15 Jun 2020

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Recommended Citation

M. Fei et al., "Vanishing Porosity Limit of the Coupled Stokes-Brinkman System," *Journal of Mathematical Analysis and Applications*, vol. 486, no. 2, article no. 123895, Elsevier, Jun 2020.
The definitive version is available at https://doi.org/10.1016/j.jmaa.2020.123895

J. Math. Anal. Appl. 486 (2020) 123895

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Vanishing porosity limit of the coupled Stokes-Brinkman system

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A R T I C L E I N F O

Article history:
Received 6 November 2019
Available online 24 January 2020
Submitted by T. Yang

Keywords:
Porosity
Stokes equations
Brinkman equations
Beavers-Joseph-Saffman-Jones interface condition

A B S T R A C T

We investigate the small porosity asymptotic behavior of the coupled Stokes-Brinkman system in the presence of a curved interface between the Stokes region and the Brinkman region. In particular, we derive a set of approximate solutions, validated via rigorous analysis, to the coupled Stokes-Brinkman system. Of particular interest is that the approximate solution satisfies a generalized Beavers-Joseph-Saffman-Jones interface condition (1.9) with the constant of proportionality independent of the curvature of the interface.

1. Introduction

The coupling of flow and transport in free-zone and those in porous media is ubiquitous in nature, science and engineering. These coupled problems are difficult to study due to the disparate time and spatial scales (fast in the free-zone, large scale for the porous media), the different physics based on governing equations (Stokes system or Navier-Stokes system for the free-zone, Darcy's equation or its variants for the porous media), the associated uncertainty of the porous media, and the coupling of two different systems (see Fig. 1). Interested readers are referred to Nield and Bejan's treatise [30] for more background material.

For flows at relatively small Reynolds number, it is well-accepted that the Stokes system is a valid governing system for the free-zone for common Newtonian fluids. On the other hand, Darcy's equation is the extensively used governing law in porous media. The coupling of the two systems at the interface is a non-trivial issue. One irrefutable physical interface boundary condition is the continuity of the normal velocity which is required for conservation of mass. The other interface boundary conditions needed for the well-posedness of the system is less obvious. It was Beavers and Joseph who first proposed the celebrated Beavers-Joseph interface boundary condition in their seminal work [5]. They observed that there might be

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https://doi.org/10.1016/j.jmaa.2020.123895

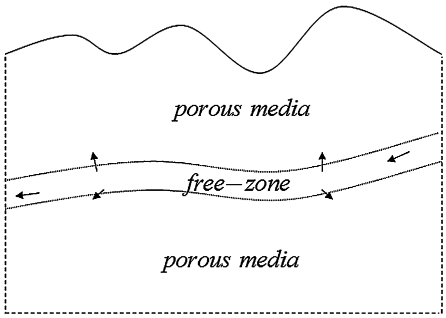

Fig. 1. Schematic domain.

a gap in the tangential velocity at the interface if one neglects a thin transition layer, and the interfacial boundary condition that bears their names can be interpreted as the viscous force that resists such a gap is linearly proportional to the size of the gap with the constant of proportionality itself positively proportional to the viscosity and inversely proportional to the square root of the permeability. Later on, Saffman [31] argued that the tangential velocity on the porous media side is usually much smaller than that in the free flow, hence it is reasonable to omit the tangential component of the porous media flow in the Beavers-Joseph interface boundary condition. On the other hand, Jones [24] reasoned that it is better to use the tangential component of the normal stress than the normal derivative of the tangential velocity in the free-flow in the Beavers-Joseph interface boundary condition. The reduced interface condition is commonly referred to as the Beavers-Joseph-Saffman-Jones interface condition (BJSJ), see equation (10c') of [36] for instance. It turns out that the BJSJ interface condition is exactly the Navier slip boundary condition proposed by Navier almost 200 years ago [29]. An additional interface boundary condition is needed to complete the coupled system. The well-accepted condition within the mathematical circle is the balance of the normal component of the normal stress (see the second equation of (1.4)), while the popular condition among groundwater study community is the continuity of pressure or hydraulic head, see for instance [6]. We will address this discrepancy between the two communities in a separate work. See [28] for a heuristic study in this direction. There is a lot of recent attention to the coupled Stokes-Darcy system, see for instance [11,7,8,10,12,13,19,26,27] among many others.

In this work, we pay attention to the constant of proportionality in the BJSJ condition. Most researchers believe that the constant of proportionality in the BJSJ condition must be a true constant in the sense that it is independent of the location on the interface, at least in the case of flat interface as was investigated by Beavers and Joseph in their original work. This belief is supported by the mathematical derivation via homogenization by Jagers and Mikelic, also for a flat interface [20–23]. (Their rigorous homogenization theory approach contains the undesirable assumption that the obstacles/sand particles in the porous media do not touch each other.) However, for a curved interface, Dobberschutz [14] argued, also via homogenization theory albeit no rigorous theorem was provided, that the constant of proportionality in the BJSJ interface condition should depend on the local geometry. Hence there is a controversial here in terms of the dependence on the local geometry in the BJSJ interface boundary condition.

Here we are interested in investigating the role of the curvature on the asymptotic behavior near the interface and hence provides a clear answer to the dependence on the local geometry (see Lemma 4.1 for details). Our approach is the following. Instead of using the Darcy's equation as the governing equation for flows in porous media, we utilize the Brinkman system which is one of the commonly used alternatives to the Darcy system if one is interested in retaining the original viscous term in the Stokes flow in the porous media [4,30]. In another word, the Brinkman system is considered the "true" system here. With the adoption of the Brinkman system which retains the viscous term, it is easy to postulate the interface boundary conditions, namely, the continuity of the velocity and the continuity of the normal component of the normal stress (balance of force). Heuristically, the viscous term is relatively small when compared to the Darcy damping term at small Darcy number/permeability/porosity. Therefore, we expect to recover the Stokes-Darcy system as the leading order non-trivial dynamics at vanishing Darcy number/permeability/porosity together with the associated interface boundary conditions. The curved interface necessitates the introduction of a curvilinear coordinates as those adopted in earlier works, see for instance [3,35]. (See [9] for a related result but with a flat interface.) Due to the renowned Carman-Kozeny empirical formula

$$\mathbf{\Pi} = \mathbf{\Pi}_0 \frac{\chi^3}{180(1-\chi)^2} \tag{1.1}$$

which relates the permeability $\mathbf{\Pi}$ to the porosity χ and the reference permeability $\mathbf{\Pi}_0$ [4,30], the small Darcy number limit is equivalent to the small porosity number limit if we assume that the reference porosity in the Carman-Kozeny relationship is a constant. This is the approach that we take here. Furthermore, we will assume for simplicity that

$$\mathbf{\Pi} = \chi^3.$$

This is a valid approximation for small porosity by assuming the reference permeability is 1.

We also remark that the Stokes-Brinkman approach is related to the so-called one-domain approach.

