01 Jul 2004

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Available online at www.sciencedirect.com
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Applied Mathematics Letters 17 (2004) 821–825
Applied Mathematics Letters
www.elsevier.com/locate/aml

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(Received and accepted July 2003)

Communicated by R. Temam

Abstract—We establish the validity of the infinite Prandtl number model as an approximation of the Boussinesq system at large Prandtl number on finite and infinite time interval, as well as in some statistical sense. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Rayleigh-Bénard convection, Boussinesq system, Infinite Prandtl number model, Prandtl number, Rayleigh number, Nusselt number.

1. INTRODUCTION

Most realistic fluid phenomena involve heat transfer. Here, we consider the Rayleigh-Bénard setting of a horizontal layer of fluids confined by two parallel planes a distance h apart and heated at the bottom plane at temperature T_2 and cooled at the top plane at temperature T_1 ($T_2 > T_1$). Hot fluids at the bottom tend to rise while cool fluids on top tend to sink by gravity force. If the relative change of density is small with respect to a background density, we may ignore density variation in the system except a buoyancy force proportional to the local temperature in the momentum balance, and we arrive at the so-called Boussinesq approximation of the Rayleigh-Bénard convection. The dynamic model consists of the heat advection-diffusion of the temperature coupled with the incompressible Navier-Stokes equations via a buoyancy force proportional to the temperature [1–3].

Since we are interested in convection, i.e., motion of the fluid induced by buoyancy, the standard/appropriate nondimensional form of the system is achieved by using the units of the layer depth h as the typical length scale, the thermal diffusion time h^2/κ as the typical time, the ratio

The author wishes to acknowledge the financial support from the National Science Foundation, New York University, Institute for Advanced Study at Princeton, Fudan University, and Iowa State University. He also wishes to acknowledge helpful conversations with P. Constantin, C. Doering, A. Majda, J. Shen, T. Spencer, R. Temam and E. Titi.

0893-9659/04/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi:10.1016/j.aml.2004.06.012

of typical length over typical time, i.e., κ/h as typical velocity, the temperature on a scale where the top plane is kept at zero and the bottom plane kept at one. The nondimensional form of the *Boussinesq equations* then take the form, after taking into account the effect of rotation, normalizing the background density to one,

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)+\frac{1}{Ek}\mathbf{k}\times\mathbf{u}+\nabla p=\Delta\mathbf{u}+\mathrm{Ra}\,\mathbf{k}T, \qquad \nabla\cdot\mathbf{u}=0,$$
$$\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T=\Delta T,$$

equipped with the following boundary conditions:

$$T|_{z=1}=0, \qquad T|_{z=0}=1, \qquad \mathbf{u}|_{z=0,1}=0$$

together with periodic boundary conditions in the horizontal directions with period L_x and L_y. Here, the control parameters are the Prandtl number $\mathrm{Pr}=\nu/\kappa$, the Rayleigh number $\mathrm{Ra}=g\alpha(T_2-T_1)h^3/\nu\kappa$, and the Ekman number $\mathrm{Ek}=\nu/2\Omega h^2$. Here ν is the kinematic viscosity of the fluid, κ is the thermal diffusivity, g is the gravitational constant, α is the thermal expansion coefficient, and Ω is the rotation rate [1–4].

The behavior of the Boussinesq system is extremely complex ranging from pure conduction at low Rayleigh number to pattern formation after and convective turbulence [1–4]. Thus, we naturally look for regimes where simplification can be made. One popular approach is to consider the case with large Prandtl number which is relevant for fluids such as the earth's mantle, silicone oil, and many gases under high pressure [1,4–6]. If we formally set the Prandtl number in the Boussinesq system to infinity, we arrive at the so-called *infinite Prandtl number model* for Rayleigh-Bénard convection

$$\frac{1}{Ek}\mathbf{k}\times\mathbf{u}^0+\nabla p^0=\Delta\mathbf{u}^0+\mathrm{Ra}\,\mathbf{k}T^0, \qquad \nabla\cdot\mathbf{u}^0=0,$$
$$\frac{\partial T^0}{\partial t}+\mathbf{u}^0\cdot\nabla T^0=\Delta T^0,$$

together with the boundary conditions

$$T^0|_{z=1}=0, \qquad T^0|_{z=0}=1, \qquad \mathbf{u}^0|_{z=0,1}=0.$$

The infinite Prandtl number model is a much simpler model with the velocity field linearly slaved by the temperature field. It has been used in many fruitful investigations of convection [1,4,5,7,8].

