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A PARALLEL ROBIN–ROBIN DOMAIN DECOMPOSITION METHOD FOR THE STOKES–DARCY SYSTEM*

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Abstract. We propose a new parallel Robin–Robin domain decomposition method for the coupled Stokes–Darcy system with Beavers–Joseph–Saffman–Jones interface boundary condition. In particular, we prove that, with an appropriate choice of parameters, the scheme converges geometrically independent of the mesh size.

Key words. Stokes and Darcy equations, Robin–Robin domain decomposition, geometric convergence

AMS subject classifications. 65N55, 65N30, 35M20, 35Q35

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1. Introduction.

The Darcy equation is a well accepted model for flow in porous media such as often found in the subsurface. Thus, discretized versions of this equation are often used to simulate both groundwater and petroleum flows. However, in these settings, one often finds that the porous media do not completely cover subsurface regions of interest. For example, in petroleum applications one often finds pockets of oil, and in karst aquifers one finds conduits in which water flows freely. In such regions, the flow of the liquid cannot be accurately modeled by the Darcy equation, even though often, for expediency, that is exactly what is done in practice. A more accurate description of the flow of liquids in cavities and conduits is given by the Navier–Stokes equations. Due to the relative slow flows often encountered in such situations, one can simplify matters and use the linear Stokes equations instead. Of course, flows in conduits and cavities are coupled to the flow in the surrounding porous media so that conditions along the interfaces separating free flows and porous media flows must be imposed to affect the coupling. Several such interface conditions have been proposed; see, e.g., [2, 36]. Once a coupled Stokes–Darcy system has been defined, the remaining tasks are to first define discrete systems whose solutions accurately approximate the exact solution of the continuous model, and then develop efficient methods for solving the discrete equations. These are the tasks that we address in this paper.

Here we consider a coupled Stokes–Darcy system on a bounded domain $\Omega = \Omega_p \bigcup \Omega_f \subset \mathbb{R}^d$ $(d = 2, 3)$. In the porous media region Ω_p, the governing equations are

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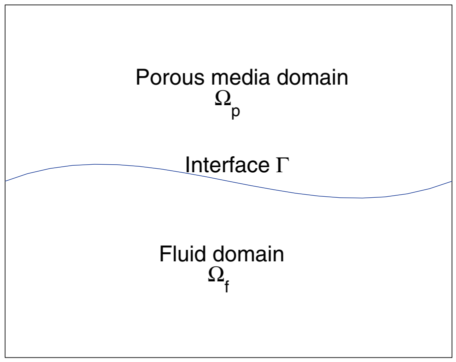

FIG. 1.1. *The free flow and porous media domains Ω_f and Ω_p, respectively, an the interface Γ.*

the Darcy equations

$$\begin{aligned} \mathbf{u}_p &= -\mathbb{K}\nabla\phi_p, \\ \nabla\cdot\mathbf{u}_p &= 0, \end{aligned}$$

where $\mathbf{u}_p$ is the fluid velocity in the porous media, $\mathbb{K}$ is the hydraulic conductivity tensor, and ϕ_p is the hydraulic head. In the fluid region Ω_f, the fluid flow is assumed to satisfy the Stokes equations

$$\begin{aligned} -\nabla\cdot\mathbb{T}(\mathbf{u}_f, p_f) &= \mathbf{f}, \\ \nabla\cdot\mathbf{u}_f &= 0, \end{aligned}$$

where $\mathbf{u}_f$ is the fluid velocity, p_f is the kinematic pressure, $\mathbf{f}$ is the external body force, ν is the kinematic viscosity of the fluid, $\mathbb{T}(\mathbf{u}_f, p_f) = 2\nu\mathbb{D}(\mathbf{u}_f) - p_f\mathbb{I}$ is the stress tensor, and $\mathbb{D}(\mathbf{u}_f) = 1/2(\nabla\mathbf{u}_f + \nabla^T\mathbf{u}_f)$ is the deformation tensor.

Let $\overline{\Gamma} = \overline{\Omega}_p \cap \overline{\Omega}_f$ denote the interface between the fluid and porous media regions; see Figure 1.1. Along the interface Γ, we require

$$\mathbf{u}_f\cdot\mathbf{n}_f = -\mathbf{u}_p\cdot\mathbf{n}_p, \tag{1.1}$$

$$-\boldsymbol{\tau}_j\cdot(\mathbb{T}(\mathbf{u}_f, p_f)\cdot\mathbf{n}_f) = \alpha\boldsymbol{\tau}_j\cdot\mathbf{u}_f, \tag{1.2}$$

$$-\mathbf{n}_f\cdot(\mathbb{T}(\mathbf{u}_f, p_f)\cdot\mathbf{n}_f) = g\phi_p, \tag{1.3}$$

where $\mathbf{n}_f$ and $\mathbf{n}_p$ denote the unit outer normal to the fluid and the porous media regions at the interface Γ, respectively; $\boldsymbol{\tau}_j$ ($j = 1, \ldots, d-1$) denote mutually orthogonal unit tangential vectors to the interface Γ; and the constant parameter α depends on ν and $\mathbb{K}$. The second condition (1.2) is referred to as the Beavers–Joseph–Saffman–Jones (BJSJ) interface condition [25, 36], which is an approximation of the Beavers–Joseph interface boundary condition [2]. The BJSJ boundary condition is also related to the Navier slip boundary condition.

To enable direct comparisons with the results of [16], and for simplicity, we assume that the hydraulic head ϕ_p and the fluid velocity $\mathbf{u}_f$ satisfy a homogeneous Dirichlet boundary condition except on Γ; i.e., $\phi_p = 0$ on the boundary $\partial\Omega_p\backslash\Gamma$ and $\mathbf{u}_f = 0$ on the boundary $\partial\Omega_f\backslash\Gamma$.

The spaces that we utilize are

$$
\begin{aligned}
&X_f = \{\mathbf{v}_f \in [H^1(\Omega_f)]^d \mid \mathbf{v}_f = 0 \text{ on } \partial\Omega_f \backslash \Gamma\},\\
&Q_f = L^2(\Omega_f),\\
&X_p = \{\psi_p \in H^1(\Omega_p) \mid \psi_p = 0 \text{ on } \partial\Omega_p \backslash \Gamma\}.
\end{aligned}
$$

For the domain D ($D = \Omega_f$ or Ω_p), $(\cdot,\cdot)_D$ denotes the L^2 inner product on the domain D, and $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product on the interface Γ or the duality pairing between $H^{-1/2}(\Gamma)$ and $H_{00}^{1/2}(\Gamma)$.

With this notation, the weak formulation of the coupled Stokes–Darcy problem is given as follows [6, 16]: find $(\mathbf{u}_f, p_f) \in X_f \times Q_f$ and $\phi_p \in X_p$ such that

$$
\begin{aligned}
&a_f(\mathbf{u}_f,\mathbf{v}_f) + b_f(\mathbf{v}_f,p_f) + g a_p(\phi_p,\psi_p) + \langle g\phi_p, \mathbf{v}_f\cdot\mathbf{n}_f\rangle - \langle g\mathbf{u}_f\cdot\mathbf{n}_f, \psi_p\rangle\\
&\qquad + \alpha\langle P_\tau \mathbf{u}_f, P_\tau \mathbf{v}_f\rangle = (\mathbf{f},\mathbf{v}_f)_{\Omega_f} \quad \forall\, \mathbf{v}_f \in X_f \;\; \psi_p \in X_p, \qquad (1.4)
\end{aligned}
$$

$$
b_f(\mathbf{u}_f, q_f) = 0 \qquad \forall\, q_f \in Q_f, \qquad (1.5)
$$

where the bilinear forms are defined as

$$
\begin{aligned}
a_p(\phi_p,\psi_p) &= (\mathbb{K}\nabla\phi_p, \nabla\psi_p)_{\Omega_p},\\
a_f(\mathbf{u}_f,\mathbf{v}_f) &= 2\nu(\mathbb{D}(\mathbf{u}_f), \mathbb{D}(\mathbf{v}_f))_{\Omega_f},\\
b_f(\mathbf{v}_f,q) &= -(\nabla\cdot\mathbf{v}_f, q)_{\Omega_f},
\end{aligned}
$$

and P_τ denotes the projection onto the tangent space on Γ, i.e.,

$$
P_\tau \mathbf{u} = \sum_{j=1}^{d-1} (\mathbf{u}\cdot\boldsymbol{\tau}_j)\boldsymbol{\tau}_j.
$$

It is easy to see that the system (1.4), (1.5) is well posed for $\mathbf{f} \in [L^2(\Omega_f)]^d$ [6, 16].

Because the governing equations are different for the fluid and the porous media regions, it is natural to utilize a domain decomposition method (DDM) so that off-the-shelf efficient solvers for the Darcy system and the Stokes system can be utilized [16]. The central issue is then to determine the convergence constraints or conditions and to find the associated convergence rate. The main contribution of this paper is the development and analysis of a *new parallel domain decomposition method based on Robin boundary conditions that converges with a rate that, for appropriate choices of acceleration parameters, is independent of the mesh size.*

For classical second-order elliptic problems, a Robin-type DDM was introduced in [27], where it was also proved that the solution of the Robin DDM converges weakly to the solution of the elliptic problems with respect to the H^1 norm. In [11, 12], new, updated techniques for the Robin data were introduced, and it was proved that the weak convergence in H^1 could induce strong convergence. In [18], it was pointed out that a convergence rate $1 - O(h^{1/2})$ can be achieved in the case of two subdomains. Recently, a rigorous analysis for the case of many subdomains was given in [31, 32] where it was proved, in certain cases such as for a small number of subdomains, that the convergence rate could be $1 - O(h^{1/2}H^{-1/2})$, where H is the size of the subdomain and h is the size of the finite element grid. In particular, the new term "winding number" was proposed in [32] to describe the depth of subdomains, and in the case of many subdomains, it was shown that the convergence rate not only depends on the mesh size h and the size H of the subdomains, but also on the winding number. Recently, in [41], it was also proved that optimized Schwarz methods with Robin transmission conditions *cannot* converge geometrically in the case of continuous second-order elliptic problems.

