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A uniformly dissipative scheme for stationary statistical properties of the infinite Prandtl number model

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Abstract

The purpose of this short communication is to announce that a class of numerical schemes, uniformly dissipative approximations, which uniformly preserve the dissipativity of the continuous infinite dimensional dissipative complex (chaotic) systems possess desirable properties in terms of approximating stationary statistics properties. In particular, the stationary statistical properties of these uniformly dissipative schemes converge to those of the continuous system at vanishing mesh size. The idea is illustrated on the infinite Prandtl number model for convection and semi-discretization in time, although the general strategy works for a broad class of dissipative complex systems and fully discretized approximations. As far as we know, this is the first result on rigorous validation of numerical schemes for approximating stationary statistical properties of general infinite dimensional dissipative complex systems.

Keywords: Prandtl number model; Dissipative scheme; Complex systems

1. Introduction

Many dynamical systems arising in physical applications are dissipative complex systems in the sense that they possess a compact global attractor and the dynamics on the global attractor are complex/chaotic [16] with generic sensitive dependence on data. Therefore, it is hardly meaningful to discuss long time behavior of a single trajectory for this kind of complex system. Instead, we should study statistical properties of the system since they are physically much more relevant than single trajectories [14,6,13]. If the system reaches some kind of stationary state, the objects that characterize the stationary statistical properties are the invariant measures or stationary statistical solutions of the system.

The numerical study of stationary statistical properties of complex system is a very challenging task since it involves long time integration (so that the statistical averaging is computed utilizing the time averaging under the assumption of ergodicity) and computation of large number of trajectories (if no ergodicity is assumed). In terms of trajectory approximations, we are not aware of any effective long time integrator for dissipative complex/chaotic systems in general unless the long time dynamics is trivial or the trajectory under approximation is stable [9,8]. It is not

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at all clear that those numerical methods that provide efficient and accurate approximations of a continuous complex dynamical system on a finite time interval are able to provide meaningful approximation for stationary statistical properties of the system, since small errors (truncation and rounding) may accumulate and grow over a long time (think about the usual error estimates with a coefficient that grows exponentially in time). Even if the numerical scheme is long time stable (solution asymptotically bounded for all time), there is still no guarantee that stationary statistical properties will converge. As we will demonstrate in this work, a class of numerical schemes, *uniformly dissipative schemes*, which preserve the dissipativity of the dissipative dynamical system under approximation uniformly in terms of the mesh size, are able to asymptotically capture the stationary statistical properties of the continuous complex/chaotic dynamical system at vanishing mesh size.

We will illustrate the idea of uniform dissipativity on the infinite Prandtl number model for convection and consider semi-discretization in time only although the methodology works for many more complex/chaotic dynamical systems [16] and fully discretized approximations. The choice of the infinite Prandtl number model is both for its physical significance and for the sake of simplicity in exposition.

To the best of our knowledge, the convergence of stationary statistical properties of the uniformly dissipative approximations to those of the continuous complex/chaotic system has never been explored before (on any infinite dimensional dissipative complex system). Therefore, our work is the first establishing the usefulness of uniformly dissipative schemes in approximating stationary statistical properties. We hope that our work will stimulate further work on statistical properties of uniformly dissipative schemes and numerical experiments utilizing such schemes.

Previous works on uniform dissipativity largely focused on the two-dimensional incompressible Navier–Stokes system (see [8,10,15,17] among others). The uniform dissipativity is called long time stability in these works. The authors' focuses there were uniform (in mesh size) bounds on the discrete solutions exclusively without any discussion on statistical properties. As far as we know, our work is the first on convergence of statistical properties for numerical schemes.

The work is organized as follows: we give an introduction in the first section; in Section 2 we propose a semi-discrete (discrete in time) scheme for the infinite Prandtl number model and introduce the main result on uniformly dissipativity and convergence of stationary statistical properties; we then provide conclusion and remarks in the third section.