We recall the governing steady state Stokes-Brinkman system in a two-dimensional domain $\Omega := \Omega_c \cup \Omega_m$. The Stokes system in the fluid region Ω_c takes the form

$$\begin{cases} -\nabla \cdot \mathbb{T}(\mathbf{u}_c, p_c) = \mathbf{f}_c, \\ \nabla \cdot \mathbf{u}_c = 0, \\ \text{ periodic in } x\text{-direction}, \\ \mathbf{u}_c \mid_{\Gamma_c} = 0, \end{cases} \tag{1.2}$$

and the Brinkman system in the porous media Ω_m takes the form

$$\begin{cases} -\nabla \cdot \mathbb{T}(\mathbf{u}_m, p_m) + \frac{\mu_m}{\chi^3}\mathbf{u}_m = \mathbf{f}_m, \\ \nabla \cdot \mathbf{u}_m = 0, \\ \text{ periodic in } x\text{-direction}, \\ \frac{\partial u_m}{\partial x_2} \mid_{\Gamma_m} = v_m \mid_{\Gamma_m} = 0, \end{cases} \tag{1.3}$$

where $\mathbb{T}(\mathbf{u}_c, p_c) = 2\mu_c \mathbb{D}(\mathbf{u}_c) - p_c\mathbb{I}, \mathbb{T}(\mathbf{u}_m, p_m) = \frac{2\mu_m}{\chi}\mathbb{D}(\mathbf{u}_m) - p_m\mathbb{I}, \mathbb{D}(\mathbf{u}_j) = \frac{\nabla \mathbf{u}_j + (\nabla \mathbf{u}_j)^t}{2}, \mathbf{u}_j = (u_j, v_j)$ and $\mathbf{f}_j = (f_{j1}, f_{j2}), j \in \{c, m\}$. Here we have adopted a popular version of the stress tensor in the porous media as presented in the treatise of Nield and Bejan [30]. Other choices are available, see for instance Allaire [1].

The two systems are coupled through the following conditions on the interface Γ_{cm}

$$\begin{cases} \mathbf{u}_c = \mathbf{u}_m, \\ \mathbf{n}_{cm}\mathbb{T}(\mathbf{u}_c, p_c) = \mathbf{n}_{cm}\mathbb{T}(\mathbf{u}_m, p_m), \end{cases} \tag{1.4}$$

where the first interface boundary condition represents the continuity of velocity and the second interface boundary condition stands for the continuity of normal component of stress tensor while $\mathbf{n}_{cm}$ represents the unit normal vector at the interface pointing from the free flow zone to the porous media zone, see Fig. 2.

For simplicity, we take $\mu_c = \mu_m = 1$, assume periodicity in x-periodic, and $\Omega = \Omega_c \cup \Omega_m = [-1, 1] \times [-1, 1]$ with a curved interface Γ_{cm}, upper boundary Γ_c and lower boundary Γ_m. To be specific, the interface Γ_{cm} is described by a regular C^3 curve: $\gamma = \gamma(s) = (\gamma_1(s), \gamma_2(s))$, where s is the arc-length parameter of $\Gamma_{cm}, 0 \le s \le L$. The tangential vector fields of Γ_{cm} is $\boldsymbol{\tau}_{cm} = (\gamma_1'(s), \gamma_2'(s))$ and the unit outward normal

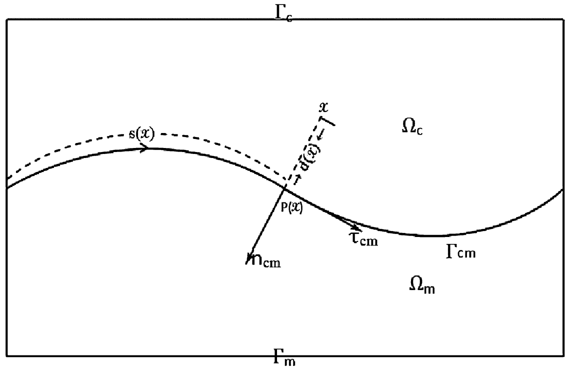

Fig. 2. Curvilinear coordinates.

vector of Γ_{cm} pointing from Ω_c to Ω_m is $\mathbf{n}_{cm} = (\gamma_2'(s), -\gamma_1'(s))$. Generally speaking, $\mathbf{u}_m$ and $\mathbf{u}_c$ could have boundary layers near the interface Γ_{cm} and the boundary of the porous media domain Γ_m since small viscosity is somewhat equivalent to small viscosity in the porous media as we shall see below.

We adopt classical function spaces of fluid mechanics. The definitions of all of our function spaces reflect that we are working with a domain which is periodic in the horizontal direction. $H^m = H^m_{per}(\Omega), m$ a nonnegative integer, is the Sobolev space consisting of all functions in Ω whose weak derivatives up to order m are square integrable and whose weak derivative up to order $m-1$ are periodic in the horizontal direction, with the usual Sobolev norm. For instance,

$$\begin{aligned}
&\mathcal{V} = \mathcal{V}(\Omega) := \{\mathbf{v} = (v_1, v_2) \in C^\infty_{per}(\Omega)^2 : div\mathbf{v} = 0, \mathbf{v} \mid_{\Gamma_c} = 0; \frac{\partial v_1}{\partial x_2} \mid_{\Gamma_m} = v_2 \mid_{\Gamma_m} = 0\},\\
&H = H(\Omega) := \text{ the closure of } \mathcal{V} \text{ in } L^2(\Omega),\\
&V = V(\Omega) := \text{ the closure of } \mathcal{V} \text{ in } H^1(\Omega),
\end{aligned} \tag{1.5}$$

where $C^\infty_{per}(\Omega)$ represents the space of smooth function that is periodic with respect to the horizontal variable.

For system (1.2)-(1.4), the existence of weak solutions can be proved in a similar fashion as [2] for instance. For completeness, we state it in the following:

Proposition 1.1. *Assume* $(\mathbf{f}_c, \mathbf{f}_m) \in H(\Omega)$*, there exists a unique weak solution* $(\mathbf{u}_c, \mathbf{u}_m) \in V(\Omega)$ *to system* (1.2)-(1.3) *with the interface boundary condition* (1.4)*.*

The main purpose of this manuscript is to investigate the asymptotic behavior of the Stokes-Brinkman system when the porosity χ approaches zero (equivalent to vanishing Darcy number with other parameters fixed). This is a singular perturbation problem involving delicate boundary layer analysis. The main result of this manuscript is summarized in the following theorem.

Theorem 1.2. *Assume that* $(\mathbf{f}_c, \mathbf{f}_m) \in H^k, k > 2$ *and the compatibility condition*

$$\partial_{x_2} f_{m1} \mid_{\Gamma_m} = \partial_{x_1} f_{m2} \mid_{\Gamma_m}, \tag{1.6}$$

then there exists an approximate solution $(\mathbf{u}_j^{app}, p_j^{app})$ *with* $j \in \{c, m\}$ *defined in* (4.2) *and* (4.3) *satisfying the following convergence estimates:*

$$\|\mathbf{u}_c - \mathbf{u}_c^{app}\|_{H^1(\Omega_c)} \leq C\chi^{\frac{3}{2}}, \tag{1.7}$$

$$\|\mathbf{u}_m - \mathbf{u}_m^{app}\|_{H^1(\Omega_m)} \leq C\chi^{2}, \tag{1.8}$$

where C is a generic constant independent of χ. Furthermore, the approximate solution $(\mathbf{u}_c^{app}, p_c^{app})$ and $(\mathbf{u}_m^{app}, p_m^{app})$ satisfy the approximated BJSJ interface condition on Γ_{cm}

$$\begin{aligned} -\boldsymbol{\tau}_{cm}\cdot(\mathbb{T}(\mathbf{u}_c^{app}, p_c^{app})\mathbf{n}_{cm}) &= \frac{1}{\chi^2}\boldsymbol{\tau}_{cm}\cdot\mathbf{u}_c^{app} + \chi^2\mathcal{H}(x), \\ \mathbf{n}_{cm}\cdot(\mathbb{T}(\mathbf{u}_c^{app}, p_c^{app})\mathbf{n}_{cm}) &= -p_m^{app} + \chi\mathcal{G}(x), \end{aligned} \tag{1.9}$$

where $\mathcal{H}(x)$ and $\mathcal{G}(x)$ defined in (4.11) are uniformly bounded with respect to χ.

