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Analysis of Nonlinear Spectral Eddy-Viscosity Models of Turbulence

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Abstract Fluid turbulence is commonly modeled by the Navier-Stokes equations with a large Reynolds number. However, direct numerical simulations are not possible in practice, so that turbulence modeling is introduced. We study artificial spectral viscosity models that render the simulation of turbulence tractable. We show that the models are well posed and have solutions that converge, in certain parameter limits, to solutions of the Navier-Stokes equations. We also show, using the mathematical analyses, how effective choices for the parameters appearing in the models can be made. Finally, we consider temporal discretizations of the models and investigate their stability.

Keywords Navier-Stokes equations · Turbulence · Eddy-viscosity models · Spectral methods

Dedicated to the memory of David Gottlieb.

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1 Introduction

Fluid turbulence in three dimensions is commonly modeled by the Navier-Stokes equations (*NSE*) with a large Reynolds number. Currently, the simulation of the *NSE* in that regime is a formidable task due to the need to resolve the small scale fluctuations or *eddies* that have subtle effects on the large-scale dynamics of the fluid. To make this problem computationally tractable, these effects must be modeled whereas the large-scale motions are simulated nearly faithfully. In one approach, the velocity field is averaged over a small radius to derive equations in terms of the averaged velocity. In this process, the problem of *closure* arises in that the average of the nonlinear term in the *NSE*, which is called the *Reynolds stress*, must be approximated and expressed solely in terms of averaged quantities. The way in which this is done gives rise to a variety of models. The approach we consider, called the *eddy-viscosity method*, treats the Reynolds stress as a viscous effect caused by the transport and dissipation of energy due to the small-scale eddies. For this reason, this additional viscosity is called the *eddy-viscosity* or *turbulent viscosity*. The turbulence model of Smagorinsky [23] belongs to this type. For an overall survey on issues related to these models, see [12, 15]. Unfortunately, a straightforward application of this approach leads to the over-smearing of the large-scale structures in the fluid. To remedy this unwanted effect, it has been proposed that the eddy-viscosity be added only to the *subgrid scales*. In this way, one hopes to prevent the large-scale structure from being smeared away. Here, we examine a particular class of models of this type called *spectral eddy-viscosity models* in which the scales are defined in terms of Fourier modes. The subgrid viscosity is simply realized as an addition of the artificial viscosity only to the high-frequency modes. A simple implementation of this approach is to insert a high-pass spectral filter into the standard artificial viscosity.

We consider two types of eddy-viscosities: hyperviscosity and nonlinear viscosity. Hyperviscosity models are considered by various researchers [9, 18] because of the simplicity of the idea. An example of the nonlinear viscosity is the Smagorinsky model mentioned before. A more general nonlinear viscosity model which includes the Smagorinsky model as a special case is given by the modified *NSE* of Ladyzhenskaya [14].

The difficulty of turbulence manifests itself mathematically in the yet to be resolved question of the well-posedness of *NSE* in three dimensions. Hyperviscosity and typical nonlinear viscosity models, on the other hand, can be shown to be well posed. However, these models involve several parameters, e.g., eddy-viscosity coefficient, strength of the viscosity operator, and the cut-off frequency that distinguishes the small scales from the large scales. Thus, one would like to find some guiding principles for how these parameters should be chosen.

A rigorous justification of a turbulence model is difficult, partly because one does not have a good physical understanding of turbulence phenomena. However, given that the weak solution of the *NSE* models turbulence accurately, in ideal settings in which one has infinite computational power one can split the problem of turbulence modeling into two parts. The first is to ensure that the turbulence model models the fluid well. This can be undertaken by showing that the model is consistent with the *NSE* in some way. For instance, this was undertaken in [9] in which it is proved that the solution to the hyperviscosity model, under certain constraints on the parameters, converges to a suitable solution of the *NSE* that satisfies the strongest partial regularity proved to date [2]. Another type of consistency result is to assume that the solution of the *NSE* is smooth and then show that, in this regime, the solution of the turbulence model is "close" to the solution of the *NSE*. Intuitively, the cut-off filter plays an important role here, as it gives us a spectral convergence rate to the solution of *NSE*, assuming that the solution is smooth. The second part is to ensure that the turbulence model is tractable for numerical simulations. This means that it should be well-posed, and if so, the numerical method used to simulate it should be stable. A by-product of these investigations is the insight gained into how various parameters introduced into the equation such as the cut-off frequency, the strength of nonlinearity or hyperviscosity, affect consistency and stability.

The plan of the paper is as follows. In Sect. 2, we introduce various notations and formal definitions of the turbulence models we consider. In Sect. 3, the question of well posedness is tackled. Section 4 investigates the consistency question. We estimate the degree by which the nonlinear viscosity model is a perturbation of the *NSE* by estimating the error rate as the perturbation goes to zero. From this analysis, we realize that certain parameters depend on others and hence can be eliminated. In Sect. 5, we prove that for some specific values of parameters, the hyperviscosity and nonlinear viscosity models have effectively finite-dimensional dynamics in that any two solutions under the same forcing that agree in their low-frequency part are exponentially contracted to a single path in phase space, i.e., the high-frequency modes becomes irrelevant to the dynamics of the solutions. The hyperviscosity model has already been shown [26] to possess such an exponential contraction property. It is interesting that we can also estimate the dimension of such a finite-dimensional *attractor* in terms of the model parameters. Temporal discretization of the nonlinear viscosity model is considered in Sect. 6. We derive a uniform-in-time stability estimate for bounded power input. This gives us some insight as to the reason why we should not choose too strong a nonlinear viscosity.

2 Spectral Eddy-Viscosity Models

We introduce two spectral eddy-viscosity models and discuss various mathematical problems they inspire. In this work, we consider a three-dimensional domain that is periodic in all directions, thus limiting ourselves to the investigation of isotropic turbulence.

2.1 The Navier-Stokes Equations

We begin with the Navier-Stokes equations (*NSE*)

$$\partial_t w - \nu \Delta w + w \cdot \nabla w + \nabla q = f, \qquad \nabla \cdot w = 0, \tag{1}$$

where w denotes the velocity field, q the pressure, f the body force per unit mass, and, if the equations have been non-dimensionalized, $\nu = Re^{-1}$ is the inverse of the Reynolds number. There exist different notions with regards to what constitutes a solution of (1). A pair (w, q) is referred to as a *classical solution* if it possesses one temporal and two spatial derivatives. In the case of *NSE*, it is not known whether classical solutions exist for all time; however, it is known that there is at most one classical solution. On the other hand, *weak solutions*, i.e., pairs (u, p) that satisfy

$$\int \langle \partial_t w, \phi \rangle \, dt + \iint \Big(\nu \nabla w : \nabla \phi + w \cdot \nabla w \cdot \phi + q \nabla \cdot \phi \Big) \, dx \, dt = \iint f \phi \, dx dt$$

$$\iint (\nabla \cdot w) \psi = 0,$$

for all test vector fields ϕ and test scalar functions ψ that are smooth and compactly supported in space and in time,[1] are known to exist but it is not known if they are unique for all time. The velocity w is now only required to have just one weak spatial derivative whereas the time derivative is allowed to be even more irregular in that it can be a measure. The specific solution spaces are introduced in Sect. 3; they basically fall out naturally from the a priori analysis of the *NSE*.

2.2 Eddy-Viscosity Models

We briefly describe how a typical eddy-viscosity model is derived. The large-scale structure of the velocity field can be extracted by filtering out small fluctuations as in

$$u_l(x) = g_\delta * w = \int g_\delta(x-y)w(y)\,dy$$

where g_δ denotes a smooth function of compact support of radius $\delta > 0$. If we average the *NSE* using this function, we obtain

$$\begin{aligned} \partial_t u_l - \nu\Delta u_l + \nabla\cdot(u_l\otimes u_l) + \nabla\cdot(g_\delta*(w\otimes w) - u_l\otimes u_l) + \nabla\pi_l &= g_\delta * f \\ \nabla\cdot u_l &= 0. \end{aligned} \tag{2}$$

Note how the averaging process introduces the additional stress term

$$R_\delta(w,w) = g_\delta*(w\otimes w) - u_l\otimes u_l,$$

that is referred to as the *Reynolds stress*. Clearly, we cannot solve (2) for the average fields (u_l,π_l) because $g_\delta*(w\otimes w)$ contains interactions between the large-scale structures and the small-scale eddies, where the latter is something we would like to avoid computing. Thus, the problem of turbulence modeling is that of designing a *closure* of (2): we would like to find an appropriate approximation $S_\delta(u_l,u_l)$ to $R_\delta(w,w)$ such that the computable solution pair (u,π) satisfying

$$\begin{aligned} \partial_t u - \nu\Delta u + \nabla\cdot(u\otimes u) + \nabla\cdot S_\delta(u,u) + \nabla\pi &= f \\ \nabla\cdot u &= 0 \end{aligned}$$

is a good approximation of the solution pair (u_l,π_l) of (2). Here, f represents the spatial average of the original body force. We adopt this notation for simplicity since it does not affect the arguments below.

The idea behind eddy-viscosity models is that the small eddies dissipate energy; therefore, their effect on the average velocity field can be modeled by a viscous dissipation term, i.e.,

$$\nabla\cdot S_\delta(u,u) = -\nabla\cdot(\nu_T\nabla u),$$

where ν_T is referred to as the *eddy-viscosity*. Ladyzhenskaya [14] proposed $\nu_T = |\nabla u|^{p-2}$ so that

$$\begin{aligned} \partial_t u - \nu\Delta u + u\cdot\nabla u - \nabla\cdot(\epsilon_\delta(|\nabla u|^{p-2}\nabla u)) + \nabla\pi &= f \\ \nabla\cdot u &= 0; \end{aligned} \tag{3}$$

[1] $A : B$ denotes the componentwise inner product for the matrix A and B and $\langle\cdot,\cdot\rangle$ denotes the duality coupling.

Smagorinsky [23] independently proposed the same model with $p = 3$, which has been widely used. We will refer to (3) as the *nonlinear viscosity model.* The dependence of ϵ_δ on δ should be determined appropriately by dimensional analysis by comparing with the dimensions of the Reynolds stress. We will circumvent this issue for now and simply denote this coefficient as ϵ.

Note that the eddy-viscosity term is modeled as a diffusion effect that is proportional in strength to the velocity gradient. The degree of such a proportionality is quantified by the parameter p. The nonlinear diffusion operator introduced here is referred to as the *p-Laplacian* and it is used extensively for modeling non-Newtonian fluid dynamics as well as in many other areas. The mathematical properties of weak solutions of (3) was investigated by Ladyzhenskays [14]. It is known that, for $p \geq \frac{11}{5}$, a globally unique strong solution of (3) exists on a periodic domain [20]. Even though (3) is well-posed in this sense, the p-Laplacian adds some difficulty. In particular, one is unable to talk about the solution in the classical sense unless one knows that the gradient of the velocity is continuous; to our knowledge, such a result has not been proven. However, a local Hölder regularity for the equation $\partial_t u - \nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0$ is known [5]; the proof depends on maximum principle type arguments. The extension of this result to (3) is obstructed by the fact that maximum principle techniques, which are often the method of choice in proving regularity results, do not apply because of the inherently global effect introduced by the incompressibility condition.

Another turbulence model that attains well-posedness is the *hyperviscosity model* analyzed by Lions [18]:

$$\begin{aligned} \partial_t u - \nu\Delta u + u\cdot\nabla u - \epsilon(-\Delta)^\alpha u + \nabla\pi = f \\ \nabla\cdot u = 0. \end{aligned} \tag{4}$$

The fractional differentiation operator is defined in the frequency domain by[2] $\widehat{(-\Delta)^\alpha u}(k) = |k|^{2\alpha}\widehat{u}(k)$, where $\widehat{u}$ denotes the kth Fourier coefficient of u. One can consider this as a certain type of eddy-viscosity model in which $S(u,u) = |\nabla|^{2\alpha-2}\nabla u$. The unique strong solution is known to exist for $\alpha > 5/4$ [18]. However, unlike for (3), it is known that the strong solution of (4) is a classical solution.

The models (3) and (4) are designed to obtain well-posedness, a property that is lacking for the three-dimensional *NSE* (1). Unfortunately, it is computationally observed that these models tend to be over-diffusive, i.e., they smear large-scale structures too much.

2.3 Spectral Eddy-Viscosity Models

In order to preserve large-scale structures, we would like to limit the regularization effect to the small-scales. In fact, this is the essential idea of the spectral viscosity method due to Tadmor [4] for hyperbolic conservation laws, where artificial viscosity is added only at high frequencies. Inspired by this idea, we could propose the following spectral viscosity model for the *NSE*:[3]

$$\partial_t u - \nu\Delta u + u\cdot\nabla u - \epsilon Q\nabla\cdot(Q(\nabla u)) + \nabla\pi = f \tag{5}$$

where Q is a *high-pass filter*, i.e., it erases all the low-frequency modes of the input. Therefore, damping is applied only to the high frequency part of the solution. An important point is

[2] In fact, it is illuminating to express this equation as an evolution equation in the frequency space.

[3] To avoid needless repetition, we sometimes omit the continuity equation $\nabla\cdot u = 0$ which, of course, continues to hold.

that for a smooth solution that can be expressed in terms of low modes, the spectral-viscosity operator becomes zero. In fact, low-mode solutions to *NSE* do exist for appropriately chosen forcing term f. This indicates how the filtered viscosity model tries to consistently model the large-scale structures in the fluid.

In (5), the turbulent viscosity coefficient is constant; unfortunately, this does not make the model well posed although this version of the spectral viscosity model was implemented in [11] with some good effect. For this reason, we will combine the spectral viscosity and eddy viscosity ideas. Basically, we modify the hyperviscosity and nonlinear viscosity models so that regularization only affects the high-frequency modes:

$$\partial_t u - \nu \Delta u + u \cdot \nabla u - \epsilon(-\Delta)^\alpha Q u + \nabla \pi = f \tag{6}$$

and

$$\partial_t u - \nu \Delta u + u \cdot \nabla u - \epsilon Q \nabla \cdot (|\nabla Q u|^{p-2} \nabla Q u) + \nabla \pi = f, \tag{7}$$

respectively. As it turns out, it is beneficial to stabilize (7) by a linear filtered viscosity as well:

$$\partial_t u - \nu \Delta u + u \cdot \nabla u - \epsilon Q \nabla \cdot ((1 + |Q(\nabla u)|^{p-2}) Q(\nabla u)) + \nabla \pi = f. \tag{8}$$

The model (8) is used for the computational simulation of turbulence in [10] in the setting of a two-grid finite element method; the cut-off operator is defined by an appropriate orthogonal projection onto the fine scales. Note that the spectral viscosity models inevitably introduce several free parameters. These parameters are important in quantifying trade-offs when modeling turbulence. Large nonlinear or hyperviscosity exponents and low cut-offs give the model more stability at the expense of increasing modeling error, whereas small nonlinear or hyperviscosity coefficients and high cut-off increase consistency at the expense of decreased stability.

In this paper, we analyze (6) and (8) and mathematically investigate such trade-offs. It is also worth noting that the model (6) is natural in our periodic setting because the equation can be formulated in frequency space. This makes the analysis of (6) somewhat simpler than that for (8) for which it is difficult to interpret the spectral nonlinear viscosity either in frequency or physical space.

2.4 Formal Definitions

We now formalize the models (6) and (8). Let $\mathbb{T}^3$ denote the unit box $[0,1]^3$ with identification of the planes $x_i = 0$ with $x_i = 1$, $i = 1, 2, 3$, and let $I = [0, T]$. Then, $Q_T = I \times \mathbb{T}^3$ denotes the time-space cylinder over a periodic domain. We define the projection operator

$$P_M(f) = \sum_{|k|_\infty \le M} \widehat{f}(k) e^{ik\cdot x}, \quad \text{where } f = \sum \widehat{f}(k) e^{ik\cdot x}.$$

Let P denote the Leray projector, i.e., an orthogonal projector onto the space of divergence-free vector fields. Define[4] $X_M = P_M(L^2)$, $V_M = P P_M((L^2)^3)$, and the filter $Q_M = I - P_M$.

[4] We use standard Sobolev space notation for function spaces. However, for economy of notation, we usually omit the spatial domain in the designation the spaces, e.g., we use L^2 instead of $L^2(\mathbb{T}^3)$, $W^{1,p}$ instead of $W^{1,p}(\mathbb{T}^3)$, etc.

Then, the filtered nonlinear viscosity model is given by

$$\begin{aligned}\partial_t u - \nu\Delta u + u\cdot\nabla u - \epsilon Q_M \nabla\cdot((1+|Q_M(\nabla u)|^{p-2})Q_M(\nabla u)) + \nabla\pi &= f\\ \nabla\cdot u &= 0.\end{aligned} \tag{9}$$

Clearly, this model is determined by the three parameters M, p, and ϵ; we refer to this model as $NV(\epsilon, p, M)$ or simply as NV.

The filtered hyperviscosity model is given by

$$\begin{aligned}\partial_t u - \nu\Delta u + u\cdot\nabla u - \epsilon(-\Delta)^\alpha Q_M u + \nabla\pi &= f\\ \nabla\cdot u &= 0.\end{aligned} \tag{10}$$

Clearly, this model is determined by the three parameters M, α, and ϵ; we refer to this model as $HV(\epsilon, \alpha, M)$ or simply as HV.[5]

Now, even though we can show that a classical solution exists for (10), this is not necessarily so for (9). Therefore, (9) does not make sense as stated. Hence, we must introduce the notion of weak solutions of NV and do so as well for HV.