In [16], two Robin DDMs for the Stokes–Darcy equations, one a serial version (sRR) and the other a parallel version (pRR), are considered and compared with the Dirichlet–Neumann DDMs [14, 15]; mesh-independent convergence rates were observed for serial Robin DDMs numerically but were not proved rigorously. In addition to providing a rigorous analysis, in this paper we treat the more general case of the BJSJ interface boundary condition instead of the further simplified interface boundary conditions considered in [16]. However, the full Beavers–Joseph interface boundary condition is not treated here, since well posedness in the steady state case is established only for particular choices of parameters [6]. Other algorithms that combine ideas from multigrid and DDMs can be found in [29], where the authors proposed to solve the coupled problem directly on a coarse grid (with mesh size h_{coarse}) and then use the coarse solution to provide boundary conditions for the Stokes and Darcy systems at the interface so that they may be solved separately on a finer mesh (with mesh size of the order of $h_{coarse}^{\frac{3}{2}}$). For DDMs under other settings, and especially for the parallel Robin–Robin DDMs, one may refer to [11, 12, 14, 15, 16, 27, 30, 31, 32, 33, 37, 38] and the references cited therein. Application of finite element methods to the Stokes–Darcy system has been a very active research area recently. There are many more interesting works besides the references mentioned above (see, for instance, [1, 3, 4, 5, 7, 9, 10, 13, 17, 20, 21, 22, 24, 26, 28, 34, 35, 39, 40]). In particular, the methodology in this paper has been generalized and modified to the development of parallel DDMs for the Stokes–Darcy system with Beavers–Joseph interface boundary condition in [4].

The rest of the paper is organized as follows. In section 2, we propose the Robin boundary conditions at the interface for the Stokes and Darcy systems. A necessary and sufficient condition on the equivalence of the Stokes–Darcy system with BJSJ interface boundary condition and the new decoupled Stokes and Darcy systems with Robin boundary conditions is derived. In section 3, we propose our new parallel Robin–Robin DDM. We establish the convergence of the new scheme for the case of equal acceleration parameters $\gamma_f = \gamma_p$ and the case of $\gamma_f < \gamma_p$, with a convergence rate for appropriate choices of the acceleration parameters. In section 4, we invoke the von Neumann method (or semianalytic method) to analyze the detailed convergence behavior of the proposed algorithms on a rectangular domain with appropriate boundary conditions. We prove that the iteration cannot converge for $\gamma_f > \gamma_p$, and convergence results can be obtained for $\gamma_f \leq \gamma_p$ cases similar to those obtained in section 3. Finite element approximations are considered in section 5. In particular, we derive a convergence rate of $1-O(h)$ for the case of equal acceleration parameters and the case of $\gamma_f > \gamma_p$ (provided that γ_f and γ_p are close enough). Although the convergence rate is derived for globally regular triangulations only, we may easily generalize the result to mortar elements so that solvers with different mesh sizes may be utilized for the fluid and the porous media regions. We present our results of some computational experiments in section 6. These results are in accordance with our analyses.

2. Robin boundary conditions. In order to solve the coupled Stokes–Darcy problem utilizing the domain decomposition idea, we naturally consider (partial) Robin boundary conditions for the Stokes and the Darcy equations because Robin boundary conditions are more general and embody both the Neumann- and Dirichlet-type conditions in (1.1)–(1.3).

Let us consider the following Robin condition for the Darcy system: for a given constant $\gamma_p > 0$ and a given function η_p defined on Γ,

$$\gamma_p \mathbb{K} \nabla \widehat{\phi}_p \cdot \mathbf{n}_p + g \widehat{\phi}_p = \eta_p \quad \text{on } \Gamma. \tag{2.1}$$

Then, the corresponding weak formulation for the Darcy system is given by the following: for $\eta_p \in L^2(\Gamma)$, find $\widehat{\phi}_p \in X_p$ such that

$$\gamma_p a_p(\widehat{\phi}_p, \psi_p) + \langle g\widehat{\phi}_p, \psi_p \rangle = \langle \eta_p, \psi_p \rangle \quad \forall\, \psi_p \in X_p.$$

Similarly, we propose the following Robin-type condition for the Stokes equations: for a given constant $\gamma_f > 0$ and a given function η_f defined on Γ,

$$\mathbf{n}_f \cdot (\mathbb{T}(\widehat{\mathbf{u}}_f, \widehat{p}_f) \cdot \mathbf{n}_f) + \gamma_f \widehat{\mathbf{u}}_f \cdot \mathbf{n}_f = \eta_f \qquad \text{on } \Gamma. \tag{2.2}$$

Then, the corresponding weak formulation for the Stokes system is given by the following: for $\eta_f \in L^2(\Gamma)$, find $\widehat{\mathbf{u}}_f \in X_f$ and $\widehat{p}_f \in Q_f$ such that

$$\begin{aligned} & a_f(\widehat{\mathbf{u}}_f, \mathbf{v}_f) + b_f(\mathbf{v}_f, \widehat{p}_f) + \gamma_f \langle \widehat{\mathbf{u}}_f \cdot \mathbf{n}_f, \mathbf{v}_f \cdot \mathbf{n}_f \rangle \\ & \qquad + \alpha \langle P_\tau \widehat{\mathbf{u}}_f, P_\tau \mathbf{v}_f \rangle = (\mathbf{f}, \mathbf{v}_f)_{\Omega_f} + \langle \eta_f, \mathbf{v}_f \cdot \mathbf{n}_f \rangle \quad \forall\, \mathbf{v}_f \in X_f, \\ & b_f(\widehat{\mathbf{u}}_f, q_f) = 0 \quad \forall\, q_f \in Q_f. \end{aligned} \tag{2.3}$$

We can combine the Stokes and Darcy systems with Robin boundary conditions into one system. Indeed, for any positive constant ω, it is easy to see that if $\eta_p \in L^2(\Gamma)$ and $\eta_f \in L^2(\Gamma)$ are given, then there exists a unique solution $(\widehat{\phi}_p, \widehat{u}_f, \widehat{p}_f)$ $\in X_p \times X_f \times Q_f$ such that

$$\begin{aligned} & a_f(\widehat{\mathbf{u}}_f, \mathbf{v}_f) + b_f(\mathbf{v}_f, \widehat{p}_f) + \omega\gamma_p a_p(\widehat{\phi}_p, \psi_p) + \omega \langle g\widehat{\phi}_p, \psi_p \rangle + \gamma_f \langle \widehat{\mathbf{u}}_f \cdot \mathbf{n}_f, \mathbf{v}_f \cdot \mathbf{n}_f \rangle \\ & \qquad + \alpha \langle P_\tau \widehat{\mathbf{u}}_f, P_\tau \mathbf{v}_f \rangle = (f, \mathbf{v}_f)_{\Omega_f} + \langle \eta_f, \mathbf{v}_f \cdot \mathbf{n}_f \rangle + \omega \langle \eta_p, \psi_p \rangle \qquad \forall\, \psi_p \in X_p, \mathbf{v}_f \in X_f, \\ & b_f(\widehat{\mathbf{u}}_f, q_f) = 0 \qquad \forall\, q_f \in Q_f. \end{aligned} \tag{2.4}$$

Remark 2.1. Note that the solution $(\widehat{\phi}_p, \widehat{\mathbf{u}}_f, \widehat{p}_f)$ is independent of the parameter ω.

Our next goal is to show that, for appropriate choices of γ_f, γ_p, η_f, and η_p, (smooth) solutions of the Stokes–Darcy system are equivalent to solutions of (2.4), and hence we may solve the latter system (2.4) instead of the former.

LEMMA 2.2. *Let* $(\phi_p, \mathbf{u}_f, p_f)$ *be the solution of the coupled Stokes–Darcy system* (1.4)–(1.5)*, and let* $(\widehat{\phi}_p, \widehat{\mathbf{u}}_f, \widehat{p}_f)$ *be the solution of the decoupled Stokes and Darcy systems with Robin boundary conditions at the interface* (2.4)*. Then,* $(\widehat{\phi}_p, \widehat{\mathbf{u}}_f, \widehat{p}_f) = (\phi_p, \mathbf{u}_f, p_f)$ *if and only if* γ_f*,* γ_p*,* η_f*, and* η_p *satisfy the following compatibility conditions:*

$$\eta_p = \gamma_p \widehat{\mathbf{u}}_f \cdot \mathbf{n}_f + g\widehat{\phi}_p, \tag{2.5}$$

$$\eta_f = \gamma_f \widehat{\mathbf{u}}_f \cdot \mathbf{n}_f - g\widehat{\phi}_p. \tag{2.6}$$

Proof. For the necessity, we set $\psi_p = 0$ in the Stokes–Darcy system (1.4)–(1.5) and deduce that $(\phi_p, \mathbf{u}_f, p_f)$ solves (2.3) if

$$\langle \eta_f - \gamma_f \mathbf{u}_f \cdot \mathbf{n}_f + g\phi_p, \mathbf{v}_f \cdot \mathbf{n}_f \rangle = 0 \quad \forall\, \mathbf{v}_f \in X_f, \tag{2.7}$$

which implies (2.6). The necessity of (2.5) can be derived in a similar fashion.

As for the sufficiency, by setting $\omega = g/\gamma_p$ in (2.4) and substituting the compatibility conditions (2.5)–(2.6), we easily see that $(\widehat{\phi}_p, \widehat{\mathbf{u}}_f, \widehat{p}_f)$ solves the coupled Stokes–Darcy system (1.4)–(1.5).

Since the solution to the Stokes–Darcy system is unique, we have $(\widehat{\phi}_p, \widehat{\mathbf{u}}_f, \widehat{p}_f) = (\phi_p, \mathbf{u}_f, p_f)$. ∎

3. Robin–Robin domain decomposition methods.

3.1. The Robin–Robin domain decomposition algorithm. Now we propose the following parallel Robin–Robin DDM for solving the coupled Stokes–Darcy system.

