner's theorem. Hence, $|A_3| = |A_3^O| + |A_3^I| = |A_3^{O(D)}| + (|A_3^{O(S)}| + |A_3^I|) \leq [\binom{s}{\lceil s/2 \rceil} - 2] + \binom{s}{\lfloor s/2 \rfloor} = 2\binom{s}{\lceil s/2 \rceil} - 2$.

Subcase 3.4. $|O^c((1,[i^*]))| = \lceil \frac{s}{2} \rceil$ and $|O^c((2,[i^*]))| = \lfloor \frac{s}{2} \rfloor$ for some $[i^*] \in A_3^{I(D)}$.

This follows from Subcase 3.3 by the Duality Lemma.

Subcase 3.5. $|O^c((1,[i]))| = |O^c((2,[i]))| = \lceil \frac{s}{2} \rceil$ and $|O^c((1,[j]))| = |O^c((2,[j]))| = \lfloor \frac{s}{2} \rfloor$ for some $[i] \in A_3^{O(D)}$ and $[j] \in A_3^{I(D)}$.

Note by definition of $A_3^{O(D)}$ that $O^c((3,[i]))$ is equal to at most one of $O^c((1,[i]))$ and $O^c((2,[i]))$, say $O^c((3,[i])) \neq O^c((1,[i]))$. Also, for any $[k] \in A_3^O - \{[i]\}$, $d_D((1,[1,k]),(1,[i])) = 3$ implies $O^c((3,[k])) \neq O^c((1,[i]))$. It follows that $|B_3^O| \leq \binom{s}{\lceil s/2 \rceil} - 1$.

Similarly, by definition of $A_3^{I(D)}$, $I^c((3,[j]))$ is equal to at most one of $I^c((1,[j]))$ and $I^c((2,[j]))$, say $I^c((3,[j])) \neq I^c((1,[j]))$. Also, for any $[k] \in A_3^I - \{[j]\}$, $d_D((1,[j]),(1,[1,k])) = 3$ implies $I^c((3,[k])) \neq I^c((1,[j]))$. It follows that $|B_3^I| \leq \binom{s}{\lceil s/2 \rceil} - 1$. Hence, $|A_3| = |B_3^O| + |B_3^I| \leq 2[\binom{s}{\lceil s/2 \rceil} - 1] = 2\binom{s}{\lceil s/2 \rceil} - 2$.

In view of the above, it is intuitive to construct an optimal orientation D of $\mathcal{T}$ with $B_3^O \cup B_3^I \subseteq \binom{(\mathbb{N}_s, \mathsf{c})}{\lceil s/2 \rceil}$. To this end, we recall Definition 3.7.

($\Leftarrow$) If $|A_{\geq 3}| \leq \binom{s}{\lceil s/2 \rceil} - 1$, then by Corollary 3.8(i), $\mathcal{T} \in \mathscr{C}_0$. Hence, we assume $|A_{\geq 3}| \geq \binom{s}{\lceil s/2 \rceil}$ hereafter, on top of the hypothesis that $|A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$. If $|A_3| \geq \binom{s}{\lceil s/2 \rceil}$, define $A_3^\diamond = A_3$. Otherwise, let $A_3^\diamond = A_3 \cup A^*$, where A^* is an arbitrary subset of $A_{\geq 4}$ such that $|A_3^\diamond| = \binom{s}{\lceil s/2 \rceil}$. Then, let $A_4^\diamond = A_3 \cup A_{\geq 4} - A_3^\diamond$. Furthermore, assume without loss of generality that $A_3^\diamond = \{[i] \mid i \in \mathbb{N}_{|A_3^\diamond|}\}$ and $A_4^\diamond = \{[i] \mid i \in \mathbb{N}_{|A_3^\diamond| + |A_4^\diamond|} - \mathbb{N}_{|A_3^\diamond|}\}$.

Let $\mathcal{H} = T(t_1, t_2, \ldots, t_n)$ be the subgraph of $\mathcal{T}$, where $t_{\mathsf{c}} = s$, $t_{[i]} = 3$ for all $[i] \in \mathcal{T}(A_3^\diamond)$, $t_{[j]} = 4$ for all $[j] \in \mathcal{T}(A_4^\diamond)$ and $t_v = 2$ otherwise. We will use A_j for $\mathcal{H}(A_j)$ for the remainder of this proof. Define an orientation D of $\mathcal{H}$ as follows.

$$\begin{aligned}
&(3,[i]) \rightarrow \{(1,[\alpha,i]),(2,[\alpha,i])\} \rightarrow \{(1,[i]),(2,[i])\},\\
&\bar{\lambda}_1 \rightarrow (1,[i]) \rightarrow \lambda_1 \rightarrow (2,[i]) \rightarrow \bar{\lambda}_1, \text{ and}\\
&\lambda_{i+1} \rightarrow (3,[i]) \rightarrow \bar{\lambda}_{i+1}
\end{aligned}$$

for all $1 \leq i \leq \binom{s}{\lceil s/2 \rceil} - 1$ and all $1 \leq \alpha \leq \deg_T([i]) - 1$, i.e., excluding λ_1, the $\lceil \frac{s}{2} \rceil$-sets λ_i's are used as 'in-sets' to construct B_3^I.

$$\begin{aligned}
&\{(1,[j]),(2,[j])\} \rightarrow \{(1,[\beta,j]),(2,[\beta,j])\} \rightarrow (3,[j]),\\
&\lambda_1 \rightarrow (1,[j]) \rightarrow \bar{\lambda}_1 \rightarrow (2,[j]) \rightarrow \lambda_1, \text{ and}\\
&\bar{\lambda}_{j+2-\binom{s}{\lceil s/2 \rceil}} \rightarrow (3,[j]) \rightarrow \lambda_{j+2-\binom{s}{\lceil s/2 \rceil}}
\end{aligned}$$

for all $\binom{s}{\lceil s/2 \rceil} \leq j \leq |A_3|$ and all $1 \leq \beta \leq \deg_T([j]) - 1$, i.e., the $\lceil \frac{s}{2} \rceil$-sets $\lambda_2, \lambda_3, \ldots, \lambda_{|A_3|+2-\binom{s}{\lceil s/2 \rceil}}$ are used as 'out-sets' to construct B_3^O.

$$\begin{aligned}
&(2,[\gamma,k]) \rightarrow \{(2,[k]),(4,[k])\} \rightarrow (1,[\gamma,k]) \rightarrow \{(1,[k]),(3,[k])\} \rightarrow (2,[\gamma,k]),\\
&\text{and } \bar{\lambda}_1 \rightarrow \{(1,[k]),(4,[k])\} \rightarrow \lambda_1 \rightarrow \{(2,[k]),(3,[k])\} \rightarrow \bar{\lambda}_1
\end{aligned}$$

for all $[k] \in A_4$ and all $1 \leq \gamma \leq \deg_T([k]) - 1$.

$$\lambda_1 \rightarrow \{(1,[l]),(2,[l])\} \rightarrow \bar{\lambda}_1$$

for all $[l] \in E$. (See Figure 9 for D when $s = 3$.)

It can be verified that $d(D) = 4$; this part of the proof is omitted for brevity and we refer the interested reader to [24] for details. Since every vertex lies in a directed C_4 for D and $d(D) = 4$, $\bar{d}(\mathcal{T}) \leq \max\{4, d(D)\}$ by Lemma 1.3, and thus $\bar{d}(\mathcal{T}) = 4$. □