Remark 1.3. If we take $\mathbb{T}(\mathbf{u}_m, p_m) = 2\mu_m\mathbb{D}(\mathbf{u}_m) - p_m\mathbb{I}$ as suggested by the rigorous homogenization work [1], and we set $\varepsilon = \chi^{\frac{3}{2}}$, then the Brinkman system in the porous media Ω_m takes the form

$$\begin{cases} -\nabla\cdot\mathbb{T}(\mathbf{u}_m, p_m) + \frac{\mu_m}{\varepsilon^2}\mathbf{u}_m = \mathbf{f}_m, \\ \nabla\cdot\mathbf{u}_m = 0, \\ \text{periodic in } x\text{-direction}, \\ \frac{\partial u_m}{\partial x_2}\mid_{\Gamma_m} = v_m\mid_{\Gamma_m} = 0. \end{cases} \tag{1.10}$$

In this case the corresponding approximated BJSJ interface condition on Γ_{cm} read

$$\begin{aligned} -\boldsymbol{\tau}_{cm}\cdot(\mathbb{T}(\mathbf{u}_c^{app}, p_c^{app})\mathbf{n}_{cm}) &= \frac{1}{\varepsilon}\boldsymbol{\tau}_{cm}\cdot\mathbf{u}_c^{app} + \varepsilon\hat{\mathcal{H}}(x), \\ \mathbf{n}_{cm}\cdot(\mathbb{T}(\mathbf{u}_c^{app}, p_c^{app})\mathbf{n}_{cm}) &= -p_m^{app} + \varepsilon\hat{\mathcal{G}}(x), \end{aligned} \tag{1.11}$$

where $\hat{\mathcal{H}}(x)$ and $\hat{\mathcal{G}}(x)$ are uniformly bounded with respect to ε.

By virtue of (1.9), we easily observe that the leading order behavior of the approximate solutions does not depend on the local geometry of the interface. One can perform a similar asymptotic expansion in small porosity for the Stokes-Darcy system and arrive at the same asymptotic modulo the internal boundary layer. Hence, this theorem suggests that the leading order non-trivial behavior of the coupled Stokes-Brinkman system is captured by the coupled Stokes-Darcy system with a BJSJ type interface condition, and the constant of proportionality in the BJSJ condition is *independent of the local geometry even in the case of a curved interface*. Therefore, our result is in accordance with classical results but incompatible with Dobberschutz's calculation [14]. Another interesting observation is that we need to adopt the version of the Brinkman equation consistent with Allaire's work under special assumption on the size of the obstacles in order to arrive at the commonly adopted BJSJ condition where the constant of proportionality is proportional to $\chi^{-3/2}$ as in (1.11) since $\mathbf{\Pi} \approx \chi^{-3/2}$ at small porosity.

The rest of the manuscript is organized as follows. In the next section we perform the formal asymptotic expansion in small porosity number for the Stokes-Brinkman system. We investigate the solvability of the outer problems and the correctors in section 3. Approximate solutions will be constructed in section 4, and the rigorous error estimates are presented in section 5. There is a conclusion in section 6. The derivation of the boundary layer (inner transition layer) system is included in Appendix A.

2. Formal asymptotic expansions

Before we study the asymptotic behavior, we introduce some preliminaries facts. For any $x = (x_1, x_2)$ and some small constant $\rho > 0$ in $\Gamma_\rho := \{x \in \Omega : dist(x, \Gamma_{cm}) \le \rho\}$, we denote by $d(x)$ the signed distance from x to Γ_{cm}, i. e. $d(x) > 0$ when $x \in \Omega_c$, and $d(x) < 0$ if $x \in \Omega_m$, and by $P(x)$ the point on Γ_{cm} that is the closest to x. Then we denote by $s(x)$ the corresponding arc-length parameter of $P(x)$ on the interface Γ_{cm}, i.e.,

$$\gamma(s(x)) = P(x), \quad x \in \Gamma_\rho.$$

From Appendix B of [15], we easily know that $P(x)$ is uniquely defined for any $x \in \Gamma_\rho$, when ρ is small enough, so are $d(x)$ and $s(x)$.

For any $x \in \Gamma_\rho$, we will use $\mathbf{n}$ and $\boldsymbol{\tau}$ instead of $\mathbf{n}_{cm}$ and $\boldsymbol{\tau}_{cm}$ in the next paragraph for simplicity. Then we have

$$\gamma(s(x)) - x = d(x)\mathbf{n}(x). \tag{2.1}$$

Furthermore, recall that $\boldsymbol{\tau}(x) = (\gamma_1'(s(x)), \gamma_2'(s(x)))$, $\mathbf{n}(x) = (\gamma_2'(s(x)), -\gamma_1'(s(x)))$ and $|\boldsymbol{\tau}| = 1$, we easily obtain by taking derivatives in (2.1) that

$$\nabla_x d(x) = -\mathbf{n}(x),\ \nabla_x s(x) = h(x)^{-1}\boldsymbol{\tau}(x), \tag{2.2}$$

where $h(x) = 1 + d(x)\kappa(s(x))$ and $\kappa(s) = \gamma_1'(s)\gamma_2''(s) - \gamma_2'(s)\gamma_1''(s)$ being the (principle) curvature of Γ_{cm}. Therefore, for any function $f(x)$ and vector field $\mathbf{v}(x)$ defined in Γ_ρ, direct computations show that

$$\begin{aligned} &\nabla_x f = h(x)^{-1}\boldsymbol{\tau}(x)\partial_s f - \mathbf{n}(x)\partial_d f, \\ &\nabla_x \cdot \mathbf{v} = h(x)^{-1}\boldsymbol{\tau}(x)\cdot\partial_s \mathbf{v} - \mathbf{n}(x)\cdot\partial_d \mathbf{v}. \end{aligned} \tag{2.3}$$

Here for $f(x)$ defined in Γ_ρ, the corresponding function of (s, d) is still written by f for simplicity and the trace of $f(x)$ at the interface Γ_{cm} by $f(s, 0)$ throughout the manuscript.

Based on the previous preliminaries, we consider the following Ansatz

$$\begin{aligned} &\mathbf{u}_j^{app}(t,x) = \sum_{j=0}^{l}\chi^i(\mathbf{u}_j^{(i)}(t,x) + \tilde{\mathbf{u}}_j^{(i)}(t, s(x), \frac{d(x)}{\chi})), \\ &p_j^{app}(t,x) = \sum_{j=0}^{l}\chi^i(p_j^{(i)}(t,x) + \tilde{p}_j^{(i)}(t, s(x), \frac{d(x)}{\chi})), \end{aligned} \tag{2.4}$$

where $j \in \{c, m\}$. It should be mentioned that (2.4) holds under the assumption that $\Gamma_\rho = \Omega$. Throughout this manuscript, we will assume that $\Gamma_\rho = \Omega$ for simplicity. In the general case when Γ_ρ is a subset of Ω, one can introduce a truncation function of $d(x)$ and utilize the stream function to preserve the incompressibility, see [18,32–34,25] for an example. Another possible way is to introduce local curvilinear coordinates as adopted in [35] for instance.