1. Initial values of η_p^0 and η_f^0 are guessed. They may be taken to be zero.
2. For $k = 1, 2, \ldots,$ independently solve the Stokes and Darcy systems with Robin boundary conditions. More precisely, $\phi_p^m \in X_p$ is computed from

$$\gamma_p a_p(\phi_p^m, \psi_p) + \langle g\phi_p^m, \psi_p\rangle = \langle \eta_p^m, \psi_p\rangle \qquad \forall\, \psi_p \in X_p, \tag{3.1}$$

and $\mathbf{u}_f^m \in X_f$ and $p_f^m \in Q_f$ are computed from

$$a_f(\mathbf{u}_f^m, \mathbf{v}_f) + b_f(\mathbf{v}_f, p_f^m) + \gamma_f\langle \mathbf{u}_f^m \cdot \mathbf{n}_f, \mathbf{v}_f \cdot \mathbf{n}_f\rangle + \alpha\langle P_\tau \mathbf{u}_f^m, P_\tau \mathbf{v}_f\rangle$$
$$= \langle \eta_f^m, \mathbf{v}_f \cdot \mathbf{n}_f\rangle + (\mathbf{f}, \mathbf{v}_f)_{\Omega_f} \quad \forall\, \mathbf{v}_f \in X_f, \tag{3.2}$$

$$b_f(\mathbf{u}_f^m, q_f) = 0 \quad \forall\, q_f \in Q_f. \tag{3.3}$$

3. η_p^{m+1} and η_f^{m+1} are updated in the following manner:

$$\eta_f^{m+1} = a\eta_p^m + bg\phi_p^m, \tag{3.4}$$

$$\eta_p^{m+1} = c\eta_f^m + d\mathbf{u}_f^m \cdot \mathbf{n}_f, \tag{3.5}$$

where the coefficients a, b, c, d are chosen as follows:

$$a = \frac{\gamma_f}{\gamma_p}, \qquad b = -1 - a, \tag{3.6}$$

$$c = -1, \qquad d = \gamma_f + \gamma_p. \tag{3.7}$$

In the special case for which $\gamma_f = \gamma_p = \gamma$, we have

$$a = 1, \qquad b = -2, \qquad c = -1, \qquad d = 2\gamma.$$

The relations (3.6)–(3.7) are necessary to ensure the convergence of the scheme. Indeed, suppose that η_f^m and η_p^m converge to η_f^* and η_p^*, respectively, and that $\phi_p^m, \mathbf{u}_f^m$ also converge to the true solution ϕ_p^* and $\mathbf{u}_f^*$, respectively. Then, by (3.4)–(3.5) and Lemma 2.2, we see that the following relationships hold:

$$\eta_f^* = a\eta_p^* + bg\phi_p^* = \gamma_f \mathbf{u}_f^* \cdot \mathbf{n}_f - g\phi_p^*, \tag{3.8}$$

$$\eta_p^* = c\eta_f^* + d\mathbf{u}_f^* \cdot \mathbf{n}_f = \gamma_p \mathbf{u}_f^* \cdot \mathbf{n}_f + g\phi_p^*. \tag{3.9}$$

This leads to

$$\begin{pmatrix} bg\phi_p^* \\ d\mathbf{u}_f^* \cdot \mathbf{n}_f \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} g\phi_p^* \\ \mathbf{u}_f^* \cdot \mathbf{n}_f \end{pmatrix} = \begin{pmatrix} 1 & -a \\ -c & 1 \end{pmatrix} \begin{pmatrix} \eta_f^* \\ \eta_p^* \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -a \\ -c & 1 \end{pmatrix} \begin{pmatrix} -1 & \gamma_f \\ 1 & \gamma_p \end{pmatrix} \begin{pmatrix} g\phi_p^* \\ \mathbf{u}_f^* \cdot \mathbf{n}_f \end{pmatrix}, \tag{3.10}$$

which implies the consistency equations (3.6)–(3.7) on the coefficients a, b, c, d and γ_f, γ_p.

These relationships (3.6)–(3.7) among the parameters are used in the convergence analysis of the Robin–Robin DDM.

Remark 3.1. If the updating strategy (3.4)–(3.5) is changed to

$$\eta_f^{m+1} = a_1\eta_p^m + b_1 g\phi_p^m + c_1\mathbf{u}_f^m \cdot \mathbf{n}_f,$$
$$\eta_p^{m+1} = a_2\eta_f^m + b_2 g\phi_p^m + c_2\mathbf{u}_f^m \cdot \mathbf{n}_f,$$

then the "consistency" conditions change to

$$\begin{pmatrix} -a_1 & 1 \\ 1 & a_2 \end{pmatrix} \begin{pmatrix} 1 & \gamma_p \\ -1 & \gamma_f \end{pmatrix} = \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix}.$$

In this case, we have more flexibility. However, the convergence analysis is somewhat more complicated and will be addressed elsewhere.

The parallel Robin–Robin domain decomposition algorithm proposed here is related to the serial version (sRR algorithm) of [16]. As a matter of fact, the sRR algorithm can be obtained by implementing our algorithm serially as follows. Initialize η_p^0, and, for $k = 0, 1, \dots,$

1. find ϕ_p^m by solving the Darcy system (3.1);
2. set $\eta_f^m = a\eta_p^m + bg\phi_p^m$ and find $\mathbf{u}_f^m$ and p_f^m by solving the Stokes system (3.2)–(3.3);
3. set $\eta_p^{m+1} = c\eta_f^m + d\mathbf{u}_f^m \cdot \mathbf{n}_f$.

In [16], it is proved, for $\gamma_f = \gamma_p = \gamma$, that the solution of the sRR algorithm converges to the solution of the Darcy–Stokes system. Here, we are able to prove a similar convergence result. Moreover, we are able to prove that the convergence could be geometric for an appropriate choice of $\gamma_f < \gamma_p$.

3.2. Convergence of the parallel Robin–Robin DDM. We follow the elegant energy method proposed in [27] to demonstrate the convergence of the parallel Robin–Robin DDM for appropriate choice of parameters γ_p and γ_f.

To this end, let $(\phi_p, \mathbf{u}_f, p_f)$ denote the solution of the coupled Stokes–Darcy system (1.4)–(1.5). Then, we have that $(\phi_p, \mathbf{u}_f, p_f)$ solves the equivalent decoupled system (2.4) with $\gamma_f, \gamma_p, \eta_p, \eta_f$ satisfying the compatibility conditions (2.5)–(2.6), with the hats removed.

Next, we define the error functions

$$\epsilon_p^m = \eta_p - \eta_p^m, \quad \epsilon_f^m = \eta_f - \eta_f^m, \quad e_\phi^m = \phi_p - \phi_p^m, \quad \mathbf{e}_u^m = \mathbf{u}_f - \mathbf{u}_f^m, \quad e_p^m = p_f - p_f^m.$$

Then, the error functions satisfy the following error equations:

$$\gamma_p a_p(e_\phi^m, \psi_p) + \langle g e_\phi^m, \psi_p\rangle = \langle \epsilon_p^m, \psi_p\rangle \quad \forall\, \psi_p \in X_p, \tag{3.11}$$

$$a_f(\mathbf{e}_u^m, \mathbf{v}_f) + b_f(\mathbf{v}_f, e_p^m) + \gamma_f\langle \mathbf{e}_u^m \cdot \mathbf{n}_f, \mathbf{v}_f \cdot \mathbf{n}_f\rangle + \alpha\langle P_\tau \mathbf{e}_u^m, P_\tau \mathbf{v}_f\rangle$$
$$= \langle \epsilon_f^m, \mathbf{v}_f \cdot \mathbf{n}_f\rangle \quad \forall\, \mathbf{v}_f \in X_f, \tag{3.12}$$

$$b_f(\mathbf{e}_u^m, q_f) = 0 \quad \forall\, q_f \in Q_f, \tag{3.13}$$

and, along the interface Γ, we have

$$\epsilon_f^{m+1} = a\epsilon_p^m + bg e_\phi^m, \tag{3.14}$$

$$\epsilon_p^{m+1} = c\epsilon_f^m + d\mathbf{e}_u^m \cdot \mathbf{n}_f. \tag{3.15}$$

Equation (3.15) leads to

$$\|\epsilon_p^{m+1}\|_\Gamma^2 = c^2\|\epsilon_f^m\|_\Gamma^2 + d^2\|\mathbf{e}_u^m \cdot \mathbf{n}_f\|_\Gamma^2 + 2cd\langle \epsilon_f^m, \mathbf{e}_u^m \cdot \mathbf{n}_f\rangle.$$

Setting $\mathbf{v}_f = \mathbf{e}_u^m$ in (3.12), we deduce

$$\langle \epsilon_f^m, \mathbf{e}_u^m \cdot \mathbf{n}_f\rangle = a_f(\mathbf{e}_u^m, \mathbf{e}_u^m) + \gamma_f\|\mathbf{e}_u^m \cdot \mathbf{n}_f\|_\Gamma^2 + \alpha\|P_\tau \mathbf{e}_u^m\|_\Gamma^2,$$

and, hence, combining the last two equations, we have

$$\|\epsilon_p^{m+1}\|_\Gamma^2 = c^2\|\epsilon_f^m\|_\Gamma^2 + (d^2+2cd\gamma_f)\|\mathbf{e}_u^m\cdot\mathbf{n}_f\|_\Gamma^2 + 2cd\, a_f(\mathbf{e}_u^m, \mathbf{e}_u^m) + 2cd\,\alpha\|P_\tau\mathbf{e}_u^m\|_\Gamma^2. \tag{3.16}$$

Similarly, (3.14) implies

$$\|\epsilon_f^{m+1}\|_\Gamma^2 = a^2\|\epsilon_p^m\|_\Gamma^2 + b^2\|g e_\phi^m\|_\Gamma^2 + 2ab\langle \epsilon_p^m, g e_\phi^m\rangle.$$

Setting $\psi_p = g e_\phi^m$ in (3.11), we have

$$\langle \epsilon_p^m, g e_\phi^m\rangle = \gamma_p a_p(e_\phi^m, g e_\phi^m) + \langle g e_\phi^m, g e_\phi^m\rangle.$$

Combining the last two equations, we deduce

$$\|\epsilon_f^{m+1}\|_\Gamma^2 = a^2\|\epsilon_p^m\|_\Gamma^2 + (b^2+2ab)\|g e_\phi^m\|_\Gamma^2 + 2ab\gamma_p g\, a_p(e_\phi^m, e_\phi^m). \tag{3.17}$$

Substituting (3.6)–(3.7) into (3.16) and (3.17), we have the following result.

LEMMA 3.2. *The error functions satisfy*

$$\begin{aligned}\|\epsilon_p^{m+1}\|_\Gamma^2 &= \|\epsilon_f^m\|_\Gamma^2 + (\gamma_p^2-\gamma_f^2)\|\mathbf{e}_u^m\cdot\mathbf{n}_f\|_\Gamma^2 \\ &\quad - 2(\gamma_f+\gamma_p)a_f(\mathbf{e}_u^m,\mathbf{e}_u^m) - 2(\gamma_f+\gamma_p)\,\alpha\|P_\tau\mathbf{e}_u^m\|_\Gamma^2, \\ \|\epsilon_f^{m+1}\|_\Gamma^2 &= \left(\frac{\gamma_f}{\gamma_p}\right)^2\|\epsilon_p^m\|_\Gamma^2 + \left(1-\left(\frac{\gamma_f}{\gamma_p}\right)^2\right)\|g e_\phi^m\|_\Gamma^2 \\ &\quad - 2\gamma_f\left(1+\frac{\gamma_f}{\gamma_p}\right) g\, a_p(e_\phi^m, e_\phi^m).\end{aligned}$$

We are now ready to demonstrate the convergence of our parallel Robin–Robin DDM. The convergence analyses for $\gamma_f = \gamma_p$ and $\gamma_f \neq \gamma_p$ are different and will be treated separately.