2. A uniformly dissipative scheme for the infinite Prandtl number model

Here we consider a simplified model for Rayleigh–Bénard convection at large Prandtl number, the **infinite Prandtl number model (non-dimensional)** which is formally derived by setting the Prandtl number to infinity in the Boussinesq system for convection

$$\nabla p = \Delta \mathbf{u} + Ra\,\mathbf{k}T, \quad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}|_{z=0,1} = 0, \tag{1}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \Delta T, \quad T|_{z=0} = 1, T|_{z=1} = 0 \tag{2}$$

where $\mathbf{u}$ is the Eulerian velocity of the fluid, p represents the kinematic pressure of the fluid, T is the temperature of the fluid, $\mathbf{k}$ is a unit vector in the z direction, and Ra is the Rayleigh number measuring the ratio of differential heating over overall dissipations. We assume that the fluids occupy the (non-dimensionalized) region $\Omega = [0, L_x] \times [0, L_y] \times [0, 1]$ with periodicity imposed in the horizontal directions for simplicity. It is well known that convection at large Rayleigh number is chaotic/turbulent. The interested reader is referred to [1–3,5,7,11,18,20–22] for more on convection at large Rayleigh number.

It is well known that the infinite Prandtl number system linearized about the pure conduction state $T = 1 - z$ is linearly unstable. Moreover, numerical evidence suggests that the mean temperature profile is of boundary layer type. Therefore, we decompose the temperature field into the mean background profile $\tau(z)$ (to be specified below) and the perturbation θ away from this mean, i.e., $T = \theta + \tau$. It is easy to see that θ satisfies the following equation:

$$\frac{\partial \theta}{\partial t} + Ra\,A^{-1}(\mathbf{k}\theta) \cdot \nabla\theta + Ra\,A^{-1}(\mathbf{k}\theta)_3 \tau'(z) = \Delta\theta + \tau''(z), \quad \theta|_{z=0,1} = 0, \tag{3}$$

and we are searching for solution in the space $H^1_{0,\mathrm{per}}$ (the subspace of H^1 with zero trace in the z direction and periodic in the horizontal directions). Here $A^{-1}(\mathbf{k}\theta)_3$ represents the third component (vertical velocity) of $A^{-1}(\mathbf{k}\theta)$ where A is the Stokes operator with the associated boundary conditions.

The **uniformly dissipative scheme** that we propose is the following *semi-implicit semi-discrete in time scheme*

$$\frac{\theta^{n+1}-\theta^n}{k} + Ra\, A^{-1}(\mathbf{k}\theta^n)\cdot\nabla\theta^{n+1} + Ra\, A^{-1}(\mathbf{k}\theta^{n+1})_3\tau'(z) = \Delta\theta^{n+1} + \tau''(z) \tag{4}$$

where θ^n denotes the approximate solution at time kn where k is the time step.

The background temperature profile τ will be taken as a locally smoothed (mollified) version of the following function:

$$\tau(z) = \begin{cases} 1-\dfrac{z}{2\delta}, & 0\le z\le\delta, \\ \dfrac{1}{2}, & \delta\le z\le 1-\delta,\ \delta=(4Ra)^{-\frac{1}{2}} \\ \dfrac{1-z}{2\delta} & 1-\delta\le z\le 1. \end{cases} \tag{5}$$

The choice of τ (or δ) given here is not optimal. A near optimal choice would be $\delta\sim Ra^{-\frac{1}{3}}$ but the control on the linear destabilizing term is much longer [3,5,22]. We use this simple one since an optimal bound is not our goal here.

The well-posedness of the discrete scheme follows from Lax–Milgram theorem [12]. The scheme is also consistent, convergent and uniformly dissipative. We summarize the preliminaries in the following lemma

Lemma 1 (*Well-posedness, Consistency, Convergence and Uniform Bound/dissipativity*). *The scheme* (4) *is well-posed in the sense that for any given* $\theta^n\in L^2$ *there exists a unique* $\theta^{n+1}\in H^1_{0,\mathrm{per}}$ *that satisfies*

$$\|\nabla\theta^{n+1}\|\le c\|\theta^n\|, \quad \forall n\ge 0. \tag{6}$$

The scheme is also consistent and convergent in the sense that for $\theta_0\in H^1_{0,\mathrm{per}}\bigcap H^2$ *and* $T^*>0$ *we have*

$$\|\theta^{n+1}-\theta^n\|\le ck, \quad \forall n\ge 0. \tag{7}$$

$$\|\theta_k-\theta\|_{L^2(0,T^*;L^2)}\to 0, \quad \text{as } k\to 0, \tag{8}$$

where $\theta_k(t)=\theta^n$, $t\in[nk,(n+1)k)$, $nk<T^*$. *Moreover, the scheme is uniformly dissipative in the sense that there exists an absorbing ball in* H^1 *that attracts all bounded sets in* L^2 *uniformly for all* k.

Our main result is the convergence of stationary statistical properties and a specific statistical property, the Nusselt number, that quantifies heat transport in the vertical direction.