Define the stretched coordinate $\eta = \frac{d(x)}{\chi}$ and in the new coordinate (s, η) there holds

$$\Delta\tilde{\mathbf{u}}_j^{(i)} = \frac{1}{\chi^2}\partial_{\eta\eta}^2\tilde{\mathbf{u}}_j^{(i)} - \frac{1}{\chi}\frac{\kappa\partial_\eta\tilde{\mathbf{u}}_j^{(i)}}{h} + \frac{1}{h^2}\partial_{ss}^2\tilde{\mathbf{u}}_j^{(i)} - \chi\frac{\eta\kappa'(s)}{h^3}\partial_s\tilde{\mathbf{u}}_j^{(i)}. \tag{2.5}$$

We plug (2.4) into (1.2)-(1.4), collect terms of the same order of χ and then get the equation of every order. In this section we only present the equations up to the order we will need and the corresponding boundary conditions.

(1) $(\mathbf{u}_c^{(i)}, p_c^{(i)}), i \in \mathbb{N} \cup \{0\}$ are the outer solutions in Ω_c. Particularly, $(\mathbf{u}_c^{(0)}, p_c^{(0)})$ satisfies

$$\begin{cases} -\nabla \cdot \mathbb{T}(\mathbf{u}_c^{(0)}, p_c^{(0)}) = \mathbf{f}_c \quad \text{in } \Omega_c, \\ \nabla \cdot \mathbf{u}_c^{(0)} = 0 \quad \text{in } \Omega_c, \\ \text{periodic in } x\text{-direction}, \\ \mathbf{u}_c^{(0)} \mid_{\Gamma_c} = 0, \\ \mathbf{u}_c^{(0)} \mid_{\Gamma_{cm}} = 0, \end{cases} \tag{2.6}$$

and $(\mathbf{u}_c^{(i)}, p_c^{(i)})(i = 1, 2, 3, 4)$ satisfies

$$\begin{cases} -\nabla \cdot \mathbb{T}(\mathbf{u}_c^{(i)}, p_c^{(i)}) = 0 \quad \text{in } \Omega_c, \\ \nabla \cdot \mathbf{u}_c^{(i)} = 0 \quad \text{in } \Omega_c, \\ \text{periodic in } x\text{-direction}, \\ \mathbf{u}_c^{(i)} \mid_{\Gamma_c} = 0, \\ \mathbf{u}_c^{(i)} \mid_{\Gamma_{cm}} = \tilde{\mathbf{u}}_m^{(i)}(s, 0) + \mathbf{u}_m^{(i)}(s, 0). \end{cases} \tag{2.7}$$

(2) $(\mathbf{u}_m^{(i)}, p_m^{(i)}), i \in \mathbb{N} \cup \{0\}$ are the outer solutions in Ω_m. More concretely,

$$\mathbf{u}_m^{(i)} = 0, \; i = 0, 1, 2, \; \text{in } \Omega_m, \tag{2.8}$$

and $(\mathbf{u}_m^{(3)}, p_m^{(0)})$ is the solution of

$$\begin{cases} \nabla p_m^{(0)} + \mathbf{u}_m^{(3)} = \mathbf{f}_m \quad \text{in } \Omega_m, \\ \nabla \cdot \mathbf{u}_m^{(3)} = 0 \quad \text{in } \Omega_m, \\ \text{periodic in } x\text{-direction}, \\ \partial_{x_2} p_m^{(0)} \mid_{\Gamma_m} = f_{m2} \mid_{\Gamma_m}, \\ p_m^{(0)} \mid_{\Gamma_{cm}} = p_c^{(0)} \mid_{\Gamma_{cm}} . \end{cases} \tag{2.9}$$

And $(\mathbf{u}_m^{(4)}, p_m^{(1)})$ satisfies

$$\begin{cases} \nabla p_m^{(1)} + \mathbf{u}_m^{(4)} = 0 \quad \text{in } \Omega_m, \\ \nabla \cdot \mathbf{u}_m^{(4)} = 0 \quad \text{in } \Omega_m, \\ \text{periodic in } x\text{-direction}, \\ \partial_{x_2} p_m^{(1)} \mid_{\Gamma_m} = 0, \\ p_m^{(1)} \mid_{\Gamma_{cm}} = p_c^{(1)}(s, 0) - \tilde{p}_m^{(1)}(s, 0) - \boldsymbol{\tau} \cdot \partial_{sd}^2 \mathbf{u}_c^{(0)}(s, 0). \end{cases} \tag{2.10}$$

(3) $(\tilde{\mathbf{u}}_c^{(i)}, \tilde{p}_c^{(i)})(s, \eta) = (\tilde{u}_c^{(i)}, \tilde{v}_c^{(i)}, \tilde{p}_c^{(i)})(s, \frac{d}{\chi}), i \in \mathbb{N} \cup \{0\}$ are the correctors to the outer solutions, i.e., the Prandtl-type boundary layer in the domain $\Omega_c^\infty := (-1, 1) \times (0, \infty)$. In fact, we have

$$\tilde{\mathbf{u}}_c^{(l)} = \tilde{p}_c^{(l)} = 0, \tag{2.11}$$

where $l = \mathbb{N} \cup \{0\}$.

(4) $(\tilde{\mathbf{u}}_m^{(i)}, p_m^{(i)})(s,\eta) = (\tilde{u}_m^{(i)}, \tilde{v}_m^{(i)}, \tilde{p}_m^{(i)})(s, \frac{d}{\chi}), i \in \mathbb{N} \cup \{0\}$ are the correctors to the outer solutions, i.e., Prandtl-type boundary layers in the domain $\Omega_m^\infty := (-1,1) \times (-\infty, 0)$. To be specific,

$$\tilde{\mathbf{u}}_m^{(0)} = \tilde{\mathbf{u}}_m^{(1)} = 0. \tag{2.12}$$

And $(\tilde{\mathbf{u}}_m^{(2)}, \tilde{p}_m^{(0)})$ satisfies

$$\begin{cases} -\dfrac{\partial^2}{\partial \eta^2}\tilde{\mathbf{u}}_m^{(2)} + \tilde{\mathbf{u}}_m^{(2)} + \mathbf{n}\partial_\eta \tilde{p}_m^{(0)} = 0 \quad \text{in } \Omega_m^\infty, \\ \partial_\eta \tilde{\mathbf{u}}_m^{(2)} \cdot \mathbf{n} = 0 \quad \text{in } \Omega_m^\infty, \\ \partial_\eta \tilde{\mathbf{u}}_m^{(2)} \mid_{\Gamma_{cm}} = \partial_d \mathbf{u}_c^{(0)}(s,0), \\ \text{periodic in } \ x\text{-direction}, \\ \tilde{u}_m^{(2)}(s,\eta) \to 0 \text{ as } \eta \to -\infty, \end{cases} \tag{2.13}$$

with $\tilde{p}_m^{(0)} = 0$.
Also, $(\tilde{\mathbf{u}}_m^{(3)}, \tilde{p}_m^{(1)})$ satisfies

$$\begin{cases} -\dfrac{\partial^2}{\partial \eta^2}\tilde{\mathbf{u}}_m^{(3)} + \tilde{\mathbf{u}}_m^{(3)} + \mathbf{n}\partial_\eta \tilde{p}_m^{(1)} = -\kappa \partial_\eta \tilde{\mathbf{u}}_m^{(2)} \quad \text{in } \Omega_m^\infty, \\ \mathbf{n} \cdot \partial_\eta \tilde{\mathbf{u}}_m^{(3)} = -h^{-1}\boldsymbol{\tau} \cdot \partial_s \tilde{\mathbf{u}}_m^{(2)} \quad \text{in } \Omega_m^\infty, \\ \partial_\eta \tilde{\mathbf{u}}_m^{(3)} \mid_{\Gamma_{cm}} = \partial_d \mathbf{u}_c^{(1)} - \mathbf{n}(\boldsymbol{\tau} \cdot \partial_{sd}^2 \mathbf{u}_c^{(0)})(s,0), \\ \text{periodic in } \ x\text{-direction}, \\ \tilde{u}_m^{(3)}(s,\eta) \to 0 \text{ as } \eta \to -\infty. \end{cases} \tag{2.14}$$