We refer to u as a weak solution of $NV(\epsilon, p, M)$ if $\nabla\cdot u = 0$ almost everywhere and, for almost all time,[6,7]

$$\begin{aligned}&\int \Big(\partial_t u\cdot\phi + \nu\nabla u : \nabla\phi + u\cdot\nabla u\cdot\phi + \epsilon(1+|\nabla\widetilde{u}|^{p-2})\nabla\widetilde{u} : \nabla\widetilde{\phi}\Big)dx\\ &\quad = \int f\cdot\phi\, dx\end{aligned} \tag{11}$$

for all $\phi \in (C^\infty(Q_T))^3$ such that $\int\phi = 0$ and $\nabla\cdot\phi = 0$. For notational convenience, we have set $\widetilde{u} = Q_M u$ and $\overline{u} = P_M u$ and refer to the former as the *high-frequency part* of u and the latter as the *low-frequency part* of u and similarly for ϕ.

Similarly, u is called a weak solution of $HV(\epsilon, \alpha, M)$ if $\nabla\cdot u = 0$ almost everywhere and, for almost all time,

$$\int \partial_t u\cdot\phi + \nu\nabla u : \nabla\phi + u\cdot\nabla u\cdot\phi + \epsilon|\nabla|^\alpha\widetilde{u}\cdot|\nabla|^\alpha\widetilde{\phi}\, dx = \int f\cdot\phi\, dx$$

for all $\phi \in (C^\infty(Q_T))^3$ such that $\int\phi = 0$ and $\nabla\cdot\phi = 0$, where $|\nabla|^\alpha$ is defined through the Fourier transform as

$$\widehat{|\nabla|^\alpha u}(k) = |k|^\alpha\widehat{u}(k).$$

Our objective is to solve these equations by approximating them by a finite-dimensional system that solves a similar problem. To this end, we refer to u_N as a solution of the problem

[5]The operator $(-\Delta)^\alpha$ is defined as a multiplier in the Fourier space: $\widehat{(-\Delta)^\alpha} = |k|^{2\alpha}$, where k is the wave number. Note that this is basically a generalization of differential operators to the setting with fractional index. One caveat is that such operators, unlike the Laplacian, are global (in the physical space) for fractional index, hence presenting some additional subtlety in the analysis.

[6]Unless specifically noted, all integrals are over the domain $\mathbb{T}^3$.

[7]We also need certain regularity on u which is evident from the weak formation so that the expressions make sense. These regularity conditions will be stated explicitly in the proof below.

$NV_N(\epsilon, p, M)$ if $u_N \in L^2((0,T):V_N), \widetilde{u}_N \in L^p((0,T):V_N)$, $\partial_t u_N \in L^s((0,T):V_N)$ for some $s > 1$, and, for all $\phi \in L^p((0,T):V_N)$ and for almost all time,

$$\int \Big(\partial_t u_N \cdot \phi + \nu \nabla u_N : \nabla \phi + u_N \cdot \nabla u_N \cdot \phi + \epsilon(1 + |\nabla \widetilde{u}_N|^{p-2}) \nabla \widetilde{u}_N : \nabla \widetilde{\phi}\Big) dx$$
$$= \int f \cdot \phi \, dx. \tag{12}$$

Similarly, we refer to u_N as a solution of the problem $HV_N(\epsilon, \alpha, M)$ if for almost all time,

$$\int \partial_t u_N \cdot \phi + \nu \nabla u_N : \nabla \phi + u_N \cdot \nabla u_N \cdot \phi + \epsilon |\nabla|^\alpha \widetilde{u}_N \cdot |\nabla|^\alpha \widetilde{\phi} \, dx = \int f \cdot \phi \, dx.$$

When convenient, we simply refer to $NV_N(\epsilon, p, M)$ and $HV_N(\epsilon, \alpha, M)$ as NV_N and HV_N, respectively.

The needed formal definitions are now in place so that, in the next section, we discuss important mathematical question about the well-posedness of our models.

3 Well-Posedness

In this section, we discuss well-posedness issues for the spectral viscosity models. In general, the *well-posedness* of a given partial differential equation means that it possesses the following properties:

- a solution to the equation exists in an appropriate (weak) sense;
- this solution is unique;
- the solution is regular in an appropriate sense; and
- the solution should depend continuously on the initial data.

The classic result in the direction of showing well-posedness for the Navier-Stokes equations (1) is the following result of Leray [16] (see also [3, 25]) on the existence of weak solutions.

Theorem 3.1 *Let $f \in L^2(Q_T)$, $u_0 \in H^1$, and $\nabla \cdot u_0 = 0$. Then, there exists a (weak) solution $u \in L^2(I; H^1) \cap L^\infty(I; L^2)$ and $\partial_t u \in L^{\frac{4}{3}}(I; H^{-1})$ such that $\nabla \cdot u = 0$ almost everywhere and, for all $\phi \in C^\infty(Q_T)$ with $\nabla \cdot \phi = 0$ and $\int \phi \, dx = 0$ and for almost all time,*

$$\int_{Q_T} \partial_t u \phi + \nu \nabla u : \nabla \phi + u \cdot \nabla u \cdot \phi \, dx dt = \int_{Q_T} f \cdot \phi \, dx dt.$$

It is currently not known if weak solutions are unique. The question of uniqueness is intimately tied to the regularity question as there are a host of results that show that, if a solution possesses a sufficient regularity, then it is unique. Examples of results in this direction are found in, e.g., [21, 22].

3.1 Energy Dissipation Estimate

Note that one of the difficulties associated with proving well posedness for the Navier-Stokes equations is its lack of sufficient globally controlled quantities. The extra dissipation of spectral eddy-viscosity models adds another globally controlled quantity to those for the Navier-Stokes equations so to enable a proof of global well-posedness.

Suppose *NV* and *HV* possess smooth solutions. Then, if we test *NV* and *HV* by u, we obtain the most important global differential inequality, referred to as the *energy dissipation estimate*:

$$\frac{1}{2}\partial_t\|u\|^2+\nu\|\nabla u\|^2+\epsilon(\|\nabla\widetilde{u}\|^2+\|\nabla\widetilde{u}\|_p^p)=(f,u)$$

for *NV* and

$$\frac{1}{2}\partial_t\|u\|^2+\nu\|\nabla u\|^2+\epsilon\||\nabla|^\alpha\widetilde{u}\|^2=(f,u)$$

for *HV*. This implies the energy inequality

$$\|u(t)\|^2+\nu\int_0^t\|\nabla u\|^2+2\epsilon\int_0^t(\|\nabla\widetilde{u}\|^2+\|\nabla\widetilde{u}\|_p^p)\le\|u(0)\|^2+\frac{C_p^2}{\nu}\int_0^t\|f\|^2 \tag{13}$$

for $NV(\epsilon,p,M)$ and, for $HV(\epsilon,\alpha,M)$,

$$\|u(t)\|^2+\nu\int_0^t\|\nabla u\|^2+2\epsilon\int_0^t\||\nabla|^\alpha\widetilde{u}\|^2\le\|u(0)\|^2+\frac{C_p^2}{\nu}\int_0^t\|f\|^2, \tag{14}$$

where C_p is a constant depending on p only. Note the additional control on the norms of $\widetilde{u}$ added to the usual energy balance equation for the *NSE*.

Another important bound for both models as well as for the *NSE* is that, for uniformly bounded forcing, the L^2 norm of the solution remains bounded. This can be seen from the inequality

$$\partial_t\|u\|^2+\nu\|u\|^2\le C\|f\|^2$$

that follows from the energy dissipation estimate and the application of the Poincaré inequality to the viscosity term.

3.1.1 Why Is It Difficult to Show Well-Posedness for NSE?

In [24], Tao uses dimensional heuristics to indicate why the well-posedness problem for the *NSE* is difficult to prove. Suppose we take the forcing $f=0$ and that u has order N support in the frequency domain so that, by the uncertainty principle, it has a physical support of order N^{-1}. Then, the energy dissipation estimate tells us that the global quantities $\int_{\mathbb{T}^d}|u|^2\sim U^2N^{-d}$ and $\int_0^T\int_{\mathbb{T}^d}|\nabla u|^2\sim TN^2U^2N^{-d}$ are controlled, where d denotes the space dimension. Due to the skew-symmetry of the nonlinear convection term, its effect on globally conserved quantities is nil; however, it can have a local effect of order $\int_0^T\int_\Omega u\cdot\nabla u\cdot u\sim TU^3NN^{-d}$. Now, let us assume that $\|u(0)\|\sim O(1)$; then, $\|u(t)\|^2\sim U^2N^{-d}\sim O(1)$ so $U\sim N^{d/2}$. It follows that the dissipative effect is of order $\int_0^T\int_{\mathbb{T}^d}\|\nabla u\|^2\sim TN^2$ whereas the nonlinear convective effect is of order $\int_0^T\int_\Omega u\cdot\nabla u\cdot u\sim TN^{1+d/2}$.

For $d=2$, the linear term and the nonlinear term have the same dimension. Hence, we say that the energy dissipation bound is *critical* for the two-dimensional *NSE*. For $d=3$, the nonlinearity dominates and hence, in three dimensions, we say that the energy bound is *supercritical*. The difficulty with three-dimensional turbulence is that the global energy bound is *supercritical* and does not provide enough control on the size of u to prevent local instabilities due to the nonlinear convective term from happening.

Note that energy dissipation also says that $TN^2\sim O(1)$ or $T\sim N^{-2}$, which is the well-known parabolic space-time scaling. This roughly means that the solution can have the Nth mode staying large only for a time interval of length N^{-2}. This implies that the nonlinear convective term is of order $TN^{1+d/2} \sim N^{d/2-1}$. The exponent of N is positive for $d \geq 3$. Hence, it allows the possibility that energy is cascaded to higher and higher frequencies, only staying at any frequency over time of intervals of length N^{-2} and meanwhile increasing the size of the nonlinear convective term. But sum of N^{-2} as $N \to \infty$ is finite; therefore, the solution can blow up in finite time due to ever increasing destabilizing effect of the nonlinear convective term. However, in practice, no such blowup has been observed and hence proving the well posedness of the *NSE* remains an important open problem.

Let us apply the heuristics of Tao to the *NV* and *HV* models that contain additional dissipative terms. For the *NV* model, we have $\int_0^T \int_{\mathbb{T}^d} |\nabla \widetilde{u}|^p \sim TU^p N^p N^{-d} \sim TN^{d(\frac{p}{2}-1)+p}$ which, in three dimensions, becomes critical at $p = \frac{11}{5}$. Upon actual analysis, this equation is shown to possess a strong solution at this value of p.

For the *HV* model, we have $\int_0^T \int_{\mathbb{T}^d} ||\nabla|^\alpha \widetilde{u}|^2 \sim TU^2 N^{2\alpha} N^{-d} \sim TN^{2\alpha}$ which becomes critical at $\alpha = \frac{5}{4}$, a value beyond which the well-posedness can be established.

Note how simple dimensional analysis is a powerful tool for predicting the criticality of the models, and therefore gives us foresight into the well-posedness question. In any case, from the dimensional analysis, the *NV* and *HV* turbulence models can be thought of as effecting sufficient control on the high-frequency modes so as to prohibit blow-up due to cascading. However, due to filtering, the nonlinear convective term remains free to act on the low-frequency part of the solution and hence we expect that the large-scale behavior solutions to the models remain accurate.

3.1.2 A Simple Interpolation Result

Note that the energy balance equation roughly says that $\|u\|_{L^\infty((0,T):L^2)}$ and $\nu^{1/2} \times \|\nabla u\|_{L^2((0,T):L^2)}$ are of the same order. This implies that we should have a bound with respect to spaces that are *in between* these spaces. This is the content of the following interpolation lemma.

Lemma 3.2 *If u is a smooth solution to NV or HV and $2 \leq r \leq 6$, then*

$$\|u\|_{L^{\frac{4r}{3(r-2)}}((0,T):L^r)} \lesssim \|u\|_{L^\infty((0,T):L^2)}^{\frac{6-r}{2r}} \|\nabla u\|_{L^2((0,T):L^2)}^{\frac{3(r-2)}{2r}}.$$

Proof Let $s = \frac{4r}{3r-6}$. Then,

$$\int_0^t \|u\|_r^s \, dt \leq \int_0^t \|u\|_2^{s\frac{6-r}{2r}} \|u\|_6^{s\frac{3(r-2)}{2r}} \, dt \lesssim \|u\|_{L^\infty((0,T):L^2)}^{\frac{2(6-r)}{3(r-2)}} \|\nabla u\|_{L^2((0,T):L^2)}^2,$$

where we have used the Hölder and Sobolev inequalities. □

Here and below, $\lesssim$ means the left-hand-side of an expression is dominated by a constant multiple of the right-hand-side with the constant independent of the function or parameter under consideration. However, the value of the constant may change from place to place.

3.2 Existence of a Weak Solution for *NV*

For the *HV* model, the well posedness, i.e., the existence, uniqueness, and regularity, of a weak solution and its convergence to a weak solution of the *NSE* have been established by Guermond [9]. Thus, here, we focus on the *NV* model.

In this section, we show that precisely beyond the index $p \geq \frac{11}{5}$ that was identified by the heuristical dimensional analysis, the *NV* model possesses a strong solution. Subsequently, we prove the global uniqueness of weak solutions of the *NV* model and its convergence to a weak solution of *NSE* as $\epsilon \to 0$. To these ends, we proceed in a standard fashion by deriving a series of a priori estimates. Besides the energy dissipation estimate (13), we must show regularity in time and in space, beginning with regularity in time.

Lemma 3.3 *Let $\frac{1}{q} + \frac{1}{p} = 1$. If u_N is a solution to NV_N, then $\partial_t u_N \in L^{\min\{4/3,q\}}(I; W^{-1,q})$.*

Proof If $\phi \in V_N$, then

$$\begin{aligned}
&(\partial_t u_N, \phi) \\
&\quad = (f + \nu \Delta u_N - u_N \cdot \nabla u_N + \epsilon Q_M \nabla \cdot ((1 + |\nabla Q_M u_N|^{p-2}) \nabla Q_M u_N), \phi) \\
&\quad \lesssim \big(\|f\|_{-1} + \nu \|\nabla u_N\| + \|u_N\|_4^2\big) \|\nabla \phi\|_2 + \epsilon \|\nabla Q_M u_N\| \|\nabla Q_M \phi\| \\
&\qquad + \epsilon \|\nabla Q_M u_N\|_p^{p-1} \|\nabla Q_M \phi\|_p,
\end{aligned}$$

where we have used the (p, p)-estimate (A.6) for the operator $Q_M = I - P_M$. After time integration,

$$\begin{aligned}
\int_I (\partial_t u_N, \phi)\, dt \lesssim{}& \big(\|f\|_{L^2(I;H^{-1})} + \nu \|\nabla u_N\|_{L^2(I,L^2)}\big) \|\nabla \phi\|_{L^\infty(I;L^2)} \\
&+ \epsilon \|\nabla Q_M u_N\|_{L^2(I;L^2)} \|\nabla Q_M \phi\|_{L^\infty(I;L^2)} \\
&+ \|u_N\|_{L^\infty(I;L^2)}^{\frac{1}{2}} \|u_N\|_{L^{4/3}(I;H^1)}^{\frac{3}{2}} \|\nabla \phi\|_{L^4(I;L^2)} \\
&+ \epsilon \|\nabla Q_M u_N\|_{L^p(I;L^p)}^{p-1} \|\nabla \phi\|_{L^p(I;L^p)}.
\end{aligned}$$

This inequality implies that $\partial_t u_N \in L^{\min\{4/3,q\}}(I; W^{-1,q})$. □

Now, in order to prove existence, we show that the Galerkin projection satisfies the a spatial regularity result uniformly in N. The proof of this result is almost identical to one given in [20], except that, for our case, we have a cut-off filter that needs to be handled in a special way.

Lemma 3.4 *Let u_N denote the solution of $NV_N(\epsilon, p, M)$, where $p \geq \frac{11}{5}$. Let $\lambda = \frac{2(3-p)}{3p-5}$. Then, if $p < 3$,*

$$\begin{aligned}
\|\nabla u_N(t)\|^2 \lesssim{}& \|\nabla u_N(0)\|^2 + \int_0^t \left(\frac{\|f\|^2}{\nu} + \epsilon M^2 \|\nabla u_N\|_p^p \right) ds \\
&+ \left(\int_0^t \epsilon^{-\frac{2\lambda}{3}} \|\nabla u_N\|_p^p \, ds \right)^{1/(1-\lambda)}
\end{aligned}$$

whereas, if $p \geq 3$,

$$\|\nabla u_N(t)\|^2 \lesssim \|\nabla u_N(0)\|^2 + \int_0^t \left(\frac{\|f\|^2}{\nu} + \|\nabla u_N\|_3^3 \right) ds.$$

Moreover, we have $u_N \in L^p(I; W^{1,3p}) \cap L^2(I; H^2) \cap L^\infty(I; H^1)$.