We first observe that the numerical scheme can be viewed as a *discrete time dynamical system* on the phase space L^2 with the notation

$$\theta^{n+1}=F_k(\theta^n). \tag{9}$$

The discrete dynamical system is uniformly (in k) dissipative thanks to the uniform estimates. Therefore, these discrete dynamical systems possess compact global attractors in H^1 which attract all bounded sets in L^2. This leads to the existence of invariant measures via a classical Krylov–Bogliubov argument [19,6] for the numerical scheme (or the discrete dynamical system).

We recall the definition of invariant measures.

Definition 1 (*Invariant Measures*). A Borel probability measure μ_k on L^2 is called an *invariant measure* for F_k if

$$\int_{L^2}\Phi(F_k(\theta))\mathrm{d}\mu_k = \int_{L^2}\Phi(\theta)\mathrm{d}\mu_k \tag{10}$$

for all reasonable test functionals Φ. The set of all invariant measures for F_k is denoted as $\mathcal{IM}_k$.

We also recall that a Borel probability measure μ on L^2 is an *invariant measure, or stationary statistical solution* for the infinite Prandtl number model for convection, if

1.
$$\int_{L^2} \|\nabla\theta\|^2 \, \mathrm{d}\mu(\theta) < \infty, \tag{11}$$
2.
$$\int_{L^2} \left\langle -Ra\, A^{-1}(\mathbf{k}\theta)\cdot\nabla\theta - Ra\, A^{-1}(\mathbf{k}\theta)_3\tau'(z) + \Delta\theta + \tau''(z),\, \Phi'(\theta)\right\rangle \mathrm{d}\mu(\theta) = 0 \tag{12}$$
for any cylindrical test functional $\Phi(\theta) = \phi((\theta, w_1), \ldots, (\theta, w_m))$ where ϕ is a C^1 function on R^m, $\{w_j, j \geq 1\}$ are the eigenfunctions of Δ which form an orthonormal basis for L^2 and $w_j \in H^1_{0,\mathrm{per}} \bigcap C^2, \forall j$, and $\langle , \rangle$ denotes the H^{-1}, $H^1_{0,\mathrm{per}}$ duality.

3.
$$\int_{L^2}\int_{\Omega} \{|\nabla\theta|^2 + Ra(A^{-1}(\mathbf{k}\theta))_3\theta\tau' - \tau''\theta\} \, \mathrm{d}\mathbf{x}\, \mathrm{d}\mu(\theta) \leq 0. \tag{13}$$

The set of all stationary statistical solutions for the infinite Prandtl number model is denoted as $\mathcal{IM}$.

We recall the definition of the *Nusselt number*.

Definition 2 (*Nusselt Number*). The Nusselt number Nu for the infinite Prandtl number model and the Nusselt number Nu_k for the numerical scheme (4) are defined as

$$Nu = 1 + Ra \sup_{\theta_0\in L^2} \limsup_{t\to\infty} \frac{1}{tL_xL_y} \int_0^t \int_{\Omega} A^{-1}(\mathbf{k}\theta(\mathbf{x}, s))_3\theta(\mathbf{x}, s) \, \mathrm{d}\mathbf{x}\mathrm{d}s, \tag{14}$$

$$Nu_k = 1 + Ra \sup_{\theta_0\in L^2} \limsup_{N\to\infty} \frac{1}{NL_xL_y} \sum_{n=1}^{N} \int_{\Omega} A^{-1}(\mathbf{k}\theta^n(\mathbf{x}))_3\theta^n(\mathbf{x}) \, \mathrm{d}\mathbf{x}. \tag{15}$$

Our main result is the following theorem.

Theorem 1 (*Convergence of Stationary Statistical Properties*). *Let μ_k be an arbitrary invariant measure of the numerical scheme* (4) *with time step k, i.e. $\mu_k \in \mathcal{IM}_k$, and let Nu_k be the Nusselt number characterizing the heat transport in the vertical direction for the scheme with time step k defined in* (15). *Then each subsequence of μ_k must contain a subsubsequence (still denoted as μ_k) and an invariant measure μ of the infinite Prandtl number model so that μ_k weakly converges to μ, i.e.,*

$$\mu_k \rightharpoonup \mu, \quad \text{as } k \to 0. \tag{16}$$

Moreover, the Nusselt number converges in an upper semi-continuous fashion in the sense that

$$\limsup_{k\to 0} Nu_k \leq Nu. \tag{17}$$

3. Conclusions and remarks

Our main result clearly demonstrated the usefulness of uniformly dissipative schemes in terms of approximating stationary statistical properties of (possibly) complex/chaotic dissipative dynamical systems since the stationary statistical properties of the scheme converge to those of the continuous in time model. To the best of our knowledge, this is the first rigorous result proving convergence of stationary statistical properties of numerical schemes to those of the continuous in time dynamical system. We would like to emphasize that the methodology here can be applied to much more general dissipative systems (with chaotic behavior for relevance), although we have treated the infinite Prandtl number model only.