Case 1: $\gamma_f = \gamma_p = \gamma$. In this case, we have

$$\begin{aligned}\|\epsilon_p^{m+1}\|_\Gamma^2 &= \|\epsilon_f^m\|_\Gamma^2 - 4\gamma a_f(\mathbf{e}_u^m,\mathbf{e}_u^m) - 4\gamma\alpha\|P_\tau\mathbf{e}_u^m\|_\Gamma^2, \\ \|\epsilon_f^{m+1}\|_\Gamma^2 &= \|\epsilon_p^m\|_\Gamma^2 - 4\gamma g a_p(e_\phi^m, e_\phi^m).\end{aligned}$$

Adding the two equations and summing over k from $k=0$ to N, we deduce

$$\|\epsilon_p^{N+1}\|_\Gamma^2 + \|\epsilon_f^{N+1}\|_\Gamma^2 = \|\epsilon_p^0\|_\Gamma^2 + \|\epsilon_f^0\|_\Gamma^2 - 4\gamma\sum_{k=0}^{N}(a_f(\mathbf{e}_u^m,\mathbf{e}_u^m) + g\, a_p(e_\phi^m,e_\phi^m) + \alpha\|P_\tau\mathbf{e}_u^m\|_\Gamma^2).$$

This implies that $\|\epsilon_p^{N+1}\|_\Gamma^2 + \|\epsilon_f^{N+1}\|_\Gamma^2$ is bounded from above by $\|\epsilon_p^0\|_\Gamma^2 + \|\epsilon_f^0\|_\Gamma^2$ and that $\mathbf{e}_u^m$ and e_ϕ^m tend to zero in $(H^1(\Omega_f))^d$ and $H^1(\Omega_p)$, respectively. The convergence of e_ϕ^m together with the error equation (3.11) implies the convergence of ϵ_p^m in $H^{-\frac{1}{2}}(\Gamma)$. Combining the convergence of ϵ_p^m and e_ϕ^m and the error equation on the interface

(3.14), we deduce the convergence of ϵ_f^m in $H^{-\frac{1}{2}}(\Gamma)$. The convergence of the pressure then follows from the inf-sup condition and (3.12)–(3.13). Note that we have no rate of convergence here. Hence, we have proved the following result.

THEOREM 3.3. *If $\gamma_p = \gamma_f = \gamma$, then*

$$\phi_p^m \xrightarrow{X_p} \phi_p, \qquad \mathbf{u}_f^m \xrightarrow{X_f} \mathbf{u}_f, \qquad p_f^m \xrightarrow{Q_f} p_f,$$

and

$$\eta_p^m \xrightarrow{H^{-\frac{1}{2}}(\Gamma)} \gamma \mathbf{u}_f \cdot \mathbf{n}_f + g\phi_p = -\gamma \mathbb{K}\nabla\phi_p \cdot \mathbf{n}_p + g\phi_p,$$

$$\eta_f^m \xrightarrow{H^{-\frac{1}{2}}(\Gamma)} \gamma \mathbf{u}_f \cdot \mathbf{n}_f - g\phi_p = \mathbf{n}_f \cdot (\mathbb{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}_f) + \gamma \mathbf{u}_f \cdot \mathbf{n}_f.$$

Case 2: $\gamma_f < \gamma_p$. In this case, because $e_\phi^m \in X_p$, we deduce that, thanks to the trace theorem and the Poincaré inequality, there exists a constant C_p (independent of $\mathbb{K}$) such that

$$\|e_\phi^m\|_\Gamma^2 \le C_p \|\mathbb{K}^{-1}\| a_p(e_\phi^m, e_\phi^m). \tag{3.18}$$

Thus, if $\gamma_f < \gamma_p$ and

$$\frac{1}{\gamma_f} - \frac{1}{\gamma_p} \le \frac{2}{gC_p\|\mathbb{K}^{-1}\|}, \tag{3.19}$$

we have

$$\begin{aligned}
&\left(1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right)\|ge_\phi^m\|_\Gamma^2 - 2\gamma_f\left(1 + \frac{\gamma_f}{\gamma_p}\right) g a_p(e_\phi^m, e_\phi^m) \\
&\le \gamma_f g \left(1 + \frac{\gamma_f}{\gamma_p}\right)\left(\left(\frac{1}{\gamma_f} - \frac{1}{\gamma_p}\right) gC_p\|\mathbb{K}^{-1}\| - 2\right) a_p(e_\phi^m, e_\phi^m) \\
&\le 0.
\end{aligned}$$

Similarly, thanks to the trace theorem and Korn's inequality, there exists a constant C_f such that

$$\|\mathbf{e}_u^m \cdot \mathbf{n}_f\|_\Gamma^2 \le C_f \int_{\Omega_f} |\mathbb{D}(\mathbf{e}_u^m)|^2 \mathrm{d}\mathbf{x}. \tag{3.20}$$

Therefore, under the additional constraint

$$\gamma_p - \gamma_f \le \frac{4\nu}{C_f},$$

we have

$$(\gamma_p^2 - \gamma_f^2)\|\mathbf{e}_u^m \cdot \mathbf{n}_f\|_\Gamma^2 \le 2(\gamma_f + \gamma_p) a_f(\mathbf{e}_u^m, \mathbf{e}_u^m). \tag{3.21}$$

This combined with Lemma 3.2 gives

$$\|\epsilon_p^{m+1}\|_\Gamma^2 \le \|\epsilon_f^m\|_\Gamma^2, \qquad \|\epsilon_f^{m+1}\|_\Gamma^2 \le \left(\frac{\gamma_f}{\gamma_p}\right)^2 \|\epsilon_p^m\|_\Gamma^2,$$

which leads to

$$\|\epsilon_p^{m+1}\|_\Gamma \le \frac{\gamma_f}{\gamma_p}\|\epsilon_p^{k-1}\|_\Gamma, \qquad \|\epsilon_f^{m+1}\|_\Gamma \le \frac{\gamma_f}{\gamma_p}\|\epsilon_f^{k-1}\|_\Gamma.$$

This implies the convergence of the η, which further implies the convergence of the velocity $\mathbf{e}_u^m$, the pressure e_p^m, and the hydraulic head e_ϕ^m through the error equations (3.11)–(3.13).

Hence we have derived the following geometric convergence result.

THEOREM 3.4. *Assume that the parameters γ_p and γ_f are chosen so that*

$$0 < \gamma_p - \gamma_f \le \frac{4\nu}{C_f}, \qquad \frac{1}{\gamma_f} - \frac{1}{\gamma_p} \le \frac{2}{gC_p\|\mathbb{K}^{-1}\|}, \tag{3.22}$$

where C_p and C_f are the constants in (3.18) *and* (3.20)*, respectively. Then, the solutions of the parallel Robin–Robin DDM converge to the solution of the Stokes–Darcy system. Moreover,*

$$a_p(e_\phi^m, e_\phi^m) + a_f(\mathbf{e}_u^m, \mathbf{e}_u^m) + \|e_p^m\|_\Gamma^2 + \|\epsilon_p^m\|_\Gamma^2 + \|\epsilon_f^m\|_\Gamma^2 \le C\left(\frac{\gamma_f}{\gamma_p}\right)^{\lfloor\frac{k}{2}\rfloor}\left(\|\epsilon_p^0\|_\Gamma^2 + \|\epsilon_f^0\|_\Gamma^2\right).$$

The last inequality follows from the geometric convergence of $\epsilon_p^m, \epsilon_f^m$, the error relationship at the interface, and the error equations.

4. Von Neumann analysis. In this section, we use the von Neumann method to analyze the convergence of the Robin–Robin DDM scheme for one special problem. We will demonstrate that the convergence behavior is essentially the same as that obtained in the previous section.

Let $\Omega_p = (0,\pi)\times(-1,0)$, $\Omega_f = (0,\pi)\times(0,1)$, and $\Gamma = \{0 \le x \le \pi, y = 0\}$. The normal directions are $\mathbf{n}_f = (0,-1)$ and $\mathbf{n}_p = (1,0)$. Along the interface Γ, the interface conditions (1.1)–(1.3) can be rewritten as follows:

$$u_{f,2} = -\mathbb{K}\nabla\phi_p \cdot \mathbf{n}_p, \tag{4.1}$$

$$\nu\left(\frac{\partial u_{f,1}}{\partial y} + \frac{\partial u_{f,2}}{\partial x}\right) = \alpha u_{f,1}, \tag{4.2}$$

$$2\nu\frac{\partial u_{f,2}}{\partial y} - p = -g\phi_p, \tag{4.3}$$

where $u_{f,1}$ and $u_{f,2}$ are the two components of the velocity $\mathbf{u}_f$. The right-hand force $\mathbf{f}$ vanishes since only the error equations are needed in the analysis. For simplicity, we assume that $\mathbb{K} = K\mathbb{I}$ and $g = 1$ in this section. By using von Neumann analysis, the solutions are assumed to be periodic in the x-direction and have single wave formulations, that is

$$\phi_p = e^{ikx}\bar{\phi}(y), \quad \mathbf{u}_f = e^{ikx}\bar{\mathbf{u}}(y), \quad p_f = e^{ikx}\bar{p}(y), \tag{4.4}$$

where the integer k is the wave number and $i = \sqrt{-1}$. The assumptions can hold if the initial guesses of η_f and η_p have the same single wave formulations of $\eta_f = e^{ikx}\bar{\eta}_f$ and $\eta_p = e^{ikx}\bar{\eta}_p$.