Proof We set $\phi = -\Delta u_N$ in (12) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_N\|^2 + \nu\|\Delta u_N\|^2 - b(u_N, u_N, \Delta u_N) + \epsilon(\nabla\cdot(1+|\nabla\widetilde{u}_N|^{p-2})\nabla\widetilde{u}_N, \Delta\widetilde{u}_N) \le \frac{\nu}{4}\|\Delta u_N\|^2 + \frac{1}{\nu}\|f\|^2,$$

where $b(u, v, w) \equiv (u\cdot\nabla v, w)$. By applying Green's identity, we have $b(u_N, u_N, -\Delta u_N) \le \|\nabla u_N\|_3^3$. Let $I_p(\widetilde{u}_N) = \sum_{i,j,k}\int |\nabla\widetilde{u}_N|^{p-2}(\partial_{kj}\widetilde{u}_{N_i})^2\,dx$. The monotonicity formula (A.13) yields

$$I_p(\widetilde{u}_N) \le (\nabla\cdot|\nabla\widetilde{u}_N|^{p-2}\nabla\widetilde{u}_N, \Delta\widetilde{u}_N).$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_N\|_2^2 + \nu\|\Delta u_N\|^2 + \epsilon I_p(\widetilde{u}_N) + \epsilon\|\Delta\widetilde{u}_N\|^2 \lesssim \frac{1}{\nu}\|f\|^2 + \|\nabla u_N\|_3^3. \tag{15}$$

The $p \ge 3$ case basically follows from this inequality.

We must work a little bit harder for $p < 3$. Note that, by interpolation,

$$\|\nabla u_N\|_3 \le \|\nabla u_N\|_2^{\frac{2(p-1)}{3p-2}}\|\nabla u_N\|_{3p}^{\frac{p}{3p-2}},$$

and

$$\|\nabla u_N\|_3 \le \|\nabla u_N\|_p^{\frac{p-1}{2}}\|\nabla u_N\|_{3p}^{\frac{3-p}{2}}.$$

Thus, given $0 < \sigma < 1$, we have

$$\|\nabla u_N\|_3^3 \le \|\nabla u_N\|_2^{q_1}\|\nabla u_N\|_p^{q_2}\|\nabla u_N\|_{3p}^{q_3},$$

where $q_1 = 3\sigma\frac{2(p-1)}{3p-2}$, $q_2 = 3(1-\sigma)\frac{p-1}{2}$, and $q_3 = 3(1-\sigma)\frac{3-p}{2} + 3\sigma\frac{p}{3p-2}$. Then,

$$\begin{aligned}
\|\nabla u_N\|_3^3 &\le \|\nabla u_N\|_2^{q_1}\|\nabla u_N\|_p^{q_2}\|\nabla u_N\|_{3p}^{q_3}\\
&\lesssim \|\nabla u_N\|_2^{q_1}\|\nabla u_N\|_p^{q_2}(\|\nabla\widetilde{u}_N\|_{3p} + M^{\frac{2}{p}}\|\nabla u_N\|_p)^{q_3}\\
&\lesssim \|\nabla u_N\|_2^{q_1}\|\nabla u_N\|_p^{q_2}(I_p(\widetilde{u}_N)^{\frac{1}{p}} + M^{\frac{2}{p}}\|\nabla u_N\|_p)^{q_3}\\
&\lesssim \left(\epsilon^{-\frac{q_3}{p}}\|\nabla u_N\|_2^{q_1}\|\nabla u_N\|_p^{q_2}\right)^{\frac{p}{p-q_3}} + \frac{\epsilon}{2}\left(I_p(\widetilde{u}_N)^{\frac{1}{p}} + M^{\frac{2}{p}}\|\nabla u_N\|_p\right)^p,
\end{aligned} \tag{16}$$

where we have used the Bernstein inequality (A.7), Lemma A.14, and Young's inequality. It is now clear how σ must be chosen. First, we would like to set $q_2\frac{p}{p-q_3} = p$. We then obtain $1-\sigma = \frac{p(3p-5)}{6(p-1)}$ and $\sigma = \frac{(3-p)(3p-2)}{6(p-1)}$. Thus, $q_1 = 3-p$, $q_2 = \frac{p(3p-5)}{4}$, and $q_3 = \frac{(3-p)3p}{4}$ which imply $\frac{p}{p-q_3} = \frac{p}{q_2} = \frac{4}{3p-5}$. Now, let $\lambda = \frac{q_1}{2}\cdot\frac{p}{p-q_3} = \frac{2(3-p)}{3p-5}$ so that $\frac{q_3}{p}\cdot\frac{p}{p-q_3} = \frac{3(3-p)}{3p-5} = \frac{2\lambda}{3}$. Now, we apply the above parameters to (16) and substitute into (15) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u_N\|^2 + \nu\|\Delta u_N\|^2 + \epsilon I_p(Q_M u_N) + \epsilon\|\Delta(Q_M u_N)\|^2\\
&\quad\lesssim \frac{1}{\nu}\|f\|^2 + \epsilon^{-\frac{2\lambda}{3}}\|\nabla u_N\|_2^{2\lambda}\|\nabla u_N\|_p^p + \frac{\epsilon}{2}I_p(Q_M u_N) + \frac{\epsilon}{2}M^2\|\nabla u_N\|_p^p.
\end{aligned}$$

Therefore, integrating in time yields

$$\|\nabla u_N(t)\|^2 \le \|\nabla u_N(0)\|^2 + C\int_0^t \left(\frac{1}{\nu}\|f\|^2 + \epsilon^{-\frac{2\lambda}{3}}\|\nabla u_N\|_2^{2\lambda}\|\nabla u_N\|_p^p + \frac{\epsilon}{2}M^2\|\nabla u_N\|_p^p\right) ds. \quad (17)$$

Let $A = \int_0^t (\frac{1}{\nu}\|f\|^2 + \frac{1}{2}\epsilon M^2\|\nabla u_N\|_p^p) ds$ and $B = \epsilon^{-\frac{2\lambda}{3}}\|\nabla u_N\|_p^p$. Letting $g = \|\nabla u_N\|^2$, (17) then has the form

$$g(t) - g(0) \le A + \int_0^t B g^\lambda \, ds.$$

Solving this inequality results in

$$g(t) \le \left((g(0)+A)^{1-\lambda} + (1-\lambda)\int_0^t B\, ds\right)^{\frac{1}{1-\lambda}} \lesssim g(0) + A + \left(\int_0^t B\, ds\right)^{\frac{1}{1-\lambda}}.$$

Applying this to (17) results in

$$\|\nabla u_N(t)\|^2 \le \|\nabla u_N(0)\|^2 + C\int_0^t \left(\frac{\|f\|^2}{\nu} + \epsilon M^2\|\nabla u_N\|_p^p\right) ds + C\left(\int_0^t \epsilon^{-\frac{2\lambda}{3}}\|\nabla u_N\|_p^p ds\right)^{\frac{1}{1-\lambda}} ds.$$

This inequality implies that $\nabla u_N \in L^\infty(I; L^2)$ if $\lambda \le 1$ which translates to $p \ge \frac{11}{5}$. This completes the proof of the lemma. $\square$

From the energy dissipation estimate and the regularity of the time derivative, we have that $u_N \in L^\infty(I; L^2) \cap L^2(I; H^1)$ and $\partial_t u_N \in L^{\min\{4/3,q\}}(I; W^{-1,q})$. We use the Aubin-Lions compactness theorem to obtain an appropriate subsequence (which we still refer to as u_N) such that $u_N \rightharpoonup u$ weakly in $L^2(I; H^1)$ and $\nabla u_N \to \nabla u$ strongly in $L^2(I; L^2)$. The regularity results of Lemma 3.4 show that, in fact, $u_N \in L^p(I; W^{1,3p}) \cap L^\infty(I; H^1) \cap L^2(I; H^2)$ uniformly in N, and therefore the same holds for u. In particular, the Aubin-Lions theorem implies the strong convergence in $L^2(I; W^{1,q})$ for $q < \min\{6, 3p\} = 6$.

Let $\psi_j \in X_j$ then clearly the weak convergence and incompressibility condition yield

$$(\partial_t u_N, \psi_j) \to \langle \partial_t u, \psi_j \rangle \quad \text{and} \quad \nu(\nabla u_N, \nabla \psi_j) \to \nu\langle \nabla u, \nabla \psi_j \rangle.$$

Finally, since $\nabla u_N \rightharpoonup \nabla u$ in $L^2(I; L^2)$ and $u_N \to u$ strongly in $L^2(I; L^2)$, we have

$$(u_N \cdot \nabla u_N, \psi_j) \to \langle u \cdot \nabla u, \psi_j \rangle.$$

Showing the convergence of the p-Laplacian part is slightly more challenging. If $p < 6$, then note that due to (A.12),

$$\int \langle |\nabla \widetilde{u}_N|^{p-2}\nabla \widetilde{u}_N - |\nabla \widetilde{u}|^{p-2}\nabla \widetilde{u}, \nabla \widetilde{\psi_j} \rangle \le (p-2)\|\nabla \widetilde{\psi_j}\|_p \|\nabla(\widetilde{u}_N - \widetilde{u})\|_p (\|\nabla \widetilde{u}\|_p + \|\nabla \widetilde{u}_N\|_p)^{p-2}.$$

Due to the fact that $p < 6$, $u_N \to u$ in $L^2(I; W^{1,p})$ strongly. The Jackson inequality (A.9) tells us that Q_M is an (p, p) type operator. Therefore, $\widetilde{u}_N \to \widetilde{u}$ in $L^2(I; W^{1,p})$. Thus, we have the required convergence.

For $p \geq 6$, we consider a different approach. First, since $\nabla\widetilde{u}_N \to \nabla\widetilde{u}$ in $L^2(Q_T)$, from (A.15) there exists a subsequence N_i such that $\nabla\widetilde{u}_{N_i} \to \nabla\widetilde{u}$ almost everywhere. Consequently, due to (A.12), $|\nabla\widetilde{u}_{N_i}|^{p-2}\nabla\widetilde{u}_{N_i} \to |\nabla\widetilde{u}|^{p-2}\nabla\widetilde{u}$ almost everywhere. Note then that for any set $M \subset Q_T$, we have

$$\int_M |\nabla\widetilde{u}_{N_i}|^{p-2}\nabla\widetilde{u_{N_i}} : \nabla\psi_j dxdt \;\leq\; |M|^{\frac{1}{p}} \int_0^T \|\nabla\widetilde{u}_{N_i}\|_p^{p-1}\|\nabla\psi_j\|_\infty.$$

Thus, the p-Laplacian term is uniformly integrable. Thus, the required convergence follows from (A.16).

The following theorem summarizes the results obtained.

Theorem 3.5 *Let $u(0) \in H^1$ and $f \in L^2$. If $p \geq \frac{11}{5}$, there exists a weak solution u to $NV(\epsilon, p, M)$. In fact, u possesses the further regularity $u \in L^p(I; W^{1,3p}) \cap L^2(I; H^2) \cap L^\infty(I; H^1)$.*

3.3 Further Regularity

In this section, we show that, if $u(0) \in W^{1,p}$, then $\partial_t u \in L^2(I; L^2)$ and $u \in L^\infty(I; L^p)$. With this additional result, u satisfies a host of regularity results so that we may refer to such a solution the *strong solution* of NV.

Theorem 3.6 *Let $p \geq \frac{11}{5}$, $f \in L^2$, and $u(0) \in W^{1,p}$. With the same hypothesis as in the previous section and suppose in addition that $u(0) \in W^{1,p}$. Then, weak solutions of NV possess the further regularity $\partial_t u \in L^2(I; L^2)$ and $u \in L^\infty(I; L^p)$.*

Proof We let $\phi = \partial_t u$ in (11) to obtain

$$\begin{aligned}
&\int_I \Big(\|\partial_t u\|^2 + \frac{\nu}{2}\|\nabla u\|^2 + b(u, u, \partial_t u) + \epsilon\langle(1 + |\nabla\widetilde{u}|^{p-2})\nabla\widetilde{u}, \partial_t \nabla\widetilde{u}\rangle\Big)\, dt \\
&\quad \leq \int_I \left(\frac{1}{4}\|\partial_t u\|^2 + 4\|f\|^2\right) dt.
\end{aligned}$$

Using (A.12), we have

$$\begin{aligned}
&\frac{1}{2}\int_I \|\partial_t u\|^2 dt + \nu\|\nabla u(t)\|^2 + \epsilon(\|\nabla\widetilde{u}(t)\|^2 + \|\nabla\widetilde{u}(t)\|_p^p) \lesssim \nu\|\nabla u(0)\|^2 \\
&\quad + \epsilon(\|\nabla\widetilde{u}(0)\|^2 + \|\nabla\widetilde{u}(0)\|_p^p) + \int_I \left(\int |u\cdot\nabla u|^2 dx + 4\|f\|^2\right) dt.
\end{aligned}$$

The Hölder inequality implies

$$\int |u\cdot\nabla u|^2\, dx \leq \|u\|^2_{\frac{6p}{3p-2}} \|\nabla u\|^2_{3p} dx \lesssim \|\nabla u\|_2^2 \|\nabla u\|^2_{3p},$$

where the last inequality is due to the Sobolev inequality and the fact that $\frac{6p}{3p-2} \leq 6$. Thus, we have

$$\frac{1}{2}\int_I \|\partial_t u\|^2\,dt + \nu\|\nabla u(t)\|^2 + \epsilon(\|\nabla\widetilde{u}(t)\|^2 + \|\nabla\widetilde{u}(t)\|_p^p)$$
$$\lesssim \nu\|\nabla u(0)\|^2 + \epsilon\widetilde{U}(0) + \|\nabla u\|^2_{L^\infty(I;L^2)}\|\nabla u\|^2_{L^2(I;L^{3p})} + 4\int_I \|f\|^2\,dt,$$

where $U(0) \equiv \|\nabla\widetilde{u}(0)\|^2 + \|\nabla\widetilde{u}(0)\|_p^p$. The right-hand side is bounded due to the regularity result of the previous section. Therefore, the theorem follows. □

3.4 Uniqueness and Stability

In this section, we show that the solution of $NV(\epsilon, p, M)$ is unique. We do this by deriving a stability estimate. Such an estimate also holds for sufficiently regular solutions of the three-dimensional *NSE*. The key point is that for NV, the needed regularity is proved in the previous section.

Theorem 3.7 *Let u_1 and u_2 be two distinct solutions to $NV(\epsilon, p, M)$ with $p \geq \frac{11}{5}$. Then, we have*

$$\|(u_1 - u_2)(t)\|^2 \leq \|u_1(0) - u_2(0)\|^2 e^{\int_0^t \min\{\nu^{-3}\|\nabla u_1\|^4, \epsilon M^2 + \epsilon^{-3}\|\nabla u_1\|^4\}\,ds}.$$

Because $u_1 \in L^\infty(I; H^1)$ for $p \geq \frac{11}{5}$, the solution to $NV(\epsilon, p, M)$ is unique.

Proof Because, for $i = 1, 2$,

$$\partial_t u_i - \nu\Delta u_i - \epsilon P Q_M \nabla\cdot((1 + |\nabla\widetilde{u}_i|^{p-2})\nabla\widetilde{u}_i) + P(u_i\cdot\nabla u_i) = f,$$

we have that $w = u_1 - u_2$ satisfies

$$\partial_t w - \nu\Delta w - \epsilon P Q_M \nabla\cdot((1 + |\nabla\widetilde{u}_1|^{p-2})\nabla\widetilde{u}_1 - ((1 + |\nabla\widetilde{u}_2|^{p-2})\nabla\widetilde{u}_2)$$
$$+ P(u_1\cdot\nabla u_1 - u_2\cdot\nabla u_2) = 0.$$

We test this equation against w and use the monotonicity result (A.12) to obtain

$$\frac{1}{2}\partial_t\|w\|^2 + \nu\|\nabla w\|^2 + \epsilon(\|\nabla\widetilde{w}\|^2 + \gamma\|\nabla\widetilde{w}\|_p^p) \leq -(w\cdot\nabla u_1, w). \tag{18}$$

Note that due to the Gagliardo-Nirenberg inequality, we have $\|w\|_4 \leq \frac{4}{3\sqrt{3}}\|w\|^{1/4}\|\nabla w\|^{3/4}$ so that

$$\begin{aligned}(w\cdot\nabla u_1, w) &\leq \|\nabla u_1\|\|\nabla\widetilde{w}\|^{3/2}\|\widetilde{w}\|^{1/2} + \|\nabla u_1\|\|\overline{w}\|_4^2\\ &\leq \frac{\epsilon}{12}\|\nabla\widetilde{w}\|^2 + c\epsilon^{-3}\|\nabla u_1\|^4\|\widetilde{w}\|^2 + M^{(1/2-1/4)3\cdot 2}\|\nabla u_1\|\|\overline{w}\|^2\\ &\leq \frac{\epsilon}{12}\|\nabla\widetilde{w}\|^2 + (M^{3/2}\|\nabla u_1\| + \epsilon^{-3}\|\nabla u_1\|^4)\|w\|^2,\end{aligned} \tag{19}$$

where we have used the Bernstein inequality (A.7). We also have that

$$(w\cdot\nabla u_1, w) \leq \|\nabla u_1\|\|\nabla w\|^{3/2}\|w\|^{1/2} \leq c\nu\|\nabla w\|^2 + c\nu^{-3}\|\nabla u_1\|^4\|w\|^2. \tag{20}$$

By applying (19) and (20) to (18) and using the Gronwall inequality, the result holds. □

3.5 Convergence to a Weak Solution of the *NSE*

We have shown that the *NV* model is well-posed. This indicates, at least in part, that solving the *NV* equation numerically is tractable. We must, however, also show that somehow *NV* models turbulence well. To this end, we show that the solution to *NV* converges to a solution of *NSE*. The first issue we must clarify, however, is that there are two senses in which the sequence of solutions to *NV* can be possibly taken to converge to a *NSE* solution: one is to take $M \to \infty$ and the other $\epsilon \to 0$. In fact, we are also interested in cases in which both of these occur at the same time. In light of the fact that we do not know much about the regularity of weak solutions of the *NSE*, it is unlikely that as $M \to \infty$ independent of ϵ, the p-Laplacian term goes to zero. Therefore, the most sensible thing to do is to take the sequence as $\epsilon \to 0$, and perhaps let $M \to \infty$ as a function of ϵ. Thus, we will show that the sequence u_ϵ parameterized by ϵ contains a subsequence that converges to a weak solution of *NSE*, and let M depend on ϵ by expressing it as $M(\epsilon)$.

