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A bound on the vertical transport of heat in the 'ultimate' state of slippery convection at large Prandtl numbers

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An upper bound on the rate of vertical heat transport is established in three dimensions for stress-free velocity boundary conditions on horizontally periodic plates. A variation of the background method is implemented that allows negative values of the quadratic form to yield 'small' ($O(1/Pr)$) corrections to the subsequent bound. For large (but finite) Prandtl numbers this bound is an improvement over the 'ultimate' $Ra^{1/2}$ scaling and, in the limit of infinite Pr, agrees with the bound of $Ra^{5/12}$ recently derived in that limit for stress-free boundaries.

Key words: Bénard convection, mantle convection, turbulent convection

1. Introduction

Since Lord Rayleigh's mathematical description of convection in Rayleigh (1916), and computation of the onset of convective instability, his idealized model (since called Rayleigh–Bénard convection after his seminal description and the experimental work of Henri Bénard) has been extensively studied. Lord Rayleigh demonstrated that the inert, conductive solution's stability is dependent on a single non-dimensional number that is proportional to the forcing imposed on the system (since called the Rayleigh number). While Rayleigh's work was restricted to two dimensions and stress-free vertical boundaries, the onset of convective instability was later investigated for a variety of boundary conditions; see Sparrow, Goldstein & Jonsson (1963) for one example. As the Rayleigh number is increased beyond this critical value (dependent on boundary conditions), roll-like structures appear, followed by more complicated dynamics and eventually turbulence; see Busse, Swinney & Gollub (1985) for an overview of this phenomenon.

Of more physical relevance to the geophysical and astrophysical community is when the Rayleigh number Ra is asymptotically larger than the critical Rayleigh number Ra_c at which the conductive profile becomes unstable (Ahlers 2009). Recent experimental and numerical investigations have focused on the 'ultimate' regime of

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strongly forced convection, in the hope of determining functional relationships between the relevant statistical quantities, the Rayleigh number, and other material parameters of the system; see Ahlers, Grossmann & Lohse (2009), Funfschilling, Bodenschatz & Ahlers (2009), Urban, Musilová & Skrbek (2011) and He *et al.* (2012) for some experimental results, and Amati *et al.* (2005), Johnston & Doering (2009), Shishkina *et al.* (2010), Stevens, Verzicco & Lohse (2010) and Stevens, Lohse & Verzicco (2011) for numerical explorations. Of key interest is the enhancement of vertical heat flux due to convection, as measured by the dimensionless Nusselt number Nu. Modulo the impact of the geometry (it is generally accepted that the dependence on aspect ratio of a cylindrical or horizontally periodic container, is small), the Nusselt number is believed to depend on Ra and Pr. Typically this dependence is depicted as a series of power laws whose coefficients vary according to the dynamic state of the flow (Grossmann & Lohse 2000, 2001, 2011).

To formulate the problem explicitly, we will follow Lord Rayleigh (Rayleigh 1916) and consider the rate of vertical heat transport in Rayleigh Bénard convection, as described by the classical (non-dimensional) Boussinesq equations with stress-free vertical boundaries:

$$\frac{1}{Pr}\left(\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right)+\nabla p=\Delta\boldsymbol{u}+Ra\,\boldsymbol{k}T, \quad \nabla\cdot\boldsymbol{u}=0, \tag{1.1}$$

$$\frac{\partial T}{\partial t}+\boldsymbol{u}\cdot\nabla T=\Delta T, \tag{1.2}$$

$$\left.\frac{\partial u_j}{\partial z}\right|_{z=0,1}=u_3|_{z=0,1}=0, \quad j=1,2 \tag{1.3}$$

$$T|_{z=0}=1, \quad T|_{z=1}=0, \tag{1.4}$$

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0, \quad T|_{t=0}=T_0, \tag{1.5}$$

where $\boldsymbol{u}=(u_1,u_2,u_3)$ is the fluid velocity field, p is the kinematic pressure, T is the (scaled) temperature field, $\boldsymbol{k}$ is the unit upward vector, $Ra=(g\alpha\Delta T h^3)/\nu\kappa$ is the Rayleigh number where g is the force of gravity, α is the thermal expansion coefficient, ΔT is the dimensional measure of the enforced temperature gradient, h is the dimensional height of the box, and ν and κ are the kinematic viscosity and thermal diffusivity respectively. The Prandtl number is defined as $Pr=\nu/\kappa$. The domain in the non-dimensional coordinates is the (non-dimensionalized) region

$$\Omega=[0,L_x]\times[0,L_y]\times[0,1], \tag{1.6}$$

where periodicity in the horizontal directions is assumed, implying the flow has mean zero (this is consistent with the horizontal momentum equations, and guarantees the applicability of the Poincaré-type inequality: see e.g. Constantin & Foias 1988), that is,

$$\int_\Omega u_j\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0, \quad j=1,2, \tag{1.7}$$

and for all $t\geqslant 0$, if this is satisfied initially.

The Nusselt number is defined (Otero 2002) as

$$Nu=1+\left\langle\int u_3T\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right\rangle, \tag{1.8}$$

where

$$\langle f(\cdot)\rangle = \limsup_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s \tag{1.9}$$

is the long time average and u_3 is the vertical component of velocity.

Approximately half a century after Rayleigh's mathematical definition of the problem, Malkus (1954) addressed the problem of establishing the rate of heat transport as a function of Ra, considering the contribution of marginally stable thermal boundary layers to the transport of heat in fully developed turbulence, postulating that $Nu \sim Ra^{1/3}$. Kraichnan (1962) then proposed that instead of the boundary layers dictating the rate of heat transport, the rate-limiting factor would be the flow through the bulk, indicating that $Nu \sim Ra^{1/2}$ (within logarithmic corrections, and dependence on Pr). Recently Grossmann & Lohse (2000, 2001, 2011) and Stevens *et al.* (2013) have developed a comprehensive theory that predicts different effective scaling laws dependent on the relative size of the thermal and viscous boundary layers. For moderately large Rayleigh numbers ($Ra \lesssim 10^{12}$) experiments and simulations agree well with the Grossmann and Lohse theory. However, to date, the only experiments claiming to observe this 'ultimate' state predicted in Grossmann & Lohse (2011) come from the same experimental apparatus (Ahlers *et al.* 2012; He *et al.* 2012), although these results must be weighed against the observations of Urban *et al.* (2012) that a 'spurious cross-over ... that might be misinterpreted as a transition to the ultimate Kraichnan regime' may be a result of measuring the mean temperature of the sample as the arithmetic mean of the top and bottom plate temperatures. With these considerations in mind, it is beneficial to consider the impact that analysis can have on the problem.

Variational bounds on the convective heat transport were first formed by Howard (1963) and elucidated in Busse (1969). Some time later, Doering & Constantin (1996) applied the background method to the problem, finding that $Nu \leqslant cRa^{1/2}$ for no-slip, fixed-temperature boundaries, although it is clear from their argument that the same bound can be applied to stress-free, fixed-temperature convection as well. Further investigation using the background method included variations in the temperature boundary conditions (Otero *et al.* 2002; Wittenberg 2010) and numerical solutions of the underlying Euler–Lagrange equations (Plasting & Kerswell 2003), but an improvement over the $Ra^{1/2}$ bound has not been made for $Pr \leqslant \infty$ in three dimensions.