A natural question to ask then is if such an approximation is valid, i.e.,

$$(\mathbf{u},T)\to\left(\mathbf{u}^0,T^0\right), \qquad \text{as } \mathrm{Pr}\to\infty?$$

Since the velocity field in the infinite Prandtl number model is linearly slaved by the temperature field, no initial data can be prescribed for the velocity in this simplified model. This is in contrast to the Boussinesq system where initial data must be prescribed for both the temperature and the velocity fields. We then observe that the large Prandtl number limit of the Boussinesq system is a *singular perturbation* problem involving an initial layer.

2. DERIVATION OF THE EFFECTIVE DYNAMICS

In order to derive the effective dynamics for the Boussinesq system at large Prandtl number, we recognize that the large Prandtl number problem is really a problem involving two time scales, namely, the fast viscous time scale of $1/\mathrm{Pr}$ (h^2/ν before nondimensionalization) and the slow (thermal) time scale of 1 (h^2/κ before nondimensionalization). This suggests that we should take the two time scale approach and introduce the fast time scale

$$\tau = \mathrm{Pr}\cdot t = \frac{t}{\varepsilon}, \qquad \text{with } \varepsilon = \frac{1}{\mathrm{Pr}}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau}$$

and postulate the formal asymptotic expansion

$$\mathbf{u} = \mathbf{u}^{(0)}(t,\tau) + \varepsilon \mathbf{u}^{(1)}(t,\tau) + h.o.t., \qquad T = T^{(0)}(t,\tau) + \varepsilon T^{(1)}(t,\tau) + h.o.t.,$$

with the usual sublinear growth condition.

Following the standard two time scale approach (see, for instance, [9]) we arrive at the following *effective dynamics*:

$$\Delta T^{(0)} = \frac{\partial T^{(0)}}{\partial t} + \mathbf{u}^{(0)} \cdot \nabla T^{(0)},$$
$$\mathbf{u}^{(0)}(t,\tau) = \mathrm{Ra}\, A^{-1}\left(\mathbf{k}T^{(0)}(t)\right) + e^{-A\tau}\mathbf{u}_0 - \mathrm{Ra}\, e^{-A\tau}A^{-1}(\mathbf{k}T_0),$$

where A is the solution operator for the velocity equation in the infinite Prandtl number model. It is easy to see that the effective dynamics is closely related to the infinite Prandtl number model

$$\mathbf{u}^{(0)} = \mathbf{u}^0 + \text{initial layer} + \text{lower-order terms}, \qquad T^{(0)} = T^0 + \text{lower-order terms}.$$

3. FINITE TIME CONVERGENCE RESULT

With the compelling formal asymptotic given in the previous section, we are able to prove the following convergence result.

THEOREM 1. *For any given T^*, there exists a constant κ independent of ε such that*

$$\left\|\mathbf{u} - \mathbf{u}^{(0)}\right\|_{L^\infty(0,T^*;L^2(\Omega))} \le \kappa\varepsilon,$$
$$\left\|T - T^{(0)}\right\|_{L^\infty(0,T^*;L^2(\Omega))} \le \kappa\varepsilon.$$

This convergence result rigorously establishes the validity of the infinite Prandtl number model as an effective model for Boussinesq system at large Prandtl number modulo an initial layer. The convergence rate of ε is optimal and the reader may consult [10] for more details.