Note that $u_\epsilon \in L^2(I; H^1) \cap L^\infty(I; L^2)$ and $\partial_t u_\epsilon \in L^{\min\{4/3,q\}}(I; W^{-1,q})$ uniformly in ϵ and therefore we can use the Aubin-Lions theorem to obtain the weak limit u. We want to show that such a u is a weak solution of the *NSE*. Note that

$$(\epsilon|\nabla Q_{M(\epsilon)}u|^{p-2}\nabla Q_{M(\epsilon)}u, \nabla\psi) \le \epsilon^{1/p}(\epsilon^{1/p}\|\nabla Q_{M(\epsilon)}u\|_p)^{p-1}\|\nabla\psi\|_p.$$

We know that $\int \epsilon^{1/p}\|\nabla Q_{M(\epsilon)}u\|_p dt$ is uniformly bounded in ϵ due to (13). Thus, we may take $\epsilon \to 0$ and see that the nonlinear term goes to zero so that the limit of the solution to $NV(\epsilon, p, M(\epsilon))$ as $\epsilon \to 0$ satisfies the weak formulation for *NSE* for any choice of $M(\epsilon)$.

Theorem 3.8 *Let u_ϵ be the solution to $NV(\epsilon, p, M(\epsilon))$. Then, there exists u such that a subsequence of u_ϵ converges weakly $u_{\epsilon_j} \rightharpoonup u$ in $L^2(I; H^1)$ and $u_{\epsilon_j} \to u$ strongly in $L^2(L^2)$ as $\epsilon_j \to 0$ and u is a weak solution of the NSE.*

In this section, we proved some preliminary results concerning the well-posedness of the NV model. In Sect. 6, we investigate the *NV* model further by discretizing the equation in time. We will derive a certain stability result that allows us to address the question of choosing the appropriate parameters for the model.

4 Convergence Rate Estimate

In this section, we bound the norm of the difference between solutions of the three-dimensional *NSE* and of $NV(\epsilon, p, M)$ in terms of the difference between the initial data and the regularity of the solution to the *NSE*. As was discussed in Sect. 3, we examine the error rate in terms of ϵ. There, we indicated that M is allowed to depend on ϵ and go to infinity when $\epsilon \to 0$. One question is to find whether there exists an optimal choice of $M(\epsilon)$. The estimation of the convergence rate offers one possibility to choose $M(\epsilon)$, as we will see that the optimization of the error estimate naturally shows us how M should depend on certain inverse polynomials of ϵ. We also note the role played by the parameter p. Raising p stabilizes the system; therefore, in the presence of sufficient regularity, we obtain a better rate. On the other hand, higher p implies that *NV* is a significantly perturbed version of the *NSE*, and therefore if the solution of the *NSE* does not have the required regularity, the rate does not apply. We also prove that a uniform in time rate estimate is possible for $p = \frac{5}{2}$, although the rate in that case depends on the exponential power of $\nu^{-\frac{5}{2}}$. In this way, we have indicated how M should depend on ϵ and how p should be chosen, effectively determining the parameter choices.

4.1 The Nonlinear Viscosity Case

4.1.1 An Estimate Uniform in the Viscosity

We first assume that $p \geq \frac{5}{2}$. We would like to obtain an estimate that is completely independent of ν. The price we have to pay is that we do not have an uniform in time error bound. The estimate we obtain, however, gives insight into the nature of the trade-off between the cut-off frequency M and the artificial viscosity coefficient ϵ.

Let u_{NS} denote a strong solution of the *NSE* and u_{NV} denote a solution of $NV(\epsilon, p, M)$ such that their initial conditions agree, i.e., $u_{NS}(0,x) = u_{NV}(0,x)$. We know that u_{NS} satisfies

$$\partial_t u_{NS} - \nu \Delta u_{NS} + P(u_{NS} \cdot \nabla u_{NS}) = f$$

whereas u_{NV} satisfies

$$\partial_t u_{NV} - \nu \Delta u_{NV} + P(u_{NV} \cdot \nabla u_{NV}) - \epsilon P Q_M \nabla \cdot ((1 + |\nabla \widetilde{u}_{NV}|^{p-2}) \nabla \widetilde{u}_{NV}) = f.$$

Then, $w = u_{NS} - u_{NV}$ satisfies

$$\begin{aligned}
&\partial_t w - \nu \Delta w - \epsilon P Q_M \nabla \cdot \big((1 + |\nabla \widetilde{u}_{NS}|^{p-2}) \nabla \widetilde{u}_{NS} \\
&\quad - (1 + |\nabla \widetilde{u}_{NV}|^{p-2}) \nabla \widetilde{u}_{NV}\big) + P(u_{NS} \cdot \nabla u_{NS} - u_{NV} \cdot \nabla u_{NV}) \\
&\quad = -\epsilon P Q_M \nabla \cdot \big((1 + |\nabla \widetilde{u}_{NS}|^{p-2}) \nabla \widetilde{u}_{NS}\big).
\end{aligned}$$

Testing this equation against w results in

$$\begin{aligned}
&\frac{1}{2} \partial_t \|w\|^2 + \nu \|\nabla w\|^2 + \epsilon (\|\nabla \widetilde{w}\|^2 + \gamma \|\nabla \widetilde{w}\|_p^p) \\
&\quad \leq \epsilon (\|\nabla \widetilde{u}_{NV}\| \|\nabla \widetilde{w}\| + \|\nabla \widetilde{u}_{NV}\|_p^{p-1} \|\nabla \widetilde{w}\|_p) - (w \cdot \nabla u_{NV}, w) \\
&\quad \leq c\epsilon (\|\nabla \widetilde{u}_{NS}\|^2 + \|\nabla \widetilde{u}_{NS}\|_p^p) \\
&\qquad + \frac{\epsilon}{12} (\|\nabla \widetilde{w}\|^2 + \gamma \|\nabla \widetilde{w}\|_p^p) + |(w \cdot \nabla u_{NV}, w)|.
\end{aligned} \tag{21}$$

We estimate the high-frequency contribution to the convective term as

$$\begin{aligned}
(w \cdot \nabla \widetilde{u}_{NV}, w) &\leq c \, \|\nabla \widetilde{u}_{NV}\|_p (\|\overline{w}\|^2_{\frac{2p}{p-1}} + \|\widetilde{w}\|^2_{\frac{2p}{p-1}}) \\
&\leq \frac{\epsilon}{12} \|\nabla \widetilde{w}\|^2 + c\, \epsilon^{-\frac{3}{2p-3}} \|\nabla \widetilde{u}_{NV}\|_p^{\frac{2p}{2p-3}} \|\widetilde{w}\|^2 + M^{\frac{3}{2p}} \|\nabla \widetilde{u}_{NV}\|_p \|w\|^2,
\end{aligned}$$

where we have used (A.7), Gagliado-Nirenberg and Young's inequality. For the low-frequency contribution, we have

$$(w \cdot \nabla \overline{u}_{NV}, w) \leq M^{\frac{5}{2}} \|u_{NV}\| \|w\|^2.$$

Gathering the above inequalities leads us to

$$\begin{aligned}
&\partial_t \|w\|^2 + \nu \|\nabla w\|^2 + \epsilon (\|\nabla \widetilde{w}\|^2 + \gamma \|\nabla \widetilde{w}\|_p^p) \\
&\quad \leq c\epsilon (\|\nabla \widetilde{u}_{NS}\|^2 + \|\nabla \widetilde{u}_{NS}\|_p^p) \\
&\qquad + C(\epsilon^{-\frac{3}{2p-3}} \|\nabla \widetilde{u}_{NV}\|_p^{\frac{2p}{2p-3}} + M^{\frac{5}{2}} \|u_{NV}\| + M^{\frac{3}{2p}} \|\nabla \widetilde{u}_{NV}\|_p) \|w\|^2.
\end{aligned}$$

Letting $f(t) = C(\epsilon^{-\frac{3}{2p-3}} \|\nabla\widetilde{u}_{NV}\|_p^{\frac{2p}{2p-3}} + M^{\frac{5}{2}} \|u_{NV}\| + M^{\frac{3}{2p}} \|\nabla\widetilde{u}_{NV}\|_p)$, we have

$$\begin{aligned}&\partial_t(e^{-f(t)}\|w\|^2) + e^{-f(t)}(\nu\|\nabla w\|^2 + \epsilon(\|\nabla\widetilde{w}\|^2 + \gamma\|\nabla\widetilde{w}\|_p^p))\\ &\quad\lesssim e^{-f(t)}\epsilon(\|\nabla\widetilde{u}_{NS}\|^2 + \|\nabla\widetilde{u}_{NS}\|_p^p).\end{aligned}$$

We also have

$$\begin{aligned}&\int_0^t \left(\epsilon^{-\frac{3}{2p-3}} \|\nabla\widetilde{u}_{NV}\|_p^{\frac{2p}{2p-3}} + M^{\frac{5}{2}} \|u_{NV}\| + M^{\frac{3}{2p}} \|\nabla\widetilde{u}_{NV}\|_p\right) ds\\ &\quad\le \epsilon^{-\frac{5}{2p-3}} C_{0,f}^{\frac{4}{2p-3}} T^{\frac{2p-5}{2p-3}} + M^{\frac{5}{2}} T C_{0,f} + M^{\frac{3}{2p}} T \epsilon^{-\frac{1}{p}} C_{0,f}^{\frac{1}{p}} T,\end{aligned}$$

where $C_{0,f}$ is a constant depending on the initial data and the external forcing term f only. It can be seen that by choosing

$$M(\epsilon) = \epsilon^{-\frac{2}{2p-3}}, \tag{22}$$

the terms can be balanced to have the same order dependence on ϵ^{-1}. Therefore, the above expression is bounded by $c(f, u_0)\max\{t, t^{\frac{2p-5}{2p-3}}\}$, where $c(f, u_0)$ is a constant that depends on the initial condition and the forcing function. Thus,

$$\begin{aligned}&\|(u_{NS} - u_{NV})(t)\|^2\\ &\quad\le e^{\epsilon^{-\frac{5}{2p-3}} c(f,u_0) \max\left\{t, t^{\frac{2p-5}{2p-3}}\right\}} \int_0^t \epsilon(\|\nabla\widetilde{u}_{NS}\|^2 + \|\nabla\widetilde{u}_{NS}\|_p^p)\, ds.\end{aligned} \tag{23}$$

We can summarize the above result in the following theorem.

Theorem 4.1 *Let u_{NS} denote a strong solution to the three-dimensional NSE such that $u_{\mathrm{NS}} \in L^p(I; W^{1,p})$ and let u_{NV} denote a solution to $NV(\epsilon, p, \epsilon^{-\frac{2}{2p-3}})$ such that their initial conditions agree and the forcing function $f \in L^1(I : L^2)$. Then, for $t \le T$, the estimate (23) holds.*

Note that the error estimate depends on the L^p norm in time of the solution. Therefore, the error rate for the nonlinear viscosity is sensitive to singularities in the solution of the *NSE* that may develop in time. This is intuitive since the nonlinear viscosity should, in principle, regularize such singularities and hence keep the nonlinear viscosity turbulence model away from the singular *NSE* solution. We also note that larger p implies smaller power dependence on ϵ^{-1}. Thus, for smooth *NSE* solutions, the stability of larger p implies that the *NV* solution stays close to the *NSE* solution.

Another notable byproduct of the estimate (23) is that the optimization of the estimate gives us the best value for M. The result is that larger p means we can choose a smaller cut-off frequency. This is consistent with the fact that when the solution is smooth, we can apply stabilization to more modes.

4.1.2 Uniform in Time Estimate for $p = \frac{5}{2}$

For the special case of $p = \frac{5}{2}$, we can obtain a uniform in time error estimate. The price paid is that the estimate is no longer uniform in ν. Again, let u_{NS} and u_{NV} denote a solution to

NSE and to *NV*, respectively, and let $w = u_{NS} - u_{NV}$. Then, in (21), by setting $p = \frac{5}{2}$, we can bound the nonlinear convective term as

$$|(w \cdot \nabla \widetilde{u}_{NV}, w)| \le \|\nabla \widetilde{u}_{NV}\|_{\frac{5}{2}} \|w\|^2_{\frac{10}{3}} \le \|\nabla \widetilde{u}_{NV}\|_3 \|w\|^{\frac{4}{5}} \|\nabla w\|^{\frac{6}{5}}$$
$$\le \nu^{\frac{-3}{2}} \|\nabla \widetilde{u}_{NV}\|^{\frac{5}{2}}_{\frac{5}{2}} \|w\|^2 + \frac{\nu}{12} \|\nabla w\|^2.$$

Theorem A.7 yields

$$|(w \cdot \nabla \overline{u_{NV}}, w)| \le \|\nabla \overline{u_{NV}}\|_3 \|w\|_3^2 \le \|\nabla \overline{u_{NV}}\|_3 \|w\| \|\nabla w\|$$
$$\le \nu^{-1} M \|\nabla u_{NV}\|_2^2 \|w\|^2 + \frac{\nu}{12} \|\nabla w\|^2.$$

It is easy to see that the energy estimate gives

$$\int_0^t \left(\nu^{-1} M \|\nabla u_{NV}\|_2^2 + \nu^{-\frac{3}{2}} \|\nabla \widetilde{u}_{NV}\|^{\frac{5}{2}}_{\frac{5}{2}} \right) ds$$
$$\le \nu^{-2} M C_{0,f}^2 + \nu^{-\frac{3}{2}} \epsilon^{-1} C_{0,f}^2 \tag{24}$$

so that

$$\|w(t)\|^2 \le e^{(\nu^{-2} M C_{0,f}^2 + \nu^{-\frac{3}{2}} \epsilon^{-1} C_{0,f}^2)} \|w(0)\|^2. \tag{25}$$

Theorem 4.2 *Let $p = \frac{5}{2}$ and let $u_{NS} \in L^p(I; W^{1,p})$ denote a solution of the NSE and u_{NV} a solution of NV such that their initial conditions agree. Let the forcing function $f \in L^1(I; L^2)$. Then, uniform in time estimate (25) holds.*

The above theorem states that as long as the solution to the *NSE* is smooth and its total fluctuation is bounded in a certain appropriate time-space norm, then the error between the *NSE* solution and *NV* solution is bounded in time. Thus, they will stay within a “tube” of constant radius about the origin.

4.2 The Hyperviscosity Case

We now bound the norm of the difference between solutions of the three-dimensional *NSE* and of $HV(\epsilon, \alpha, M)$ in terms of the difference between the initial data and the regularity of the solution to the *NSE*. Let u_{NS} and u_{HV} denote a strong solution of *NSE* and a solution of $HV(\epsilon, \alpha, M)$, respectively, such that their initial conditions agree, i.e., $u_{NS}(0, x) = u_{HV}(0, x)$. The solution u_{NS} satisfies

$$\partial_t u_{NS} - \nu \Delta u_{NS} + P(u_{NS} \cdot \nabla u_{NS}) = f$$

whereas u_{HV} satisfies

$$\partial_t u_{HV} - \nu \Delta u_{HV} + P(u_{HV} \cdot \nabla u_{HV}) - \epsilon(-\Delta)^\alpha \widetilde{u}_{HV} = f.$$

Then, $w = u_{NS} - u_{HV}$ satisfies

$$\partial_t w - \nu \Delta w - \epsilon(-\Delta)^\alpha \widetilde{w} + P(u_{NS} \cdot \nabla u_{NS} - u_{HV} \cdot \nabla u_{HV}) = -\epsilon(-\Delta)^\alpha \widetilde{u_{NS}}.$$

Testing against w, we obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t \|w\|^2 + \nu \|\nabla w\|^2 + \epsilon \||\nabla|^\alpha \widetilde{w}\|^2 \\
&\quad \le \epsilon \||\nabla|^\alpha \widetilde{u_{NS}}\| \||\nabla|^\alpha \widetilde{w}\| - (w \cdot \nabla u_{NS}, w) \\
&\quad \le c\,\epsilon \||\nabla|^\alpha \widetilde{u_{NS}}\|^2 + \frac{\epsilon}{12}\||\nabla|^\alpha \widetilde{w}\|^2 + |(w \cdot \nabla u_{NS}, w)|.
\end{aligned}$$

The Gagliardo-Nirenberg inequality yields

$$\|w\|_4 \le \|w\|^\theta \||\nabla|^\alpha w\|^{1-\theta}$$

for $\theta = \frac{4\alpha-3}{4\alpha}$. Then, using (A.7) and (A.8), we obtain

$$\begin{aligned}
(w \cdot \nabla u_{NS}, w) &\le \|\nabla u_{NS}\| \||\nabla|^\alpha w\|^{2(1-\theta)} \|w\|^{2\theta} \\
&\le \frac{\epsilon}{12}\||\nabla|^\alpha w\|^2 + c\epsilon^{-\frac{1-\theta}{\theta}} \|\nabla u_{NS}\|^{\frac{1}{\theta}} \|w\|^2 \\
&\le \frac{\epsilon}{12}\||\nabla|^\alpha \widetilde{w}\|^2 + (\epsilon M^{2\alpha} + \epsilon^{-\frac{3}{4\alpha-3}} \|\nabla u_{NS}\|^{\frac{4\alpha}{4\alpha-3}}) \|w\|^2.
\end{aligned}$$

We can also bound this nonlinear convective term using the physical viscosity term as for regularizing in the same was as in the previous section for the nonlinear viscosity case. Consequently, we obtain

$$\begin{aligned}
&\partial_t \|w\|^2 + \nu \|\nabla w\|^2 + \epsilon \||\nabla|^\alpha \widetilde{w}\|^2 \\
&\quad \le C_1 \epsilon \||\nabla|^\alpha \widetilde{u_{NS}}\|^2 + C_2(\epsilon M^{2\alpha} + \epsilon^{-\frac{3}{4\alpha-3}} \|\nabla u_{NS}\|^{\frac{4\alpha}{4\alpha-3}}) \|w\|^2.
\end{aligned}$$

We can again balance the terms so that they can be bounded by a common power of ϵ^{-1}. To this end, we choose $M \sim \epsilon^{\frac{-2}{4\alpha-3}}$. Solving the above equation results in

$$\|(u_{NS} - u_{HV})(t)\|^2 \le e^{\epsilon^{-\frac{3}{4\alpha-3}} \int_0^t (1+\|\nabla u_{NS}\|^{\frac{4\alpha}{4\alpha-3}})\, ds} \int_0^t \epsilon \||\nabla|^\alpha \widetilde{u_{NS}}\|^2. \tag{26}$$

In summary, we have the following theorem.