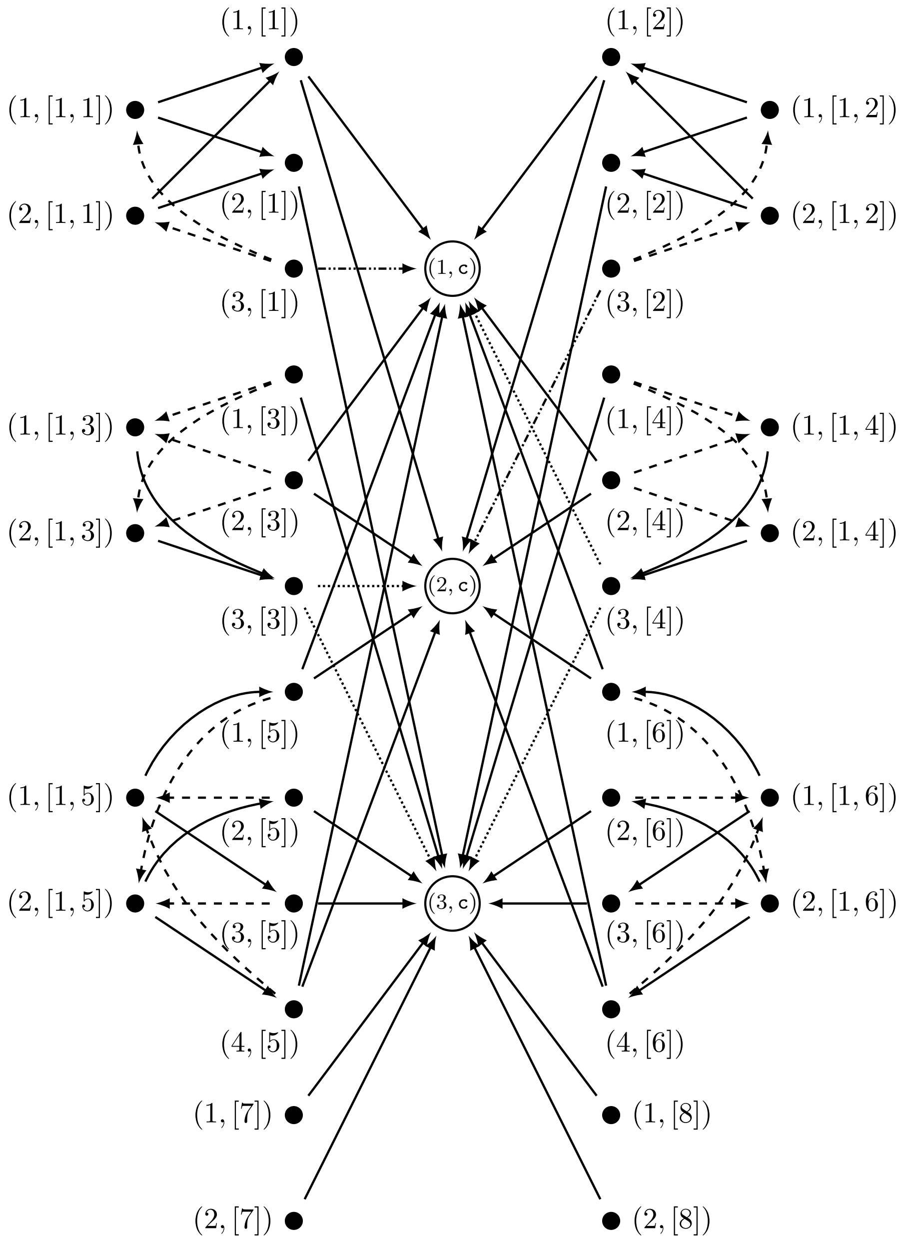

Figure 9: Orientation D for $\mathcal{H}$ for $s = 3$,
$A_3 = \{[1], [2], [3], [4]\}$, $A_4 = \{[5], [6]\}$, E={[7],[8]}.

Proposition 4.3. *Suppose $s \geq 3$ is odd, $A_2 \neq \emptyset$, $A_3 \neq \emptyset$, and $A_{\geq 4} = \emptyset$ for a $\mathcal{T}$. Then, $\mathcal{T} \in \mathscr{C}_0$ if and only if*

$$\begin{cases} |A_2| \leq \binom{s}{\lceil s/2 \rceil} - 1, & \textit{if } |A_3| = 1, \\ (i)\ 2|A_2| + |A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2, \ \textit{or} & \\ (ii)\ 2|A_2| + |A_3| = 2\binom{s}{\lceil s/2 \rceil} - 1, |A_2| \geq \lceil \frac{s}{2} \rceil \lfloor \frac{s}{2} \rfloor \textit{ and } s \geq 5, & \textit{if } |A_3| \geq 2. \end{cases}$$

Proof: ($\Rightarrow$) Since $\mathcal{T} \in \mathscr{C}_0$, there exists an orientation D of $\mathcal{T}$, where $d(D) = 4$. As $A_2 \neq \emptyset$ and $A_3 \neq \emptyset$, we assume (3.1)-(3.5) here, unless stated otherwise.

Case 1: $|A_3| = 1$.

Let $A_3 = \{[j]\}$. By Lemma 2.13(b), either $|O((1, [j]))| = 1$ or $|I((1, [j]))| = 1$. If $|O((1, [1, j]))| = 1$, say $O((1, [1, j])) = \{(1, [j])\}$ (instead of (3.4)), then by Lemma 2.15,

$\{O^{\mathsf{c}}((1,[i])) \mid [i] \in A_2 \cup A_3\}$ is an antichain. So, $|A_2|+|A_3| \leq \binom{s}{\lfloor s/2 \rfloor}$ by Sperner's theorem, i.e., $|A_2| \leq \binom{s}{\lfloor s/2 \rfloor} - 1$. If $|I((1,[j]))| = 1$, then this case follows by the Duality Lemma.

Remark 4.4. The outline of Case 2 is largely similar to Proposition 4.1; we shall skip most of the similar parts. Note that Remark 3.3 applies here. We first show that for any $[i] \in A_2 \cup A_3$ and any $p = 1,2,3$ (wherever applicable), if $|O^{\mathsf{c}}((p,[i]))|$ is too big ($> \lceil \frac{s}{2} \rceil$) or too small ($< \lfloor \frac{s}{2} \rfloor$), then $2|A_2| + |A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$. Here, observe that $B_2^O \cup B_3^O$ ($B_2^I \cup B_3^I$ resp.) plays an analogous role of B_3^O (B_3^I resp.) in Proposition 4.1.

Case 2: $|A_3| \geq 2$.

Note that $B_2^O \cup B_3^O$ and $B_2^I \cup B_3^I$ are antichains by Lemma 2.15 and its dual respectively. With some modifications to the argument in Proposition 4.1, we may assume $|B_3^O| > 0$ and $|B_3^I| > 0$, otherwise $2|A_2| + |A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$ and we are done. Furthermore, by proceeding similarly to Cases 1 and 2 of Proposition 4.1, it suffices to consider the case where $\lfloor \frac{s}{2} \rfloor \leq |O^{\mathsf{c}}((p,[i]))| \leq \lceil \frac{s}{2} \rceil$ for all $1 \leq i \leq \deg_T(\mathsf{c})$ and all $p = 1,2,3$. Partition A_3^O (A_3^I resp.) into $A_3^{O(D)}$ and $A_3^{O(S)}$ ($A_3^{I(D)}$ and $A_3^{I(S)}$ resp.) as in (3.14).

Assumption 1: $|A_3^{O(D)}| \geq 1$ and $|A_3^{I(D)}| \geq 1$.

Suppose $A_3^{O(D)} = \emptyset$, i.e., $O^{\mathsf{c}}((1,[i])) = O^{\mathsf{c}}((2,[i]))$ for all $[i] \in A_3^O$. By the dual of Lemma 2.15, $\{I^{\mathsf{c}}((1,[i])) \mid [i] \in A_3^O\} \cup B_2^I \cup B_3^I$ is an antichain. By Sperner's theorem, $|A_2| + |A_3| = |B_3^O| + |B_2^I| + |B_3^I| = |\{I^{\mathsf{c}}((1,[i])) \mid [i] \in A_3^O\} \cup B_2^I \cup B_3^I| \leq \binom{s}{\lfloor s/2 \rfloor}$. Since $|A_3| \geq 2$, this implies $|A_2| \leq \binom{s}{\lfloor s/2 \rfloor} - 2$. Therefore, $2|A_2| + |A_3| \leq \binom{s}{\lfloor s/2 \rfloor} + [\binom{s}{\lfloor s/2 \rfloor} - 2] = 2\binom{s}{\lceil s/2 \rceil} - 2$. A similar argument follows if $A_3^{I(D)} = \emptyset$.

Remark 4.5. At this stage of Proposition 4.1, we invoked Theorems 2.4 and 2.5 to conclude $B_3^O \cup B_3^I \subseteq \binom{(\mathbb{N}_s,\mathsf{c})}{\lceil s/2 \rceil}$. However, the two theorems cannot apply here because for $[i] \in A_2$, and $d_D((1,[i]),(2,[i])) \leq 4$, it is not necessary that $O^{\mathsf{c}}((1,[i])) \cap I^{\mathsf{c}}((2,[i])) \neq \emptyset$. Consequently, $B_2^O \cup B_3^O$ and $B_2^I \cup B_3^I$ may not be cross-intersecting. Fortunately, by exhausting all possibilites through some easy but tedious computations that we omitted (see [24]), it remains to consider $B_2^O \cup B_3^O \cup B_2^I \cup B_3^I \subseteq \binom{(\mathbb{N}_s,\mathsf{c})}{\lceil s/2 \rceil}$ as desired.

