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Abstract—We present a rigorous result regarding the boundary layer associated with the incompressible Newtonian channel flow with injection and suction. © 2000 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION AND THE MAIN RESULT

The purpose of this article is to present a new result regarding the boundary layer associated with Navier-Stokes equations for incompressible Newtonian fluids in a channel with injection and suction at the boundary, i.e., when the boundary in noncharacteristic.

$$\frac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon \cdot \nabla) u^\varepsilon - \varepsilon \Delta u^\varepsilon + \nabla p^\varepsilon = f, \qquad \text{in } \Omega \times (0, T), \tag{1.1}$$

$$\operatorname{div} u^\varepsilon = 0, \qquad \text{in } \Omega \times (0, T). \tag{1.2}$$

Here u^ε is the velocity field in the Eulerian representation, p^ε is the pressure which enforces the incompressibility condition, $\varepsilon > 0$ is the kinematic viscosity, f is the external body force, and

$$\Omega = (0, L_1) \times (0, L_2) \times (0, h) \tag{1.3}$$

is the channel occupied by the fluid.

For realistic fluids like air and water, the kinematic viscosity ε is very small, and hence, we may formally set $\varepsilon = 0$ in the Navier-Stokes equations and arrive at the Euler equations for incompressible inviscid fluids:

$$\frac{\partial u^0}{\partial t} + \left(u^0 \cdot \nabla\right) u^0 + \nabla p^0 = f, \qquad \text{in } \Omega \times (0,T), \tag{1.4}$$

$$\operatorname{div} u^0 = 0, \qquad \text{in } \Omega \times (0,T). \tag{1.5}$$

The natural question is then if such a formal procedure can be justified.

In this article, we provide an affirmative answer to this problem, for short time, in the case when there are injections or suction at the boundary. More precisely, we specify the boundary condition for the viscous problem to be

$$u^\varepsilon|_{z=h} = (0,0,-U_t(x,y)), \tag{1.6}$$

satisfying (for injection at the top):

$$U_t(x,y) \geq a > 0, \qquad \text{for all } (x,y) \in (0,L_1) \times (0,L_2), \tag{1.7}$$

and

$$u^\varepsilon|_{z=0} = (0,0-U_b(x,y)), \tag{1.8}$$

satisfying (for suction at the bottom):

$$U_b(x,y) \geq b > 0, \qquad \text{for all } (x,y) \in (0,L_1) \times (0,L_2), \tag{1.9}$$

and periodicity in x and y.

Thanks to the boundary conditions, the top $(0,L_1) \times (0,L_2) \times \{h\}$ and the bottom $(0,L_1) \times (0,L_2) \times \{0\}$ are not streamlines of the flow. Hence, it is also called the noncharacteristic boundary condition.

Physically, such boundary conditions could occur in the control of fluids when there are injection and suction at the boundary or in the situation of a moving domain after applying a Galilean transformation, or in geophysical fluid dynamics for flows in a limited domain; see also [1].

The corresponding boundary condition for the Euler system is a little bit unconventional. Instead of specifying the normal velocity only at the boundary, we need to specify the whole velocity at the inlet (top) of the channel

$$u^0\big|_{z=h} = (0,0,-U_t(x,y)), \tag{1.10}$$

and at the outlet (bottom)

$$u^0_3\big| z = 0 = -U_b(x,y). \tag{1.11}$$

The physical reason for the boundary condition (1.10) is that we have flux at the boundary, and thus, we need to specify the whole velocity field at the upwind direction ($z=h$) in order to have a well-posed inviscid problem [2].

The approximation of u^ε by u^0 cannot be uniform due to the presence of the boundary layer (see for instance [3]). The uniform convergence can be established only with the aid of a corrector φ^ε. Our corrector φ^ε will be an approximate solution to the model boundary layer problem proposed in [3].

More precisely we have the following.

THEOREM. Let u_0, f, U_t, U_b be smooth functions satisfying certain compatibility conditions (see [4,5]) including

$$u_0 + (0,0,U_b) = 0, \qquad \text{at } z = 0, \tag{1.12}$$

and (1.7) and (1.9).