(5) The interface conditions on Γ_{cm} are very helpful to build up the relationship between the outer solutions and the correctors. We have

$$\begin{cases} \mathbf{u}_c^{(i)}(s,0) = \tilde{\mathbf{u}}_m^{(i)}(s,0) = 0, \ i = 0,1, \\ \mathbf{u}_c^{(2)}(s,0) = \tilde{\mathbf{u}}_m^{(2)}(s,0), \\ \mathbf{u}_c^{(i)}(s,0) = \tilde{\mathbf{u}}_m^{(i)}(s,0) + \mathbf{u}_m^{(i)}(s,0), \ i = 3,4, \end{cases} \tag{2.15}$$

and

$$\begin{cases} \partial_\eta \tilde{\mathbf{u}}_m^{(2)}(s,0) = \partial_d \mathbf{u}_c^{(0)}(s,0), \quad p_c^{(0)}(s,0) = p_m^{(0)}(s,0), \\ \partial_\eta \tilde{\mathbf{u}}_m^{(3)}(s,0) = \partial_d \mathbf{u}_c^{(1)}(s,0) - \mathbf{n}(\boldsymbol{\tau} \cdot \partial_{sd}^2 \mathbf{u}_c^{(0)})(s,0), \\ [p_m^{(1)} + \tilde{p}_m^{(1)} - p_c^{(1)}](s,0) = -\boldsymbol{\tau} \cdot \partial_{sd}^2 \mathbf{u}_c^{(0)}(s,0). \end{cases} \tag{2.16}$$

The equations in Ω_c or Ω_m and boundary conditions on γ_c or γ_m in (2.6)-(2.11) can be derived directly and the details are omitted. The detailed calculations of getting (2.12)–(2.16) will be represented in the Appendix. The process of solving these closed systems will be shown in next section.

3. Solvability of the outer solutions and correctors

In this section, we will state the well-posedness of solutions to the outer systems and the boundary layer systems. First, we focus on the outer systems. The corresponding closed systems are given in section 2 and then the well-posedness results follow in this section.

(1) $(\mathbf{u}_c^{(0)}, p_c^{(0)})$ satisfies (2.6).
(2) $(\mathbf{u}_c^{(1)}, p_c^{(1)})$ satisfies

$$\begin{cases} \nabla \cdot \mathbb{T}(\mathbf{u}_c^{(1)}, p_c^{(1)}) = 0, \\ \nabla \cdot \mathbf{u}_c^{(1)} = 0, \\ \text{periodic in } x\text{-direction}, \\ \mathbf{u}_c^{(1)} \mid_{\Gamma_c} = 0, \\ \mathbf{u}_c^{(1)}(s,0) = 0. \end{cases} \tag{3.1}$$

From the above system, we easily deduce that

$$\mathbf{u}_c^{(1)} = p_c^{(1)} = 0.$$

In fact, we mention that $p_c^{(1)}$ is a constant according to the system (3.2). For simplicity, we take $p_c^{(1)} = 0$.
(3) $(\mathbf{u}_c^{(2)}, p_c^{(2)})$ satisfies

$$\begin{cases} \nabla \cdot \mathbb{T}(\mathbf{u}_c^{(2)}, p_c^{(2)}) = 0, \\ \nabla \cdot \mathbf{u}_c^{(2)} = 0, \\ \text{periodic in } x\text{-direction}, \\ \mathbf{u}_c^{(2)} \mid_{\Gamma_c} = 0, \\ \mathbf{u}_c^{(2)}(s,0) = \partial_d \mathbf{u}_c^{(0)}(s,0). \end{cases} \tag{3.2}$$

(4) $(\mathbf{u}_c^{(3)}, p_c^{(3)})$ satisfies

$$\begin{cases} \nabla \cdot \mathbb{T}(\mathbf{u}_c^{(3)}, p_c^{(3)}) = 0, \\ \nabla \cdot \mathbf{u}_c^{(3)} = 0, \\ \text{periodic in } x\text{-direction}, \\ \mathbf{u}_c^{(3)} \mid_{\Gamma_c} = 0, \\ \mathbf{u}_c^{(3)}(s,0) = \partial_d \mathbf{u}_c^{(1)}(s,0) - 2\mathbf{n}(\boldsymbol{\tau} \cdot \partial_{sd}^2 \mathbf{u}_c^{(0)})(s,0) + \mathbf{u}_m^{(3)}(s,0). \end{cases} \tag{3.3}$$

It is then easy to deduce from Theorem 1.1 and Theorem 5.2 of [16] that

Theorem 3.1. *Given $\mathbf{f}_c \in H^k(\Omega_c)$ for some given k, there exists a strong solution $(\mathbf{u}_c^{(0)}, p_c^{(0)})$ to (2.6) and then get solutions $(\mathbf{u}_c^{(i)}, p_c^{(i)})(i = 2, 3)$ to (3.2) and (3.3), respectively. Furthermore, it follows that $(\mathbf{u}_c^{(0)}, p_c^{(0)}) \in H^{k+2}(\Omega_c) \times H^{k+1}(\Omega_c)$ and $(\mathbf{u}_c^{(2)}, p_c^{(2)}) \in H^{k+1}(\Omega_c) \times H^k(\Omega_c)$ and $(\mathbf{u}_c^{(3)}, p_c^{(3)}) \in H^k(\Omega_c) \times H^{k-1}(\Omega_c)$.*

(5) $(\mathbf{u}_m^{(3)}, p_m^{(0)})$ satisfies (2.9).
Now we are in the position to solve the system (2.9). Specifically, applying $\nabla \cdot \mathbf{u}_m^{(3)} = 0$ to the first equation of (2.9), then we arrive at

$$\begin{cases} \Delta p_m^{(0)} = \nabla \cdot \mathbf{f}_m, \\ p_m^{(0)} \mid_{\Gamma_{cm}} = p_c^{(0)}(s,0), \\ \text{periodic in } x\text{-direction}, \\ \partial_{x_2} p_m^{(0)} \mid_{\Gamma_m} = f_{m2} \mid_{\Gamma_m} . \end{cases} \tag{3.4}$$

For the above elliptic system, we easily obtain the solvability and regularity as the classical theory of [17]. We will summarize it below.