We shall establish a series of claims on the structure of D, from which we will derive $2|A_2|+|A_3| \leq 2\binom{s}{\lceil s/2 \rceil}-2$ if any one fails to hold. In other words, $2|A_2|+|A_3| = 2\binom{s}{\lceil s/2 \rceil}-1$ is only possible in the last scenario where all these claims hold (see Remark 4.2).

Subcase 2.1. $|O^{\mathsf{c}}((1,[i]))| = |O^{\mathsf{c}}((2,[i]))| = \lfloor \frac{s}{2} \rfloor$ for some $[i] \in A_3^{O(D)}$.

This is similar to Subcase 3.1 in Proposition 4.1.

Subcase 2.2. $|O^{\mathsf{c}}((1,[i]))| = |O^{\mathsf{c}}((2,[i]))| = \lceil \frac{s}{2} \rceil$ for some $[i] \in A_3^{I(D)}$.

This subcase follows from the Duality Lemma and Subcase 2.1.

Subcase 2.3. $|O^{\mathsf{c}}((1,[i^*]))| = \lfloor \frac{s}{2} \rfloor$ and $|O^{\mathsf{c}}((2,[i^*]))| = \lceil \frac{s}{2} \rceil$ for some $[i^*] \in A_3^{O(D)}$.

For any $[j] \in A_2 \cup A_3^I$, $d_D((1,[i^*]),(1,[1,j])) = 3$ implies $X \neq I^{\mathsf{c}}((1,[i^*]))$ for all $X \in B_2^I \cup B_3^I$. It follows that $|A_2|+|A_3^I| = |B_2^I \cup B_3^I| \leq \binom{s}{\lceil s/2 \rceil}-1$. Therefore, $2|A_2|+|A_3| = (|B_2^O| + |B_3^O|) + (|B_2^I| + |B_3^I|) \leq [\binom{s}{\lceil s/2 \rceil} - 1] + \binom{s}{\lceil s/2 \rceil} = 2\binom{s}{\lceil s/2 \rceil} - 1$.

Subcase 2.3 is done for $2|A_2| + |A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$. Now, our aim is to prove $|A_2| \geq \lceil \frac{s}{2} \rceil \lfloor \frac{s}{2} \rfloor$ and $s \geq 5$ in the event that $2|A_2| + |A_3| = 2\binom{s}{\lceil s/2 \rceil} - 1$. The following claims will help us to achieve the said aim.

Suppose $2|A_2| + |A_3| = 2\binom{s}{\lceil s/2\rceil} - 1$. Equivalently,

$$|B_2^O| + |B_3^O| = \binom{s}{\lceil s/2\rceil} \text{ and } |B_2^I| + |B_3^I| = \binom{s}{\lceil s/2\rceil} - 1. \tag{4.1}$$

Then, Claims 2A-6A can be proved similarly to Assumptions 2A-6A.
Claim 2A: $|O^c((1,[i]))| = \lfloor \frac{s}{2} \rfloor$ and $|O^c((2,[i]))| = \lceil \frac{s}{2} \rceil$ for all $[i] \in A_3^{O(D)}$.
Claim 3A: $O^c((1,[i])) = O^c((1,[i^*]))$ for all $[i] \in A_3^{O(D)}$.
Claim 4A: $O^c((2,[i])) = O^c((3,[i]))$ for all $[i] \in A_3^{O(D)}$.
Claim 5A: $|O^c((1,[i]))| = |O^c((2,[i]))| = \lfloor \frac{s}{2} \rfloor$ for all $[i] \in A_3^{I(D)}$.
Claim 6A: $O^c((1,[i])) = O^c((1,[i^*]))$ and $O^c((2,[i])) = O^c((3,[i]))$ for all $[i] \in A_3^{I(D)}$.

Suppose $s = 3$. By Claims 5A-6A, $|O^c((1,[j])) \cup O^c((2,[j]))| \le \lceil \frac{s}{2} \rceil$ for any $[j] \in A_3^{I(D)}$. Since $B_2^O \cup B_3^O = \binom{\mathbb{N}_{s,c}}{\lceil s/2\rceil}$, we have $O^c((1,[j])) \cup O^c((2,[j])) \subseteq X$ for some $X \in B_2^O \cup B_3^O$, a contradiction to Lemma 2.16(b). Hence, $s \ge 5$.

Let $[j] \in A_3^{I(D)}$, $[i] \in A_2 \cup A_3^O$ and $\delta = 1$, if $[i] \in A_2$; $\delta = 3$, if $[i] \in A_3^O$. Then, Claim 6A and $d_D((1,[1,j]),(\delta,[i])) = 3$ imply $O^c((1,[i^*])) \cup O^c((2,[j])) = O^c((1,[j])) \cup O^c((2,[j])) \neq O^c((\delta,[i]))$, i.e., $O^c((2,[j]))$ cannot be a $\lfloor \frac{s}{2} \rfloor$-set whose union with $O^c((1,[i^*]))$ forms a $\lceil \frac{s}{2} \rceil$-set. So, $\{O^c((3,[j])) \mid [j] \in A_3^{I(D)}\} \cap R = \{O^c((2,[j])) \mid [j] \in A_3^{I(D)}\} \cap R = \emptyset$, where $R = \{X \in \binom{\mathbb{N}_{s,c}}{\lfloor s/2\rfloor} \mid |X \cup O^c((1,[i^*]))| = \lceil \frac{s}{2} \rceil\}$. It is easy to see that $|R| = \lceil \frac{s}{2} \rceil \lfloor \frac{s}{2} \rfloor$.

Furthermore, $d_D((1,[i^*]),(1,[1,j])) = 3$ and the definition of R imply $O^c((1,[i^*])) \notin \{O^c((3,[j])) \mid [j] \in A_3^{I(D)}\} \cup R$. It follows that $\{O^c((3,[j])) \mid [j] \in A_3^{I(D)}\} \cup R \subseteq \binom{\mathbb{N}_{s,c}}{\lfloor s/2\rfloor} - \{O^c((1,[i^*]))\}$. So, $|A_3^{I(D)}| \le \binom{s}{\lceil s/2\rceil} - 1 - |R| = \binom{s}{\lceil s/2\rceil} - 1 - \lceil \frac{s}{2} \rceil \lfloor \frac{s}{2} \rfloor$. Since $\{O^c((1,[i])) \mid [i] \in A_3^{I(S)}\} \cup B_2^O \cup B_3^O$ is an antichain by Lemma 2.15, $|A_3^{I(S)}| + |B_2^O| + |B_3^O| \le \binom{s}{\lceil s/2\rceil}$. So, $|A_2| + |A_3| = |A_3^{I(D)}| + (|A_3^{I(S)}| + |B_3^O| + |B_2^O|) \le 2\binom{s}{\lceil s/2\rceil} - 1 - \lceil \frac{s}{2} \rceil \lfloor \frac{s}{2} \rfloor$. Using $2|A_2| + |A_3| = 2\binom{s}{\lfloor s/2\rfloor} - 1$, we derive $|A_2| \ge \lceil \frac{s}{2} \rceil \lfloor \frac{s}{2} \rfloor$.

Remark 4.6. In addition to Claims 2A to 6A, the following claim can be shown too (see [24] for details).
Claim 7A: $A_3^O = A_3^{O(D)}$ and $A_3^I = A_3^{I(D)}$.

Subcase 2.4. $|O^c((1,[i^*]))| = \lceil \frac{s}{2} \rceil$ and $|O^c((2,[i^*]))| = \lfloor \frac{s}{2} \rfloor$ for some $[i^*] \in A_3^{I(D)}$.
This follows from Subcase 2.3 by the Duality Lemma.

Subcase 2.5. $|O^c((1,[i]))| = |O^c((2,[i]))| = \lceil \frac{s}{2} \rceil$ and $|O^c((1,[j]))| = |O^c((2,[j]))| = \lfloor \frac{s}{2} \rfloor$ for some $[i] \in A_3^{O(D)}$ and $[j] \in A_3^{I(D)}$.
This is similar to Subcase 3.5 in Proposition 4.1.