In contrast, the infinite Prandtl number problem, which is a valuable approximation for fluids such as some silicone oils, the earth's mantle as well as many gases under high pressure (e.g. Busse 1989, Getling 1998 and Bodenschatz, Pesch & Ahlers 2000), has yielded several interesting results. For no-slip, fixed-temperature boundaries it can rigorously be shown that to within a logarithmic correction, $Nu \leqslant Ra^{1/3}$ (Constantin & Doering 1999; Doering, Otto & Reznikoff 2006; Otto & Seis 2011). The work of Doering *et al.* (2006) can be combined with Wittenberg (2010) to yield similar bounds for no-slip convection with varying temperature boundary conditions (Whitehead & Wittenberg 2013). For no-slip, fixed-temperature boundaries some of these results were verified (at infinite Pr) by the asymptotic and numerical calculations of Plasting & Ierley (2005) and Ierley, Kerswell & Plasting (2006). The results of Constantin & Doering (1999) and Doering *et al.* (2006) were extended to the case of finite but large Pr in Wang (2008*a*,*b*) respectively, indicating that the rigorous estimates for infinite Prandtl numbers are indicative of bounds on heat transport even when the limit (of $Pr = \infty$) is not reached.

In Plasting & Ierley (2005) and Ierley *et al.* (2006) the authors also investigated the behaviour of infinite Pr convection with stress-free, fixed-temperature boundaries and found that $Nu \lesssim Ra^{5/12}$, identifying the optimal background profile achieving this scaling as piecewise linear (albeit non-monotonic). Using these results and a similar calculation performed for finite Pr in two dimensions (Otero 2002) as motivation, Whitehead & Doering (2011, 2012) proved that for stress-free, fixed-temperature boundaries, either at finite Pr in two dimensions or $Pr = \infty$ in three dimensions, $Nu \lesssim Ra^{5/12}$. This paper will extend the work of Whitehead & Doering (2012) to large but finite Prandtl numbers for stress-free, fixed-temperature boundaries.

These particular boundary conditions (stress-free on the horizontal plates, and periodic sidewalls) are not well suited to experiment, and likely induce very different dynamics than what would be seen in the more traditional setting of no-slip plates. Even so, it is likely that the physically interesting situations where convection is asymptotically strong (and the ultimate state occurs) will not have exact no-slip boundaries either. In particular asymptotically strong convection in stellar hydrodynamics (convective cells in the outer envelope of the sun for example) would be better modelled by either stress-free boundaries or more appropriately by the Navier-slip condition (an interpolation between no-slip and stress-free). In conjunction with Whitehead & Doering (2011, 2012) this paper indicates that these variations in boundary conditions (that are physically relevant) need to be considered more carefully, particularly with respect to the theory of Grossmann & Lohse (2011).

At sufficiently large Prandtl number, we can formally consider the infinite Prandtl number limit as

$$\nabla p^0 = \Delta \boldsymbol{u}^0 + Ra\boldsymbol{k}T^0, \quad \nabla \cdot \boldsymbol{u}^0 = 0, \tag{1.10}$$

$$\frac{\partial T^0}{\partial t} + \boldsymbol{u}^0 \cdot \nabla T^0 = \Delta T^0, \tag{1.11}$$

$$\left.\frac{\partial u_j^0}{\partial z}\right|_{z=0,1} = u_3^0|_{z=0,1} = 0, \quad j = 1, 2 \tag{1.12}$$

$$T^0|_{z=0} = 1, \quad T^0|_{z=1} = 0. \tag{1.13}$$

This simplification removes the nonlinear and time derivative terms from the momentum equation, *slaving* the temperature field to the velocity. This slaving was key in previous bounds on the Nusselt number, allowing (as described above) stricter bounds than in the finite Pr case.

Following Wang (2008*a*,*b*), we consider the full Boussinesq system as a perturbation of the infinite Prandtl number model, implying that the velocity field is only a perturbation from a linear slaving with the temperature field. This near-linear relationship is exploited to bound the Nusselt number as

$$Nu \leqslant 0.3546\, Ra^{5/12} + cGr^2 Ra^{1/4} \tag{1.14}$$

when the Grashof number $Gr = Ra/Pr$ is small, i.e. $Pr/Ra \geqslant c_0$ (where c_0 is given in (2.25)). It is pertinent to point out at this point that the bound $Nu \lesssim Ra^{5/12}$ derived here is valid only for a finite range of Rayleigh numbers, $Ra \leqslant Pr/c_0$.

Throughout this manuscript, we assume the physically important case of high Rayleigh number $Ra \gg 1$ so that we may have non-trivial dynamics. We also follow the mathematical tradition of denoting the small parameter as ε, i.e.

$$\varepsilon = \frac{1}{Pr}. \tag{1.15}$$

Here c will denote a generic non-dimensional constant independent of the Rayleigh number and Prandtl number.

The rest of the manuscript is organized as follows. In § 2 we recall a few *a priori* estimates on the solutions to the Boussinesq system at large Prandtl number. In § 3, we derive the $Ra^{5/12}$ upper bound for the Nusselt number at large Prandtl number. In § 4, we offer concluding remarks.

2. *A priori* estimates

Some of the estimates contained in this section (or at least closely related estimates) are contained in Wang (2007) and Constantin & Doering (1996). For completeness we re-derive all the estimates needed for the current result. The mathematical inequalities referred to are listed (with references) in the Appendix.

The bulk of this paper is concerned with the long time average of certain relevant statistical quantities. To that end it is important to note that for long time averages, one can change the initial time without affecting the average. More specifically, we have

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s = \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t+t_0} f(s)\,\mathrm{d}s \tag{2.1}$$

for any bounded (measurable) function f, and any $t_0 > 0$. Because the L^2 norms of the velocity and temperature are uniformly bounded in time for this particular system (even for Leray weak solutions), then the long time average of the vertical heat transport always exists. It can be shown that (suitable weak) solutions to this specific system become regular after an initial period of time (the so-called eventual regularity) provided that the Prandtl number is sufficiently large (Wang 2005, 2007, 2008*b*). Therefore, we focus only on the (sufficiently large) time interval over which the weak solutions are regular enough to justify all the operations involved below. The eventual regularity of the (suitable weak) solutions is implicitly implied in all of the *a priori* estimates below. In this sense, we will assume sufficient regularity on $\boldsymbol{u}(t)$ and $T(t)$ to derive the following estimates.

2.1. *Maximum principle and previous estimates*

Throughout this manuscript, we assume that the range of initial temperature T_0 is contained in the unit interval [0, 1]. Hence we deduce by the maximum principle that the range of T is contained in [0, 1] for all time, i.e.

$$\|T\|_{L^\infty} \leqslant 1 \tag{2.2}$$

where we will use the notation $\|\cdot\|_{L^\infty}$ to refer to the L^∞ norm. Following the background method developed in Doering & Constantin (1996) and Constantin & Doering (1996), we will decompose the temperature field as $T(x, y, z, t) = \theta(x, y, z, t) + \tau(z)$ and assume that the background temperature profile $\tau(z)$ under consideration is always contained in the unit interval [0, 1] as well (see (3.27)). Therefore, the fluctuation temperature field $\theta = T - \tau$ satisfies the same estimate,

$$\|\theta\|_{L^\infty} \leqslant 1. \tag{2.3}$$

Note that the estimates on the Nusselt number derived in Constantin & Doering (1996) (again noting that these bounds are valid for stress-free boundaries as well as no-slip) imply that

$$\langle\|\nabla\boldsymbol{u}\|^2\rangle \leqslant cRa^{3/2}, \tag{2.4}$$

$$\langle \|\nabla T\|^2 \rangle \leqslant c Ra^{1/2}, \tag{2.5}$$

for all suitable weak solutions of the Boussinesq system with arbitrary Prandtl number where $\langle \cdot \rangle$ represents long time average as defined in (1.9) and we let $\| \cdot \| = \| \cdot \|_{L^2}$, where

$$\|f\|_{L^p} = \left(\int f^p(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \right)^{1/p} \tag{2.6}$$

defines the L^p norms.