Theorem 4.3 *Let u_{NS} denote a strong solution to the three-dimensional NSE such that $u_{NS} \in L^2((0,T); H^\alpha) \cap L^{\frac{4\alpha}{4\alpha-3}}((0,T); H^1)$ and u_{HV} denote a solution to $HV(\epsilon, \alpha, \epsilon^{\frac{-2}{4\alpha-3}})$ such that their initial conditions agree. Then, for $t \le T$, the estimate (26) holds.*

Note that we have reduced the exponential dependence on ϵ^{-1} which is significant for small ϵ. Also, the balancing of term allowed us to choose the cut-off $M \sim \epsilon^{\frac{-2}{4\alpha-3}}$ which is smaller than that for the nonlinear viscosity method.

5 Contraction in Phase Space

It is known that for the two-dimensional *NSE* there is a frequency scale beyond which molecular dissipation becomes dominant and the exponential contraction of the phase space results; no such result is known to hold for the *NSE* in three dimensions. In this section, for special values of p and α, we show that the *NV* and *HV* models in three dimensions also possess such a characteristic frequency scale if the cut-off frequencies exceed such a scale. In this section, we will estimate the dimension of this finite-dimensional attractor for *NV* and *HV* models in three dimensions. We assume that the forcing is bounded, i.e., $f \in L^\infty(I, L^2)$.

5.1 Contraction for the *NV* model at $p = \frac{5}{2}$

We obtain the phase space contraction estimate for the *NV* model for the case $p = \frac{5}{2}$. This is essentially done by considering two distinct solutions of the *NV* problem that differ only in their high-frequency parts. Because dissipation acts much more strongly on high-frequency modes, we can show that such solutions converge to each other exponentially fast in time. This implies that the dynamics of the solutions to *NV* is mainly concentrated in the low modes. Note that lowering the dimension of this *exponential attractor* without appreciably affecting the large-scale dynamics is exactly the goal of turbulence modeling.

For the *NV* model, the dimension of the exponential attractor can be obtained for $p = \frac{5}{2}$ for the same reason that we were able to obtain a uniform in time bounds in Sect. 4.1.2. We show that the dimension of the attractor is bounded by $\epsilon^{-\frac{3}{2}}\nu^{-\frac{1}{2}}$.

We denote the low and high frequency parts of the solution u of the *NV* model by $P_M u = \overline{u}$ and $Q_M u = \widetilde{u}$, respectively. Then,

$$\partial_t \widetilde{u} - (\nu + \epsilon)\Delta \widetilde{u} + Q_M P(u \cdot \nabla u) - \epsilon Q_M P \nabla \cdot (|\nabla \widetilde{u}|^{p-1} \nabla \widetilde{u}) = 0. \tag{27}$$

Assume there exists a function v that solves (27) with $v(0) = \widetilde{u}(0)$ and forced by the low-frequency part $\overline{u}$, i.e.,

$$\begin{aligned} &\partial_t v - (\nu + \epsilon)\Delta v + Q_M P(v \cdot \nabla v + \overline{u} \cdot \nabla v + v \cdot \nabla \overline{u} + \overline{u} \cdot \nabla \overline{u}) \\ &\quad - \epsilon Q_M P \nabla \cdot (|\nabla v|^{p-1} \nabla v) = 0. \end{aligned} \tag{28}$$

Note that if we apply P_M to this equation, then we see that the low-frequency part $\overline{v}$ of v satisfies

$$\partial_t \overline{v} - (\nu + \epsilon)\Delta \overline{v} = 0.$$

Then, since $\overline{v}(0, \cdot) = 0$ we have that $\overline{v}(t, \cdot) = 0$ for all $t \geq 0$. Subtracting (28) from (27), we obtain, for $w = \widetilde{u} - v$,

$$\begin{aligned} &\partial_t w - (\nu + \epsilon)\Delta w + Q_M P(\widetilde{u} \cdot \nabla \widetilde{u} - v \cdot \nabla v + \overline{u} \cdot \nabla w + w \cdot \nabla \overline{u}) \\ &\quad - \epsilon Q_M P \nabla \cdot (|\nabla \widetilde{u}|^{p-1} \nabla \widetilde{u} - |\nabla v|^{p-1} \nabla v) = 0. \end{aligned}$$

Testing this equation with w results in

$$\frac{1}{2}\partial_t \|w\|^2 + (\nu + \epsilon)\|\nabla w\|^2 + \gamma \epsilon \|\nabla w\|_p^p + (w \cdot \nabla \widetilde{u}, w) + (w \cdot \nabla \overline{u}, w) \leq 0.$$

Due to the fact that $w = \widetilde{u} - v = \widetilde{u} - \widetilde{v}$ lives in the high-frequency space, we have $\|\nabla w\| \geq M\|w\|$ which is used to obtain

$$(w \cdot \nabla \widetilde{u}, w) \leq \|\nabla \widetilde{u}\|_{\frac{5}{2}} \|w\|_{\frac{10}{3}}^2 \lesssim \|\nabla \widetilde{u}\|_{\frac{5}{2}} \|w\|^{\frac{4}{5}} \|\nabla w\|^{\frac{6}{5}}$$

$$\lesssim \frac{\|w\|^2 \|\nabla\widetilde{u}\|_{\frac{5}{2}}^{\frac{5}{2}}}{(\epsilon+\nu)^{\frac{3}{2}}} + \frac{\nu+\epsilon}{12}\|\nabla w\|^2.$$

Then,

$$\frac{1}{2}\partial_t\|w\|^2 + (\nu+\epsilon)\|\nabla w\|^2 \le \frac{\epsilon+\nu}{2}\|\nabla w\|^2 + \left(\frac{\|\nabla\widetilde{u}\|_{\frac{5}{2}}^{\frac{5}{2}}}{(\epsilon+\nu)^{\frac{3}{2}}} + \frac{\|\nabla\overline{u}\|_3^2}{\epsilon+\nu}\right)\|w\|^2$$

and therefore,

$$\partial_t\|w\|^2 \le -\left((\epsilon+\nu)M^2 - 2(\epsilon+\nu)^{-\frac{3}{2}}\|\nabla\widetilde{u}\|_{\frac{5}{2}}^{\frac{5}{2}} - 2(\epsilon+\nu)^{-1}M\|\nabla u\|^2\right)\|w\|^2.$$

Now, let

$$g(t) = ((\epsilon+\nu)M^2 - \frac{1}{t}\int_0^t \left(2(\epsilon+\nu)^{-\frac{3}{2}}\|\nabla\widetilde{u}\|_{\frac{5}{2}}^{\frac{5}{2}} + 2M(\epsilon+\nu)^{-1}\|\nabla u\|^2\right)ds.$$

Using (24), we know that

$$t^{-1}\int_0^t \|\nabla\widetilde{u}\|_{\frac{5}{2}}^{\frac{5}{2}} \le \epsilon^{-1}C_{f,\nu,0}^2,$$

where $C_{f,\nu,0}^2 = \|u_0\|^2 + \frac{1}{\nu}\|f\|_{L^\infty([0,\infty),L^2)}^2$. Therefore,

$$g(t) \ge (\nu+\epsilon)M^2 - \left(2(\epsilon+\nu)^{-\frac{3}{2}}\epsilon^{-1}C_{f,\nu,0}^2 + 2M(\epsilon+\nu)^{-1}\nu^{-1}C_{f,\nu,0}^2\right).$$

Using the fact that $C_{f,\nu,0} \sim \nu^{-\frac{1}{2}}$, we obtain

$$(\nu+\epsilon)M^2 - 2(\nu+\epsilon)^{-\frac{3}{2}}(\epsilon\nu)^{-1} - 2M(\epsilon+\nu)^{-1}\nu^{-2} \lesssim g(t).$$

Thus, we see that for large enough M, there exists a $\delta > 0$ such that $g(t) \ge \delta$. Then, we have

$$\partial_t(e^{g(t)t}\|w\|^2) \le 0.$$

We can summarize the above discussion in the following theorem.

Theorem 5.1 *Let u solve $NV(\epsilon, \frac{5}{2}, M)$ and v solve the high-frequency equation (28) for $NV(\epsilon, p, M)$ with low-frequency forcing by $\overline{u}$. Then, for large enough M, there exists a $\delta > 0$ so that we have the contraction estimate*

$$\|\widetilde{u}(t) - v(t)\|^2(t) \le e^{-t\delta}\|\widetilde{u}_0 - v_0\|^2.$$

5.2 Attractor for HV with $\alpha = \frac{3}{2}$

We now consider the hyperviscosity model for $\alpha = \frac{3}{2}$ and show that, in this case, we can obtain a contraction estimate even in three dimension and therefore we can estimate the dimension of the attractor. The high-frequency part of the solution satisfies

$$\partial_t\widetilde{u} - \nu\Delta\widetilde{u} - \epsilon(-\Delta)^\alpha\widetilde{u} + Q_M P(u\cdot\nabla u) = 0. \tag{29}$$

Assume there exists a function v that solves this equation with $v(0) = \widetilde{u}(0)$ and forced by the low-frequency part $\overline{u}$, i.e.,

$$\partial_t v - \nu\Delta v - \epsilon(-\Delta)^\alpha v + Q_M P(v\cdot\nabla v + \overline{u}\cdot\nabla v + v\cdot\nabla\overline{u} + \overline{u}\cdot\nabla\overline{u}) = 0. \tag{30}$$

Note that if we apply P_M to this equation, we then see that the low-frequency part $\overline{v}$ of v satisfies

$$\partial_t\overline{v} - \nu\Delta\overline{v} - \epsilon(-\Delta)^\alpha\overline{v} = 0.$$

Then, since $\overline{v}(0,\cdot) = 0$ we have that $\overline{v}(t,\cdot) = 0$ for all $t \geq 0$. Subtracting (30) from (29), we obtain, for $w = \widetilde{u} - v$,

$$\partial_t w - \nu\Delta w - \epsilon(-\Delta)^\alpha w + Q_M P(\widetilde{u}\cdot\nabla\widetilde{u} - v\cdot\nabla v + \overline{u}\cdot\nabla w + w\cdot\nabla\overline{u}) = 0.$$

Testing this equation with w results in

$$\frac{1}{2}\partial_t\|\widetilde{w}(t)\|^2 + \epsilon\||\nabla|^\alpha\widetilde{w}\|^2 + (w\cdot\nabla u, w) = 0.$$

Note that

$$\|\nabla w\|_{\frac{6}{5-2\alpha}} \leq \||\nabla|^\alpha w\|_2 \quad\text{and}\quad \|w\|_{\frac{3}{\alpha}} \lesssim \|w\|_2^\theta\||\nabla|^\alpha w\|^{1-\theta}, \tag{31}$$

where $\theta = \frac{4\alpha-3}{2\alpha}$ with $\alpha \leq \frac{3}{2}$. By the Hölder inequality and (31), we obtain

$$\begin{aligned}(w\cdot\nabla u, w) &= -(w\cdot\nabla w, u) \lesssim \|u\|_6\|w\cdot\nabla w\|_{\frac{6}{5}} \lesssim \|u\|_6\|w\|_{\frac{3}{\alpha}}\||\nabla|^\alpha w\|_2\\ &\leq \|u\|_6\|w\|^\theta\||\nabla|^\alpha w\|^{2-\theta} \leq \epsilon^{-\frac{2-\theta}{\theta}}\|u\|_6^{\frac{2}{\theta}}\|w\|^2 + \epsilon\||\nabla|^\alpha w\|^2,\end{aligned}$$

where $\frac{\alpha}{3} + \frac{5-2\alpha}{6} = \frac{5}{6}$. Note that due to the Poincare (forward) inequality and the fact that $w \in Ran(Q_M)$, we have $M^\alpha\|w\| \lesssim \||\nabla|^\alpha w\|$ which yields

$$\partial_t\|w(t)\|^2 \lesssim -(\epsilon M^{2\alpha} - 2\|u\|_6^{\frac{4\alpha}{4\alpha-3}}\epsilon^{-\frac{3}{4\alpha-3}})\|w\|^2.$$

Then, to respect the energy dissipation we must set $\alpha = \frac{3}{2}$ so that

$$\left(t^{-1}\int_0^t\|u\|_6^{\frac{4\alpha}{4\alpha-3}}\right)^{\frac{4\alpha-3}{4\alpha}} \lesssim \left(t^{-1}\int_0^t\|\nabla u\|_2^2\right)^{1/2}.$$

Thus, we have arrived at the following result.

Theorem 5.2 *Let u solve $HV(\epsilon, \frac{3}{2}, M)$ and v solve the high-frequency equation* (28) *for $HV(\epsilon, \frac{3}{2}, M)$ with low-frequency forcing by $\overline{u}$. Then, for large enough M, there exists a $\delta > 0$ so that we have the contraction estimate*

$$\|\widetilde{u}(t) - v(t)\|^2(t) \leq e^{-t\delta}\|\widetilde{u}_0 - v_0\|^2.$$

Again, note the small dependence of M on the power of ν. We can see from this estimate that the dimension of the exponential attractor is rather manageable. For more information about attractors and its finite dimensionality in the setting of hyperviscosity models, see [26].

6 Stability and Convergence for Semi-Discrete and Fully-Discrete Approximations for $p = 3$

In this section, we consider a semi-implicit Euler time-stepping scheme to solve the nonlinear spectral viscosity method for $p = 3$.[8] The method is semi-implicit in the sense that all terms are treated in an implicit manner, including the eddy viscosity term, except that nonlinear inertial term is partially lagged.

Assume that $f \in L^2((0,T):L^2) \cap L^1((0,T):L^2)$. Let δt denote the time step, K the number of time steps, and, for $i = 1,\dots,K$, $t_i = i\delta t$ and $f^{(i)} = f(t_i,\cdot)$. Given $u_0 \in W^{1,3}$, seek $u_{\delta t}^{(i)} : \{i = 1,\dots,K\} \times \mathbb{T}^3 \to \mathbb{R}$ such that $\nabla \cdot u_{\delta t}^{(i)} = 0$ and, for $i = 1,\dots,K$,

$$\int \Big(\big(u_{\delta t}^{(i)} - u_{\delta t}^{(i-1)}\big)\cdot\phi + \delta t\big(\nu\nabla u_{\delta t}^{(i)} : \nabla\phi + u_{\delta t}^{(i-1)}\cdot\nabla u_{\delta t}^{(i)}\cdot\phi\big)\Big)dx \\ + \delta t \int \epsilon(1 + |\nabla\widetilde{u}_{\delta t}^{(i)}|)\nabla\widetilde{u}_{\delta t}^{(i)} : \nabla\widetilde{\phi}\,dx - \delta t\int f^{(i)}\cdot\phi\,dx = 0. \tag{32}$$

We refer to this system as $NV_{\delta t}(\epsilon, 3, M)$. Note that the convection velocity is lagged in the nonlinear term so that we are essentially solving a sequence of Oseen-type equations with nonlinear dissipation. For the case $M = 0$, this semi-implicit Euler scheme is analyzed in [6]. However, their emphasis was on the more challenging case of smaller values of p.

We also discretize in space to obtain a fully-discrete system. To this end, we use the Galerkin projection of $u_{\delta t}^{(i)}$ much as was done in the previous sections. Let $f_N^{(i)} = P_N f(t_i)$. We solve for $u_{\delta t,N}^{(i)} \in V_N$ such that for each $\phi \in V_N$ and for $i = 1,\dots,K$,

$$\int u_{\delta t,N}^{(i)}\cdot\phi\,dx - \int u_{\delta t,N}^{(i-1)}\cdot\phi\,dx + \delta t\int \nu\nabla u_{\delta t,N}^{(i)} : \nabla\phi dx \\ + \delta t\int \big(u_{\delta t,N}^{(i-1)}\cdot\nabla u_{\delta t,N}^{(i)}\cdot\phi + \epsilon(1 + |\nabla\widetilde{u}_{\delta t,N}^{(i)}|)\nabla\widetilde{u}_{\delta t,N}^{(i)} : \nabla\widetilde{\phi}\big)dx \\ - \delta t\int f_N^{(i)}\cdot\phi\,dx = 0. \tag{33}$$

We refer to (33) as $NV_{\delta t,N}(\epsilon, 3, M)$.