($\Leftarrow$) By Corollary 3.8, $\mathcal{T} \in \mathscr{C}_0$ if $|A_2| \le \binom{s}{\lceil s/2\rceil} - 2$ and $|A_3| = 1$, or $|A_2| + |A_3| \le \binom{s}{\lceil s/2\rceil} - 1$ and $|A_3| \ge 2$. Hence, it suffices to consider the following three cases. Let $\mathcal{H} = T(t_1, t_2, \ldots, t_n)$ be the subgraph of $\mathcal{T}$, where $t_c = s$, $t_{[i]} = 3$ for all $[i] \in \mathcal{T}(A_3)$ and $t_v = 2$ otherwise. We will use A_j for $\mathcal{H}(A_j)$ for the remainder of this proof.

Case 1. $|A_2| = \binom{s}{\lceil s/2\rceil} - 1$ and $|A_3| = 1$.
Assume without loss of generality that $A_3 = \{[1]\}$ and $A_2 = \{[i] \mid i \in \mathbb{N}_{\binom{s}{\lceil s/2\rceil}}\} - A_3$.

Define an orientation D_1 of $\mathcal{H}$ as follows.

$$\begin{aligned}&(2,[i]) \to \{(1,[\alpha,i]),(2,[\alpha,i])\} \to (1,[i]), \text{ and}\\&\bar{\lambda}_i \to (1,[i]) \to \lambda_i \to (2,[i]) \to \bar{\lambda}_i\end{aligned}$$

for all $[i] \in A_2$ and all $1 \le \alpha \le \deg_T([i]) - 1$, i.e., excluding λ_1, the $\lceil \frac{s}{2} \rceil$-sets λ_i's are used as 'in-sets' ('out-sets' resp.) to construct B_2^I (B_2^O resp.).

$$\begin{aligned}&\{(1,[1]),(2,[1])\} \to (1,[\beta,1]) \to (3,[1]),\\&\{(1,[1]),(3,[1])\} \to (2,[\beta,1]) \to (2,[1]), \text{ and}\\&\bar{\lambda}_1 \to \{(2,[1]),(3,[1])\} \to \lambda_1 \to (1,[1]) \to \bar{\lambda}_1\end{aligned}$$

for all $1 \le \beta \le \deg_T([1]) - 1$. Furthermore,

$$\lambda_1 \to \{(1,[j]),(2,[j])\} \to \bar{\lambda}_1$$

for all $[j] \in E$.

Case 2. $|A_3| \ge 2$ and $2|A_2| + |A_3| \le 2\binom{s}{\lceil s/2 \rceil} - 2$.

By Corollary 3.8, we may assume $|A_2| + |A_3| \ge \binom{s}{\lceil s/2 \rceil}$. Furthermore, assume without loss of generality that $A_2 = \{[i] \mid i \in \mathbb{N}_{|A_2|}\}$, and $A_3 = \{[i] \mid i \in \mathbb{N}_{|A_2|+|A_3|} - \mathbb{N}_{|A_2|}\}$. Define an orientation D_2 of $\mathcal{H}$ as follows.

$$\begin{aligned}&(2,[i]) \to \{(1,[\alpha,i]),(2,[\alpha,i])\} \to (1,[i]), \text{ and}\\&\bar{\lambda}_{i+1} \to (1,[i]) \to \lambda_{i+1} \to (2,[i]) \to \bar{\lambda}_{i+1},\end{aligned}$$

for all $1 \le i \le |A_2|$ and all $1 \le \alpha \le \deg_T([i]) - 1$, i.e., the $\lceil \frac{s}{2} \rceil$-sets $\lambda_2, \lambda_3, \ldots, \lambda_{|A_2|+1}$ are used as 'in-sets' ('out-sets' resp.) to construct B_2^I (B_2^O resp.).

$$\begin{aligned}&(3,[j]) \to \{(1,[\beta,j]),(2,[\beta,j])\} \to \{(1,[j]),(2,[j])\},\\&\bar{\lambda}_1 \to (1,[j]) \to \lambda_1 \to (2,[j]) \to \bar{\lambda}_1, \text{ and}\\&\lambda_{j+1} \to (3,[j]) \to \bar{\lambda}_{j+1}\end{aligned}$$

for all $|A_2| + 1 \le j \le \binom{s}{\lceil s/2 \rceil} - 1$ and all $1 \le \beta \le \deg_T([j]) - 1$, i.e., the $\lceil \frac{s}{2} \rceil$-sets $\lambda_{|A_2|+2}, \lambda_{|A_2|+3}, \ldots, \lambda_{\binom{s}{\lceil s/2 \rceil}}$ are used as 'in-sets' to construct B_3^I.

$$\begin{aligned}&\{(1,[k]),(2,[k])\} \to \{(1,[\gamma,k]),(2,[\gamma,k])\} \to (3,[k]),\\&\lambda_1 \to (1,[k]) \to \bar{\lambda}_1 \to (2,[k]) \to \lambda_1, \text{ and}\\&\bar{\lambda}_{k-\binom{s}{\lceil s/2 \rceil}+|A_2|+2} \to (3,[k]) \to \lambda_{k-\binom{s}{\lceil s/2 \rceil}+|A_2|+2}\end{aligned}$$

for all $\binom{s}{\lceil s/2 \rceil} \le k \le |A_2| + |A_3|$ and all $1 \le \gamma \le \deg_T([k]) - 1$, i.e., the $\lceil \frac{s}{2} \rceil$-sets $\lambda_{|A_2|+2}, \lambda_{|A_2|+3}, \ldots, \lambda_{2|A_2|+|A_3|+2-\binom{s}{\lceil s/2 \rceil}}$ are used as 'out-sets' to construct B_3^O.

$$\lambda_1 \to \{(1,[l]),(2,[l])\} \to \bar{\lambda}_1$$

for all $[l] \in E$.

Case 3. $|A_3| \geq 2$, $|A_2| \geq \lceil \frac{s}{2} \rceil \lfloor \frac{s}{2} \rfloor$, $2|A_2| + |A_3| = 2\binom{s}{\lceil s/2 \rceil} - 1$, and $s \geq 5$.

Let $\psi = (\mathbb{N}_{\lfloor \frac{s}{2} \rfloor}, \mathsf{c})$ and $I_\psi = \{\lambda \in \binom{(\mathbb{N}_s,\mathsf{c})}{\lceil s/2 \rceil} \mid |\lambda \cap \psi| = 1\}$. It is easy to check $|I_\psi| = \lceil \frac{s}{2} \rceil \lfloor \frac{s}{2} \rfloor$ and $\bar{\psi} \notin I_\psi$. Let $O_\psi = \{\lambda \in \binom{(\mathbb{N}_s,\mathsf{c})}{\lceil s/2 \rceil} \mid \psi \subset \lambda\}$ and observe that $|O_\psi| = \lceil \frac{s}{2} \rceil$. Our aim is to design an orientation in which the elements of I_ψ and O_ψ are used as $I^{\mathsf{c}}((2,[i]))$ and $O^{\mathsf{c}}((1,[i]))$ respectively, where $[i] \in A_2$, i.e., $I_\psi \subseteq B_2^I$ and $O_\psi \subseteq B_2^O$. To achieve this, we introduce two new listings of the elements in $\binom{(\mathbb{N}_s,\mathsf{c})}{\lceil s/2 \rceil}$. Let $\binom{(\mathbb{N}_s,\mathsf{c})}{\lceil s/2 \rceil} = \{\gamma_1, \gamma_2, \ldots, \gamma_{\binom{s}{\lceil s/2 \rceil}}\} = \{\mu_1, \mu_2, \ldots, \mu_{\binom{s}{\lceil s/2 \rceil}}\}$ such that $\bar{\psi} = \gamma_{\binom{s}{\lceil s/2 \rceil}}$, $I_\psi = \{\gamma_1, \gamma_2, \ldots, \gamma_{\lceil \frac{s}{2} \rceil \lfloor \frac{s}{2} \rfloor}\}$ and $O_\psi = \{\mu_1, \mu_2, \ldots, \mu_{\lceil \frac{s}{2} \rceil}\}$. The denotation of the remaining γ_i's and μ_j's can be arbitrary. Assume further that $A_2 = \{[i] \mid i \in \mathbb{N}_{|A_2|}\}$ and $A_3 = \{[i] \mid i \in \mathbb{N}_{|A_2|+|A_3|} - \mathbb{N}_{|A_2|}\}$. Define an orientation D_3 of $\mathcal{H}$ as follows.