2.2. *Long time supremum*

For the first estimate derived in this paper, multiply the velocity equation (1.1) by $\boldsymbol{u}$, and after integrating over the domain (including several integrations by parts) and rearranging some terms, we see that

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}(t)\|^2 + \|\nabla \boldsymbol{u}(t)\|^2 = Ra \int_{\Omega} u_3(t) T(t)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z. \tag{2.7}$$

Applying the Cauchy–Schwarz inequality (A 1) to the right-hand side, followed by the Poincaré inequality (A 2) applied to $\boldsymbol{u}(t)$ (noting that $\|u_3(t)\| \leqslant \|\boldsymbol{u}(t)\|$, and using the maximum principle as discussed above) we arrive at

$$\begin{aligned} \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}(t)\|_{L^2}^2 + \|\nabla \boldsymbol{u}(t)\|^2 &\leqslant Ra \|T(t)\| \|u_3(t)\| \\ &\leqslant c\, Ra \|\nabla \boldsymbol{u}(t)\|. \end{aligned} \tag{2.8}$$

Applying the Young inequality (A 3) to the right-hand side yields

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}(t)\|^2 + \|\nabla \boldsymbol{u}(t)\|^2 \leqslant \frac{c^2}{2} Ra^2 + \frac{1}{2}\|\nabla \boldsymbol{u}(t)\|^2. \tag{2.9}$$

Rearranging, and again making use of the Poincaré inequality, we obtain the bound

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}(t)\|^2 \leqslant c Ra^2 - \frac{c_p^2}{2}\|\boldsymbol{u}(t)\|^2. \tag{2.10}$$

Applying the Gronwall inequality (A 5) and considering the long time limit, we deduce that

$$\limsup_{t\to\infty} \|\boldsymbol{u}(t)\| \leqslant \frac{c_p}{c} Ra. \tag{2.11}$$

2.3. *Large Prandtl number estimates*

2.3.1. *Bounds on the Stokes operator*

For the next estimate, multiply the velocity equation (1.1) by $A\boldsymbol{u}(t)$, where A denotes the Stokes operator with viscosity one and the associated boundary conditions (formally the Stokes operator is the Laplacian operator followed by the Leray projection onto divergence-free fields: see e.g. Constantin & Foias 1988), and integrate (by parts when necessary) over the domain, again applying Cauchy–Schwarz throughout and using the L^∞ norm on the nonlinear term, yielding

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}(t)\|^2 + \|A\boldsymbol{u}(t)\|^2 \leqslant Ra \|T(t)\| \|A\boldsymbol{u}(t)\| + \varepsilon \|\nabla \boldsymbol{u}(t)\| \|A\boldsymbol{u}(t)\| \|\boldsymbol{u}(t)\|_{L^\infty}. \tag{2.12}$$

Applying the Agmon inequality (A 6) and employing the maximum principle on the temperature field $T(t)$ results in

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}(t)\|^2 + \|A\boldsymbol{u}(t)\|^2 \leqslant Ra|\varOmega|^{1/2}\|A\boldsymbol{u}(t)\| + c_A\varepsilon\|\nabla \boldsymbol{u}(t)\|^{3/2}\|A\boldsymbol{u}(t)\|^{3/2}$$
$$\leqslant \frac{1}{2}\|A\boldsymbol{u}(t)\|^2 + Ra^2|\varOmega| + \frac{27}{4}c_A^4\varepsilon^4\|\nabla \boldsymbol{u}(t)\|^6, \tag{2.13}$$

where the generalized Young inequality (A 3) was used twice to obtain the last line, first with $a = \|A\boldsymbol{u}(t)\|/2, p = 2$, and $b = 2Ra|\varOmega|^{1/2}, q = 2$ and second with $a = (1/3)^{3/4}\|A\boldsymbol{u}(t)\|^{3/2}, p = 4/3$, and $b = 3^{3/4}c_A\varepsilon\|\nabla \boldsymbol{u}(t)\|^{3/2}, q = 4$ (following the notation in the Appendix). Finally the Poincaré inequality on $\nabla \boldsymbol{u}(t)$ with constant c_p is used to arrive at

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}(t)\|^2 + c_p^2\|\nabla \boldsymbol{u}(t)\|^2 \leqslant 2|\varOmega|Ra^2 + \frac{27}{2}c_A^4\varepsilon^4\|\nabla \boldsymbol{u}(t)\|^6. \tag{2.14}$$

It follows that the ball of radius $2|\varOmega|^{1/2}Ra/c_p$ is invariant for $\|\nabla \boldsymbol{u}(t)\|$ if the following large Prandtl number (small Grashof number) condition holds:

$$Gr = \frac{Ra}{Pr} \leqslant \frac{c_p^{3/2}}{2\,3^{3/4}c_A|\varOmega|^{1/2}}. \tag{2.15}$$

That is, if (2.15) is satisfied and $\|\nabla \boldsymbol{u}(t_0)\| \leqslant 2|\varOmega|^{1/2}Ra/c_p$ at some time t_0, then $\|\nabla \boldsymbol{u}(t)\| \leqslant 2|\varOmega|^{1/2}Ra/c_p$ for all $t \geqslant t_0$. On the other hand, estimate (2.4) implies that for Ra sufficiently large, any orbit will enter this ball of radius $2|\varOmega|^{1/2}Ra/c_p$ (provided that the initial time is chosen after any irregularities are present). Hence this is an absorbing ball and

$$\limsup_{t\to\infty} \|\nabla \boldsymbol{u}(t)\| \leqslant 2Ra|\varOmega|^{1/2}/c_p. \tag{2.16}$$

Inserting this into (2.13) and taking the long time average (relying on (2.15)), we have the following estimate:

$$\langle\|A\boldsymbol{u}\|^2\rangle \leqslant cRa^2|\varOmega|. \tag{2.17}$$

2.3.2. *Bounds on the time derivative*

Next, we need an estimate on the time derivative of the velocity. For this purpose we differentiate the velocity (1.1) in time to reach

$$\varepsilon\left(\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \left(\frac{\partial \boldsymbol{u}}{\partial t}\boldsymbol{\cdot}\nabla\right)\boldsymbol{u} + (\boldsymbol{u}\boldsymbol{\cdot}\nabla)\frac{\partial \boldsymbol{u}}{\partial t}\right) + \nabla\frac{\partial p}{\partial t} = \varDelta\frac{\partial \boldsymbol{u}}{\partial t} + Ra\boldsymbol{k}\frac{\partial T}{\partial t}. \tag{2.18}$$

Multiplying this equation by $\partial \boldsymbol{u}/\partial t$, integrating over $\varOmega$ and applying Cauchy–Schwarz and the generalized Hölder inequalities (A 1) and (A 8) (applied to the nonlinear term specifically), we deduce that for t sufficiently large to avoid any regularity issues in the initial phase:

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|^2 \leqslant Ra\left\|\frac{\partial T(t)}{\partial t}\right\|_{H^{-1}}\left\|\nabla\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|$$
$$+\varepsilon\|\boldsymbol{u}(t)\|_{L^3}\left\|\nabla\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|\left\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|_{L^6}, \tag{2.19}$$