Our goal is to prove stability and convergence results for the semi-discrete and fully-discrete systems (32) and (33). The case $p = 3$ turns out to be especially nice because, for the equation that is satisfied by $\nabla u_{\delta t}^{(i)}$, the L^3 norm of the gradient has exactly the same dimension as the nonlinear form. We will use this property to derive estimates that allows us to show convergence, as $\delta t \to 0$, to the solution of $NV(\epsilon, 3, M)$ and show the rate at which the convergence takes place.

6.1 Existence of Fully-Discrete Approximations

We first show that the finite-dimensional fully-discrete system (33) is solvable.

For each $i = 1,\dots,K$, we can define the mapping $\mathbf{f}^{(i)} : \mathbb{R}^N \to \mathbb{R}^N$ by setting its components $\mathbf{f}^{(i)}(\widehat{u}_{\delta t,N}^{(i)})_\ell$, $\ell = 1,\dots,N$, to be the left-hand-side of (33) for each basis function

[8] Recall that $p = 3$ is the Smagorinsky case.

$\phi_\ell \in V_N$. Then, (33) can be expressed as an algebraic problem, i.e., find $u^{(i)}_{\delta t,N}$ such that $\mathbf{f}^{(i)}(\widehat{u}^{(i)}_{\delta t,N}) = 0$. Setting $\phi = u^{(i)}_{\delta t,N}$ in (33), we obtain

$$\begin{aligned}\mathbf{f}_i(\widehat{u}^{(i)}_{\delta t,N})\cdot\widehat{u}^{(i)}_{\delta t,N} &\geq \frac{1}{2}\|u^{(i)}_{\delta t,N}\|^2 + \delta t\nu\|\nabla u^{(i)}_{\delta t,N}\|^2 + \delta t\epsilon(\|\nabla\widetilde{u}^{(i)}_{\delta t,N}\|^2 \\ &\quad + \|\nabla\widetilde{u}^{(i)}_{\delta t,N}\|_3^3) - \frac{1}{2}\|u^{(i-1)}_{\delta t,N}\|^2 - \delta t\|f^{(i)}_N\|\|u^{(i)}_{\delta t,N}\| \\ &\geq \frac{1}{2}(\|u^{(i)}_{\delta t,N}\|^2 - \|u^{(i-1)}_{\delta t,N}\|^2) - \delta t\|f^{(i)}_N\|\|u^{(i)}_{\delta t,N}\|.\end{aligned}$$

Recall the following topological fixed-point theorem (or *hairy-ball* theorem) taken from [7] and which is a consequence of the Brouwer fixed point theorem.

Lemma 6.1 *Assume the continuous function* $\mathbf{g}:\mathbb{R}^N \to \mathbb{R}^N$ *satisfies* $\mathbf{g}(v)\cdot v \geq 0$ *if* $|v| = r$ *for some* $r > 0$. *Then, there exists a point* $v \in B(0,r)$ *such that* $\mathbf{g}(v) = 0$.

Proof Using this lemma with $r = \|u^{(i-1)}_{\delta t,N}\|^2 + 2\delta t\|f^{(i)}_N\|\|u^{(i)}_{\delta t,N}\|$ together with the assumption that δt be sufficiently small, we can prove, for each $i = 1,\ldots,K$, the existence of $u^{(i)}_{\delta t,N}$ that solves (33) and for which

$$\begin{aligned}&\|u^{(i)}_{\delta t,N}\|^2 + 2\delta t\big(\nu\|\nabla u^{(i)}_{\delta t,N}\|^2 + \epsilon(\|\nabla\widetilde{u}^{(i)}_{\delta t,N}\|^2 + \|\nabla\widetilde{u}^{(i)}_{\delta t,N}\|_3^3)\big) \\ &\quad\leq \|u^{(i-1)}_{\delta t,N}\|^2 + 2\delta t\|f_N\|\|u^{(i)}_{\delta t,N}\|.\end{aligned} \tag{34}$$

□

6.2 A Priori Energy Estimate

Having shown the existence of solutions for the finite-dimensional fully-discrete system (33), we are now interested in seeing what happens as $N \to \infty$. To this end, we need to show the existence of a solution of (33) that satisfies a discrete energy dissipation estimate.

Theorem 6.2 *Let* $C_{0,f} = (\|u^{(0)}_N\|^2 + \delta t\frac{C_P^2}{\nu}\sum_{i=1}^{j}\|f^{(i)}_N\|)^{\frac{1}{2}}$. *Then, there exists a solution* $u^{(i)}_{\delta t,N}$ *of* (33) *that satisfies*

$$\|u^{(j)}_{\delta t,N}\|^2 + \delta t\sum_{i=1}^{j}\Big(\nu\|\nabla u^{(i)}_{\delta t,N}\|^2 + 2\epsilon(\|\nabla\widetilde{u}^{(i)}_{\delta t,N}\|^2 + \|\nabla\widetilde{u}^{(i)}_{\delta t,N}\|_3^3)\Big) \leq C_{0,f}^2. \tag{35}$$

Proof We apply Cauchy-Schwarz, Poincaré inequality, and sum (34) over $i = 1,\ldots,j$ to obtain

$$\begin{aligned}&\|u^{(j)}_{\delta t,N}\|^2 + \sum_{i=1}^{j}\delta t\Big(\nu\|\nabla u^{(i)}_{\delta t,N}\|^2 + 2\epsilon(\|\nabla\widetilde{u}^{(i)}_{\delta t,N}\|^2 + \|\nabla\widetilde{u}^{(i)}_{\delta t,N}\|_3^3)\Big) \\ &\quad\leq \|u^{(0)}_N\|^2 + \sum_{i=1}^{j}\delta t\frac{C_P^2}{\nu}\|f^{(i)}_N\|^2.\end{aligned}$$

□

6.3 Regularity

In order to show existence of solutions of the semi-discrete system (32) by taking limits as $N \to \infty$ of solutions of the fully-discrete system (33), we use a compactness method. To be able to do this, we must show that solutions of (33) possess additional regularity.

6.3.1 Space Regularity

Theorem 6.3 *Let $C_{sp} = (\epsilon^{-1} + C_P^{-2})C_{0,f}^2 + C_{0,f}^3 M^{\frac{5}{2}}\nu^{-1}$. Then, solutions of (33) satisfy*

$$\begin{aligned}&\|\nabla u_{\delta t,N}^{(i)}\|^2 + \sum \delta t(2\epsilon(\|\Delta\widetilde{u}_{\delta t,N}^{(i)}\|^2 + \|\widetilde{u}_{\delta t,N}^{(i)}\|_9^3) + \nu\|\Delta u_{\delta t,N}^{(i)}\|^2)\\ &\quad\lesssim \|\nabla u_N^{(0)}\|^2 + C_{sp}.\end{aligned}$$

Proof In (33), set $\phi = -\Delta u_{\delta t,N}^{(i)}$ and use

$$\begin{aligned}&\int u_{\delta t,N}^{(i-1)} \cdot \nabla u_{\delta t,N}^{(i)} \cdot (-\Delta u_{\delta t,N}^{(i)})\\ &\quad= \sum_{l,k,j}\int \partial_l u_{\delta t,N;k}^{(i-1)} \partial_k u_{\delta t,N;j}^{(i)} \partial_l u_{\delta t,N;j}^{(i)} \le \frac{2}{3}\|\nabla u_{\delta t,N}^{(i)}\|_3^3 + \frac{1}{3}\|u_{\delta t,N}^{(i-1)}\|_3^3,\end{aligned}$$

where the last subscript in, e.g., $u_{\delta t,N;\cdot}^{(1)}$, denotes the spatial components of the velocity field, to obtain

$$\begin{aligned}&\frac{1}{2}\|\nabla u_{\delta t,N}^{(i)}\|_2^2 + \delta t(\nu\|\Delta u_{\delta t,N}^{(i)}\|^2 + \epsilon(\|\Delta\widetilde{u}_{\delta t,N}^{(i)}\|^2 + I_3(\widetilde{u}_{\delta t,N}^{(i)})))\\ &\quad\le \frac{1}{2}\|\nabla u_{\delta t,N}^{(i-1)}\|^2 + \delta t(\|\nabla u_{\delta t,N}^{(i)}\|_3^3 + \|\nabla u_{\delta t,N}^{(i-1)}\|_3^3) + \delta t\|f_N^{(i)}\|\|\Delta u_{\delta t,N}^{(i)}\|.\end{aligned}$$

Now we have

$$\|\nabla\overline{u}\|_3^3 \lesssim M^{\frac{3}{2}}\|\nabla\overline{u}\|_2^3 \lesssim M^{\frac{5}{2}}\|\nabla\overline{u}\|^2\|\overline{u}\|.$$

Therefore, we have

$$\begin{aligned}&\|\nabla u_{\delta t,N}^{(i)}\|_2^2 + \delta t\Big(\nu\|\Delta u_{\delta t,N}^{(i)}\|^2 + 2\epsilon\big(\|\Delta\widetilde{u}_{\delta t,N}^{(i)}\|^2 + I_3(\widetilde{u}_{\delta t,N}^{(i)})\big)\Big)\\ &\quad\lesssim \|\nabla u_{\delta t,N}^{(i-1)}\|^2 + 2\delta t\left(\frac{2}{3}\|\nabla\widetilde{u}_{\delta t,N}^{(i)}\|_3^3 + \frac{1}{3}\|\nabla\widetilde{u}_{\delta t,N}^{(i-1)}\|_3^3\right) + \delta t M^{\frac{5}{2}}\left(\frac{2}{3}\|\nabla u_{\delta t,N}^{(i)}\|^2\right.\\ &\qquad\left.+\frac{1}{3}\|\nabla u_{\delta t,N}^{(i-1)}\|^2\right)\max_j\|u_{\delta t,N}^{(j)}\| + \frac{1}{\nu}\delta t\|f_N^{(i)}\|^2.\end{aligned}$$

We now sum over i to obtain

$$\begin{aligned}&\|\nabla u_{\delta t,N}^{(i)}\|^2 + \delta t\sum\Big(\nu\|\Delta u_{\delta t,N}^{(i)}\|^2 + 2\epsilon\big(\|\Delta\widetilde{u}_{\delta t,N}^{(i)}\|^2 + I_3(\widetilde{u}_{\delta t,N}^{(i)})\big)\Big)\\ &\quad\lesssim \|\nabla u_N^{(0)}\|^2 + 2\delta t\sum\|\nabla\widetilde{u}_{\delta t,N}\|_3^3 + M^{\frac{5}{2}}\max_j\|u_{\delta t,N}(j)\|\sum\|\nabla u_{\delta t,N}\|^2\\ &\qquad+\frac{1}{\nu}\delta t\sum\|f_N^{(i)}\|^2.\end{aligned}$$

Consequently, using Theorem 6.2,

$$\|\nabla u^{(i)}_{\delta t,N}\|^2 + \delta t \sum \big(\nu\|\Delta u^{(i)}_{\delta t,N}\|^2 + 2\epsilon(\|\Delta\widetilde{u}^{(i)}_{\delta t,N}\|^2 + \|\widetilde{u}^{(i)}_{\delta t,N}\|_9^3)\big)$$
$$\lesssim \|\nabla u^{(0)}_N\|^2 + \epsilon^{-1}C^2_{0,f} + C^3_{0,f}M^{\frac{5}{2}}\nu^{-1} + C_P^{-2}C^2_{0,f}. \qquad \square$$

6.3.2 Regularity in Time

To use the compactness method, we also need $u_{\delta t,N}$ to have some regularity in time. The following lemma provides such a result.

Lemma 6.4 *Let* $C_{tm} = (2\epsilon)^{-1}(\|\nabla u_{\delta t,N}(0)\|^2 + C_{sp}) + T(\delta t)^{\frac{1}{2}}(2\nu)^{-1}C^5_{0,f} + M^{\frac{3}{2}}C^{\frac{5}{2}}_{0,f}\nu^{-1} + 2\nu C_P^{-2}C^2_{0,f}$. *Then,*

$$\sum_i^n \frac{\|u^{(i)}_{\delta t,N} - u^{(i-1)}_{\delta t,N}\|^2}{2\delta t} + \frac{\nu}{2}\|\nabla u_{\delta t,N}(n)\|^2 + \frac{\epsilon}{2}\|\nabla\widetilde{u}_{\delta t,N}(i)\|^2 + \frac{\epsilon}{3}\|\nabla\widetilde{u}_{\delta t,N}(i)\|_3^3$$
$$\lesssim \frac{\nu}{2}\|\nabla u_{\delta t,N}(0)\|^2 + \epsilon\left(\frac{1}{2}\|\nabla\widetilde{u}_{\delta t,N}(0)\|^2 + \frac{1}{3}\|\nabla\widetilde{u}_{\delta t,N}(0)\|_3^3\right) + C_{tm}.$$

Proof Set $\phi = (u^{(i)}_{\delta t,N} - u^{(i-1)}_{\delta t,N})/\delta t$ in (33). Note that

$$\int |\nabla u|\nabla u : \nabla v \le \frac{2}{3}\|\nabla u\|_3^3 + \frac{1}{3}\|\nabla v\|_3^3$$

and, by the Hölder, Gagliardo-Nirenberg, and Sobolev inequalities,

$$\int \nabla u|u|^2\,dx \le \|u\|^2_{\frac{9}{4}}\|\nabla u\|_9 \le \frac{2}{3}\|u\|^6_{\frac{9}{4}} + \frac{1}{3}\|\nabla u\|_9^3 \lesssim \|u\|^5\|\nabla u\| + \|\nabla u\|_9^3.$$

We also have, by Sobolev and Bernstein (see (A.7)) inequalities,

$$\int \nabla\overline{u}|u|^2\,dx \le \|u\|_2^2\|\nabla\overline{u}\|_\infty \le c_2\|u\|^2\|\nabla u\|M^{\frac{3}{2}}.$$

Therefore, we have

$$\frac{1}{2\delta t}\|u^{(i)}_{\delta t,N} - u^{(i-1)}_{\delta t,N}\|^2 + \frac{\nu}{2}\|\nabla u^{(i)}_{\delta t,N}\|^2 + \frac{\epsilon}{2}\|\nabla\widetilde{u}_{\delta t,N}(i)\|^2 + \frac{\epsilon}{3}\|\nabla\widetilde{u}_{\delta t,N}(i)\|_3^3$$
$$\lesssim \frac{\nu}{2}\|\nabla u^{(i-1)}_{\delta t,N}\|^2 + \frac{\epsilon}{2}\|\nabla\widetilde{u}^{(i-1)}_{\delta t,N}\|^2 + \frac{\epsilon}{3}\|\nabla\widetilde{u}^{(i-1)}_{\delta t,N}\|_3^3$$
$$+ \delta t\big(\|\nabla\widetilde{u}_{\delta t,N}(i)\|_9^3 + \|\nabla u^{(i-1)}_{\delta t,N}\|\|u^{(i-1)}_{\delta t,N}\|^5$$
$$+ M^{\frac{3}{2}}\|u^{(i-1)}_{\delta t,N}\|^2\|\nabla u^{(i)}_{\delta t,N}\| + \|f_i\|^2\big).$$

If we sum this successively, we obtain

$$\sum_i^n \frac{\|u^{(i)}_{\delta t,N} - u^{(i-1)}_{\delta t,N}\|^2}{2\delta t} + \frac{\nu}{2}\|\nabla u_{\delta t,N}(n)\|^2 + \frac{\epsilon}{2}\|\nabla\widetilde{u}_{\delta t,N}(i)\|^2 + \frac{\epsilon}{3}\|\nabla\widetilde{u}_{\delta t,N}(i)\|_3^3$$

$$\lesssim \frac{\nu}{2}\|\nabla u_{\delta t,N}(0)\|^2 + \frac{\epsilon}{2}\|\nabla \widetilde{u}_{\delta t,N}(0)\|^2 + \frac{\epsilon}{3}\|\nabla \widetilde{u}_{\delta t,N}(0)\|_3^3$$
$$+ \delta t \sum \|\nabla \widetilde{u}_{\delta t,N}\|_9^3 + \max_j \|u_{\delta t,N}(j)\|^5 \delta t \sum \|\nabla u^{(i)}_{\delta t,N}\|$$
$$+ M^{\frac{3}{2}} \delta t \max_j \|u_{\delta t,N}(j)\|^2 \sum \|\nabla u^{(i)}_{\delta t,N}\| + 2\delta t \sum \|f^{(i)}_N\|^2.$$

We are now left, on the right-hand side, terms that are bounded due to Theorem 6.2. Substituting the bounds completes the proof. □

6.4 Existence and Convergence of Semi-Discrete Approximations

Having shown the existence of solutions of (33), we now show that some subsequence of $u^{(i)}_{\delta t,N}$ converges to a solution of (32). We have shown that $u^{(i)}_{\delta t,N} \in W^{1,9} \cap H^2$ for each i. We construct a subsequence by successively reducing the sequence to a convergent sequence that converges at each of the time steps up to the current iteration. We start this process at $i = 1$. By compactness, there exists $u^{(1)}_{\delta t} \in W^{1,9} \cap H^2$ and a subsequence such that $u^{(1)}_{\delta t,N_k}$ converges to $u^{(1)}_{\delta t}$ strongly in $W^{1,3}$ and weakly in H^2. We now again take the subsequence of $u_{\delta t,N_k}$ to obtain $u^{(2)}_{\delta t}$ such that this subsequence converges to it. This process will continue for all $i = 1, \dots, n$. In this way, we can obtain a subsequence $u_{\delta t,N_l}$ and a function $u_{\delta t}\{i = 1, \dots, K\} \times \mathbb{T}^3 \to \mathbb{R}$ such that for each discrete time steps $i = 1, \dots, K$, $u^{(i)}_{\delta t,N_l}$ converges to $u^{(i)}_{\delta t}$ strongly in $W^{1,3}$ and weakly in H^2. It remains to show that each $u^{(i)}_{\delta t}$ satisfies (32). This is done in almost exactly the same manner as was done in Sect. 3. Thus we have the following result.