(6) $(\mathbf{u}_m^{(4)}, p_m^{(1)})$ satisfies

$$\begin{cases} \nabla p_m^{(1)} + \mathbf{u}_m^{(4)} = 0, \\ \nabla \cdot \mathbf{u}_m^{(4)} = 0, \\ \text{periodic in } x\text{-direction}, \\ \partial_{x_2} p_m^{(1)} \mid_{\Gamma_m} = 0, \\ p_m^{(1)} \mid_{\Gamma_{cm}} = p_c^{(1)}(s,0) - \tilde{p}_m^{(1)}(s,0) - \boldsymbol{\tau} \cdot \partial_{sd}^2 \mathbf{u}_c^{(0)}(s,0). \end{cases} \tag{3.5}$$

For system (3.5), we have the same technique to deal with as the above (2.9). Then we arrive at the following theorem:

Theorem 3.2. *Given $\mathbf{f}_m \in H^k(\Omega_m)$ for some given k, there exists a strong solution $(\mathbf{u}_m^{(3)}, p_m^{(0)})) \in H^k(\Omega_m) \times H^{k+1}(\Omega_m)$ to system (2.9). Also, there exists a strong solution $(\mathbf{u}_m^{(4)}, p_m^{(1)})) \in H^{k-1}(\Omega_m) \times H^k(\Omega_m)$ to system (3.5).*

The following boundary condition on Γ_m will ensure the boundary condition in (4.5) of the approximate solution defined in (4.3).

Proposition 3.3. *If the compatibility condition (1.6) holds, then*

$$\partial_{x_2} u_m^{(3)} \mid_{\Gamma_m} = v_m^{(3)} \mid_{\Gamma_m} = 0. \tag{3.6}$$

Proof. Due to (2.9) and

$$\partial_{x_2} p_m^{(0)} \mid_{\Gamma_m} = f_{m2} \mid_{\Gamma_m},$$

one has

$$v_m^{(3)} \mid_{\Gamma_m} = 0. \tag{3.7}$$

It follows from (1.6) and (2.9) that

$$\begin{aligned} \partial_{x_2} u_m^{(3)} \mid_{\Gamma_m} &= \partial_{x_2} f_{m1} \mid_{\Gamma_m} - \partial_{x_2 x_1} p_m^{(0)} \mid_{\Gamma_m} \\ &= \partial_{x_2} f_{m1} \mid_{\Gamma_m} - \partial_{x_1} f_{m2} \mid_{\Gamma_m} = 0. \end{aligned} \tag{3.8}$$

Hence we end the proof of this proposition. □

Second, we are ready to investigate the boundary layer systems satisfied by $\tilde{\mathbf{u}}_m^{(2)}$ and $\tilde{\mathbf{u}}_m^{(3)}$.

(7) $(\tilde{\mathbf{u}}_m^{(2)}, \tilde{p}_m^{(0)})$ satisfies (2.13).

By using $\tilde{p}_m^{(0)} \equiv 0$ we get $-\partial_{\eta\eta}^2 \tilde{\mathbf{u}}_m^{(2)} + \tilde{\mathbf{u}}_m^{(2)} = 0$ with the corresponding boundary conditions and then can write the formula of solutions as the following

Theorem 3.4. *Assume that* $\mathbf{f}_c \in H^k(\Omega_c)$ *for some given* k. *Then there exists* $\tilde{\mathbf{u}}_m^{(2)} \in H^{k+\frac{1}{2}}(\Omega_m^\infty)$ *with the form of*

$$\begin{cases} \tilde{\mathbf{u}}_m^{(2)}(s,\eta) = \partial_d \mathbf{u}_c^{(0)}(s,0)\ e^{\eta}, \\ \tilde{p}_m^{(0)}(s,\eta) = 0. \end{cases} \tag{3.9}$$

(8) $(\tilde{\mathbf{u}}_m^{(3)}, \tilde{p}_m^{(1)})$ satisfies (2.14).

Similarly we can write the formula of the solutions as

Theorem 3.5. *Assume that* $\mathbf{f}_c \in H^k(\Omega_c)$ *for some given* k, *then there exists* $\tilde{\mathbf{u}}_m^{(3)} \in H^{k-\frac{1}{2}}(\Omega_m^\infty)$ *with the form of*

$$\begin{cases} \tilde{\mathbf{u}}_m^{(3)}(s,\eta) = \dfrac{\kappa}{2}\partial_d \mathbf{u}_c^{(0)} \eta e^{\eta} + A e^{\eta}, \\ \tilde{p}_m^{(1)}(s,\eta) = 0, \end{cases} \tag{3.10}$$

where

$$A = \partial_d \mathbf{u}_c^{(1)}(s,0) - 2\mathbf{n}(\boldsymbol{\tau} \cdot \partial_{sd}^2 \mathbf{u}_c^{(0)})(s,0).$$

4. Approximate solutions

With the outer solutions and correctors $(\mathbf{u}_j^{(i)}, p_j^{(i)}), (\tilde{\mathbf{u}}_j^{(i)}, \tilde{p}_j^{(i)}), j \in \{c, m\}$ and $i \in \mathbb{N} \cup \{0\}$ in hand, we are now in a position to construct approximate solutions with the given Ansatz (2.4).

Before that, we first introduce a cut-off function to ensure that the approximate solutions $\mathbf{u}_j^{app}, j \in \{c, m\}$, given below, satisfying the boundary conditions as the original system (1.2)-(1.3). Let $\varrho(z)$ be a smooth cut-off function with

$$\varrho(z) = \begin{cases} 0, & z \in [-\frac{3}{4}, -1], \\ 1, & z \in [-\frac{1}{4}, 0]. \end{cases} \tag{4.1}$$

Then we define the modified approximate solutions as

$$\begin{aligned} \mathbf{u}_c^{app}(s,d) &= \mathbf{u}_c^{(0)}(s,d) + \chi \mathbf{u}_c^{(1)}(s,d) + \chi^2 \mathbf{u}_c^{(2)}(s,d) + \chi^3 \mathbf{u}_c^{(3)}(s,d), \\ p_c^{app}(s,d) &= p_c^{(0)}(s,d) + \chi p_c^{(1)}(s,d) + \chi^2 p_c^{(2)}(s,d), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \mathbf{u}_m^{app}(s,d) &= \chi^2 \varrho(d) \tilde{\mathbf{u}}_m^{(2)}(s, \frac{d}{\chi}) + \chi^3 (\mathbf{u}_m^{(3)}(s,d) + \varrho(d) \tilde{\mathbf{u}}_m^{(3)}(s, \frac{d}{\chi})), \\ p_m^{app}(s,d) &= p_m^{(0)}(s,d) + \chi p_m^{(1)}(s,d), \end{aligned} \tag{4.3}$$

where we used the facts that $\tilde{\mathbf{u}}_c^{(i)} = \tilde{p}_c^{(j)} = \tilde{\mathbf{u}}_m^{(0)} = \tilde{\mathbf{u}}_m^{(1)} = \mathbf{u}_m^{(j)} = \tilde{p}_m^{(0)} = \tilde{p}_m^{(1)} = 0$ with $i = 0,1,2,3$ and $j = 0,1,2$ from (2.11), (2.12), (3.9) and (3.10), respectively.