Other issues such as spatial discretization, high order schemes, schemes utilizing possible strong/exponential mixing of the underlying dynamical system, generalizations to random dynamical systems, convergence of the global attractors are all under investigation.

More details of the proof and discussions can be found in [4].

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References

[1] E. Bodenschatz, W. Pesch, G. Ahlers, Recent developments in Rayleigh–Bénard convection, Annu. Rev. Fluid Mech. 32 (2000) 709–778.
[2] P. Constantin, C.R. Doering, Heat transfer in convective turbulence, Nonlinearity 9 (1996) 1049–1060.
[3] P. Constantin, C.R. Doering, Infinite Prandtl number convection, J. Stat. Phys. 94 (1–2) (1999) 159–172.
[4] W. Cheng, X. Wang, Uniformly dissipative approximations of stationary statistical properties of infinite dimensional dissipative complex/chaotic systems, SINUM (submitted for publication).
[5] C.R. Doering, F. Otto, M.G. Reznikoff, Bounds on vertical heat transport for infinite Prandtl number Rayleigh–Bénard convection, J. Fluid Mech. 560 (2006) 229–241.
[6] C. Foias, O. Manley, R. Rosa, R. Temam, Navier–Stokes equations and turbulence, in: Encyclopedia of Mathematics and its Applications, vol. 83, Cambridge University Press, Cambridge, 2001.
[7] A.V. Getling, Rayleigh–Bénard convection. Structures and dynamics, in: Advanced Series in Nonlinear Dynamics, vol. 11, World Scientific Publishing Co., Inc., River Edge, NJ, 1998.
[8] T. Geveci, On the convergence of a time discretization scheme for the Navier–Stokes equations, Math. Comp. 53 (1989) 43–53.
[9] J.G. Heywood, R. Rannacher, Finite element approximation of the nonstationary Navier–Stokes problem. II. Stability of solutions and error estimates uniform in time, SIAM J. Numer. Anal. 23 (4) (1986) 750–777.
[10] N. Ju, On the global stability of a temporal discretization scheme for the Navier–Stokes equations, IMA J. Numer. Anal. 22 (2002) 577–597.
[11] L.P. Kadanoff, Turbulent heat flow: structures and scaling, Phys. Today 54 (8) (2001) 34–39.
[12] P.D. Lax, Functional Analysis, Wiley, New York, 2002.
[13] A.J. Majda, X. Wang, Nonlinear Dynamics and Statistical Theory for Basic Geophysical Flows, Cambridge University Press, Cambridge, England, 2006.
[14] A.S. Monin, A.M. Yaglom, Statistical Fluid Mechanics; Mechanics of Turbulence, MIT Press, Cambridge, MA, 1975 (English ed. updated, augmented and rev. by the authors).
[15] J. Shen, Long time stabilities and convergences for the fully discrete nonlinear Galerkin methods, Appl. Anal. 38 (1990) 201–229.
[16] R.M. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, 2nd ed., Springer-Verlag, New York, 1997.
[17] F. Tone, D. Wirosoetisno, On the long-time stability of the implicit Euler scheme for the two-dimensional Navier–Stokes equations, SIAM J. Numer. Anal. 44 (1) (2006) 29–40.
[18] D.J. Tritton, Physical Fluid Dynamics, Oxford Science Publishing, 1988.
[19] M.I. Vishik, A.V. Fursikov, Mathematical Problems of Statistical Hydromechanics, Kluwer Acad. Publishers, Dordrecht, Boston, London, 1988.
[20] X. Wang, Infinite Prandtl Number Limit of Rayleigh–Bénard Convection, Comm. Pure Appl. Math. 57 (10) (2004) 1265–1282.
[21] X. Wang, Asymptotic behavior of global attractors to the Boussinesq system for Rayleigh–Bénard convection at large Prandtl number, Comm. Pure Appl. Math. 60 (9) (2007) 1293–1381.
[22] X. Wang, Stationary statistical properties of Rayleigh–Bénard convection at large Prandtl number, CPAM, in press (doi:10.1002/cpa.20214).