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where the Sobolev norm $\|\cdot\|_{H^{-1}}$ is defined as

$$\|f\|_{H^{-1}} = \|f\| + \|\nabla^{-1}f\|. \tag{2.20}$$

Using the Sobolev inequalities (A 9) we see that for large t (sufficiently large to avoid irregularities in the solution) the right-hand side of this is less than or equal to

$$\begin{aligned} Ra\left\|\frac{\partial T(t)}{\partial t}\right\|_{H^{-1}}\left\|\nabla\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\| + c_S\varepsilon\,\|\nabla \boldsymbol{u}(t)\|\left\|\nabla\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|^2 \\ \leqslant \frac{1}{4}\left\|\nabla\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|^2 + Ra^2\left\|\frac{\partial T(t)}{\partial t}\right\|_{H^{-1}}^2 + \frac{2}{c_p}Ra|\Omega|^{1/2}c_S\varepsilon\left\|\nabla\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|^2, \end{aligned} \tag{2.21}$$

where $c_S = c_{S1}c_{S2}$, and (2.16) and the Young inequality were used in the last line. This implies that

$$\left\langle\left\|\nabla\frac{\partial \boldsymbol{u}}{\partial t}\right\|^2\right\rangle \leqslant 2Ra^2\left\langle\left\|\frac{\partial T}{\partial t}\right\|_{H^{-1}}^2\right\rangle \tag{2.22}$$

provided the following large Prandtl number (small Grashof number) condition is satisfied:

$$Gr = \frac{Ra}{Pr} \leqslant \frac{c_p}{8c_S|\Omega|^{1/2}}. \tag{2.23}$$

Setting

$$c_0 = \frac{c_p}{2|\Omega|^{1/2}}\min\left\{\frac{c_p^{1/2}}{3^{3/4}\,c_A}, \frac{1}{4c_S}\right\}, \tag{2.24}$$

we combine the two large Prandtl number conditions (2.15) and (2.23) into the following large Prandtl number (small Grashof number) condition

$$Gr = \frac{Ra}{Pr} \leqslant c_0. \tag{2.25}$$

In order to express the right-hand side of (2.22) in terms of the Rayleigh number, we consider the H^{-1} norm applied to the temperature equation (1.2) to deduce

$$\begin{aligned} \left\|\frac{\partial T(t)}{\partial t}\right\|_{H^{-1}} &\leqslant \|T(t)\boldsymbol{u}(t)\| + \|\nabla T(t)\| \\ &\leqslant \|\boldsymbol{u}(t)\| + \|\nabla T(t)\|, \end{aligned} \tag{2.26}$$

where we have used the maximum principle on the temperature field T. This further implies, thanks to (2.4) and (2.5),

$$\left\langle\left\|\frac{\partial T}{\partial t}\right\|_{H^{-1}}^2\right\rangle \leqslant 2\langle\|\boldsymbol{u}\|^2 + \|\nabla T\|^2\rangle \leqslant cRa^{3/2}. \tag{2.27}$$

Inserting this back into (2.22), we have

$$\left\langle\left\|\nabla\frac{\partial \boldsymbol{u}}{\partial t}\right\|^2\right\rangle \leqslant CRa^{7/2}. \tag{2.28}$$

2.3.3. *Bounds on the nonlinear term*

For the final estimate, we need to bound $\langle\|\nabla((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\|^2\rangle$ in terms of the non-dimensional parameters of the system. With this in mind, note that the momentum equation can be rewritten as

$$A\boldsymbol{u} = RaP(\boldsymbol{k}T) - \varepsilon P\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right), \tag{2.29}$$

where P denotes the Leray–Hopf projector from the square-integrable space onto the divergence-free subspace. Classic elliptic regularity arguments applied to the Stokes operator A (see e.g. Constantin & Foias 1988) give

$$\|A^{3/2}\boldsymbol{u}\|^2 \leqslant C\left\{Ra^2\|\nabla T\|^2 + \varepsilon^2\left\|\nabla\frac{\partial \boldsymbol{u}}{\partial t}\right\|^2 + \varepsilon^2\|\nabla((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\|^2\right\}. \tag{2.30}$$

Using the product rule, and pulling out an L^∞ norm, we see that

$$\|\nabla((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\|^2 \leqslant c(\|\boldsymbol{u}\|_{L^\infty}^2\|\Delta\boldsymbol{u}\|^2 + \|\nabla\boldsymbol{u}\|_{L^4}^4). \tag{2.31}$$

Using the Agmon inequality (A 6) on the first term, and the Sobolev inequality (A 9) on the second, we bound the portion of the right-hand side in parentheses by

$$\|\nabla\boldsymbol{u}\|\,\|A\boldsymbol{u}\|\,\|\Delta\boldsymbol{u}\|^2 + \|\nabla\boldsymbol{u}\|\,\|\Delta\boldsymbol{u}\|^3, \tag{2.32}$$

which after application of the elliptic regularity of the Stokes operator (A 11) and interpolation inequality (A 12) indicates that

$$\begin{aligned}\|\nabla((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\|^2 &\leqslant c\|\nabla\boldsymbol{u}\|\,\|A\boldsymbol{u}\|^3 \\ &\leqslant C\|\nabla\boldsymbol{u}\|^{5/2}\|A^{3/2}\boldsymbol{u}\|^{3/2},\end{aligned} \tag{2.33}$$

where c and C are distinct constants.

Inserting this into (2.30) and rearranging the terms to be a bound on $\|A^{3/2}\boldsymbol{u}\|^2$ only (and allowing C to absorb all of the relevant constants) we see that (using (2.5) and (2.28))

$$\begin{aligned}\langle\|A^{3/2}\boldsymbol{u}\|^2\rangle &\leqslant C\left\langle Ra^2\|\nabla T\|^2 + \varepsilon^2\left\|\nabla\frac{\partial \boldsymbol{u}}{\partial t}\right\|^2 + \varepsilon^8\|\nabla\boldsymbol{u}\|^{10}\right\rangle \\ &\leqslant C\left(Ra^{2+1/2} + \varepsilon^2 Ra^{7/2} + \varepsilon^8 Ra^{19/2}\right).\end{aligned} \tag{2.34}$$

where we have used the following estimate that relies on (2.16) and (2.4),

$$\langle\|\nabla\boldsymbol{u}\|^{10}\rangle \leqslant \limsup_{t\to\infty}\|\nabla\boldsymbol{u}(t)\|^8\langle\|\nabla\boldsymbol{u}\|^2\rangle \leqslant C\,Ra^{19/2}, \tag{2.35}$$

valid for initial times taken beyond the point of any irregularities in the solution $\boldsymbol{u}(t)$ and $T(t)$. This also implies that

$$\begin{aligned}\langle\|\nabla((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\|^2\rangle &\leqslant C\langle\|\nabla\boldsymbol{u}\|^{5/2}\|A^{3/2}\boldsymbol{u}\|^{3/2}\rangle \\ &\leqslant C\langle\|\nabla\boldsymbol{u}\|^{10}\rangle^{1/4}\langle\|A^{3/2}\boldsymbol{u}\|^2\rangle^{3/4} \\ &\leqslant C\left(Ra^{19/2}\right)^{1/4}\left(Ra^{5/2} + \varepsilon^2 Ra^{7/2} + \varepsilon^8 Ra^{19/2}\right)^{3/4} \\ &\leqslant C\left(Ra^{17/4} + \varepsilon^{3/2}Ra^5 + \varepsilon^6 Ra^{19/2}\right),\end{aligned} \tag{2.36}$$

where we have used the Hölder inequality (A 8) on the second line.