Then utilizing the above mentioned system of $(\mathbf{u}_c^{(i)}, p_c^{(j)}), i = 0, \cdots, 3$ and $j = 0, 1, 2$ and also $(\mathbf{u}_m^{(3)}, \tilde{\mathbf{u}}_m^{(2)}, \tilde{\mathbf{u}}_m^3, p_m^{(0)}, p_m^{(1)})$, and (3.6) we find that $(\mathbf{u}_c^{app}, p_c^{app})$ and $(\mathbf{u}_m^{app}, p_m^{app})$ satisfy

$$\begin{cases} -\nabla \cdot \mathbb{T}(\mathbf{u}_c^{app}, p_c^{app}) = \mathbf{f}_c \text{ in } \Omega_c, \\ \nabla \cdot \mathbf{u}_c^{app} = 0 \text{ in } \Omega_c, \\ \mathbf{u}_c^{app} \mid_{\Gamma_c} = 0 \end{cases} \tag{4.4}$$

and

$$\begin{cases} -\nabla \cdot \mathbb{T}(\mathbf{u}_m^{app}, p_m^{app}) + \frac{1}{\chi^3}\mathbf{u}_m^{app} = \mathbf{f}_m + \mathbf{g}_m^{err} \text{ in } \Omega_m, \\ \nabla \cdot \mathbf{u}_m^{app} = \chi^3 w(x) \text{ in } \Omega_m, \\ \partial_{x_2} u_m^{app} \mid_{\Gamma_m} = v_m^{app} \mid_{\Gamma_m} = 0 \end{cases} \tag{4.5}$$

and the interface conditions on Γ_{cm}

$$\begin{cases} \mathbf{u}_c^{app}(s,0) = \mathbf{u}_m^{app}(s,0), \\ \mathbf{n}_{cm}\mathbb{T}(\mathbf{u}_c^{app}, p_c^{app}) = \mathbf{n}_{cm}\mathbb{T}(\mathbf{u}_m^{app}, p_m^{app}) - \boldsymbol{\gamma}^{err}, \end{cases} \tag{4.6}$$

where

$$w(x) = \varrho(d)h^{-1}\boldsymbol{\tau} \cdot \partial_s \tilde{\mathbf{u}}_m^{(3)} + \varrho'(d)\mathbf{n} \cdot \tilde{\mathbf{u}}_m^{(3)}, \tag{4.7}$$

$$\begin{aligned} \mathbf{g}_m^{err} &= \chi(-\nabla p_m^{(1)} - h^{-1}\boldsymbol{\tau}\partial_s \tilde{p}_m^{(1)} - \frac{1}{h^2}\partial_{ss}^2 \tilde{\mathbf{u}}_m^{(2)} - \varrho(d)\frac{\kappa\partial_\eta \tilde{\mathbf{u}}_m^{(3)}}{h} \\ &+ 2\chi^2(\varrho'(d)\partial_\eta \tilde{\mathbf{u}}_m^{(3)}) + \chi^2(\Delta \mathbf{u}_m^{(3)} + \Delta\varrho(d)\tilde{\mathbf{u}}_m^{(3)} + \frac{\eta\kappa'(s)}{h^2}\partial_s \tilde{\mathbf{u}}_m^{(2)} - \frac{\varrho}{h^2}\partial_{ss}^2 \tilde{\mathbf{u}}_m^{(3)}) + \chi^3\varrho(d)\frac{\eta\kappa'(s)}{h^2}\partial_s \tilde{\mathbf{u}}_m^{(3)}, \end{aligned} \tag{4.8}$$

$$\boldsymbol{\gamma}^{err} = \chi^2 p_c^{(2)}\boldsymbol{n}. \tag{4.9}$$

The following lemma shows, as we mentioned after Theorem 1.2, the leading order system of the approximate solutions we constructed here is inconsistent with the result of Dobberschutz [14], i.e., it does not depend on the geometry of the interface.

Lemma 4.1. *Given* $(\mathbf{f}_c, \mathbf{f}_m) \in H^k(\Omega_c) \times H^k(\Omega_m)$ *for some give* k*, then* $(\mathbf{u}_c^{app}, p_c^{app}) \in H^k \times H^{k-1}$ *and* $(\mathbf{u}_m^{app}, p_m^{app}) \in H^{k-\frac{1}{2}} \times H^k$*. Moreover,*

$$\begin{aligned} -\boldsymbol{\tau}_{cm} \cdot (\mathbb{T}(\mathbf{u}_c^{app}, p_c^{app})\mathbf{n}_{cm}) \mid_{\Gamma_{cm}} &= \frac{1}{\chi^2}\boldsymbol{\tau}_{cm} \cdot \mathbf{u}_c^{app} \mid_{\Gamma_{cm}} + \chi^2 \mathcal{H}(x), \\ \mathbf{n}_{cm} \cdot (\mathbb{T}(\mathbf{u}_c^{app}, p_c^{app})\mathbf{n}_{cm}) \mid_{\Gamma_{cm}} &= -p_m^{app} \mid_{\Gamma_{cm}} + \chi\mathcal{G}(x), \end{aligned} \tag{4.10}$$

where

$$\begin{aligned}\mathcal{H}(x) &= -\kappa(\boldsymbol{\tau}\cdot\partial_d \mathbf{u}_c^{(0)})(s,0) + \boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(2)}(s,0)\\ &\quad -\chi\kappa(\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(1)})(s,0) - \chi(\mathbf{n}\cdot\partial_s\mathbf{u}_m^{(3)})(s,0)\\ &\quad +\chi(\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(3)}(s,0) + 2\chi\partial_s(\boldsymbol{\tau}\cdot\partial^2_{sd}\mathbf{u}_c^{(0)})(s,0) \in H^{k-\frac{3}{2}}(\Gamma_{cm}),\\ \mathcal{G}(x) &= -2(\boldsymbol{\tau}\cdot\partial^2_{sd}\mathbf{u}_c^{(0)})(s,0) + 2\chi(\boldsymbol{\tau}\cdot\partial^2_{sd}\mathbf{u}_c^{(0)})(s,0)\\ &\quad -\chi p_c^{(2)}(s,0) - 2\chi^2(\boldsymbol{\tau}\cdot\partial_s\mathbf{u}_c^{(3)})(s,0)\in H^{k-\frac{3}{2}}(\Gamma_{cm}).\end{aligned} \tag{4.11}$$

Proof of Lemma 4.1. According to the definition of (4.2), (4.3) and (4.11), we easily get the regularity of $(\mathbf{u}_c^{app}, p_c^{app})$, $(\mathbf{u}_m^{app}, p_m^{app})$ and H, G respectively. Now we aim to show (4.10). For clearness, we divide the proof in the three steps.

Step I: We first prove that

$$\begin{aligned}&(\mathbf{n}\cdot\partial_s\mathbf{u}_c^{(2)})(s,0)\\ &= \partial_s(\mathbf{n}\cdot\partial_d\mathbf{u}_c^{(0)})(s,0) + \kappa(\boldsymbol{\tau}\cdot\mathbf{u}_c^{(2)})(s,0)\\ &= \partial_s(\boldsymbol{\tau}\cdot\partial_s\mathbf{u}_c^{(0)})(s,0) + \kappa(\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(0)})(s,0)\\ &= \kappa(\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(0)})(s,0).\end{aligned} \tag{4.12}$$

In fact, in the first equation, we utilize the fact that $\mathbf{u}_c^{(2)}(s,0) = \partial_d\mathbf{u}_c^{(0)}(s,0)$ and that $\partial_s\mathbf{n} = -\kappa\boldsymbol{\tau}$, where κ is the curvature of Γ_{cm}. In the second equation, we use the divergence free condition of $\mathbf{u}_c^{(0)}$, i.e., $h^{-1}\boldsymbol{\tau}\cdot\partial_s\mathbf{u}_c^{(0)} - \mathbf{n}\cdot\partial_d\mathbf{u}_c^{(0)} = 0$ and $h\mid_{\Gamma_{cm}} = 1$. Utilizing that $\mathbf{u}_c^{(0)}(s,0) = 0$, we easily $\partial_s\mathbf{u}_c^{(0)}(s,0) = 0$, which implies the last equation in (4.12).