Note that if $\mathbf{u}_f$ satisfies the divergence-free constraint, there exists a stream function $\psi(y)$ such that

$$\bar{\mathbf{u}}(y) = \left(-\frac{d\psi(y)}{dy}, ik\psi\right)^T. \tag{4.5}$$

From the Stokes equation, after substituting the single wave formulations of $\mathbf{u}_f$ and p_f, $\psi(y)$ must satisfy

$$\frac{d^4\psi}{dy} - 2k^2\frac{d^2\psi}{dy} + k^4\psi = 0, \tag{4.6}$$

and therefore $\psi(y) = C_1e^{ky} + C_2e^{-ky} + C_3ye^{ky} + C_4ye^{-ky}$. Similarly, ϕ_p satisfies the Darcy equations, and then

$$-\bar{\phi}(y) + k^2\bar{\phi}(y) = 0, \tag{4.7}$$

which means that $\bar{\phi}(y) = C_5e^{ky} + C_6e^{-ky}$. In order to determine the coefficients $C_i(i = 1,\dots,6)$, the boundary conditions should be imposed. In the Stokes domain, the Dirichlet boundary condition $\mathbf{u}_f|_{y=1} = 0$ leads to the conditions

$$\psi(1) = \frac{d\psi}{dy}(1) = 0. \tag{4.8}$$

The interface condition (4.2) becomes

$$\nu\frac{d^2\psi}{dy^2} + \alpha\frac{d\psi}{dy} + \nu k^2\psi = 0. \tag{4.9}$$

Instead of the condition (1.3) or (4.3), the Robin condition (2.2) is used in the DDM iterations and can be simplified to

$$-3\nu ki\frac{d\psi}{dy} - \frac{i\nu}{k}\frac{d^3\psi}{dy} - \gamma_f ki\psi = \bar{\eta}_f. \tag{4.10}$$

In fact, we are interested in the relationship of $\psi(0)$ and $\bar{\eta}_f$, that is, in finding R_f such that

$$\psi(0) = R_f\frac{i\bar{\eta}_f}{k}. \tag{4.11}$$

And from the boundary conditions (4.8)–(4.10),

$$R_f = (1,1,0,0)\begin{pmatrix} e^k & e^{-k} & e^k & e^{-k} \\ ke^k & -ke^{-k} & (1+k)e^k & (1-k)e^{-k} \\ 2\nu k^2 + \alpha k & 2\nu k^2 - \alpha k & 2\nu k + \alpha & -2\nu k + \alpha \\ -2\nu k + \gamma_f & 2\nu k + \gamma_f & 0 & 0 \end{pmatrix}^{-1}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{4.12}$$

After careful computations, we deduce

$$\frac{1}{R_f} = \gamma_f + \nu\frac{4(e^{-k}+e^k)^2 - \frac{2\alpha}{k\nu}(e^{2k}-e^{-2k}) + 8 + 16k^2 - 8\frac{\alpha}{\nu}}{\frac{2}{k}(e^{2k}-e^{-2k}) - \frac{\alpha}{\nu}\frac{(e^k-e^{-k})^2}{k^2} - 8 + 4\frac{\alpha}{\nu}}. \tag{4.13}$$

Let us denote the fractional part of the second term on the right-hand side as ζ_f so that $R_f = \frac{1}{\gamma_f+\nu\zeta_f}$. Since $\frac{\partial\zeta_f}{\partial\alpha} > 0$, ζ_f monotonically increases with respect to α or $\frac{\alpha}{\nu}$, so it is enough to set $\alpha = 0$ if we want to obtain the minimal value of ζ_f. ζ_f takes its minimal value at $k = 2$ and $\alpha = 0$, and $\min\zeta_f \approx 6.408$ for all positive α,ν and nonnegative integral k.

Remark 4.1. In fact, α depends on ν and $\mathbb{K}$ (see, for example, [6]) and may be replaced by $\frac{\sqrt{3}\alpha\nu}{\sqrt{\mathrm{trace}(\mathbb{K})}}$, where α is a new parameter independent of the viscosity and hydraulic conductivity. Then ζ_f is independent of ν.

Note that $\mathbf{u}_f \cdot \mathbf{n}_f = -ik\psi(0)$; then

$$\mathbf{u}_f \cdot \mathbf{n}_f = R_f \bar{\eta}_f. \tag{4.14}$$

Similarly, in the Darcy domain, we have the Dirichlet boundary condition $\bar{\phi} = 0$ at the bottom boundary and the Robin condition at the interface Γ

$$\gamma_p K \frac{d\bar{\phi}}{dy} + \bar{\phi} = \bar{\eta}_p. \tag{4.15}$$

We can also build up the relationship

$$\bar{\phi}(0) = R_p \bar{\eta}_p, \tag{4.16}$$

and

$$R_p = \frac{1}{1 + \gamma_p k K \frac{e^k + e^{-k}}{e^k - e^{-k}}}. \tag{4.17}$$

Obviously, R_p and R_f are even functions with respect to k; then only nonnegative integer k should be considered.

Now the parallel Robin–Robin DDM (3.4)–(3.5) can be written as

$$\bar{\eta}_f^{m+1} = \frac{\gamma_f}{\gamma_p}\bar{\eta}_p^m + \left(-1 - \frac{\gamma_f}{\gamma_p}\right)\bar{\phi}^m = \left(\frac{\gamma_f}{\gamma_p} + \left(-1 - \frac{\gamma_f}{\gamma_p}\right) R_p\right)\bar{\eta}_p^m \triangleq \rho_p \bar{\eta}_p^m, \tag{4.18}$$

$$\bar{\eta}_p^{m+1} = -\bar{\eta}_f^m + (\gamma_f + \gamma_p)\mathbf{u}_f^m \cdot \mathbf{n}_f = (-1 + (\gamma_f + \gamma_p)R_f)\bar{\eta}_f^m \triangleq \rho_f \bar{\eta}_f^m. \tag{4.19}$$

So the convergence rate ρ of the iteration is $\rho = \sqrt{|\rho_p \rho_f|}$. For large k, we have the asymptotic expressions

$$R_f \sim \frac{1}{\gamma_f + 2\nu k} \quad \text{and} \quad R_p \sim \frac{1}{1 + \gamma_p k K}, \tag{4.20}$$

and then

$$\rho_p \sim \frac{\gamma_f k K - 1}{\gamma_p k K + 1} \quad \text{and} \quad \rho_f \sim \frac{\gamma_p - 2\nu k}{\gamma_f + 2\nu k}. \tag{4.21}$$

So ρ_p tends to $\frac{\gamma_f}{\gamma_p}$ and ρ_f tends to -1 when k tends to infinity. Combining the above analysis with Theorem 3.3 and Theorem 3.4, we can obtain the following result.

THEOREM 4.2. *The parallel Robin–Robin DDM* (3.4)–(3.5) *converges for any* η_f^0 *and* η_p^0 *if and only if* $\gamma_f \leq \gamma_p$.

Moreover, even for $\gamma_f \leq \gamma_p$, further results can be obtained. For the case of $\gamma_p = \gamma_f = \gamma$, and for large k, the convergence rate ρ has the asymptotic estimation

$$\rho \sim \left(1 - \frac{2}{\gamma k K}\right)^{\frac{1}{2}} \left(1 - \frac{\gamma}{\nu k}\right)^{\frac{1}{2}} \sim 1 - \left(\frac{1}{\gamma K} + \frac{\gamma}{2\nu}\right)\frac{1}{k}. \tag{4.22}$$

The estimation implies that the convergence rate in the finite element context will depend on the mesh size h.

Now let us consider the case of $\gamma_f < \gamma_p$. In this case, ρ_p is bounded by

$$|\rho_p| \leq \max\left(\frac{\gamma_f}{\gamma_p}, \frac{|1-\gamma_f K|}{1+\gamma_p K}\right) \quad \forall\ k. \tag{4.23}$$

And $|\rho_p| \leq \frac{\gamma_f}{\gamma_p}$ if the following inequality holds:

$$\frac{1}{\gamma_f} - \frac{1}{\gamma_p} \leq 2K. \tag{4.24}$$

Recall that $0 \leq R_f \leq \frac{1}{\gamma_f + \nu \min \zeta_f}$ and $\min \zeta_f < 6.41$. Therefore if

$$\gamma_p - \gamma_f \leq 2\nu \min_k \zeta_f \approx 12.82\nu, \tag{4.25}$$

then $|\rho_f| \leq 1$ for all integral k. Hence for the problem under consideration, we have stronger results than Theorems 3.3 and 3.4.

THEOREM 4.3. *If $\gamma_f \leq \gamma_p$, the parallel Robin–Robin DDM* (3.4)–(3.5) *converges for any η_f^0 and η_p^0, and*

- *when $\gamma_f = \gamma_p$, the convergence rate has asymptotic expression $1 - O(\frac{1}{k})$;*
- *when $\gamma_f < \gamma_p$ and if* (4.24) *holds, the convergence rate is $\max(\frac{\gamma_f}{\gamma_p}, \frac{|1-\gamma_f K|}{1+\gamma_p K})^{\frac{1}{2}}$. Moreover if* (4.25) *also holds, then the convergence rate is $\sqrt{\frac{\gamma_f}{\gamma_p}}$.*

Remark 4.4. In the serial Robin–Robin DDM, the iteration (4.18)–(4.19) is implemented by

$$\bar{\eta}_f^{m+1} = \rho_p \bar{\eta}_p^m, \quad \bar{\eta}_p^{m+1} = \rho_f \bar{\eta}_f^{m+1}. \tag{4.26}$$

Then the convergence rate of the serial Robin–Robin DDM is $\rho_p \rho_f$, which means that the serial Robin–Robin DDM is twice as fast as the parallel Robin–Robin DDM. In sections 3 and 5, R_f and R_p become operators, but this remark still holds.

5. Finite element approximations. We next consider finite element discretization of the Robin DDM that was proposed in the previous section. One of the advantages of considering finite element approximations is that we can then derive explicit convergence rates, even for the case $\gamma_f = \gamma_p = \gamma$. Of course, the rate of convergence will depend on the size of the element h. This is different from the case of $\gamma_f < \gamma_p$, where the rate of convergence is independent of h. We will also demonstrate the convergence of the finite element approximation even in the parameter region of $\gamma_p < \gamma_f$, which may seem unlikely in view of Lemma 3.2. One of the key advantages of the finite element setting is the availability of an inverse Poincaré-type inequality [8] that allows us to control various terms.

We consider a regular triangulation $\mathcal{T}_h$ of the global domain $\overline{\Omega}_p \bigcup \overline{\Omega}_f$, which is assumed to be regular and quasi uniform. We also assume that the triangulations $\mathcal{T}_{f,h}, \mathcal{T}_{p,h}$ induced on the subdomains Ω_f and Ω_p are compatible on Γ and that the mesh on the interface Γ is quasi uniform. The induced triangulation on Γ will be denoted $\mathcal{B}_h$. Nonmatching grid or mortar cases will be considered elsewhere. We denote by $X_{p,h} \subset X_p$ a finite element space on the porous media domain Ω_p and denote by $X_{f,h} \subset X_f$ and $Q_{f,h} \subset Q_f$ finite element spaces on the fluid domain Ω_f. We use these spaces to approximate the hydraulic head in the porous media and the fluid velocity and pressure.