Theorem 6.5 *There exists a unique solution $u^{(i)}_{\delta t}$, $i = 1, \dots, K$, of the semi-discrete problem $NV_{\delta t}(\epsilon, 3, M)$.*

Can we take a sequence of problems $NV_{\delta t}$ as $\delta t \to 0$ and conclude that such a sequence converges to a function that satisfies NV? To do this, we need to clarify a few things. First, $u^{(i)}_{\delta t}$ is defined on a discrete grid and therefore not suitable when we discuss its convergence to a function that is defined continuously in time. We must interpolate this sequence between the discrete time steps to derive a function that is defined on space-time. Secondly, we should decide on the most convenient way in which $\delta t \to 0$. To this end, we consider successively refining the mesh. That is, at the kth step, we take $\delta t = 2^{-k}$, so that the mesh at the kth step is a refinement of the mesh at the $(k-1)$st step. Then, to obtain an appropriate function from the sequence $u^{(i)}_{\delta t}$ so that we can talk about it as a function in time, we interpolate $u^{(i)}_{\delta t}$'s to define:

$$u_{\delta t}(t) = \frac{t - t_{i-1}}{\delta t} u^{(i)}_{\delta t} + \frac{t_i - t}{\delta t} u^{(i-1)}_{\delta t}.$$

Note that, for any norm,

$$\|u_{\delta t}(t)\| \le \|u^{(i)}_{\delta t}\| + \|u^{(i-1)}_{\delta t}\|.$$

This implies, for instance, that

$$\left(\int \|\nabla u_{\delta t}\|^p\right)^{\frac{1}{p}} \lesssim \left(\sum_i \|\nabla u^{(i)}_{\delta t}\|^p \delta t\right)^{\frac{1}{p}}.$$

Thus, $u_{\delta t}(t) \in L^2(I; H^2) \cap L^3(I; W^{1,9}) \cap L^\infty(I; H^1)$ and

$$\int \|\partial_t u_{\delta t}(t)\|^2 = \sum (\delta t)^{-1} \|u_{\delta t}^{(i)} - u_{\delta t}^{(i-1)}\|^2.$$

Thus, due to Lemma 6.4, $\partial u_{\delta t} \in L^2(I; L^2)$ and the bound is uniform in k.

By Aubin-Lions compactness theorem, there exists a subsequence that converges to a function u strongly in $L^2(I; W^{1,3})$. This convergence also takes place for almost all time in I, say, $J \subset I$. Choose a time $t \in J \cap \{j2^{-k},\ \ j$ relatively prime to $k\}$, i.e., t is a dyadic time that belongs to J. Now, note that

$$\partial_t u_{\delta t}(t) = \frac{u_{\delta t}^{(i-1)} + u_{\delta t}^{(i)}}{\delta t},$$

whenever $t \in (t_{i-1}, t_i]$. We see, therefore, that $u_{\delta t}$ almost satisfies *NV* at time t, except that the nonlinearity has dependence on the time t_{i-1}. However, due to the convergence of the L^2 norm of $\|u_{\delta t}^{(i)} - u_{\delta t}^{(i-1)}\|$ to zero as $\delta t \to 0$, the residual term in the nonlinear term goes to zero. Since $u_{\delta t}(t)$ satisfies *NV* with residual of order $O(\delta t)$, it suffices to show that, at time t, u also satisfies *NV*. But we know that strong convergence takes place at t so it can be shown that u satisfies *NV* by an analogous manner as was shown in Sect. 3.

Theorem 6.6 *Let $u_{\delta t}^{(i)}$ be sequence of solutions to $NV_{\delta t}$ where $\delta t = 2^{-k}$. We define for each k the interpolant*:

$$u_{\delta t = 2^{-k}}(t) = \frac{t - t_{i-1}}{\delta t} u_{\delta t}^{(i)} + \frac{t_i - t}{\delta t} u_{\delta t}^{(i-1)}.$$

Then, there exists a subsequence of $u_{\delta t = 2^{-k}}$ that converges to $u \in L^2(I; H^2) \cap L^3(I; W^{1,9}) \cap L^\infty(I; H^1)$ and that u satisfies NV.

6.5 Stability and Uniqueness

We now show a stability estimate from which uniqueness follows. The following stability estimate is interesting in that if the power input is finite so that $C_{0,f} < \infty$ for all time, then two solutions stay boundedly close to each other.

Theorem 6.7 *Let $u_{1,\delta t}^{(i)}$ and $u_{2,\delta t}^{(i)}$ be two solutions to $NV_{\delta t}$. Let δt satisfy*

$$4\delta t(M + \epsilon^{-1})\nu^{-2} C_{tm} \leq 1.$$

Then,

$$\|u_{1,\delta t}^{(n)} - u_{2,\delta t}^{(n)}\|^2 \lesssim 2^{(M+\frac{1}{\epsilon})\frac{C_{0,f}^2}{\nu^2}} \|u_{1,\delta t}^{(0)} - u_{2,\delta t}^{(0)}\|^2 + O(\delta t \cdot \nu) \|\nabla(u_{1,\delta t}^{(0)} - u_{2,\delta t}^{(0)})\|^2.$$

In particular, a solution $u_{\delta t}$ to $NV_{\delta t}$ is unique.

Proof Let $w_i = u_{1,\delta t}^{(i)} - u_{2,\delta t}^{(i)}$. Then, subtracting the two equations they satisfy and testing against w_i we obtain

$$\frac{1}{2}\|w_i\|^2 + \delta t(\nu \|\nabla w_i\|^2 + \epsilon(\|\nabla \widetilde{w}_i\|^2 + \gamma \|\nabla \widetilde{w}_i\|_p^p))$$

$$\leq \delta t|(w_{i-1}\cdot\nabla u^1_{\delta t}(i), w_i)| + \frac{1}{2}\|w_{i-1,N}\|^2,$$

where we have used (A.12). Due to the Gagliardo-Nirenberg inequality, we have $\|w\|_{\frac{18}{5}} \leq 2^{\frac{4}{3}}\|w\|^{\frac{1}{3}}\|\nabla w\|^{\frac{2}{3}}$ and by the Sobolev imbedding theorem, $L^{\frac{9}{4}}$ is continuously imbedded in L^3. Thus,

$$\begin{aligned}
&|(w_{i-1}\cdot\nabla\widetilde{u}^{(i)}_{\delta t}, w_i)| \\
&\quad\leq \|\nabla\widetilde{u}^{(i)}_{\delta t}\|_{\frac{9}{4}}\|w_{i-1}\|_{\frac{18}{5}}\|w_i\|_{\frac{18}{5}} \\
&\quad\leq 2^{\frac{1}{3}}\|\nabla\widetilde{u}^{(i)}_{\delta t}\|_{\frac{9}{4}}(\|w_i\|^{\frac{2}{3}}\|\nabla w_i\|^{\frac{4}{3}} + \|w_{i-1}\|^{\frac{2}{3}}\|\nabla w_{i-1}\|^{\frac{4}{3}}) \\
&\quad\lesssim \frac{1}{\nu^2}(\|w_i\|^2 + \|w_{i-1}\|^2)\|\nabla\widetilde{u}^{(i)}_{\delta t}\|^3_{\frac{9}{4}} + \frac{\nu}{4}\|\nabla w_i\|^2 + \frac{\nu}{4}\|\nabla w_{i-1}\|^2 \\
&\quad\lesssim \frac{1}{\nu^2}(\|w_i\|^2 + \|w_{i-1}\|^2)\|\nabla\widetilde{u}^{(i)}_{\delta t}\|^3_3 + \frac{\nu}{4}\|\nabla w_i\|^2 + \frac{\nu}{4}\|\nabla w_{i-1}\|^2.
\end{aligned}$$

For the low-frequency part, note that, by Gagliardo-Nirenberg inequality,

$$\|u\|_3 \leq 2\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}
&|(w_{i-1}\cdot\nabla\overline{u}^{(i)}_{\delta t}, w_i)| \\
&\quad\leq \|\nabla\overline{u}^{(i)}_{\delta t}\|_3\|w_{i-1}\|_3\|w_i\|_3 \\
&\quad\leq 2\|\nabla\overline{u}^{(i)}_{\delta t}\|_3(\|w_{i-1}\|\|\nabla w_{i-1}\| + \|w_i\|\|\nabla w_i\|) \\
&\quad\leq \frac{3}{\nu}(\|w_i\|^2 + \|w_{i-1}\|^2)\|\nabla\overline{u}^{(i)}_{\delta t}\|^2_3 + \frac{\nu}{4}\|\nabla w_i\|^2 + \frac{\nu}{4}\|\nabla w_{i-1}\|^2 \\
&\quad\lesssim \frac{1}{\nu}(\|w_i\|^2 + \|w_{i-1}\|^2)M\|\nabla\overline{u}^{(i)}_{\delta t}\|^2_2 + \frac{\nu}{4}\|\nabla w_i\|^2 + \frac{\nu}{4}\|\nabla w_{i-1}\|^2.
\end{aligned}$$

Let $A_i = \nu^{-1}M\|\nabla u^{(i)}_{\delta t}\|^2 + \nu^{-2}\|\nabla\widetilde{u}^{(i)}_{\delta t}\|^3_3$. By summarizing our calculations, we have that

$$|(w_{i-1}\cdot\nabla u^{(i)}_{1,\delta t}, w_i)| \leq A_i\|w_i\|^2 + A_i\|w_{i-1}\|^2 + \frac{\nu}{2}\|\nabla w_i\|^2 + \frac{\nu}{2}\|\nabla w_{i-1}\|^2.$$

Then, Lemma 6.4 yields $A_i \leq (\nu^{-2}M + \nu^{-2}\epsilon^{-1})C_{tm}$. Thus, we have

$$\begin{aligned}
&\|w_i\|^2 + \delta t(2\epsilon(\|\nabla\widetilde{w}_i\|^2 + \gamma\|\nabla\widetilde{w}_i\|^3_3) + \nu\|\nabla w_i\|^2) \\
&\qquad\lesssim 2A_i\delta t\|w_i\|^2 + (1+2A_i\delta t)\|w_{i-1}\|^2 + \delta t\nu\|\nabla w_{i-1}\|^2
\end{aligned}$$

or

$$\begin{aligned}
&\|w_i\|^2 + (1-2A_i\delta t)^{-1}\delta t\left(2\epsilon(\|\nabla\widetilde{w}_i\|^2 + \gamma\|\nabla\widetilde{w}_i\|^3_3) + \nu\|\nabla w_i\|^2\right) \\
&\quad\leq (1+2A_i\delta t)(1-2A_i\delta t)^{-1}\|w_{i-1}\|^2 + (1-2A_i\delta t)^{-1}\delta t\nu\|\nabla w_{i-1}\|^2.
\end{aligned}$$

Since $1-2A_i\delta t \geq \frac{1}{2}$ by hypothesis, we have, due to Lemma A.18,

$$\|w_j\|^2 + \sum_{i=1}^{j}\left(\prod_{l=i+1}^{j}\alpha_l\right)(1-2A_i\delta t)^{-1}\delta t\left(2\epsilon(\|\nabla\widetilde{w}_i\|^2+\gamma\|\nabla\widetilde{w}_i\|_3^3)+\nu\|\nabla w_i\|^2\right)$$

$$\leq \left(\prod_{l=1}^{j}\alpha_l\right)\|w_0\|^2 + (1-2A_1\delta t)^{-1}\delta t\nu\|\nabla w_0\|^2$$

$$+\sum_{i=1}^{j}\left(\prod_{l=i+1}^{j}\alpha_l\right)(1-2A_i\delta t)^{-1}\delta t\nu\|\nabla w_{i-1}\|^2,$$

where $\alpha_l = (1+2A_l\delta t)(1-2A_l\delta t)^{-1}$. Note that for $\delta t\cdot A_i \leq \frac{1}{4}$, $(1+2\delta t A_i)(1-2\delta t A_i)^{-1} \leq 2^{2\delta t A_i}$. The theorem follows from this observation together with Lemma 6.2. □

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Certainly, David Gottlieb was a very gifted and accomplished mathematician who made many enduring and influential contributions. But Max Gunzburger, the author who knew him best, will mostly remember David for being, for over 30 years, his good and kind and thoughtful friend.

Appendix: Mathematical Background

In the Appendix, we gather a set of mathematical results that are used in our analyses.

A.1 Multiplier Theory, Filter Operators, and Fractional Differentiation and Integration

The analysis of the spectral viscosity equation involves an interaction of a rather broad range of mathematical concepts. The filter operator is defined in frequency space, while the nonlinear viscosity involves effects in physical space. Hyperviscosity brings the issue of fractional differentiation and how it interacts with the filter. In this section, we present a set of results that can serve as a common framework in which these concepts can be manipulated. Due to the use of spectral filtering, it is no surprise that the tools from harmonic analysis are used extensively.

Recall that we defined the operator P_N as follows:

$$P_N f = \sum_{|k|_\infty \leq N}\widehat{f}(k)e^{ik\cdot x}, \quad \text{where } f = \sum \widehat{f}(k)e^{ik\cdot x}.$$

Thus, we can consider P_N as an operator that is multiplied by $\chi_{|k|_\infty\leq N}$ in frequency space. An operator that is obtained in this manner is called a *multiplier* operator. For notational convenience, we denote by T_m an operator obtained by the multiplication by the given function $m(k)$ in frequency space. Therefore,

$$P_N = T_{\chi_{|k|_\infty\leq N}}.$$

We are also be interested in how this operator interacts with the fractional differentiation operator:

$$|\nabla|^s P_N = T_{|k|^s\chi_{|k|_\infty\leq N}}$$

and also how $I - P_N$ interacts with the fractional integration operator:

$$|\nabla|^{-s}(I - P_N) = T_{|k|^{-s}\chi_{|k|_\infty > N}},$$

both with $s \geq 0$. These operators become fundamental to the discussion that follows. We need to be able to manipulate these operators analytically; that is, we need estimates in various norms. We call a linear operator T that maps a measure space into another of type (p,q) if $\|T\|_{L^p \to L^q} < \infty$. Thus, the goal is to obtain (p,q) estimates for various values of p and q.

As we see later, the choice of the cube $|k|_\infty \leq N$ as the frequency region to which we project is important. We could not have chosen $|k|_2$ there for a rather deep reason in harmonic analysis. We will briefly mention this issue later. We can immediately obtain that $|\nabla|^s P_N$ is an operator of type $(2,2)$. Let d denote the dimension.

Lemma A.1 *For $s \geq 0$,*

$$\| |\nabla|^s P_N \|_{L^2 \to L^2} \lesssim N^s$$

Proof

$$\| |\nabla|^s P_N f \|_2 = \sum_{|k|_\infty \leq N} |k|^s |\widehat{f}|^2 \leq d^{s/2} N^s \|\widehat{f}\|_2 = d^{s/2} N^s \|f\|_2.$$

□

The $(2,2)$ estimate is not flexible enough for our analysis. We want to obtain (p,q) type estimates. One convenience for choosing $|k|_\infty$ is that the cut-off operator can be expressed in physical space as a convolution with a Dirichlet kernel:

$$D_N(x) = \prod_l \left(\frac{\sin((N+1/2)x_l)}{\sin(x_l/2)} \right)$$

so that,

$$P_N(f) = D_N * f.$$

Obtaining a (p,q) estimates for each values of p,q may be tedious. The well-known *Riesz-Thorin interpolation theorem* allows us to obtain (p,q) type estimates when $(1/p, 1/q)$ is a convex combination of two end-point types. Thus, we can simplify the task of obtaining an infinite number of estimates to just two [1]:

Theorem A.2 (Riesz-Thorin) *Let T be an operator satisfying*

$$\|T\|_{L^{p_1} \to L^{q_1}} < \infty \quad \textit{and} \quad \|T\|_{L^{p_2} \to L^{q_2}} < \infty.$$

Let $(\frac{1}{p}, \frac{1}{q}) = (1-\theta)(\frac{1}{p_1}, \frac{1}{q_1}) + \theta(\frac{1}{p_2}, \frac{1}{q_2})$. *Then, we have*

$$\|T\|_{L^p \to L^q} \leq \|T\|_{L^{p_1} \to L^{q_1}}^{1-\theta} \|T\|_{L^{p_2} \to L^{q_2}}^{\theta}.$$

We now show that $\|P_N\|_{L^1 \to L^\infty}$ and $\|P_N\|_{L^p \to L^p}$ can be estimated. The desired estimate follows by an interpolation. First, the former is easy to prove:

Lemma A.3

$$\|P_N g\|_\infty \leq c N^d \|g\|_1.$$

Proof We simply estimate the maximum norm of the Dirichlet kernel. Note that if $0 \le t \le \pi$, then, $|t|/\pi \le |\sin(t/2)|$ and $|\sin(t/2)| \le \min\{|t|, 1\}$ for all t. Therefore, we have

$$\left|\frac{\sin((N+1/2)t)}{\sin(t/2)}\right| \le \frac{\min\{(N+1/2)|t|, 1\}\pi}{t} \lesssim N.$$

By symmetry, this holds for all $0 \le t \le 2\pi$. Thus, we get $\|D_N\|_\infty \lesssim N^d$ from which it follows that

$$\|D_N * g\|_\infty \le \|D_N\|_\infty \|g\|_1 \le cN^d \|g\|_1.$$

□

The $L^p \to L^p$ bound is more difficult. Actually, since $\|D_N\|_1 \sim \log(N)$, we can show such a bound up to a logarithmic factor by simply using the Young's inequality. However, the logarithmic dependence can be eliminated. The trade-off is that we need to use the theory of *singular integral operators* and *multiplier theory* which implies the non-triviality of such a bound. In fact, it is a result in [8] that, had we chosen to use a cut-off filter with a square norm: $\sum_{|k|_2 \le N} \widehat{u}(k)$, no such bound can exist. However, for the partial sum operator we use, such a bound is true. The relevant tool to show this has been of historical significance in the development of harmonic analysis.