Then there exists a time

$$T_* = T_*(u_0, f, U_t, U_b, L_1, L_2, h), \tag{1.13}$$

and a constant $\kappa = \kappa(u_0, f, U_t, U_b, L_1, L_2, h)$ *(independent of* ε*) such that*

$$\left\| u^\varepsilon - u^0 \right\|_{L^\infty\left(0, T_*, (L^2)^3\right)} \leq \kappa \varepsilon^{1/2}, \tag{1.14}$$

$$\left\| u^\varepsilon - u^0 - \varphi^\varepsilon \right\|_{L^2\left(0, T_*, (H^1)^3\right)} \leq \kappa \varepsilon^{1/2}. \tag{1.15}$$

Furthermore, in the two-dimensional case (with the y *dependence suppressed), we have the following uniform* (L^∞) *estimate:*

$$\left\| u^\varepsilon - u^0 - \varphi^\varepsilon \right\|_{L^\infty\left((\Omega \times (0, T_*))\right)} \leq \kappa \varepsilon^{1/4}. \tag{1.16}$$

Here φ^ε *is a corrector (boundary layer function) defined in the following way:*

$$\varphi^\varepsilon = curl\ \psi^\varepsilon, \tag{1.17}$$

with

$$\psi^\varepsilon = (\psi_1^\varepsilon, \psi_2^\varepsilon, 0), \tag{1.18}$$

$$\psi_1^\varepsilon(x, y, z, t) = -u_2^0(x, y, 0, t)\rho\left(\frac{z}{h}\right)\frac{\varepsilon}{U_b(x, y)}\left(1 - e^{-U_b z/\varepsilon}\right), \tag{1.19}$$

$$\psi_2^\varepsilon(x, y, z, t) = u_1^0(x, y, 0, t)\rho\left(\frac{z}{h}\right)\frac{\varepsilon}{U_b(x, y)}\left(1 - e^{-U_b z/\varepsilon}\right), \tag{1.20}$$

and where the cut-off function ρ *is defined as follows:*

$$\rho \in C^\infty[0, \infty), \qquad supp\ \rho \subset [0, 1], \qquad \rho(0) = 1, \qquad \rho'(0) = \rho''(0) = \rho'''(0) = 0. \tag{1.21}$$

In other words, we proved the existence of a stable boundary layer of the form $e^{-U_b z/\varepsilon}$ at the outlet (bottom) of the channel only.

To the best of our knowledge, this is the first rigorous result on boundary layer associated with incompressible viscous fluids. The result also agrees with laboratory experiments of flow past obstacle in the region close to the front of the smooth obstacle (see for instance [6]).

Partial results (not extending to H^1 and L^∞ convergences) appear in [7], and convergence of the Navier-Stokes to the Euler equations were recently proved by Sammartino and Caflisch [8] for short time, half-space and analytic data. See [9–15] for partial results related to the characteristic boundary case, and [16–19] for results related to viscous perturbation of hyperbolic systems.

The detailed proof of the theorem for the special case of $U_t = U_b = U_\infty = $ constant will be presented elsewhere [5]. Here we just present a sketch of the proof.

2. SKETCH OF THE PROOF

In this section, we sketch the proof of our theorem.

We consider the adjusted difference

$$w^\varepsilon = u^\varepsilon - u^0 - \varphi^\varepsilon, \tag{2.1}$$

which satisfies the following equation:

$$\frac{\partial w^\varepsilon}{\partial t} - \varepsilon \Delta w^\varepsilon + (v^\varepsilon \cdot \nabla) w^\varepsilon + (w^\varepsilon \cdot \nabla) v^0 + (w^\varepsilon \cdot \nabla) \varphi^\varepsilon + \nabla (p^\varepsilon - p_0) = F^\varepsilon, \tag{2.2}$$

$$div\ w^\varepsilon = 0, \tag{2.3}$$

$$w^\varepsilon|_{z=0} = w^\varepsilon|_{z=h} = 0, \tag{2.4}$$

where

$$F^\varepsilon = -\frac{\partial \varphi^\varepsilon}{\partial t} - \varepsilon \Delta v^0 + \varepsilon \Delta \varphi^\varepsilon - (\varphi^\varepsilon \cdot \nabla)\varphi^\varepsilon - (\varphi^\varepsilon \cdot \nabla)v^0 - (v^0 \cdot \nabla)\varphi^\varepsilon. \tag{2.5}$$

We proceed with energy methods. Notice that the adjusted difference w^ε is divergence free due to the explicit construction of our divergence free corrector. Hence, the most problematic term $(v^\varepsilon \cdot \nabla)w^\varepsilon$ disappears in the basic energy estimates.