3. Bound on the Nusselt number

3.1. *The background method at large Prandtl number*

The Nusselt number (1.8) can equivalently be defined by (see e.g. Otero 2002 for detailed derivations)

$$Nu = 1 + \left\langle \int u_3\theta \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \right\rangle \tag{3.1}$$

$$= \left\langle \int |\nabla T|^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \right\rangle, \tag{3.2}$$

where $T(x, y, z, t) = \tau(z) + \theta(x, y, z, t)$ is the temperature field and $\tau(z)$ is the background temperature profile (as proposed in the theory of Constantin & Doering (1996), Doering & Constantin (1996), Constantin & Doering (1999), Doering *et al.* (2006), Whitehead & Doering (2011, 2012) as a generalization of E. Hopf's original calculation Temam (2000)), satisfying the same boundary conditions as T, and $(\boldsymbol{u}, \theta)$ are suitable weak solutions to

$$\nabla p = \Delta \boldsymbol{u} + Ra\, \boldsymbol{k}\theta + \varepsilon \mathbf{f}, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{3.3}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla\theta + u_3\tau'(z) = \Delta\theta + \tau''(z), \tag{3.4}$$

$$\left.\frac{\partial u_j}{\partial z}\right|_{z=0,1} = u_3|_{z=0,1} = 0, \quad j = 1, 2 \tag{3.5}$$

$$\theta|_{z=0,1} = 0, \tag{3.6}$$

$$\boldsymbol{f} = -\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right), \tag{3.7}$$

with appropriate initial conditions $(\boldsymbol{u}_0, \theta_0)$. The Nusselt number is a statistical property of the Boussinesq system in the sense that it is the average of $1 + (\int_\Omega u_3\theta/|\Omega|)$ over the whole phase space with respect to some appropriate invariant measure (stationary statistical solution) of the Boussinesq system (Wang 2008*b*).

Writing the momentum equation as a perturbation away from the infinite Prandtl number (1.10) ($\varepsilon = 0$) model illustrates the methodology employed in this paper. Heuristically the perturbation from the infinite Pr system is small as $\varepsilon \to 0$ or $Pr \to \infty$. Using the estimates on the Nusselt number obtained in Whitehead & Doering (2012) for infinite Prandtl number stress-free convection on this system yields bounds on the Nusselt number for large Pr within $O(\varepsilon)$. The $O(\varepsilon)$ terms can then be bounded using the estimates derived in the previous section, yielding a bound on the vertical heat transport for a restricted region of Ra so that (2.25) is satisfied, that agrees in the limit of $Pr \to \infty$ with the bounds derived in Whitehead & Doering (2011, 2012).

Using the background decomposition described above, we multiply the temperature equation (3.4) by θ and integrate over Ω to reach

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|^2 + \|\nabla\theta(t)\|^2 + \int_\Omega \tau' \frac{\partial\theta(t)}{\partial z} + \int_\Omega \tau' u_3(t)\theta(t) = 0. \tag{3.8}$$

From the definition of θ we also have

$$\|\nabla T(t)\|^2 = \|\nabla\theta(t)\|^2 + 2\int_\Omega \tau' \frac{\partial\theta(t)}{\partial z} + \|\tau'\|^2. \tag{3.9}$$

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Following Constantin & Doering (1999) and Doering *et al.* (2006), we combine these two identities to see that

$$\left\langle \|\nabla T\|^2 \right\rangle = \|\tau'\|^2 - \left\langle \int_\Omega (|\nabla\theta|^2 + 2\tau' u_3\theta) \right\rangle. \tag{3.10}$$

In traditional applications of the background method, this gives the bound on the Nusselt number

$$Nu \leqslant \|\tau'\|^2 \tag{3.11}$$

provided that

$$\mathcal{Q} = \left\langle \int_\Omega \left(|\nabla\theta|^2 + 2\tau' u_3\theta\right) \right\rangle \geqslant 0. \tag{3.12}$$

For finite Prandtl number in three dimensions, however, we do not guarantee the positivity of $\mathcal{Q}$, but we do show that all $O(1)$ terms are positive definite, showing that the only negative contribution to $\mathcal{Q}$ is $o(\varepsilon)$. With this in mind, in the following we will break $\mathcal{Q}$ into an $\varepsilon = 0$ component (for which the methods in Whitehead & Doering (2012) in the absence of the balance parameter can be applied directly), and use the estimates obtained in § 2 to bound the remaining quantities.

To estimate the indefinite term in $\mathcal{Q}$, we combine the divergence of the momentum equation with the Laplace operator applied to the evolution equation for u_3 to arrive at

$$\Delta^2 u_3 = -Ra\Delta_H\theta + \varepsilon\left(-\Delta_H f_3 + \frac{\partial^2 f_1}{\partial x\partial z} + \frac{\partial^2 f_2}{\partial y\partial z}\right), \tag{3.13}$$

$$u_3|_{z=0,1} = \left.\frac{\partial^2 u_3}{\partial z^2}\right|_{z=0,1} = 0, \quad \Delta_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{3.14}$$

Using the periodic horizontal boundary conditions, we note that this equation can be rewritten in terms of the horizontal Fourier coefficients of each variable, i.e. we let $f(x, y, z) = \sum_{\boldsymbol{k}} f_{\boldsymbol{k}}(z)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, where $\boldsymbol{k} = (k_x, k_y)^{\mathrm{T}}$ and

$$f_{\boldsymbol{k}} = (1/|L_x L_y|)\int_0^{L_x}\int_0^{L_y} f(x, y, z)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}y\,\mathrm{d}x, \tag{3.15}$$

and then (3.13) is equivalent to

$$(k^2 - D^2)^2\hat{u}_{3k} = Ra\,k^2\hat{\theta}_k + \varepsilon(k^2\hat{f}_{3k} + \mathrm{i}k_1 D\hat{f}_{1k} + \mathrm{i}k_2 D\hat{f}_{2k}), \tag{3.16}$$

where $k = |\boldsymbol{k}|$ is the length of the horizontal wavenumber $\boldsymbol{k}$, $D = \mathrm{d}/\mathrm{d}z$ is the derivative operator in the vertical direction, and $\hat{\cdot}$ indicates the Fourier coefficient of the corresponding variable.

The quadratic form $\mathcal{Q}$ can be rewritten in terms of its Fourier expansion as well, and one can see (Wittenberg 2010) that maintaining $\mathcal{Q} \geqslant 0$ is equivalent to maintaining

$$\mathcal{Q}_k = \left\langle \|D\hat{\theta}_k\|^2 + k^2\|\hat{\theta}_k\|^2 + 2\mathrm{Re}\int_0^1 \tau'\hat{u}_{3k}\hat{\theta}_k\,\mathrm{d}z \right\rangle \geqslant 0 \tag{3.17}$$

for all wavenumbers $\boldsymbol{k}$, where $\|\cdot\|$ now refers to the L^2 norm in the z-direction only and Re refers to the real part of a complex quantity. In the following we will use (3.16), showing that all $O(1)$ terms of $\mathcal{Q}_k$ are bounded below by zero, and then the $o(\varepsilon)$ remaining terms will be estimated using the results of § 2.

3.2. *Pseudo-vorticity*

Following the motivation in Whitehead & Doering (2011, 2012), we denote the pseudo-differential operator $|\boldsymbol{\nabla}_H| = \sqrt{-\boldsymbol{\nabla}_H \cdot \boldsymbol{\nabla}_H} = \sqrt{-\Delta_H}$ and introduce the pseudo-vorticity ω as

$$\Delta u_3 = |\boldsymbol{\nabla}_H|\omega, \tag{3.18}$$

which in terms of the Fourier coefficients is expressed as

$$(-k^2 + D^2)\hat{u}_{3k} = k\hat{\omega}_k. \tag{3.19}$$

The pseudo-vorticity takes the place of the scalar vorticity in two dimensions utilized in Whitehead & Doering (2011). In that paper the two-dimensional scalar vorticity was used (with the stress-free boundary conditions) to include an enstrophy balance that proved to be key in controlling the growth of the vertical velocity near the boundaries. Although the vortex-stretching term is absent at infinite Prandtl number, the vorticity is not a scalar so the same balance does not exist, and therefore Whitehead & Doering (2012) introduced the pseudo-vorticity as a quantity that imitates the role of the vorticity in two dimensions. The slaving of the vertical velocity to the temperature fluctuation at infinite Prandtl number then includes a pseudo-enstrophy balance automatically. For large but finite Prandtl number this relationship is still present, although the inertial terms (and hence vortex-stretching) are present at $O(\varepsilon)$. Eliminating $\hat{u}_{3k}$ in (3.16), we tie the pseudo-vorticity to the temperature fluctuations as

$$k(-k^2 + D^2)\hat{\omega}_k = Ra\, k^2\hat{\theta}_k + \varepsilon(k^2\hat{f}_{3k} + \mathrm{i}k_1 D\hat{f}_{1k} + \mathrm{i}k_2 D\hat{f}_{2k}). \tag{3.20}$$