4. LONG TIME PROXIMITY

Encouraged by the short time convergence presented in the previous section, we naturally ask if the solutions of the Boussinesq system are close to the solutions of the infinite Prandtl number model over a long period of time. On the other hand, such a long time orbital stability result should not be expected for such complex systems where turbulent/chaotic behavior abound. Instead, we care more about statistical properties for such systems, and hence, it is natural to ask if the statistical properties (in terms of invariant measures) as well as global attractors remain close.

The first obstacle in studying long time behavior is the well-posedness of the Boussinesq system global in time. This is closely related to the well-known problem related to 3D Navier-Stokes equations [11–13]. Fortunately, in the regime of large Prandtl number, we are able to prove the eventual regularity for suitably defined weak solutions to the Boussinesq system, which exists for all time. The suitable weak solutions will be defined as Leray-Hopf type weak solution plus suitable energy inequality which ensures certain maximum principle type estimates. More precisely, we have, in terms of the perturbative variables $(\mathbf{u}, \theta)$ where $\theta = T - (1-z)$ $((0, 1-z)$ being the conduction state), the following.

DEFINITION 1. SUITABLE WEAK SOLUTIONS. $(\mathbf{u},\theta)$ *is called a suitable weak solution to the Boussinesq equations on the time interval* $[0,T^*]$ *with given initial data* $(\mathbf{u}_0,\theta_0)$ *if the following hold:*

$$\mathbf{u}\in L^\infty(0,T^*;H)\cap L^2(0,T^*;V)\cap C_w(0,T^*;H),$$
$$\mathbf{u}'\in L^{4/3}(0,T^*;V'),\qquad \mathbf{u}(0)=\mathbf{u}_0,$$
$$\theta\in L^\infty\left(0,T^*;L^2(\Omega)\right)\cap L^2\left(0,T^*;H_0^1(\Omega)\right)\cap C_w\left(0,T^*;L^2(\Omega)\right),$$
$$\theta'\in L^{4/3}\left(0,T^*;H^{-1}(\Omega)\right),\qquad \theta(0)=\theta_0,$$

$$\frac{1}{\mathrm{Pr}}\left(\frac{d}{dt}(\mathbf{u},\mathbf{v})+b(\mathbf{u},\mathbf{u},\mathbf{v})\right)+((\mathbf{u},\mathbf{u}))=\mathrm{Ra}(\theta,v_3),\qquad \forall\,\mathbf{v}\in V,$$
$$\frac{d}{dt}(\theta,\eta)+\tilde{b}(\theta,\theta,\eta)+((\theta,\eta))=(u_3,\eta),\qquad \forall\,\eta\in H^1(\Omega),$$
$$\frac{1}{\mathrm{Pr}}|\mathbf{u}(t)|_{L^2}^2+2\int_0^t|\nabla\mathbf{u}(s)|_{L^2}^2\,ds\le\frac{1}{\mathrm{Pr}}|\mathbf{u}_0|_{L^2}^2+2\,\mathrm{Ra}\int_0^t(\theta(s),u_3(s))\,ds,$$
$$|(T-1)_+(t)|_{L^2}^2+2\int_0^t|\nabla(T-1)_+(s)|_{L^2}^2\,ds\le|(T_0-1)_+|_{L^2}^2,$$
$$|T_-(t)|_{L^2}^2+2\int_0^t|\nabla T_-(s)|_{L^2}^2\,ds\le|(T_0)_-|_{L^2}^2.$$

The existence of suitable weak solutions can be derived using hyperviscosity together with suitable Galerkin approximation.

We then have the following regularity result concerning the suitable weak solutions.

THEOREM 2. EVENTUAL REGULARITY FOR SUITABLE WEAK SOLUTIONS. *There exists an absolute constant* κ_0 *such that for* $\mathrm{Pr}/\mathrm{Ra}\ge\kappa_0$ *and any initial data* (u_0,T_0)*, there exists a constant* T^* *so that all the suitable weak solutions with the given initial data to the Boussinesq equations become regular after* T^*.