$$\begin{aligned}
&(2,[i]) \to \{(1,[\alpha,i]),(2,[\alpha,i])\} \to (1,[i]),\\
&\bar{\mu}_i \to (1,[i]) \to \mu_i, \text{ and } \gamma_i \to (2,[i]) \to \bar{\gamma}_i,
\end{aligned}$$

for all $1 \leq i \leq |A_2|$ and all $1 \leq \alpha \leq \deg_T([i]) - 1$, i.e., the $\lceil \frac{s}{2} \rceil$-sets $\gamma_1, \gamma_2, \ldots, \gamma_{|A_2|}$ ($\mu_1, \mu_2, \ldots, \mu_{|A_2|}$ resp.) are used as 'in-sets' ('out-sets' resp.) to construct B_2^I (B_2^O resp.).

$$\begin{aligned}
&\{(1,[j]),(2,[j])\} \to (1,[\beta,j]) \to (3,[j]),\\
&\{(1,[j]),(3,[j])\} \to (2,[\beta,j]) \to (2,[j]),\\
&\bar{\psi} \to (1,[j]) \to \psi, \text{ and}\\
&\bar{\mu}_j \to \{(2,[j]),(3,[j])\} \to \mu_j
\end{aligned}$$

for all $|A_2|+1 \leq j \leq \binom{s}{\lceil s/2 \rceil}$ and all $1 \leq \beta \leq \deg_T([j])-1$, i.e., the $\lceil \frac{s}{2} \rceil$-sets $\mu_{|A_2|+1}, \mu_{|A_2|+2}, \ldots, \mu_{\binom{s}{\lceil s/2 \rceil}}$ are used as 'out-sets' to construct B_3^O.

$$\begin{aligned}
&(3,[k]) \to (1,[\theta,k]) \to \{(1,[k]),(2,[k])\},\\
&(2,[k]) \to (2,[\theta,k]) \to \{(1,[k]),(3,[k])\},\\
&\bar{\psi} \to (1,[k]) \to \psi, \text{ and}\\
&\gamma_{k-\binom{s}{\lceil s/2 \rceil}+|A_2|} \to \{(2,[k]),(3,[k])\} \to \bar{\gamma}_{k-\binom{s}{\lceil s/2 \rceil}+|A_2|}
\end{aligned}$$

for all $\binom{s}{\lceil s/2 \rceil} + 1 \leq k \leq |A_2| + |A_3|$ and all $1 \leq \theta \leq \deg_T([k]) - 1$, i.e., the $\lceil \frac{s}{2} \rceil$-sets $\gamma_{|A_2|+1}, \gamma_{|A_2|+2}, \ldots, \gamma_{\binom{s}{\lceil s/2 \rceil}-1}$ are used as 'in-sets' to construct B_3^I.

$$\bar{\psi} \to \{(1,[l]),(2,[l])\} \to \psi$$

for all $[l] \in E$.

It can be verified for $i = 1,2,3$, that $d(D_i) = 4$ and every vertex lies in a directed C_4 for D_i; this part of the proof is omitted for brevity and we refer the interested reader to [24] for details. Hence, $\bar{d}(\mathcal{T}) \leq \max\{4, d(D_i)\}$ by Lemma 1.3, and we have $\bar{d}(\mathcal{T}) = 4$. □

Corollary 4.7. *Suppose $s \geq 5$ is odd, $A_{\geq 4} = \emptyset$, $|A_2| \geq \lceil \frac{s}{2} \rceil \lfloor \frac{s}{2} \rfloor$, $|A_3| \geq 2$, and $2|A_2| + |A_3| = 2\binom{s}{\lceil s/2 \rceil} - 1$ for a $\mathcal{T}$. If D is an optimal orientation of $\mathcal{T}$, then either D or $\tilde{D}$ fulfills the following, after a suitable relabelling of vertices and with A_3 partitioned into A_3^O and A_3^I.*
(I) $|O^{\mathsf{c}}((1,[i]))| = |I^{\mathsf{c}}((2,[i]))| = \lceil \frac{s}{2} \rceil$ for all $[i] \in A_2$,
(II) $|O^{\mathsf{c}}((3,[j]))| = |I^{\mathsf{c}}((3,[k]))| = \lceil \frac{s}{2} \rceil$ for all $[j] \in A_3^O$, $[k] \in A_3^I$, and

(III) $(2,[i]) \to (p,[\alpha,i]) \to (1,[i])$ *for all* $[i] \in A_2$*, all* $1 \le \alpha \le \deg_T([i]) - 1$ *and* $1 \le p \le s_{[\alpha,i]}$.
(1A) $|A_2 \cup A_3^O| = \binom{s}{\lceil s/2 \rceil}$ *and* $|A_2 \cup A_3^I| = \binom{s}{\lceil s/2 \rceil} - 1$;
(2A) $|O^c((1,[i]))| = \lfloor \frac{s}{2} \rfloor$ *and* $|O^c((2,[i]))| = \lceil \frac{s}{2} \rceil$ *for all* $[i] \in A_3^O$;
(3A) $O^c((1,[i])) = O^c((1,[j]))$ *for all* $[i],[j] \in A_3^O$;
(4A) $O^c((2,[i])) = O^c((3,[i]))$ *for all* $[i] \in A_3^O$;
(5A) $|O^c((1,[j]))| = |O^c((2,[j]))| = \lfloor \frac{s}{2} \rfloor$ *for all* $[j] \in A_3^I$;
(6A) $O^c((1,[j])) = O^c((1,[i]))$ *and* $O^c((2,[j])) = O^c((3,[j]))$ *for all* $[i] \in A_3^O$, $[j] \in A_3^I$.

Except for (III), the proof of Corollary 4.7 largely follows from the proof of Proposition 4.3; we refer the interested reader to [24] for details. The optimal orientation(s) described in Corollary 4.7 was extended to the construction D_3 in Case 3 of the proof of Proposition 4.3.

Proposition 4.8. *Suppose* $s \ge 3$ *is odd,* $A_2 \neq \emptyset$*,* $A_3 = \emptyset$ *and* $A_{\ge 4} \neq \emptyset$ *for a* $\mathcal{T}$*. Then,* $\mathcal{T} \in \mathscr{C}_0$ *if and only if* $|A_2| \le \binom{s}{\lceil s/2 \rceil} - 1$.

Proof: ($\Rightarrow$) Since $\mathcal{T} \in \mathscr{C}_0$, there exists an orientation D of $\mathcal{T}$, where $d(D) = 4$. As $A_2 \neq \emptyset$, we assume (3.1)-(3.2) here. Let $[j] \in A_{\ge 4}$. If $|O^c((1,[j]))| \ge \lceil \frac{s}{2} \rceil$, then for any $[i] \in A_2$, $d_D((1,[1,i]),(1,[j])) = 3$ implies $O^c((1,[i])) \cap I^c((1,[j])) \neq \emptyset$. Hence, by Lih's theorem, $|A_2| = |B_2^O| \le \binom{s}{\lceil s/2 \rceil} - \binom{s-|I^c((1,[i]))|}{\lceil s/2 \rceil} \le \binom{s}{\lceil s/2 \rceil} - 1$. Suppose $|O^c((1,[j]))| \le \lfloor \frac{s}{2} \rfloor$. Equivalently, $|I^c((1,[j]))| \ge \lceil \frac{s}{2} \rceil$. So, this case follows from the previous case by the Duality Lemma.

($\Leftarrow$) If $|A_2| + |A_{\ge 4}| \le \binom{s}{\lceil s/2 \rceil} - 1$, then by Corollary 3.8(i), $\mathcal{T} \in \mathscr{C}_0$. Hence, we assume $|A_2| + |A_{\ge 4}| \ge \binom{s}{\lceil s/2 \rceil}$ hereafter, on top of the hypothesis that $|A_2| \le \binom{s}{\lceil s/2 \rceil} - 1$. If $|A_2| \ge s-1$, define $A_2^\diamond = A_2$. Otherwise, $A_2^\diamond = A_2 \cup A^*$, where A^* is an arbitrary subset of $A_{\ge 4}$ such that $|A_2^\diamond| = s-1$. Then, let $A_4^\diamond = A_2 \cup A_{\ge 4} - A_2^\diamond$. Furthermore, assume without loss of generality that $A_2^\diamond = \{[i] \mid i \in \mathbb{N}_{|A_2^\diamond|}\}$ and $A_4^\diamond = \{[i] \mid i \in \mathbb{N}_{|A_2^\diamond|+|A_4^\diamond|} - \mathbb{N}_{|A_2^\diamond|}\}$.