The tool to use is to show that the question about the boundedness in L^p $1 < p < \infty$ of an operator on a torus can be reduced to the corresponding question on $\mathbb{R}^d$ due to the following transference principle which is proved in [13]:

Theorem A.4 (Transference Principle) *Let T_m be an operator associated with a multiplier $m : \mathbb{R}^d \to \mathbb{C}$ such that m is continuous at each point of $\mathbb{Z}^d$. Then the restriction $\overline{m} = m|_{\mathbb{Z}^d}$ defines a multiplier operator on $L^2(\mathbb{T}^d)$ and*

$$\|T_{\overline{m}}\|_{L^p(\mathbb{T}^d)\to L^p(\mathbb{T}^d)} \le \|T_m\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)}.$$

Thus, we can transfer the question about a multiplier operator on a torus to a corresponding question on $\mathbb{R}^n$. The boundedness of the multiplier operator on $\mathbb{R}^d$ is answered by another fundamental theorem in harmonic analysis (see [1]):

Theorem A.5 (Hormander-Mikhlin) *Let $m : \mathbb{R}^d \to \mathbb{C}$ satisfy the homogeneous symbol estimates of order* 0:

$$|\nabla^k m(\zeta)| \lesssim |\zeta|^{-k}$$

for all $\zeta \ne 0$ and $0 \le k \le d+2$. Then, $\|T_m\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \lesssim c_p$ for all $1 < p < \infty$.

We will not give details of the proof, but, using this principle, the boundedness of P_N in L^p follows by noting that it can be expressed as a combination of modulation and a Hilbert transform. Then, using the Hormander-Mikhlin theorem, the Hilbert transform can be shown to be bounded in L^p. Then, the following theorem follows by the transference principle [13]:

Theorem A.6 *If $1 < p < \infty$ we have*

$$\|P_N\|_{L^p\to L^p} \lesssim c_p.$$

The following inequality is called the *Bernstein type* inequality, or *reverse inequality*.

Theorem A.7 *Let* $1 < p \le q < \infty$ *or* $1 < p < q \le \infty$. *Then,*

$$\|P_N f\|_q \le c_p N^{d(\frac{q-p}{pq})} \|f\|_p.$$

Proof We simply interpolate between Lemma A.3 and Theorem A.6. We select $p \le r \le q$ such that $1 < r < \infty$ so that then there exists $0 \le \theta \le 1$ such that $\frac{1}{q} = \frac{\theta}{r}$ and $\frac{1}{p} = \frac{\theta}{r} + 1 - \theta$. Thus, noting that $\|P_N\|_{L^r \to L^r} \le c$ and $\|P_N\|_{L^1 \to L^\infty} \le N^n$, the conclusion follows by Theorem A.2. □

The Bernstein inequality is just the equivalence of a pair of norms in a finite dimensional vector space with an explicit constant. However, it is also instructive to view this as one manifestation of the *the uncertainty principle* which states that if a function is localized about the origin in frequency space, then it must have a large physical support. But note that larger the p, the less sensitive the L^p norm becomes to the size of the physical support. Thus, the L^p norm of the frequency localized functions becomes less sensitive to p as p becomes larger, which is indeed shown by the dependence on p of the constant in the Bernstein's inequality.

We now turn to a host of inequalities that involve fractional derivatives. We define the fractional differentiation operator using multiplication by $|k|_2^s$ in frequency space:

$$|\nabla|^s f = \sum |k|_2^s \widehat{f} e^{ik\cdot x}$$

for each f for which the right-hand-side is in L^2. We note in particular that $-\Delta = |\nabla|^2$.

Note that fractional differentiation behaves in the following manner when composed with P_N. This is also another type of Bernstein inequality.

Theorem A.8 *Let* $s \ge 0$. *Then,*

$$\||\nabla|^s P_N f\|_p \le N^s c_p \|f\|_p.$$

Proof Define

$$\phi_j(x_j) = \begin{cases} 1 & |x_j| \le 1 \\ 0 & |x_j| \ge 2, \end{cases}$$

and $\phi_j \in \mathcal{C}^\infty$. Let $\phi = \Pi_j \phi_j$. Then,

$$|k|^s \chi_{|k|_\infty \le N} = N^s (|k| N^{-1})^s \phi(N^{-1}k) \chi_{|k|_\infty \le N}.$$

Note that due to the dilation symmetry of the Fourier transform,

$$((|k|N^{-1})^s \phi(N^{-1}k))^\vee(x) = (|k|^s \phi(k))^\vee(Nx) N^n.$$

Thus, it suffices to consider the multiplier $|k|^s \phi(k)$ which is a symbol of order 0. Hence, by the Hormander-Mikhlin theorem, it is bounded in L^p. Thus,

$$\begin{aligned} \||\nabla|^s P_N\|_{L^p \to L^p} &= \|T_{N^s(|k|N^{-1})^s \phi(N^{-1}k)} P_N\|_{L^p \to L^p} \\ &\le N^s \|T_{|k|^s \phi(k)}\|_{L^p \to L^p} \|P_N\|_{L^p \to L^p} \le N^s c_p, \end{aligned}$$

and the claim follows by the transference principle [13]. □

An easy consequence of Theorem A.6 is the following inequality which is in a sense a *Jackson-type* inequality from approximation theory.

Theorem A.9

$$\|(I-P_N)f\|_p \le c_p\|f\|_p.$$

The significance of the last two inequalities can be illustrated with an example. We will use these results to prove the Gagliardo-Nirenberg inequality. The more general Besov space version of the inequality is proved in [19].

Lemma A.10 *Let* λ, μ, p, q, r, *and* θ *satisfy* $1<q, p\le r<\infty$, $0<\theta<1$, $0>\frac{r-q}{rq}n-\lambda$, $0\le\frac{r-p}{rp}n+\mu$, *and* $\theta(\lambda-\frac{n}{p}+\frac{n}{r})+(1-\theta)(\mu-\frac{n}{q}+\frac{n}{r})=0$. *Then,*

$$\|f\|_{L^r}\lesssim|\widehat{f}(0)|+\||\nabla|^\lambda f\|_{L^q}^\theta\||\nabla|^\mu(f-P_0f)\|_{L^p}^{1-\theta}.$$

Proof We introduce the operator $\widetilde{P}_{2^k}=P_{2^{k+1}}(I-P_{2^k})$. Then,

$$\|f\|_r\le\|P_0f+\sum_{k=0}^\infty\widetilde{P}_{2^k}f\|\le|\widehat{f}(0)|+\sum_{k=0}^\infty\|\widetilde{P}_{2^k}f\|_r.$$

Let $t>0$ be chosen later. We split the sum into high-frequency and low-frequency parts and estimate them differently.

First, by using the first condition in our hypothesis, $0>\frac{r-q}{rq}n-\lambda$ in the Jackson and Bernstein inequalities, we have

$$\sum_{k\ge\log t}\|\widetilde{P}_{2^k}f\|_r\lesssim\sum_{k\ge\log t}2^{k(\frac{r-q}{rq}n-\lambda)}\||\nabla|^\lambda f\|_q\lesssim t^{\frac{r-q}{rq}n-\lambda}\||\nabla|^\lambda f\|_q.$$

Analogously, for the low-frequency part we use the second condition in our hypothesis, $0\le\frac{r-p}{rp}n+\mu$, to get

$$\begin{aligned}\sum_{0\le k<\log t}\|\widetilde{P}_{2^k}f\|_r&\lesssim\sum_{0\le k<\log t}2^{k(\frac{r-p}{rp}n+\mu)}\||\nabla|^\mu(f-P_0f)\|_p\\&\lesssim t^{\frac{r-p}{rp}n+\mu}\||\nabla|^\mu(f-P_0f)\|_p.\end{aligned}$$

Let $a=-(\frac{r-q}{rq}n-\lambda)$ and $b=(\frac{r-p}{rp}n+\mu)$. Then, we minimize

$$t^{-a}\||\nabla|^\lambda f\|_q+t^b\||\nabla|^\mu f\|_p$$

with respect to t. In order optimum is obtained for $t=(\frac{a\||\nabla|^\lambda f)\|_q}{b\||\nabla|^\mu(f-P_0f)\|_q})^{\frac{1}{a+b}}$. Substituting t into the above inequalities yields

$$\|f\|_r\lesssim|\widehat{f}(0)|+\||\nabla|^\lambda f\|_q^{\frac{b}{a+b}}\||\nabla|^\mu(f-P_0f)\|_p^{\frac{a}{a+b}}.$$

We choose $\theta=\frac{b}{a+b}$ so that it satisfies $\theta(-a)+(1-\theta)b=0$ which is the third condition in our hypothesis, $\theta(\lambda-\frac{n}{p}+\frac{n}{r})+(1-\theta)(\mu-\frac{n}{q}+\frac{n}{r})=0$. □

A.2 Nonlinear Monotone Operator

We set $p \geq 2$ to be the degree of nonlinearity of our viscosity operator. Also, for convenience, we introduce the number q where $p+q=pq$. The nonlinear viscosity operator is defined as follows:

$$-\nabla\cdot(|\nabla u|^{p-2}\nabla u) = -\nabla\cdot((\nabla u:\nabla u)^{(p-2)/2}\nabla u).$$

The nonlinear viscosity operator is a type of *monotone operator*. First, we consider some algebraic inequalities for vectors; see [5].

Lemma A.11 *Let $p \geq 2$. For all $a, b \in \mathbb{R}^d$, there exists $\gamma > 0$ independent of a, b such that*

$$(|a|^{p-2}a - |b|^{p-2}b, a-b) \geq \gamma|a-b|^p$$

and

$$(|a|^{p-2}a - |b|^{p-2}b, c) \leq (p-1)|c||a-b||\,|a|+|b|\,|^{p-2}.$$

From the above inequalities, we have the following:

Lemma A.12

$$\begin{aligned}\gamma\|\nabla(u-v)\|_p^p &\leq (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u-v))\\ &\quad\times(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla w)\\ &\leq (p-1)\|\nabla w\|_p\|\nabla(u-v)\|_p(\|\nabla u\|_p^2+\|\nabla v\|_p^2)^{p-2}.\end{aligned}$$

Proof

$$\begin{aligned}&(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla w)\\ &\quad\leq (p-1)\int|\nabla w||\nabla(u-v)|(|\nabla u|+|\nabla v|)^{p-2}\,dx\\ &\quad\leq (p-1)\left(\int|\nabla w|^p\right)^{\frac{1}{p}}\left(\int|\nabla(u-v)|^p\right)^{\frac{1}{p}}\left(\int\big|\,|\nabla u|+|\nabla v|\,\big|^{\frac{(p-2)p}{p-2}}\right)^{\frac{p-2}{p}}.\end{aligned}$$ □

Another remarkable property of this monotone operator is that it remains a coercive operator when tested against $-\Delta u$ in the following sense:

Lemma A.13 *Let $u \in \mathcal{C}^2$. Then,*

$$(\nabla\cdot|\nabla u|^{p-2}\nabla u, \Delta u) \geq \sum_{i,j,k}\int|\nabla u|^{p-2}(\partial_{kj}u^i)^2.$$

Proof Note that we have

$$\begin{aligned}&(\nabla\cdot|\nabla u|^{p-2}\nabla u, \Delta u)\\ &\quad= (|\nabla u|^{p-2}\nabla u, -\nabla\Delta u)\end{aligned}$$

$$= \sum_{i,j,k} \int |\nabla u|^{p-2} (\partial_{kj} u^i)^2 + \int (p-2) |\nabla u|^{p-4} \sum_{i,j,k,l} \partial_{kl} u^m \partial_l u^m \partial_{kj} u^i \partial_j u^i \, dx$$

$$\geq \sum_{i,j,k} \int |\nabla u|^{p-2} (\partial_{kj} u^i)^2 \, dx.$$

The last inequality is due to the fact that the sum inside of the integral is non-negative. □

Define

$$I_p(u) = \sum_{i,j,k} \int |\nabla u|^{p-2} (\partial_{kj} u^i)^2.$$

We then have the following embedding result.

Lemma A.14

$$\|\nabla u\|_{3p} \leq I_p(u)^{1/p}$$

Proof By Sobolev inequality, we have

$$\int |\nabla u|^{p-2} (\partial_{kj} u^i)^2 \geq \int |\partial_k u^i|^{p-2} (\partial_{kj} u^i)^2$$

$$= \int \left(\frac{2}{p} \partial_j |\partial_k u^i|^{\frac{p}{2}} \right)^2 \gtrsim \left(\int |\partial_k u^i|^{\frac{p}{2} \cdot 6} \right)^{\frac{1}{3}}. \qquad \square$$

A.3 Compactness and Measure Theory Results

The following is a standard measure theory result in [17].

Lemma A.15 *Let u_i be a Cauchy sequence in L^1. Then, there exists $u \in L^1$ and a subsequence u_{i_j} such that $u_{i_j} \to u$ almost everywhere.*

Given a sequence of functions u_i in $L^1(\Omega)$, we call this sequence *uniformly integrable* if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $M \subset \Omega$ with $|M| < \delta$, we have $|\int_M u_i \, dx| < \epsilon$ for all i. A nice property of the uniformly integrable sequence of functions is that, if they converge pointwise, then the integral also converges.

Lemma A.16 *Let u_i be uniformly integrable and $|\Omega| < \infty$. Suppose there exists $u \in L^1(\Omega)$ such that $u_i \to u$ almost everywhere. Then,*

$$\int u_i \to \int u.$$

We also use the Aubin-Lions compactness theorem [20] extensively:

Theorem A.17 (Aubin-Lions) *Let $1 < \alpha, \beta < \infty$. Let X be a Banach space, and let X_0, X_1 be separable and reflexive Banach spaces. If $X_0 \subset\subset X \subset X_1$, then the set*

$$\left\{ v \in L^\alpha(I; X_0); \frac{dv}{dt} \in L^\beta(I; X_1) \right\}$$

is compactly embedded in $L^{\alpha}(I; X)$.

We recall the following discrete Gronwall type inequality.

Lemma A.18 *Suppose we have a sequences of real numbers* a_i, b_i, *and* μ_i *for* $i = 1, \ldots, K$ *and* c_i *for* $i = 0, \ldots, K$ *that satisfy* $c_i + a_i \le \mu_i c_{i-1} + b_i$ *for each* $i = 1, \ldots, K$. *Then, for each* $i = 1, \ldots, K$,

$$c_i + \sum_{j=1}^{i} \Big(\prod_{\ell=j+1}^{i} \mu_\ell \Big) a_j \le c_0 \Big(\prod_{\ell=1}^{i} \mu_\ell \Big) + \sum_{j=1}^{i} \Big(\prod_{\ell=j+1}^{i} \mu_\ell \Big) b_j,$$

where it is understood that $\prod_{\ell=i+1}^{i} \mu_\ell = 1$.

Proof We proceed by induction. For $i = 1$, the assertion follows from the hypothesis. Supposing that the assertion is true for $i = 1, \ldots, n-1$, we would like to prove it for $i = n$. We have

$$\begin{aligned} c_n &+ \sum_{j=1}^{n} \Big(\prod_{\ell=j+1}^{n} \mu_\ell \Big) a_j \\ &= c_n + a_n + \mu_n \sum_{j=1}^{n-1} \Big(\prod_{\ell=j+1}^{n-1} \mu_\ell \Big) a_j \\ &\le \mu_n \Big[c_{n-1} + \sum_{j=1}^{n-1} \Big(\prod_{\ell=j+1}^{n-1} \mu_\ell \Big) a_j \Big] + b_n \\ &\le \mu_n \Big[c_0 \Big(\prod_{\ell=1}^{n-1} \mu_\ell \Big) + \sum_{j=1}^{n-1} \Big(\prod_{\ell=j+1}^{n-1} \mu_\ell \Big) b_j \Big] + b_n \\ &\le c_0 \Big(\prod_{\ell=1}^{n} \mu_l \Big) + \sum_{j=1}^{n} \Big(\prod_{\ell=j+1}^{n} \mu_\ell \Big) b_j. \end{aligned}$$

□

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