Modulo the $O(\varepsilon)$ inertial terms, this is identical to the relationship exploited in Whitehead & Doering (2012). To see where the pseudo-enstrophy balance arises, consider the definition of the Nusselt number given by (3.2), and the relationship (3.10). Noting that in Fourier space

$$\|\boldsymbol{\nabla}\theta\|^2 = \sum_k \left(\|D\hat{\theta}_k\|^2 + k^2\|\hat{\theta}_k\|^2\right), \tag{3.21}$$

we see that for $k \neq 0$,

$$\begin{aligned}
k^2|\hat{\theta}_k|^2 &\geqslant \frac{1}{Ra^2}\left(|(-k^2 + D^2)\hat{\omega}_k|^2 + \varepsilon^2\left|k\hat{f}_{3k} + \mathrm{i}\frac{k_1}{k}D\hat{f}_{1k} + \mathrm{i}\frac{k_2}{k}D\hat{f}_{2k}\right|^2\right.\\
&\quad \left. - 2\varepsilon\left|(-k^2 + D^2)\hat{\omega}_k\right|\left|k\hat{f}_{3k} + \mathrm{i}\frac{k_1}{k}D\hat{f}_{1k} + \mathrm{i}\frac{k_2}{k}D\hat{f}_{2k}\right|\right)\\
&\geqslant \frac{1}{2Ra^2}\left|(-k^2 + D^2)\hat{\omega}_k\right|^2 - \frac{\varepsilon^2}{Ra^2}\left|k\hat{f}_{3k} + \mathrm{i}\frac{k_1}{k}D\hat{f}_{1k} + \mathrm{i}\frac{k_2}{k}D\hat{f}_{2k}\right|^2\\
&\geqslant \frac{1}{2Ra^2}\left|(-k^2 + D^2)\hat{\omega}_k\right|^2 - \frac{\varepsilon^2}{Ra^2}\left(\left|k\hat{f}_{3k}\right| + \left|D\hat{f}_{1k}\right| + \left|D\hat{f}_{2k}\right|\right)^2,
\end{aligned} \tag{3.22}$$

so that the L^2 norms of the temperature fluctuations θ yield a pseudo-enstrophy balance with $O(\varepsilon^2)$ corrections due to the inertial terms.

It is shown in § 5.1 of Whitehead & Doering (2012) that the use of the balance parameter is equivalent (at least in terms of the qualitative bound) to the presence of a non-monotonic piecewise linear background profile. The key notion is that either case introduces an energy balance into the variational formulation of the Nusselt number. The balance parameter does this by utilizing the equivalent definition of the Nusselt number (derived directly from the equations of motion)

$$Nu = \frac{1}{Ra}\left\langle \int |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right\rangle, \tag{3.23}$$

while the non-monotonic slope of the background profile introduces a stabilizing component to the variational formulation. To fully utilize this energy balance we need to determine the relationship between that balance and the pseudo-enstrophy. In Fourier space this is done by first letting $\hat{u}_{Hk} = (\hat{u}_{1k}, \hat{u}_{2k})^{\mathrm{T}}$. Then it follows from (3.19) and incompressibility of the flow field (expressed in Fourier space as $k_x\hat{u}_{1k} + k_y\hat{u}_{2k} + D\hat{u}_{3k} = 0$) that

$$\begin{aligned} |k|^2|\hat{\omega}_k|^2 &= |D^2\hat{u}_{3k}|^2 + |k|^2|D\hat{u}_{3k}|^2 + |k|^4|\hat{u}_{3k}|^2 \\ &= |k_xD\hat{u}_{1k} + k_yD\hat{u}_{2k}|^2 + |k|^2|D\hat{u}_{3k}|^2 + |k|^4|\hat{u}_{3k}|^2 \quad \text{(since } \boldsymbol{u} \text{ is divergence-free)} \\ &\leqslant 2|k|^2|D\hat{u}_{Hk}|^2 + |k|^2|D\hat{u}_{3k}|^2 + |k|^4|\hat{u}_{3k}|^2 \\ &\leqslant 2|k|^2(|D\hat{u}_{Hk}|^2 + |D\hat{u}_{3k}|^2 + |k|^2|\hat{u}_{3k}|^2). \end{aligned} \tag{3.24}$$

Dividing this through by $|k|^2$ we see that

$$|\hat{\omega}_k|^2 \leqslant 2(|D\hat{u}_{Hk}|^2 + |D\hat{u}_{3k}|^2 + |k|^2|\hat{u}_{3k}|^2), \tag{3.25}$$

or in real space

$$\|\omega\|^2 \leqslant 2\|\nabla \boldsymbol{u}\|^2. \tag{3.26}$$

3.3. *Bounds on heat transport at large Pr*

We now choose a specific background temperature profile following Whitehead & Doering (2012)

$$\tau'(z) = \begin{cases} 1 - \left(\dfrac{1+p}{2\delta} - p\right) z, & 0 \leqslant z \leqslant \delta, \\ \dfrac{1}{2} + p\left(z - \dfrac{1}{2}\right), & \delta \leqslant z \leqslant 1 - z, \\ \left(\dfrac{1+p}{2\delta} - p\right)(1-z), & 1-\delta \leqslant z \leqslant 1. \end{cases} \tag{3.27}$$

We focus on this piecewise linear, non-monotonic background profile, but neglect to consider the balance parameter ($b = 0$ in the notation of Whitehead & Doering 2012) in order to keep the derivation simpler, and because the balance parameter will not affect the resultant exponential dependence of *Nu* on *Ra*. With this in mind, we apply exactly the same calculations as in Whitehead & Doering (2012) in the absence of the balance parameter to see that the $O(1)$ terms of $\mathscr{Q}_k$ are positive definite, i.e.

$$\begin{aligned} &\frac{1}{2Ra^2}\|(k^2 - D^2)\hat{\omega}_k\|^2 + \|D\hat{\theta}_k\|^2 + \frac{p}{Ra}\|\hat{\omega}_k\|^2 \\ &\quad - \frac{1+p}{\delta}\mathrm{Re}\left\{\int_0^\delta \hat{u}_{3k}\bar{\hat{\theta}}_k + \int_{1-\delta}^1 \hat{u}_{3k}\bar{\hat{\theta}}_k\right\} \geqslant 0, \end{aligned} \tag{3.28}$$

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for all k, is equivalent to enforcing

$$\frac{k^4}{Ra^2} + \frac{p}{Ra} - \frac{3^{3/2}(1+p)^2 k}{2^2\, 5^2}\delta^3 \geqslant 0$$
$$\Rightarrow \frac{3^{3/2}(1+p)^2}{2^2\, 5^2}\delta^3 \leqslant \frac{k^3}{Ra^2} + \frac{p}{kRa}, \tag{3.29}$$

implying that

$$\delta \leqslant \frac{2^{4/3}\, 5^{2/3}\, p^{1/4}}{3^{3/4}(1+p)^{2/3}} \frac{1}{Ra^{5/12}}, \tag{3.30}$$

where

$$k_m = \left(\frac{p}{3}Ra\right)^{1/4} \tag{3.31}$$

is the horizontal wavenumber that saturates the bound on δ. Letting $p = 3/29$ (optimally chosen to minimize the prefactor in the final bound: see Whitehead & Doering 2012 for details) we choose the optimal size of the boundary layer as

$$\delta = \frac{5^{2/3}\, 29^{5/12}}{2^2\, 3^{1/2}} \frac{1}{Ra^{5/12}}. \tag{3.32}$$