Specifically, we choose

$$X_{p,h} = \{\psi_{p,h} \in C^0(\overline{\Omega}_p) \mid \psi_{p,h}|_T \in \mathbb{P}_2(T) \quad \forall T \in \mathcal{T}_{p,h},\ \psi_{p,h}\big|_{\partial\Omega_p\setminus\Gamma} = 0\},$$
$$X_{f,h} = \{\mathbf{v}_{f,h} \in (C^0(\overline{\Omega}_f))^d \mid \mathbf{v}_{f,h}|_T \in (\mathbb{P}_2(T))^d \quad \forall T \in \mathcal{T}_{f,h},\ \mathbf{v}_{f,h}\big|_{\partial\Omega_f\setminus\Gamma} = 0\},$$
$$Q_{f,h} = \{q_{p,h} \in C^0(\overline{\Omega}_f) \mid q_{f,h}|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_{f,h}\}.$$

The spaces $X_{f,h}$ and $Q_{f,h}$ are assumed to satisfy the discrete LBB or inf-sup condition [19, 23].

We also define the discrete trace space on the interface

$$Z_h = \{\eta_h \in C^0(\Gamma) \mid \eta_h|_\tau \in \mathbb{P}_2(\tau) \quad \forall \tau \in \mathcal{B}_h,\ \eta_h|_{\partial\Gamma} = 0\}.$$

It is easy to see that Z_h is the trace space in the sense that

$$Y_{p,h} := X_{p,h}\big|_\Gamma = Z_h,$$
$$Y_{f,h} := X_{f,h}\big|_\Gamma \cdot \mathbf{n}_f = Z_h.$$

The discrete weak formulation of the coupled Stokes–Darcy problem is then given by the following: find $(\mathbf{u}_{f,h}, p_{f,h}) \in X_{f,h} \times Q_{f,h}$ and $\phi_{p,h} \in X_{p,h}$ such that

(5.1)
$$\begin{aligned}
&a_f(\mathbf{u}_{f,h}, \mathbf{v}_f) + b_f(\mathbf{v}_f, p_{f,h}) + g a_p(\phi_{p,h}, \psi_p) + \langle g\phi_{p,h}, \mathbf{v}_f \cdot \mathbf{n}_f\rangle - \langle g\mathbf{u}_{f,h} \cdot \mathbf{n}_f, \psi_p\rangle \\
&\qquad + \alpha\langle P_\tau \mathbf{u}_{f,h}, P_\tau \mathbf{v}_f\rangle = (\mathbf{f}, \mathbf{v}_f)_{\Omega_f} \quad \forall \mathbf{v}_f \in X_{f,h}, \psi_p \in X_{p,h},\\
&b_f(\mathbf{u}_{f,h}, q_f) = 0 \quad \forall q_f \in Q_{f,h}.
\end{aligned}$$

The finite element approximation of the decoupled Stokes–Darcy system with Robin boundary conditions (2.1)–(2.2) can be formulated in the following way: for given $\eta_{p,h} \in L^2(\Gamma)$ and $\eta_{f,h} \in L^2(\Gamma)$, find $(\widehat{\phi}_{p,h}, \widehat{\mathbf{u}}_{f,h}, \widehat{p}_{f,h}) \in X_{p,h} \times X_{f,h} \times Q_{f,h}$ such that

(5.2)
$$\begin{aligned}
&a_f(\widehat{\mathbf{u}}_{f,h}, \mathbf{v}_f) + b_f(\mathbf{v}_f, \widehat{p}_{f,h}) + \omega\gamma_p a_p(\widehat{\phi}_{p,h}, \psi_p) + \omega\langle g\widehat{\phi}_{p,h}, \psi_p\rangle + \gamma_f\langle \widehat{\mathbf{u}}_{f,h} \cdot \mathbf{n}_f, \mathbf{v}_f \cdot \mathbf{n}_f\rangle \\
&\quad + \alpha\langle P_\tau \widehat{\mathbf{u}}_{f,h}, P_\tau \mathbf{v}_f\rangle = (\mathbf{f}, \mathbf{v}_f)_{\Omega_f} + \langle \eta_{f,h}, \mathbf{v}_f \cdot \mathbf{n}_f\rangle + \omega\langle \eta_{p,h}, \psi_p\rangle \quad \forall \psi_p \in X_{p,h}, \mathbf{v}_f \in X_{f,h},\\
&b_f(\widehat{\mathbf{u}}_{f,h}, q_f) = 0 \quad \forall q_f \in Q_{f,h}.
\end{aligned}$$

Similarly to the continuous case, finite element approximations of the coupled Stokes–Darcy system, defined by (5.1), and of the revised Robin approximation, defined by (5.2), are related in the following fashion.

LEMMA 5.1. *For $\eta_{p,h} \in Y_{p,h}$ and $\eta_f \in Y_{f,h}$, $(\widehat{\phi}_{p,h}, \widehat{\mathbf{u}}_{f,h}, \widehat{p}_{f,h}) = (\phi_{p,h}, \mathbf{u}_{f,h}, p_{f,h})$ if and only if $\eta_{p,h}$ and $\eta_{f,h}$ satisfy*

$$\eta_{p,h} = P_{p,h}(\gamma_p \mathbf{u}_{f,h} \cdot \mathbf{n}_f + g\phi_{p,h}), \qquad \eta_{f,h} = P_{f,h}(\gamma_f \mathbf{u}_{f,h} \cdot \mathbf{n}_f - g\phi_{p,h}),$$

where $P_{p,h}$ and $P_{f,h}$ are $L^2(\Gamma)$-projections onto the spaces $Y_{p,h}$ and $Y_{f,h}$ respectively; i.e., for $v \in L^2(\Gamma)$,

$$\langle P_{p,h} v, w_p\rangle = \langle v, w_p\rangle \quad \forall w_p \in Y_{p,h}, \qquad \langle P_{f,h} v, w_f\rangle = \langle v, w_f\rangle \quad \forall w_f \in Y_{f,h}.$$

Remark 5.2. Comparing with the Robin conditions (2.1) and (2.2), we see that we have heuristically used

$$P_{p,h}(\gamma_p \mathbb{K}\nabla\phi_{p,h} \cdot \mathbf{n}_p + g\phi_{p,h}) = \eta_{p,h},$$
$$P_{f,h}(\mathbf{n}_f \cdot (T(\mathbf{u}_{f,h}, p_{f,h}) \cdot \mathbf{n}_f) + \gamma_f \mathbf{u}_{f,h} \cdot \mathbf{n}_f) = \eta_{f,h}.$$

The choices of the spaces $Y_{p,h}$ and $Y_{f,h}$ are not unique; other choices are also possible.

The parallel Robin–Robin domain decomposition finite element method is defined as follows.

1. The initial values of $\eta^0_{p,h} \in Y_{p,h}$ and $\eta^0_{f,h} \in Y_{f,h}$ are guessed; they may be taken to be zero.
2. For $k = 1, 2, \dots,$ solve the discrete Stokes and Darcy systems with Robin conditions independently; i.e., $\phi^m_{p,h} \in X_{p,h}$ is determined from
$$\gamma_p a_p(\phi^m_{p,h}, \psi_p) + \langle g\phi^m_{p,h}, \psi_p\rangle = \langle \eta^m_{p,h}, \psi_p\rangle \quad \forall\, \psi_p \in X_{p,h},$$
and $\mathbf{u}^m_{f,h} \in X_{f,h}$ and $p^m_{f,h} \in Q_{f,h}$ are determined from
$$\begin{aligned} &a_f(\mathbf{u}^m_{f,h}, \mathbf{v}_f) + b_f(\mathbf{v}_f, p^m_{f,h}) + \gamma_f\langle \mathbf{u}^m_{f,h}\cdot\mathbf{n}_f, \mathbf{v}_f\cdot\mathbf{n}_f\rangle + \alpha\langle P_\tau \mathbf{u}^m_{f,h}, P_\tau \mathbf{v}_f\rangle \\ &\qquad = \langle \eta^m_{f,h}, \mathbf{v}_f\cdot\mathbf{n}_f\rangle + (\mathbf{f}, \mathbf{v}_f)_{\Omega_f} \quad \forall\, \mathbf{v}_f \in X_{f,h}, \\ &b_f(\mathbf{u}^m_{f,h}, q_f) = 0 \quad \forall\, q_f \in Q_{f,h}. \end{aligned}$$
3. $\eta^{m+1}_{p,h}$ and $\eta^{m+1}_{f,h}$ are updated by
$$\eta^{m+1}_{f,h} = P_{f,h}(a\eta^m_{p,h} + bg\phi^m_{p,h}), \qquad \eta^{m+1}_{p,h} = P_{p,h}(c\eta^m_{f,h} + d\mathbf{u}^m_f\cdot\mathbf{n}_f).$$

One important observation is that $\eta^m_{p,h}, \eta^m_{f,h}, \phi^m_{p,h}\big|_\Gamma, \mathbf{u}^m_f\cdot\mathbf{n}_f\big|_\Gamma \in Z_h$ for all k, provided that the initial guesses belong to Z_h. Therefore, the projections $P_{p,h}$ and $P_{f,h}$ in the implementation of the algorithm are identity operators.

We now consider the error functions for finite element approximations, just as in the continuous case studied in the previous section. Let

$$\begin{aligned} &\epsilon^m_{p,h} = \eta_{p,h} - \eta^m_{p,h}, \quad \epsilon^m_{f,h} = \eta_{f,h} - \eta^m_{f,h}, \quad e^m_{\phi,h} = \phi_{p,h} - \phi^m_{p,h}, \\ &\mathbf{e}^m_{u,h} = \mathbf{u}_{f,h} - \mathbf{u}^m_{f,h}, \quad e^m_{p,h} = p_{f,h} - p^m_{f,h}. \end{aligned}$$

It is straightforward to verify that

$$\epsilon^m_{p,h} \in Z_h, \qquad \epsilon^m_{f,h} \in Z_h, \qquad e^m_{\phi,h}|_\Gamma \in Z_h, \qquad \mathbf{e}^m_{u,h}\cdot\mathbf{n}_f \in Z_h.$$

It is also easy to see that the error functions satisfy the error equations

$$\gamma_p a_p(e^m_{\phi,h}, \psi_p) + \langle g e^m_{\phi,h}, \psi_p\rangle = \langle \epsilon^m_{p,h}, \psi_p\rangle \quad \forall\, \psi_p \in X_{p,h}, \tag{5.3}$$

$$a_f(\mathbf{e}^m_{u,h}, \mathbf{v}_f) + b_f(\mathbf{v}_f, e^m_{p,h}) + \gamma_f\langle \mathbf{e}^m_{u,h}\cdot\mathbf{n}_f, \mathbf{v}_f\cdot\mathbf{n}_f\rangle + \alpha\langle P_\tau \mathbf{e}^m_{f,h}, P_\tau \mathbf{v}_f\rangle$$
$$\qquad = \langle \epsilon^m_{f,h}, \mathbf{v}_f\cdot\mathbf{n}_f\rangle \quad \forall\, \mathbf{v}_f \in X_{f,h}, \tag{5.4}$$
$$b_f(\mathbf{e}^m_{u,h}, q_f) = 0 \quad \forall\, q_f \in Q_{f,h}, \tag{5.5}$$

and, along the interface,

$$\epsilon^{m+1}_{f,h} = P_{f,h}(a\epsilon^m_{p,h} + bge^m_{\phi,h}), \qquad \epsilon^{m+1}_{p,h} = P_{p,h}(c\epsilon^m_{f,h} + d\mathbf{e}^m_{u,h}\cdot\mathbf{n}_f).$$

It is easy to verify the following relationship for the error functions, just as in the continuous case.