Let $\mathcal{H} = T(t_1, t_2, \ldots, t_n)$ be the subgraph of $\mathcal{T}$, where $t_{\mathsf{c}} = s$, $t_{[i]} = 4$ for all $[i] \in \mathcal{T}(A_4^\diamond)$ and $t_v = 2$ otherwise. We will use A_j for $\mathcal{H}(A_j)$ for the remainder of this proof. Define an orientation D of $\mathcal{H}$ as follows.

$$(2,[i]) \to \{(1,[\alpha,i]),(2,[\alpha,i])\} \to (1,[i]), \text{ and}$$
$$\bar{\lambda}_{i+1} \to (1,[i]) \to \lambda_{i+1} \to (2,[i]) \to \bar{\lambda}_{i+1}$$

for all $1 \le i \le |A_2|$ and all $1 \le \alpha \le \deg_T([i]) - 1$, i.e., the $\lceil \frac{s}{2} \rceil$-sets $\lambda_2, \lambda_3, \ldots, \lambda_{|A_2|+1}$ are used as 'in-sets' ('out-sets' resp.) to construct B_2^I (B_2^O resp.).

$$(2,[\beta,j]) \to \{(2,[j]),(4,[j])\} \to (1,[\beta,j]) \to \{(1,[j]),(3,[j])\} \to (2,[\beta,j]),$$
$$\text{and } \bar{\lambda}_1 \to \{(1,[j]),(4,[j])\} \to \lambda_1 \to \{(2,[j]),(3,[j])\} \to \bar{\lambda}_1$$

for all $[j] \in A_4$ and all $1 \le \beta \le \deg_T([j]) - 1$.

$$\lambda_1 \to \{(1,[k]),(2,[k])\} \to \bar{\lambda}_1$$

for all $[k] \in E$.

It can be verified that $d(D) = 4$; this part of the proof is omitted for brevity and we refer the interested reader to [24] for details. Since every vertex lies in a directed C_4 for D and $d(D) = 4$, $\bar{d}(\mathcal{T}) \le \max\{4, d(D)\}$ by Lemma 1.3, and thus $\bar{d}(\mathcal{T}) = 4$. □

Corollary 4.9. *If $s \geq 3$ is odd and $|A_{\leq 3}| \leq \binom{s}{\lceil s/2 \rceil} - 1$ for a $\mathcal{T}$, then $\mathcal{T} \in \mathscr{C}_0$.*

Proof: Note that every vertex lies in a directed C_4 for the orientation D defined in Proposition 4.8, $\bar{d}(\mathcal{T}) \leq \max\{4, d(D)\}$ by Lemma 1.3. This implies $\bar{d}(\mathcal{T}) = 4$. □

Corollary 4.9 is an improvement of Corollary 3.8(i) for odd s as the count $|A_{\leq 3}|$ now excludes $|A_{\geq 4}|$, compared to the previous $|A_{\geq 2}|$.

Proposition 4.10. *Suppose $s \geq 3$ is odd, $A_2 \neq \emptyset$, $A_3 \neq \emptyset$ and $A_{\geq 4} \neq \emptyset$ for a $\mathcal{T}$. Then,*

$$\mathcal{T} \in \mathscr{C}_0 \iff \begin{cases} |A_2| \leq \binom{s}{\lceil s/2 \rceil} - 2, & \text{if } |A_3| = 1, \\ 2|A_2| + |A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2, & \text{if } |A_3| \geq 2. \end{cases}$$

Proof: ($\Rightarrow$) Since $\mathcal{T} \in \mathscr{C}_0$, there exists an orientation D of $\mathcal{T}$, where $d(D) = 4$. As $A_2 \neq \emptyset$ and $A_3 \neq \emptyset$, we assume (3.1)-(3.5) here.

Case 1. $|A_3| = 1$.

Partition $A_{\geq 4}$ into $A^O_{\geq 4}$, $A^I_{\geq 4}$, and $A^N_{\geq 4}$ as follows. Let $A^N_{\geq 4} = \{[i] \in A_{\geq 4} \mid |O((p, [\alpha, i]))| \geq 2$ and $|I((p, [\alpha, i]))| \geq 2$ for all $1 \leq \alpha \leq \deg_T([i]) - 1$ and all $1 \leq p \leq s_{[\alpha,i]}\}$, and $A^O_{\geq 4} = \{[i] \in A_{\geq 4} \mid |O((p, [\alpha, i]))| = 1$ for some $1 \leq \alpha \leq \deg_T([i]) - 1$ and some $1 \leq p \leq s_{[\alpha,i]}\}$. Furthermore, let $A^I_{\geq 4} = A_{\geq 4} - A^O_{\geq 4} \cup A^N_{\geq 4}$, i.e., for every $[i] \in A^I_{\geq 4}$, there exist some $1 \leq \alpha \leq \deg_T([i]) - 1$ and some $1 \leq p \leq s_{[\alpha,i]}$ such that $|I((p, [\alpha, i]))| = 1$.

Without loss of generality, we assume

$$\begin{aligned} (\mathbb{N}_{s_{[i]}}, [i]) - \{(4, [i])\} \rightarrow &(1, [1, i]) \rightarrow (4, [i]) \text{ if } [i] \in A^O_{\geq 4}, \\ (4, [i]) \rightarrow &(1, [1, i]) \rightarrow (\mathbb{N}_{s_{[i]}}, [i]) - \{(4, [i])\} \text{ if } [i] \in A^I_{\geq 4}, \\ \text{and } \{(1, [i]), (2, [i])\} \rightarrow &(1, [1, i]) \rightarrow \{(3, [i]), (4, [i])\} \text{ if } [i] \in A^N_{\geq 4}. \end{aligned}$$

Also, we let

$$B^O_{\geq 4} = \{O^c((4, [i])) \mid [i] \in A^O_{\geq 4}\} \text{ and } B^I_{\geq 4} = \{I^c((4, [i])) \mid [i] \in A^I_{\geq 4}\}.$$

Note that each of $B^O_2 \cup B^O_3 \cup B^O_{\geq 4}$ and $B^I_2 \cup B^I_3 \cup B^I_{\geq 4}$ is an antichain by Lemma 2.15 and its dual respectively. Hence, $|A_2| + |A^O_3| + |A^O_{\geq 4}| = |B^O_2 \cup B^O_3 \cup B^O_{\geq 4}| \leq \binom{s}{\lfloor s/2 \rfloor}$ and $|A_2| + |A^I_3| + |A^I_{\geq 4}| = |B^I_2 \cup B^I_3 \cup B^I_{\geq 4}| \leq \binom{s}{\lfloor s/2 \rfloor}$ by Sperner's theorem.

We will only consider the case when $|A^O_3| = 1$ since the case when $|A^I_3| = 1$ can be argued analogously. If $|A^O_{\geq 4}| > 0$, then $|A_2| + |A^O_3| + |A^O_{\geq 4}| \leq \binom{s}{\lfloor s/2 \rfloor}$ implies $|A_2| \leq \binom{s}{\lfloor s/2 \rfloor} - 2$. Hence, suppose $|A^O_{\geq 4}| = 0$. Note that $|A_3| = 1$ implies $|A^I_3| = |B^I_3| = 0$.

Subcase 1.1. $|O^c((q, [j]))| \geq \lceil \frac{s}{2} \rceil$ for some $[j] \in A^I_{\geq 4} \cup A^N_{\geq 4}$ and some $1 \leq q \leq s_{[j]}$.

For any $[i] \in A_2 \cup A^O_3$, $d_D((1, [1, i]), (q, [j])) = 3$ implies $X \cap I^c((q, [j])) \neq \emptyset$ for all $X \in B^O_2 \cup B^O_3$. Hence, by Lih's theorem, $|A_2| + |A^O_3| = |B^O_2 \cup B^O_3| \leq \binom{s}{\lceil s/2 \rceil} - \binom{s - |I^c((1,[i]))|}{\lceil s/2 \rceil} \leq \binom{s}{\lceil s/2 \rceil} - \binom{\lceil s/2 \rceil}{\lceil s/2 \rceil} \leq \binom{s}{\lceil s/2 \rceil} - 1$. Since $|A^O_3| = 1$, it follows that $|A_2| \leq \binom{s}{\lfloor s/2 \rfloor} - 2$.