Therefore, using (3.9),

$$\begin{aligned} Nu &= \langle \|\nabla T\|^2 \rangle \\ &= \|\tau'\|^2 - \left\langle \int_\Omega (|\nabla\theta|^2 + 2\tau' u_3\theta) \right\rangle \\ &= \|\tau'\|^2 - \left\langle \|\nabla\theta\|^2 + 2p\int_\Omega \theta u_3 - \frac{1+p}{\delta}\left\{\int_0^\delta u_3\theta + \int_{1-\delta}^1 u_3\theta\right\}\right\rangle. \end{aligned} \tag{3.33}$$

Using the identity

$$\left\langle \int_\Omega u_3\theta \right\rangle = \frac{1}{Ra}\left\langle \int_\Omega |\nabla \boldsymbol{u}|^2 \right\rangle \tag{3.34}$$

and (3.26), we can bound the Nusselt number from above as

$$\begin{aligned} Nu &\leqslant \|\tau'\|^2 - \sum_k \left\langle \|k\hat{\theta}_k\|^2 + \|D\hat{\theta}_k\|^2 + \frac{p}{Ra}\|\hat{\omega}_k\|^2 - \frac{1+p}{\delta}\mathrm{Re}\left\{\int_0^\delta \hat{u}_{3k}\bar{\hat{\theta}}_k + \int_{1-\delta}^1 \hat{u}_{3k}\bar{\hat{\theta}}_k\right\}\right\rangle \\ &\leqslant \|\tau'\|^2 + \frac{\varepsilon^2}{Ra^2}\langle \|\nabla \boldsymbol{f}\|^2\rangle - \sum_k \left\langle \frac{1}{2Ra^2}\|(k^2 - D^2)\hat{\omega}_k\|^2 + \|D\hat{\theta}_k\|^2 + \frac{p}{Ra}\|\hat{\omega}_k\|^2 \right. \\ &\quad \left. - \frac{1+p}{\delta}\mathrm{Re}\left\{\int_0^\delta \hat{u}_{3k}\bar{\hat{\theta}}_k + \int_{1-\delta}^1 \hat{u}_{3k}\bar{\hat{\theta}}_k\right\}\right\rangle \\ &\leqslant \|\tau'\|^2 + \frac{\varepsilon^2}{Ra^2}\langle \|\nabla \boldsymbol{f}\|^2\rangle, \end{aligned} \tag{3.35}$$

as long as the size of the boundary layer is chosen according to (3.32) so that (3.28) is satisfied.

The remainder term is multiplied through, and then the Young inequality is used to combine everything into two terms as

$$\frac{\varepsilon^2}{Ra^2}\langle\|\nabla \boldsymbol{f}\|^2\rangle \leqslant \frac{2\varepsilon^2}{Ra^2}\left\langle\left(\left\|\nabla\frac{\partial \boldsymbol{u}}{\partial t}\right\|^2 + \|\nabla((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\|^2\right)\right\rangle. \tag{3.36}$$

These terms are bounded by (2.28) and (2.36) respectively, to arrive at

$$\frac{\varepsilon^2}{Ra^2}\left\langle\|\nabla \boldsymbol{f}\|^2\right\rangle \leqslant C\left(\varepsilon^2 Ra^{9/4} + \varepsilon^{7/2}Ra^3 + \varepsilon^8 Ra^{15/2} + \varepsilon^2 Ra^{3/2}\right). \tag{3.37}$$

Inserting this back into the bound on Nu, calculating $\|\tau'\|^2$ explicitly, and rewriting the remainder in terms of the Grashof number Gr, we can see that

$$\begin{aligned} Nu &\leqslant \frac{2^{11}\,3^{1/2}}{5^{2/3}\,29^{29/12}}Ra^{5/12} + C\left(Gr^2Ra^{1/4} + \varepsilon^{1/2}Gr^3 + \varepsilon^{1/2}Gr^{15/2} + \varepsilon^{1/2}Gr^{3/2}\right) \\ &\sim 0.3546Ra^{5/12} + CGr^2Ra^{1/4}, \end{aligned} \tag{3.38}$$

for $Gr \leqslant c_0$ for stress-free (slippery) horizontal plates in three dimensions.

4. Concluding remarks

The bound (3.38) at small Gr is consistent with the infinite Prandtl ($Gr = 0$) bound obtained in Whitehead & Doering (2012), albeit with a less optimal prefactor. ($Nu \lesssim 0.28764\,Ra^{5/12}$ at $Pr = \infty$, although the current estimate does not consider the impact of the balance parameter, which may subtly improve the prefactor as indicated in Whitehead & Doering (2012).) The improvement provided by this result is shown in figure 1(*b*). Specifically, for sufficiently large Prandtl numbers, the lower curves in this plot indicate a tighter bound than that dictated by Doering & Constantin (1996) (the upper curve in the plot). Figure 1(*a*) compares this bound with the theoretical predictions of the Grossmann and Lohse theory (Grossmann & Lohse 2000, 2001, 2004; Stevens *et al.* 2013). To make a valid comparison, the relationship $Pr = c_0 Ra$ (marginally satisfying (2.25)) was chosen and inserted into (2.1) and (2.2) of Stevens *et al.* (2013) to derive the asymptotic relation (in the limit of $Re \to 0$):

$$Nu \sim \frac{c_4^{2/3}Ra^{1/3}}{4^{1/3}c_1^{1/3}c_0}, \tag{4.1}$$

where c_4 and c_1 are given in Stevens *et al.* (2013). One can see immediately that the Grossmann and Lohse prediction is consistent with the current bound, and understandably it is closest to the current bound if c_0 is ‘small’. It is possible that the significant difference between (3.38) and the prediction of the Grossmann and Lohse theory is because the analysis here is not sharp, even in the limit of infinite Pr. The current result and that in Whitehead & Doering (2011, 2012) indicate the need to consider the impact of velocity boundary conditions (in this case stress-free) on the predictions of the Grossmann and Lohse theory.

The Grossmann and Lohse theory is based on the premise that the effective size and influence of the thermal and viscous boundary layers, and their relationship with the bulk flow, dictate different regimes for which there are equivalent scaling laws for the Nusselt and Reynolds numbers. This theory does not depend explicitly on the flow being three-dimensional, nor does it necessarily require the presence of a no-slip boundary, although the effects of a viscous boundary layer will clearly be

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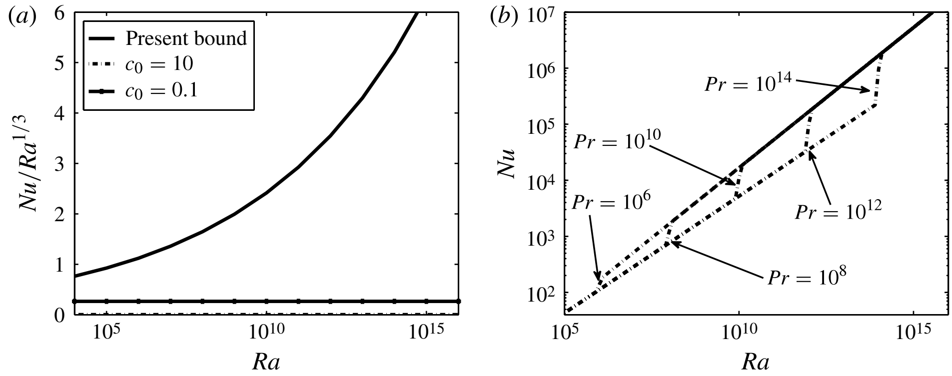