LEMMA 5.3. *The error functions satisfy*

$$\|\epsilon_{p,h}^{m+1}\|_\Gamma^2 = \|\epsilon_{f,h}^m\|_\Gamma^2 + (\gamma_p^2 - \gamma_f^2)\|\mathbf{e}_{u,h}^m \cdot n\|_\Gamma^2 - 2(\gamma_f + \gamma_p)a_f(\mathbf{e}_{u,h}^m, \mathbf{e}_{u,h}^m) - 2(\gamma_f + \gamma_p)\|P_\tau \mathbf{e}_{u,h}^m\|_\Gamma^2,$$

$$\|\epsilon_{f,h}^{m+1}\|_\Gamma^2 = \left(\frac{\gamma_f}{\gamma_p}\right)^2 \|\epsilon_{p,h}^m\|_\Gamma^2 + \left(1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right)\|ge_{\phi,h}^m\|_\Gamma^2 - 2\gamma_f\left(1 + \frac{\gamma_f}{\gamma_p}\right) g a_p(e_{\phi,h}^m, e_{\phi,h}^m).$$

The key ingredient in deriving an explicit convergence rate for the case of $\gamma_f = \gamma_p$ is the following estimate.

LEMMA 5.4.

$$\|\epsilon_{p,h}^m\|_\Gamma^2 \le C(\|\mathbb{K}^{-1}\|^{1/2} + \gamma_p h^{-1/2}\|\mathbb{K}\|^{1/2})^2 a_p(e_{\phi,h}^m, e_{\phi,h}^m).$$

Proof. The key in the proof of this lemma is an extension operator. Let $\mathcal{N}_{p,h}$ be the set of nodes for the finite element triangulation on $\overline{\Omega}_p$, and let $\mathcal{N}_{p,\Gamma} = \mathcal{N}_{p,h}|_\Gamma$. Denote by $E_{p,h}$ the zero extension operator from $Y_{p,h} = Z_h$ to $X_{p,h}$:

$$E_{p,h}\epsilon_{p,h}^m(P) = \begin{cases} \epsilon_{p,h}^m(P) & \text{if } P \in \mathcal{N}_{p,\Gamma}, \\ 0 & \text{if } P \in \mathcal{N}_{p,h}\backslash\mathcal{N}_{p,\Gamma}. \end{cases}$$

Then, we have

$$\|E_{p,h}\epsilon_{p,h}^m\|_{L^2(\Omega_p)}^2 \approx h^d \sum_{P\in\mathcal{N}_{p,h}} (E_{p,h}\epsilon_{p,h}^m(P))^2 \approx h^d \sum_{P\in\mathcal{N}_{p,\Gamma}} (\epsilon_{p,h}^m(P))^2 \approx h\|\epsilon_{p,h}^m\|_\Gamma^2. \tag{5.6}$$

Note that $E_{p,h}\epsilon_{p,h}^m \in X_{p,h}$ due to the definition of the extension operator and the fact that $\epsilon_{p,h}^m \in Z_h$. Hence, we may set $\psi_p = E_{p,h}\epsilon_{p,h}^m$ in (5.3) and utilize the Cauchy–Schwarz inequality to deduce

$$\begin{aligned} \|\epsilon_{p,h}^m\|_\Gamma^2 &= \gamma_p a_p(e_{\phi,h}^m, E_{p,h}\epsilon_{p,h}^m) + \langle ge_{\phi,h}^m, \epsilon_{p,h}^m\rangle \\ &\le \gamma_p a_p(e_{\phi,h}^m, e_{\phi,h}^m)^{1/2} a_p(E_{p,h}\epsilon_{p,h}^m, E_{p,h}\epsilon_{p,h}^m)^{1/2} + \|ge_{\phi,h}^m\|_\Gamma \|\epsilon_{p,h}^m\|_\Gamma. \end{aligned} \tag{5.7}$$

On the other hand, thanks to the inverse inequality for finite element spaces and (5.6), we deduce

$$\begin{aligned} a_p(E_{p,h}\epsilon_{p,h}^m, E_{p,h}\epsilon_{p,h}^m) &\le \|\mathbb{K}\|\|\nabla E_{p,h}\epsilon_{p,h}^m\|_{L^2(\Omega_p)}^2 \\ &\le Ch^{-2}\|\mathbb{K}\|\|E_{p,h}\epsilon_{p,h}^m\|_{L^2(\Omega_p)}^2 \le Ch^{-1}\|\mathbb{K}\|\|\epsilon_{p,h}^m\|_\Gamma^2. \end{aligned} \tag{5.8}$$

Combining the inequality (3.18) with (5.7)–(5.8), we obtain

$$\begin{aligned} \|\epsilon_{p,h}^m\|_\Gamma^2 \le\ & C\gamma_p h^{-1/2}\|\mathbb{K}\|^{1/2} a_p(e_{\phi,h}^m, e_{\phi,h}^m)^{1/2}\|\epsilon_{p,h}^m\|_\Gamma \\ &+ gC_p^{1/2}\|\mathbb{K}^{-1}\|^{1/2} a_p(e_{\phi,h}^m, e_{\phi,h}^m)^{1/2}\|\epsilon_{p,h}^m\|_\Gamma, \end{aligned}$$

which implies the lemma. ∎

Remark 5.5. Similar results are available for P_1 conforming and nonconforming elements for classical second-order elliptic problems (see [31, 32]).

Now for $\gamma_f = \gamma_p = \gamma$, by Lemma 5.3, we have

$$\|\epsilon_{p,h}^{m+1}\|_\Gamma^2 = \|\epsilon_{f,h}^m\|_\Gamma^2 - 4\gamma a_f(\mathbf{e}_{u,h}^m, \mathbf{e}_{u,h}^m) - 4\gamma\|P_\tau \mathbf{e}_{u,h}^m\|_\Gamma^2,$$
$$\|\epsilon_{f,h}^{m+1}\|_\Gamma^2 = \|\epsilon_{p,h}^m\|_\Gamma^2 - 4\gamma g a_p(e_{\phi,h}^m, e_{\phi,h}^m).$$

Combining the above equalities with Lemma 5.4, we deduce

$$\|\epsilon_{p,h}^{m+1}\|_\Gamma^2 \le \|\epsilon_{f,h}^m\|_\Gamma^2, \quad \|\epsilon_{f,h}^{m+1}\|_\Gamma^2 \le (1 - C\gamma(\|\mathbb{K}^{-1}\|^{1/2} + \gamma h^{-\frac{1}{2}}\|\mathbb{K}\|^{1/2})^{-2})\|\epsilon_{p,h}^m\|_\Gamma^2.$$

Therefore, we have proved the following lemma.

LEMMA 5.6. *For $\gamma_f = \gamma_p = \gamma$, we have*

$$\|\epsilon_{p,h}^{m+1}\|_\Gamma^2 \le (1 - C\gamma h(h^{1/2}\|\mathbb{K}^{-1}\|^{1/2} + \gamma\|\mathbb{K}\|^{1/2})^{-2})\|\epsilon_{p,h}^{m-1}\|_\Gamma^2,$$
$$\|\epsilon_{f,h}^{m+1}\|_\Gamma^2 \le (1 - C\gamma h(h^{1/2}\|\mathbb{K}^{-1}\|^{1/2} + \gamma\|\mathbb{K}\|^{1/2})^{-2})\|\epsilon_{f,h}^{m-1}\|_\Gamma^2.$$

This further implies convergence with convergence rate proportional to $1 - Ch$ for the case of $\gamma_p = \gamma_f = \gamma$.

THEOREM 5.7. *If $\gamma_p = \gamma_f = \gamma$, then*

$$a_p(e_{\phi,h}^m, e_{\phi,h}^m) + a_f(\mathbf{e}_{u,h}^m, \mathbf{e}_{u,h}^m) + \|e_{p,h}^m\|_{L^2(\Omega_f)}^2 + \|\epsilon_{p,h}^m\|_\Gamma^2 + \|\epsilon_{f,h}^m\|_\Gamma^2$$
$$\le C(1 - Ch\gamma(h^{1/2}\|\mathbb{K}^{-1}\|^{1/2} + \gamma\|\mathbb{K}\|^{1/2})^{-2})^{\lfloor \frac{m}{2} \rfloor}\left(\|\epsilon_p^0\|_\Gamma^2 + \|\epsilon_f^0\|_\Gamma^2\right).$$

Remark 5.8. Now assume that hydraulic conductivity tensor $\mathbb{K} = K\mathbb{I}$; then the convergence rate of the Robin–Robin domain decomposition finite element method is $1 - Ch$ since $(1 - ChK/(h^{1/2} + \gamma K)^{-2})^{\frac{1}{2}} = 1 - O(h)$ for small h. This result is consistent with the result (4.22).

In the case $\gamma_f \ne \gamma_p$, we have the same result as Theorem 3.4 with geometric convergence with a rate independent of h.