Subcase 1.2. $|O^c((q, [j]))| < \lfloor \frac{s}{2} \rfloor$ for some $[j] \in A^I_{\geq 4} \cup A^N_{\geq 4}$ and some $1 \leq q \leq s_{[j]}$.

For any $[i] \in A_2$, $d_D((q, [j]), (1, [1, i])) = 3$ implies $O^c((q, [j])) \cap X \neq \emptyset$ for all $X \in B^I_2$. Hence, by Lih's theorem, $|A_2| = |B^I_2| \leq \binom{s}{\lceil s/2 \rceil} - \binom{s - |O^c((q,[j]))|}{\lceil s/2 \rceil} \leq \binom{s}{\lceil s/2 \rceil} - \binom{\lceil s/2 \rceil + 1}{\lceil s/2 \rceil} = \binom{s}{\lceil s/2 \rceil} - (\lceil \frac{s}{2} \rceil + 1) \leq \binom{s}{\lceil s/2 \rceil} - 3$.

Subcase 1.3. $|O^c((q,[j]))| = \lfloor \frac{s}{2} \rfloor$ for all $[j] \in A^I_{\geq 4} \cup A^N_{\geq 4}$ and all $1 \leq q \leq s_{[j]}$.

For any $[i] \in A_2$, $d_D((q,[j]),(1,[1,i]) = 3$ implies $\bar{X} \cap O^c((q,[j])) \neq \emptyset$ for all $X \in B^I_2$. Hence, by Lih's theorem, $|A_2| = |B^I_2| \leq \binom{s}{\lceil s/2 \rceil} - \binom{s-|O^c((q,[j]))|}{\lceil s/2 \rceil} = \binom{s}{\lceil s/2 \rceil} - \binom{\lceil s/2 \rceil}{\lceil s/2 \rceil} = \binom{s}{\lceil s/2 \rceil} - 1$. By Griggs' theorem, $|A_2| = \binom{s}{\lceil s/2 \rceil} - 1$ if and only if B^I_2 consists of only $\lceil \frac{s}{2} \rceil$-sets.

If $|A_2| = \binom{s}{\lceil s/2 \rceil} - 1$, then it must follow that $O^c((p,[j])) = O^c((q,[j]))$ for all $[j] \in A^I_{\geq 4} \cup A^N_{\geq 4}$ and all $1 \leq p,q \leq s_{[j]}$. Otherwise, $d_D((r,[j]),(1,[1,i])) = 3$ for all $1 \leq r \leq s_{[j]}$ and all $[i] \in A_2$ implies $\bar{X} \neq O^c((r,[j]))$ for all $X \in B^I_2$, i.e., $B^I_2 \subseteq \binom{(\mathbb{N}_s, \mathsf{c})}{\lceil s/2 \rceil} - \{O^c((p,[j])), O^c((q,[j]))\}$. Hence, $|A_2| = |B^I_2| \leq \binom{s}{\lceil s/2 \rceil} - 2$, a contradiction.

Now, $B^O_2 \cup B^O_3 \cup \{O^c((3,[j])) \mid [j] \in A^I_{\geq 4} \cup A^N_{\geq 4}\}$ is an antichain by Lemma 2.15. So, $|A_2| + |A_3| + |A_4| = |B^O_2| + |B^O_3| + |A^I_{\geq 4}| + |A^N_{\geq 4}| \leq \binom{s}{\lceil s/2 \rceil}$. Since $|A_3| = 1$ and $|A_4| \geq 1$, it follows that $|A_2| \leq \binom{s}{\lceil s/2 \rceil} - 2$.

Case 2. $|A_3| \geq 2$.

For any $[i],[j] \in A_2 \cup A_3$, $i \neq j$, $[k] \in A_{\geq 4}$, $1 \leq \alpha \leq \deg_T([i]) - 1$, $1 \leq \gamma \leq \deg_T([k]) - 1$, $1 \leq x \leq s_{[\alpha,i]}$, and $1 \leq z \leq s_{[\gamma,k]}$, $1 \leq q \leq 3$ (where applicable), $1 \leq r \leq 4$, observe in D that the vertices $(r,[k])$ and $(z,[\gamma,k])$ do not lie on any shortest path between $(x,[\alpha,i])$ and $(q,[j])$. By the proof in Proposition 4.3, we have $2|A_2| + |A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 1$, where equality is possible only as in Subcases 2.3 and 2.4.

Suppose $2|A_2| + |A_3| = 2\binom{s}{\lceil s/2 \rceil} - 1$ holds, as in Subcase 2.3 with $[i^*]$ as given, and recall (4.1). In particular, $B^I_2 \cup B^I_3 = \binom{(\mathbb{N}_s, \mathsf{c})}{\lceil s/2 \rceil} - \{I^c((1,[i^*]))\}$ by Claims 6A and 7A.

Let $[i] \in A_{\geq 4}$. If there exists some $1 \leq p \leq s_{[i]}$ such that $|O^c((p,[i]))| \geq \lceil \frac{s}{2} \rceil$, then $X \subseteq O^c((p,[i]))$ for some $X \in B^O_2 \cup B^O_3$. This implies that $d_D((1,[1,j]),(p,[i])) > 4$ for some $[j] \in A_2 \cup A^O_3$, a contradiction. If there exists some $1 \leq p \leq s_{[i]}$ such that $|O^c((p,[i]))| < \lfloor \frac{s}{2} \rfloor$, or $|O^c((p,[i]))| = \lfloor \frac{s}{2} \rfloor$ and $O^c((p,[i])) \neq O^c((1,[i^*]))$, then $O^c((p,[i])) \subseteq \bar{X}$ for some $X \in B^I_2 \cup B^I_3$. It follows that $d_D((p,[i]),(1,[1,j])) > 4$ for some $[j] \in A_2 \cup A^I_3$, a contradiction. Thus, it remains that $O^c((p,[i])) = O^c((1,[i^*]))$ for all $[i] \in A_{\geq 4}$ and all $1 \leq p \leq s_{[i]}$. By Lemma 2.15, $Q = \{O^c((4,[i])) \mid [i] \in A^O_{\geq 4}\} \cup \{O^c((3,[i])) \mid [i] \in A^I_{\geq 4} \cup A^N_{\geq 4}\} \cup B^O_2 \cup B^O_3$ is an antichain. However, this contradicts Sperner's theorem as $|Q| > |B^O_2| + |B^O_3| = \binom{s}{\lceil s/2 \rceil}$.

A similar argument shows $2|A_2| + |A_3| = 2\binom{s}{\lceil s/2 \rceil} - 1$ does not hold as in Subcase 2.4. Hence, $2|A_2| + |A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$.

($\Leftarrow$) By Corollary 4.9, $\mathcal{T} \in \mathscr{C}_0$ if $|A_2| \leq \binom{s}{\lceil s/2 \rceil} - 2$ and $|A_3| = 1$, or $|A_2| + |A_3| \leq \binom{s}{\lceil s/2 \rceil} - 1$ and $|A_3| \geq 2$. Hence, we assume $|A_2| + |A_3| \geq \binom{s}{\lceil s/2 \rceil}$, on top of the hypothesis that $2|A_2| + |A_3| \leq 2\binom{s}{\lceil s/2 \rceil} - 2$ and $|A_3| \geq 2$. Furthermore, assume without loss of generality that $A_2 = \{[i] \mid i \in \mathbb{N}_{|A_2|}\}$, and $A_3 = \{[i] \mid i \in \mathbb{N}_{|A_2|+|A_3|} - \mathbb{N}_{|A_2|}\}$.