We illustrate our method on one of the other terms, namely $(w^\varepsilon \cdot \nabla)\varphi^\varepsilon$. In the energy estimate, we wish to bound $\int_\Omega [(w^\varepsilon \cdot \nabla)\varphi^\varepsilon] \cdot w^\varepsilon$ by the dissipative term $\varepsilon|\nabla w^\varepsilon|^2_{L^2}$ and $|w^\varepsilon|^2_{L^2}$. Roughly speaking, we have

$$
\begin{aligned}
\left| \int_\Omega (w^\varepsilon \cdot \nabla)\varphi^\varepsilon \cdot w^\varepsilon \right| &= \left| \int_\Omega (w^\varepsilon \cdot \nabla)w^\varepsilon \cdot \varphi^\varepsilon \right| \\
&\leq \left| \frac{w^\varepsilon}{z} \right|_{L^2} |\nabla w^\varepsilon|_{L^2} |z\varphi^\varepsilon|_{L^\infty} \\
&\leq \tilde{\kappa}\kappa_h \frac{\overset{\max}{\{z=0\}} \left\{ |u_1^0| + |u_2^0| \right\}}{\underset{\min}{\{z=0\}U_b}} \varepsilon|\nabla w^\varepsilon|^2_{L^2},
\end{aligned}
\tag{2.6}
$$

where κ_h (depending on h only) is the constant from the Hardy inequality (see more details in [5]) and $\tilde{\kappa}$ is a universal constant. In order to dominate the right-hand side of (2.6) by the dissipative term $\varepsilon|\nabla w^\varepsilon|_{L^2}$, we impose

$$\tilde{\kappa}\kappa_h \frac{\overset{\max}{\{z=0\}} \left\{ |u_1^0| + |u_2^0| \right\}}{\underset{\min}{\{z=0\}U_b}} \leq \frac{1}{8}. \tag{2.7}$$

This condition is guaranteed at initial time $t = 0$ by the compatibility condition (1.12). This combined with local in time well-posedness of the inviscid problem we deduce that there exists T_* such that (2.7) is valid for $t \in [0, T_*]$.

This basically explains why we have a short time result only, from the mathematical perspective. From the physical point of view, we anticipate the development of turbulence after a sufficiently long time; after this we would need to address the turbulent boundary layer problem.

The first half of the theorem then follows by similar treatment of the other terms.

For the uniform in space and in time estimates, the problem is much harder because there is no known maximum principle for the velocity field. Here, to bypass this difficulty, we observe that we have a better control on the tangential derivatives than on the normal derivative on average and we want to take advantage of this. This agrees with the classical boundary layer theory. For this purpose we consider the equations satisfied by $\frac{\partial u^\varepsilon}{\partial x}$. In the process of estimating $\frac{\partial u^\varepsilon}{\partial x}$, we encounter the same difficulty as in the study of the global regularity for solutions to the three-dimensional Navier-Stokes equations in the three-dimensional case. Thus, we restrict ourselves to the two-dimensional case.

The energy estimates (sometimes weighted as in the first half) together with the following anisotropic Sobolev imbedding (see [13])

$$\|g\|_{L^\infty(\Omega)} \leq \kappa \left(\|g\|_{H^1}^{1/2} \left\| \frac{\partial g}{\partial x} \right\|_{L^2}^{1/2} + \|g\|_{L^2}^{1/2} \left\| \frac{\partial^2 g}{\partial x \partial z} \right\|_{L^2}^{1/2} \right) \tag{2.8}$$

yield the second part of our theorem.

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