THEOREM 5.9. *If* (3.22) *is satisfied, then*

$$a_p(e_{\phi,h}^m, e_{\phi,h}^m) + a_f(\mathbf{e}_{u,h}^m, \mathbf{e}_{u,h}^m) + \|e_{p,h}^m\|_{L^2(\Omega_f)}^2 + \|\epsilon_{p,h}^m\|_\Gamma^2 + \|\epsilon_{f,h}^m\|_\Gamma^2$$
$$\le C\left(\frac{\gamma_f}{\gamma_p}\right)^m \left(\|\epsilon_{p,h}^0\|_\Gamma^2 + \|\epsilon_{f,h}^0\|_\Gamma^2\right).$$

Now we consider the case $\gamma_f > \gamma_p$ which is counterintuitive in view of Lemma 3.2. Nevertheless, at the discrete level, we are able to control the excessive growth term by the decay terms so long as the parameters γ_f and γ_p are chosen to be close (depending on $\mathbb{K}$ and h). Indeed, thanks to Lemma 5.4, we have

$$\left(\left(\frac{\gamma_f}{\gamma_p}\right)^2 - 1\right)\|\epsilon_{p,h}^m\|_\Gamma^2 - \gamma_f\left(1 + \frac{\gamma_f}{\gamma_p}\right) g\, a_p(e_{\phi,h}^m, e_{\phi,h}^m)$$
$$\le \left(\left(\left(\frac{\gamma_f}{\gamma_p}\right)^2 - 1\right) C(\|\mathbb{K}^{-1}\|^{1/2} + \gamma_p h^{-1/2}\|\mathbb{K}\|^{1/2})^2 - \gamma_f\left(1 + \frac{\gamma_f}{\gamma_p}\right) g\right) a_p(e_{\phi,h}^m, e_{\phi,h}^m)$$
$$= \gamma_f\left(1 + \frac{\gamma_f}{\gamma_p}\right)\left(\left(\frac{1}{\gamma_p} - \frac{1}{\gamma_f}\right) C(\|\mathbb{K}^{-1}\|^{1/2} + \gamma_p h^{-1/2}\|\mathbb{K}\|^{1/2})^2 - g\right) a_p(e_{\phi,h}^m, e_{\phi,h}^m)$$
$$\le 0$$

provided that the following condition holds:

$$0 \le \frac{1}{\gamma_p} - \frac{1}{\gamma_f} \le \frac{gh}{C(h^{1/2}\|\mathbb{K}^{-1}\|^{1/2} + \gamma_p\|\mathbb{K}\|^{1/2})^2}. \tag{5.9}$$

An undesirable feature here is the dependence on the mesh size h, but this constraint is natural since the scheme could be divergent for the continuous problem (see Theorem 4.2). For every small h, the γ's must be very close in order to have the above inequality satisfied. This would lead to a very slow convergence rate.

We then have, when combined with Lemma 5.3 and under assumption (5.9),

$$\|\epsilon_{p,h}^{m+1}\|_\Gamma^2 \le \|\epsilon_{f,h}^m\|_\Gamma^2 - 2(\gamma_f+\gamma_p)a_f(\mathbf{e}_{u,h}^m,\mathbf{e}_{u,h}^m) \le \|\epsilon_{f,h}^m\|_\Gamma^2 - 4\gamma_p a_f(\mathbf{e}_{u,h}^m,\mathbf{e}_{u,h}^m),$$
$$\|\epsilon_{f,h}^{m+1}\|_\Gamma^2 \le \|\epsilon_{p,h}^m\|_\Gamma^2 - \gamma_f\left(1+\frac{\gamma_f}{\gamma_p}\right) g\, a_p(e_{\phi,h}^m,e_{\phi,h}^m) \le \|\epsilon_{p,h}^m\|_\Gamma^2 - 2\gamma_f g\, a_p(e_{\phi,h}^m,e_{\phi,h}^m).$$

Summing the two inequalities for k from 0 to N, we deduce the convergence of $e_{\phi,h}^m$ and $\mathbf{e}_{u,h}^m$ that leads to the convergence (without rate) of all quantities involved. A rate of convergence can be derived just as in the case of $\gamma_p=\gamma_f$. Indeed, we may deduce, with the help of Lemma 5.4 and the last inequality above,

$$\|\epsilon_{f,h}^{m+1}\|_\Gamma^2 \le (1 - C\gamma_f h(h^{1/2}\|\mathbb{K}^{-1}\|^{1/2}+\gamma_p\|\mathbb{K}\|^{1/2})^{-2})\|\epsilon_{p,h}^m\|_\Gamma^2,$$

and hence we obtain

$$\|\epsilon_{p,h}^{m+1}\|_\Gamma^2 \le (1 - C\gamma_f h(h^{1/2}\|\mathbb{K}^{-1}\|^{1/2}+\gamma_p\|\mathbb{K}\|^{1/2})^{-2})\|\epsilon_{p,h}^{m-1}\|_\Gamma^2,$$
$$\|\epsilon_{f,h}^{m+1}\|_\Gamma^2 \le (1 - C\gamma_f h(h^{1/2}\|\mathbb{K}^{-1}\|^{1/2}+\gamma_p\|\mathbb{K}\|^{1/2})^{-2})\|\epsilon_{f,h}^{m-1}\|_\Gamma^2.$$

This further implies convergence with convergence rate proportional to $1-Ch$ for the case of $\gamma_f>\gamma_p$ under the additional constraint of (5.9). Therefore, we have proved the following theorem.

THEOREM 5.10. *For $\gamma_f>\gamma_p$, assume that the additional constraint* (5.9) *holds. Then, we have the following convergence result for our parallel Robin–Robin domain decomposition finite element method for the coupled Stokes–Darcy system:*

$$a_p(e_{\phi,h}^m,e_{\phi,h}^m)+a_f(\mathbf{e}_{u,h}^m,\mathbf{e}_{u,h}^m)+\|e_{p,h}^m\|_{L^2(\Omega_f)}^2+\|\epsilon_{p,h}^m\|_\Gamma^2+\|\epsilon_{f,h}^m\|_\Gamma^2$$
$$\le C(1-C\gamma_f h(h^{1/2}\|\mathbb{K}^{-1}\|^{1/2}+\gamma_p\|\mathbb{K}\|^{1/2})^{-2})^{\lfloor \frac{m}{2}\rfloor}\left(\|\epsilon_p^0\|_\Gamma^2+\|\epsilon_f^0\|_\Gamma^2\right).$$

6. Computational experiments.

We present some preliminary computational results based on the parallel Robin–Robin domain decomposition finite element method for the coupled Stokes–Darcy system with the BJSJ interface boundary condition.

The following example is used in the parallel Robin–Robin DDM. Let $\Omega_p=(0,\pi)\times(-1,0)$, $\Omega_f=(0,\pi)\times(0,1)$, and let $\Gamma=\{0\le x\le\pi, y=0\}$. Assume that the hydraulic conductivity is homogeneous and isotropic, i.e., $\mathbb{K}=K\mathbb{I}$, and we have the solutions

$$u_{f,1}=v'(y)\cos x, \qquad u_{f,2}=v(y)\sin x, \qquad p_f=0, \qquad \phi_p=e^y\sin x,$$

where $v(y)=-K-\frac{gy}{2\nu}+(-\frac{\alpha g}{4\nu^2}+\frac{K^2}{2})y^2$. Then these functions exactly satisfy the Stokes–Darcy system with the BJSJ interface boundary condition.

We now consider the differences between iterations of the Robin–Robin DDM and the exact solution of the discrete finite element problems; the convergence and superconvergence behaviors have been analyzed in [7]. Specifically, in Figure 6.1, we plot the relative errors $\frac{\|\mathbf{u}_{f,h}^m-\mathbf{u}_{f,h}\|_{\ell^2}}{\|\mathbf{u}_{f,h}\|_{\ell^2}}$ versus the iteration counter m; the results of the other three relative errors $\frac{\|\phi_h^m-\phi_h\|_{\ell^2}}{\|\phi_h\|_{\ell^2}}$, $\frac{\|\eta_{f,h}^m-\eta_{f,h}\|_{\ell^2}}{\|\eta_{f,h}\|_{\ell^2}}$, and $\frac{\|\eta_{p,h}^m-\eta_{p,h}\|_{\ell^2}}{\|\eta_{p,h}\|_{\ell^2}}$ are almost the same. The computational results presented confirm our theoretical convergence analysis.

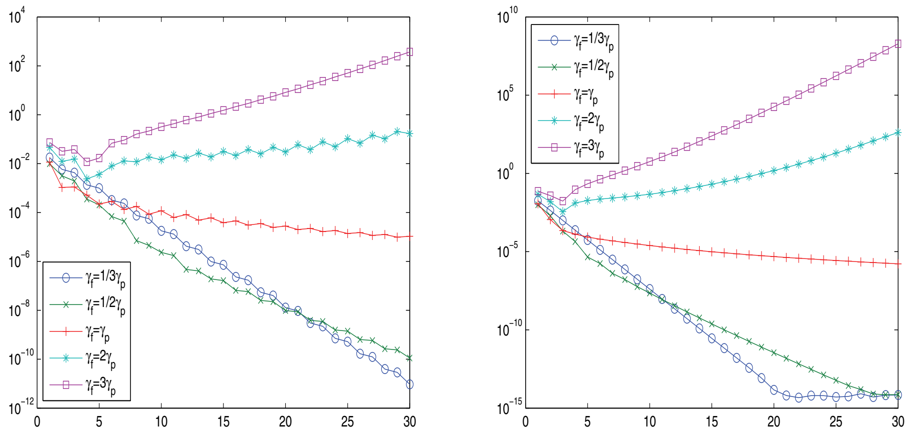

FIG. 6.1. *Relative errors* $\frac{\|\mathbf{u}_{f,h}^m - \mathbf{u}_{f,h}\|_{\ell^2}}{\|\mathbf{u}_{f,h}\|_{\ell^2}}$ *of Robin–Robin DDM versus* m. *Left: parallel version. Right: serial version.*

By setting $K = 1$, $\nu = 1$, $\alpha = 1$ and $\gamma_p = 1$, Figure 6.1 shows that, for the Robin–Robin DDMs,

- if $\gamma_f < \gamma_p(\gamma_f = \frac{1}{3}\gamma_p$ or $\frac{1}{2}\gamma_p)$, convergence is very fast;
- if $\gamma_f > \gamma_p(\gamma_f = 3\gamma_p$ or $2\gamma_p)$, the iterative method diverges;
- if $\gamma_f = \gamma_p$, the iterative method converges, but with a slow rate;
- the serial implementation is twice as fast as the parallel one.

7. Conclusions. The convergence behavior of Robin–Robin DDMs for the Stokes–Darcy equations were studied. The geometric convergence can be obtained for appropriate choices of the parameters both for the continuous PDE and the finite element, and it is possible to get the convergence rate independent of the mesh size, which is different from the recent results obtained in [41] for the classical second-order elliptic problem.

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