Furthermore, we are able to derive uniform estimates on these suitable weak solutions which further ensures the existence of a global attractor [11,12,14,15] in the sense of attracting all suitable weak solutions.

THEOREM 3. *There exists an absolute constant* κ_0 *such that for* $\mathrm{Pr}/\mathrm{Ra}\ge\kappa_0$*, one may derive uniform in time estimates for the* H^1 *norm of the solutions after a sufficiently long time which depends on the parameters and the* L^2 *norm of the initial data only. In particular, this implies that the Boussinesq system possesses a global attractor which attracts all suitable weak solutions.*

Next, we proceed to compare the global attractors for the Boussinesq system (with large Prandtl number) and the global attractor for the infinite Prandtl number model. The first difficulty that we encounter is the difference in phase spaces. This can be resolved by lifting the phase space for the infinite Prandtl number model or projecting down the phase space of the Boussinesq system. We then proceed to derive uniform H^1 as well as time derivative estimates in Pr.

THEOREM 4. PROXIMITY OF GLOBAL ATTRACTORS. *Let* $\mathcal{A}_0$ *and* $\mathcal{A}_\epsilon$ *be the global attractors for the infinite Prandtl number model and the Boussinesq equations with* $\epsilon=1/\mathrm{Pr}$*. Let* $\mathcal{L}$ *be the lift operator from the phase space for the perturbative temperature* θ *to the product space of* θ *and* $\mathbf{u}$*. Then, the global attractors are upper semicontinuous in the following sense:*

$$\lim_{\epsilon\to0}\mathrm{dist}(\mathcal{A}_\epsilon,\mathcal{L}\mathcal{A}_0)=0.$$

Recall that Prandtl number is the ratio of viscosity over thermal diffusivity. Thus, large Prandtl number implies large viscosity (in our case with the ratio of Prandtl over Rayleigh being large).

A native question is if the global attractors are trivial as for the case of Navier-Stokes equations with small Grashoff number. Numerical simulations and laboratory experiments indicate that the long time behavior of the Boussinesq system is extremely complex [1–3,6,16]. Indeed, we observe that the global attractor for the Boussinesq system must contain at least three steady states: the pure conduction state $(0, 1-z)$, two Bénard cells (different orientation) after threshold value of Rayleigh number [2,10,17]. The existence of the three steady states can be verified on the infinite Prandtl number model as well by modifying the work of Yudovitch [18] and Rabinowitz [17].

Finally, we turn to proximity of statistical properties. It seems that we are able to prove the tightness of the set of invariant measures for the Boussinesq system at large Prandtl number, which implies convergence of subsequences of invariant measures to that of the infinite Prandtl number model. However, convergence of specific statistical properties such as time averaged bulk heat transport is not known in the sense that the limiting invariant measure may not correspond to a time average for the limiting system. Nevertheless, we are able to prove a new bound on time averaged bulk heat transport in terms of Nusselt number that links to the optimal upper bound for the infinite Prandtl number model derived by Constantin and Doering [5].

Recall that the Nusselt number is defined as

$$\mathrm{Nu} = 1 + \frac{1}{L_x L_y} \lim_{t\to\infty} \frac{1}{t} \int_0^t \int_\Omega u_3(s,\mathbf{x}) T(s,\mathbf{x})\, d\mathbf{x}\, ds.$$

We then have

$$\begin{aligned} &\mathrm{Nu} \le \kappa \mathrm{Ra}^{1/3} (\ln \mathrm{Ra})^{2/3} + \mathrm{Remainder}(\mathrm{Ra}, \mathrm{Pr}), \\ &\mathrm{Remainder}(\mathrm{Ra}, \mathrm{Pr}) \to 0, \qquad \text{as } \mathrm{Pr} \to \infty. \end{aligned}$$

This is an improvement over previous results and consistent with physical predictions [6,16].

There are many other interesting problems that need to be investigated.

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