FIGURE 1. Comparison of the asymptotic bound obtained in this paper and Doering & Constantin (1996) with the theory predicted in Grossmann & Lohse (2000, 2001) and Stevens *et al.* (2013). (*a*) The $Ra^{1/3}$ bound obtained here for Prandtl numbers satisfying $Pr = c_0 Ra$ (the marginal case given by (2.25)) in comparison to the predictions of the Grossmann and Lohse theory for various values of c_0. (*b*) The equivalent bound (the current result combined with that in Doering & Constantin 1996) for a variety of Prandtl numbers, assuming $c_0 = 1$. Plot (*b*) is not compensated by Ra^{γ} as this would unnecessarily detract from the transitions at the given Prandtl numbers.

different in such a case. In light of the current work and Whitehead & Doering (2011, 2012), it is worth considering if and how the dimension of the system and the velocity boundary conditions will affect the resultant theoretical prediction. Further consideration of these effects may yield insight into how the turbulent boundary layers (thought to be realized in the 'ultimate' state as predicted in Grossmann & Lohse (2011) and observed in He *et al.* (2012)) are formed and suppressed, and the subsequent impact on the heat transport.

The extension of the current result to $Gr > c_0$ is non-trivial and cannot be accomplished by means of the current methodology, indicating perhaps that such a result is not possible and that the 'ultimate' state as described in Grossmann & Lohse (2011) holds for $Pr \lesssim Ra$. We note, however, that to date the 5/12 scaling has been demonstrated in the analysis for stress-free velocities, first predicted in Otero (2002) for two-dimensional stress-free convection, and in Plasting & Ierley (2005) and Ierley *et al.* (2006) for three-dimensional infinite Prandtl number stress-free convection. But there is no theory that predicts such a scaling and numerical simulations and experiments have not observed such a state, so it remains to be seen whether this scaling is a by-product of the analysis or if some physically relevant information can be gathered from the current derivation. Either way the similarity between the current result and that obtained in Whitehead & Doering (2012) with the asymptotically motivated numerical results of Plasting & Ierley (2005) and Ierley *et al.* (2006) is indicative that the current result is sharp with respect to the background method.

This statement may be misleading, however, as the current approach does not apply the background method as traditionally outlined. Instead, following Wang (2008*b*), we have not enforced the *spectral constraint* explicitly, meaning that we allow the quadratic form to be 'nearly' positive definite. The indefinite portion of $\mathscr{Q}$ is then estimated as $O(\epsilon)$ in order to obtain the final bound. Such variations on the background method may prove useful for additional problems, and although the current method does not appear conducive to improving upon the bound of $Ra^{1/2}$

in three dimensions at $O(1)$ Prandtl number, other modifications of the background method may yield such insight.

In addition to the derived unique (to stress-free boundaries) scaling law, we note the appearance of the saturating wavenumber (3.31), also indicated in the numerical and asymptotic calculations of Plasting & Ierley (2005) and Ierley *et al.* (2006). Such a dominant horizontal scale distinct from the size of the boundary layer has not been observed in the simulations to date, and it remains to be seen if such a scale is physically important or only a mathematical construct of the variational formulation. This indicates a dominant horizontal scale that can be used to construct asymptotic solutions (both numeric and rigorous) akin to either Chini & Cox (2009) or Corson (2011) that (according to the current analysis) should saturate the bound derived in this paper. In addition, direct numerical simulations in which forcing is added at this scale and/or careful analysis of the energy at these scales is investigated will provide additional insight into the nature of convection between slippery plates.

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Appendix. List of useful inequalities

In this appendix we provide a list of mathematical inequalities that are used in the body of the text, in particular in § 2. The same notation used in the body of the text is used here, and the constants are denoted with the same symbol. Constantin & Foias (1988) and Adams & Fournier (2003) are excellent references for the inequalities cited below. Note that sufficient smoothness (regularity) is assumed for each of the following inequalities, so it will not be stated explicitly for each.

(i) Cauchy–Schwarz inequality. For integrable functions $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$,

$$\int_{\Omega} |f(\boldsymbol{x})g(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x} \leqslant \|f(\boldsymbol{x})\|\,\|g(\boldsymbol{x})\|. \tag{A 1}$$

(ii) Poincaré inequality. For a (possibly vector-valued) function $\boldsymbol{u}(\boldsymbol{x})$ such that either $\boldsymbol{u}|_{\partial\Omega} = 0$ or $\int_{\Omega} \boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = 0$,

$$\|\nabla \boldsymbol{u}\| \geqslant c_p\|\boldsymbol{u}\| \quad \text{and} \quad \|\Delta \boldsymbol{u}\| \geqslant c_{p'}\|\nabla \boldsymbol{u}\|. \tag{A 2}$$

(iii) Young inequality. For any real a and b,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q} \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1. \tag{A 3}$$

(iv) Gronwall inequality. Let $f(t)$ be defined on the interval $[a, b]$ and let C, D be absolute constants, where

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} \leqslant Cf(t) + D. \tag{A 4}$$

Then

$$f(t) \leqslant f(a)\mathrm{e}^{C(t-a)} - \frac{D}{C}\mathrm{e}^{aC}. \tag{A 5}$$

(v) Agmon inequality. For a function $\boldsymbol{u}(\boldsymbol{x}, t)$ there exists a constant c_A such that

$$\|\boldsymbol{u}(t)\|_{L^\infty} \leqslant c_A \|\nabla \boldsymbol{u}(t)\|^{1/2} \|A\boldsymbol{u}(t)\|^{1/2}. \tag{A 6}$$

(vi) Generalized Hölder inequality. Assume that $r \in (0, \infty)$ and $p_k \in (0, \infty]$ such that

$$\sum_{k=1}^{n} \frac{1}{p_k} = \frac{1}{r}. \tag{A 7}$$

Then, for all sufficiently smooth, measurable functions f_k,

$$\left\| \prod_{k=1}^{n} f_k \right\|_r \leqslant \prod_{k=1}^{n} \|f_k\|_{p_k}, \tag{A 8}$$

where $\|\cdot\|_{p_k}$ is the L^{p_k} norm and $\prod$ is the product operator.

(vii) Sobolev inequalities in three dimensions. For a function $\boldsymbol{u}(\boldsymbol{x}, t)$ there are constants C_{S1}, C_{S2} and C such that

$$\|\boldsymbol{u}(t)\|_{L^3} \leqslant c_{S1} \|\nabla \boldsymbol{u}(t)\|, \quad \left\| \frac{\partial \boldsymbol{u}(t)}{\partial t} \right\|_{L^6} \leqslant c_{S2} \left\| \nabla \frac{\partial \boldsymbol{u}(t)}{\partial t} \right\|, \tag{A 9}$$

and

$$\|\nabla \boldsymbol{u}(t)\|_{L^4}^2 \leqslant C \|\nabla \boldsymbol{u}(t)\|^{1/2} \|\Delta \boldsymbol{u}\|^{3/2}. \tag{A 10}$$

(viii) Elliptic regularity of the Stokes operator. For $\boldsymbol{u}(\boldsymbol{x}, t)$ there exists a constant C such that

$$\|\Delta \boldsymbol{u}(t)\| \leqslant C \|A\boldsymbol{u}(t)\| \tag{A 11}$$

where A is the Stokes operator defined in the body of the paper; see also Constantin & Foias (1988).

(ix) Interpolation inequality. For the Stokes operator A and a function $\boldsymbol{u}(\boldsymbol{x}, t)$ we have

$$\|A\boldsymbol{u}(t)\| \leqslant \|\nabla \boldsymbol{u}(t)\|^{1/2} \|A^{3/2}\boldsymbol{u}(t)\|^{1/2}. \tag{A 12}$$

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