Let $\mathcal{H} = T(t_1, t_2, \ldots, t_n)$ be the subgraph of $\mathcal{T}$, where $t_{\mathsf{c}} = s$, $t_{[i]} = 3$ for all $[i] \in \mathcal{T}(A_3)$, $t_{[j]} = 4$ for all $[j] \in \mathcal{T}(A_{\geq 4})$ and $t_v = 2$ otherwise. We will use A_j for $\mathcal{H}(A_j)$ for the remainder of this proof. Define an orientation D of $\mathcal{H}$ as follows.

$$(2,[i]) \to \{(1,[\alpha,i]),(2,[\alpha,i])\} \to (1,[i]), \text{ and}$$
$$\bar{\lambda}_{i+1} \to (1,[i]) \to \lambda_{i+1} \to (2,[i]) \to \bar{\lambda}_{i+1}$$

for all $1 \leq i \leq |A_2|$ and all $1 \leq \alpha \leq \deg_T([i]) - 1$, i.e., the $\lceil \frac{s}{2} \rceil$-sets $\lambda_2, \lambda_3, \ldots, \lambda_{|A_2|+1}$ are used as 'in-sets' ('out-sets' resp.) to construct B_2^I (B_2^O resp.).

$$\begin{aligned}
&(3,[j]) \to \{(1,[\beta,j]),(2,[\beta,j])\} \to \{(1,[j]),(2,[j])\},\\
&\bar{\lambda}_1 \to (1,[j]) \to \lambda_1 \to (2,[j]) \to \bar{\lambda}_1, \text{ and}\\
&\lambda_{j+1} \to (3,[j]) \to \bar{\lambda}_{j+1}
\end{aligned}$$

for all $|A_2|+1 \le j \le \binom{s}{\lceil s/2\rceil}-1$ and all $1 \le \beta \le \deg_T([j])-1$, i.e., the $\lceil \frac{s}{2}\rceil$-sets $\lambda_{|A_2|+2}, \lambda_{|A_2|+3}, \ldots, \lambda_{\binom{s}{\lceil s/2\rceil}}$ are used as 'in-sets' to construct B_3^I.

$$\begin{aligned}
&\{(1,[k]),(2,[k])\} \to \{(1,[\gamma,k]),(2,[\gamma,k])\} \to (3,[k]),\\
&\lambda_1 \to (1,[k]) \to \bar{\lambda}_1 \to (2,[k]) \to \lambda_1, \text{ and}\\
&\bar{\lambda}_{k-\binom{s}{\lceil s/2\rceil}+|A_2|+2} \to (3,[k]) \to \lambda_{k-\binom{s}{\lceil s/2\rceil}+|A_2|+2}
\end{aligned}$$

for all $\binom{s}{\lceil s/2\rceil} \le k \le |A_2|+|A_3|$ and all $1 \le \gamma \le \deg_T([k])-1$, i.e., the $\lceil \frac{s}{2}\rceil$-sets $\lambda_{|A_2|+2}, \lambda_{|A_2|+3}, \ldots, \lambda_{2|A_2|+|A_3|+2-\binom{s}{\lceil s/2\rceil}}$ are used as 'out-sets' to construct B_3^O.

$$\begin{aligned}
&(2,[\tau,l]) \to \{(2,[l]),(4,[l])\} \to (1,[\tau,l]) \to \{(1,[l]),(3,[l])\} \to (2,[\tau,l]), \text{ and}\\
&\bar{\lambda}_1 \to \{(1,[l]),(4,[l])\} \to \lambda_1 \to \{(2,[l]),(3,[l])\} \to \bar{\lambda}_1
\end{aligned}$$

for all $[l] \in A_4$ and all $1 \le \tau \le \deg_T([l])-1$.

$$\lambda_1 \to \{(1,[m]),(2,[m])\} \to \bar{\lambda}_1$$

for all $[m] \in E$. (See Figure 10 for D when $s=3$.)

It can be verified that $d(D)=4$; this part of the proof is omitted for brevity and we refer the interested reader to [24] for details. Since every vertex lies in a directed C_4 for D and $d(D)=4$, $\bar{d}(\mathcal{T}) \le \max\{4, d(D)\}$ by Lemma 1.3, and thus $\bar{d}(\mathcal{T})=4$. □

This concludes the proof of Theorem 1.9.

5. Conclusion

In this paper, we almost completely characterise the case of even s and give a complete characterisation for the case of odd $s \ge 3$. With the current approach of searching for optimal orientation(s) in tree vertex-multiplications, the complexity and quantity of the subcases increase sharply when the subsets $O^c((p,[i]))$ are of 'middle' size ($\lfloor \frac{s}{2}\rfloor$ or $\lceil \frac{s}{2}\rceil$). For instance in Proposition 4.3, it is relatively easy to settle Subcases 2.1 and 2.2 but Subcase 2.3 is rather involved. Furthermore, the even case (see Proposition 3.12) illustrates a similar yet more complicated situation. It seems that a new approach may be needed to cut through this entanglement. Since this paper focuses on trees of diameter 4, we end off by proposing the following problem.

Problem 5.1. For trees T with $d(T)=3$, characterise the tree vertex-multiplications $T(s_1, s_2, \ldots, s_n)$ that belong to $\mathscr{C}_0$.

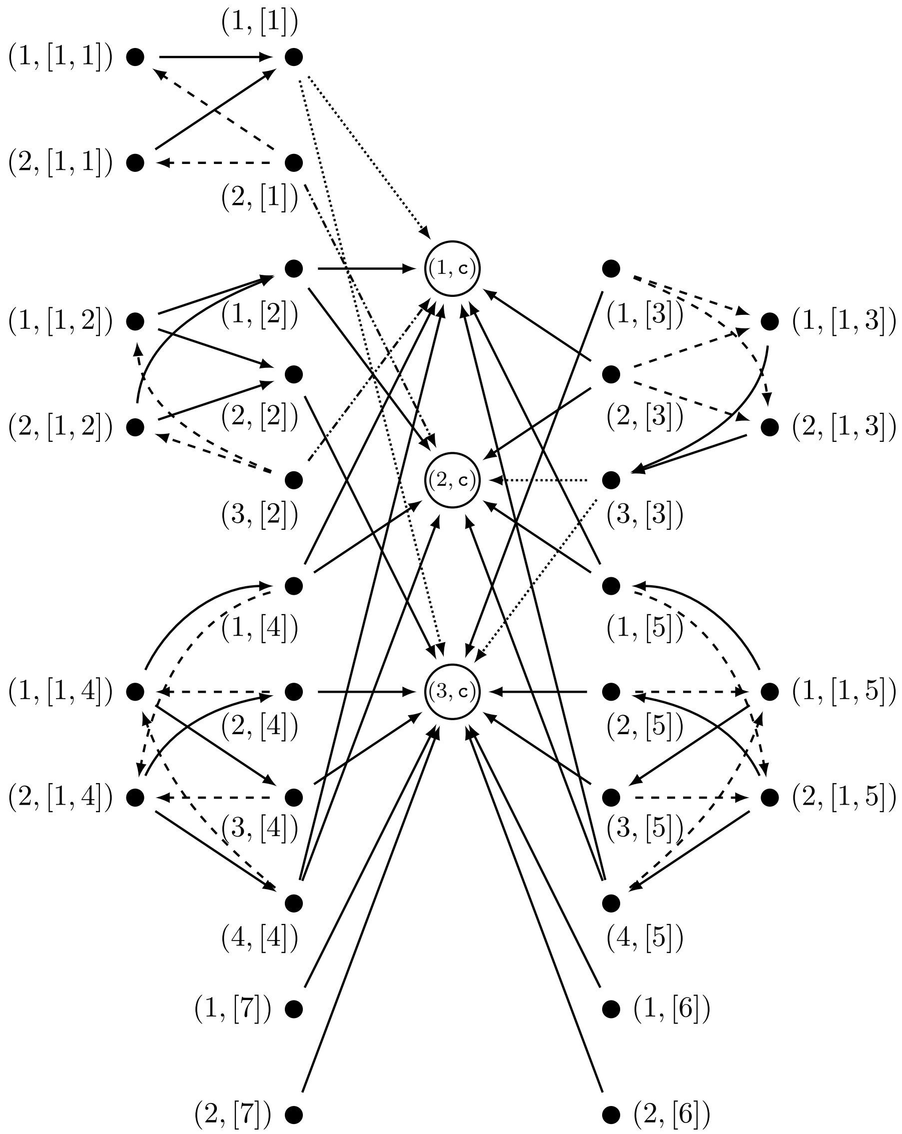

Figure 10: Orientation D for $\mathcal{H}$ for $s = 3$,
$A_2 = \{[1]\}$, $A_3 = \{[2], [3]\}$, $A_4 = \{[4], [5]\}$, $E = \{[6], [7]\}$.

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