Using the similar technique to get (4.12) and the last equation of (3.3), we can prove that

$$\begin{aligned}&(\mathbf{n}\cdot\partial_s\mathbf{u}_c^{(3)})(s,0)\\ &= \partial_s(\mathbf{n}\cdot\mathbf{u}_c^{(3)})(s,0) + \kappa(\boldsymbol{\tau}\cdot\mathbf{u}_c^{(3)})(s,0)\\ &= \partial_s(\mathbf{n}\cdot\partial_d\mathbf{u}_c^{(1)}(s,0)) - 2\partial_s(\boldsymbol{\tau}\cdot\partial^2_{sd}\mathbf{u}_c^{(0)}(s,0))\\ &\quad + \kappa\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(1)}(s,0) + (\mathbf{n}\cdot\partial_s\mathbf{u}_m^{(3)})(s,0)\\ &= -2\partial_s(\boldsymbol{\tau}\cdot\partial^2_{sd}\mathbf{u}_c^{(0)}) + \kappa\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(1)}(s,0) + (\mathbf{n}\cdot\partial_s\mathbf{u}_m^{(3)})(s,0).\end{aligned} \tag{4.13}$$

Step II: Now we focus on the left hand side of (4.10). First,

$$\begin{aligned}&\boldsymbol{\tau}_{cm}\cdot(\mathbb{T}(\mathbf{u}_c^{app}, p_c^{app})\mathbf{n}_{cm})\mid_{\Gamma_{cm}}\\ &= [h^{-1}\mathbf{n}\cdot\partial_s\mathbf{u}_c^{app} - \boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{app}]\mid_{\Gamma_{cm}}\\ &= [h^{-1}\mathbf{n}\cdot\partial_s\mathbf{u}_c^{(0)} - \boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(0)}]\mid_{\Gamma_{cm}} + \chi[h^{-1}\mathbf{n}\cdot\partial_s\mathbf{u}_c^{(1)} - \boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(1)}]\mid_{\Gamma_{cm}}\\ &\quad + \chi^2[h^{-1}\mathbf{n}\cdot\partial_s\mathbf{u}_c^{(2)} - \boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(2)}]\mid_{\Gamma_{cm}} + \chi^3[h^{-1}\mathbf{n}\cdot\partial_s\mathbf{u}_c^{(3)} - \boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(3)}]\mid_{\Gamma_{cm}}.\end{aligned} \tag{4.14}$$

Then utilizing the boundary condition that $\mathbf{u}_c^{(l)}(s,0) = 0, l = 0,1$, and the divergence free conditions $h^{-1}\boldsymbol{\tau}\cdot\partial_s\mathbf{u}_c^{(l)} - \mathbf{n}\cdot\partial_d\mathbf{u}_c^{(l)} = 0$, we derive the relationship: $\mathbf{n}\cdot\partial_d\mathbf{u}_c^{(l)}(s,0) = 0$, which ensure that the equation (4.14) has the following concise form:

$$\begin{aligned}
&\boldsymbol{\tau}_{cm}\cdot(\mathbb{T}(\mathbf{u}_c^{app},p_c^{app})\mathbf{n}_{cm})\mid_{\Gamma_{cm}}\\
&\quad=-(\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(0)})(s,0)-\chi(\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(1)})(s,0)\\
&\quad-\chi^2(\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(2)})(s,0)+\chi^2\kappa(\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(0)})(s,0)\\
&\quad+\chi^3\kappa(\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(1)}(s,0))+\chi^3(\mathbf{n}\cdot\partial_s\mathbf{u}_m^{(3)})(s,0)\\
&\quad-\chi^3(\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(3)})(s,0)-2\chi^3\partial_s(\boldsymbol{\tau}\cdot\partial_{sd}^2\mathbf{u}_c^{(0)})(s,0),
\end{aligned}\tag{4.15}$$

where we utilized (4.12) and (4.13) on the right hand of (4.15).

Secondly,

$$\begin{aligned}
\boldsymbol{\tau}\cdot\mathbf{u}_c^{app}\mid_{\Gamma_{cm}}&=(\boldsymbol{\tau}\cdot\mathbf{u}_c^{(0)})(s,0)+\chi(\boldsymbol{\tau}\cdot\mathbf{u}_c^{(1)})(s,0)+\chi^2(\boldsymbol{\tau}\cdot\mathbf{u}_c^{(2)})(s,0)+\chi^3(\boldsymbol{\tau}\cdot\mathbf{u}_c^{(3)})(s,0)\\
&=\chi^2(\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(0)})(s,0)+\chi^3(\boldsymbol{\tau}\cdot\partial_d\mathbf{u}_c^{(1)})(s,0).
\end{aligned}\tag{4.16}$$

Then from (4.15) and (4.16), we conclude that

$$-\boldsymbol{\tau}_{cm}\cdot(\mathbb{T}(\mathbf{u}_c^{app},p_c^{app})\boldsymbol{n}_{cm})\mid_{\Gamma_{cm}}=\frac{1}{\chi^2}\boldsymbol{\tau}\cdot\mathbf{u}_c^{app}\mid_{\Gamma_{cm}}+\chi^2\mathcal{H}(x).$$

Similarly, by utilizing the boundary condition $\mathbf{u}_c^{(l)}(s,0)=0, l=0,1$ and the divergence free condition $(\mathbf{n}\cdot\partial_d\mathbf{u}_c^l)(s,0)=0, l=0,1$, and also using the equations of $(2.9)_4$ and $(2.16)_3$, we easily verify that

$$\begin{aligned}
&\mathbf{n}_{cm}\cdot(\mathbb{T}(\mathbf{u}_c^{app},p_c^{app})\mathbf{n}_{cm})\mid_{\Gamma_{cm}}+p_m^{app}\mid_{\Gamma_{cm}}\\
&=[2n_1^2\partial_1u_c^{app}+2n_1n_2(\partial_1v_c^{app}+\partial_2u_c^{app})+2n_2^2\partial_2v_c^{app}-p_c^{app}+p_m^{app}]\mid_{\Gamma_{cm}}\\
&=2(\mathbf{n}\cdot\partial_d\mathbf{u}_c^{(0)})(s,0)+(p_m^{(0)}-p_c^{(0)})(s,0)+2\chi(\mathbf{n}\cdot\partial_d\mathbf{u}_c^{(1)})(s,0)\\
&\quad+\chi(p_m^{(1)}-p_c^{(1)})(s,0)+2\chi^2(\mathbf{n}\cdot\partial_d\mathbf{u}_c^{(2)})(s,0)-\chi^2p_c^{(2)}(s,0)+2\chi^3(\mathbf{n}\cdot\partial_d\mathbf{u}_c^{(3)})(s,0)\\
&=-2\chi(\boldsymbol{\tau}\cdot\partial_{sd}^2\mathbf{u}_c^{(0)})(s,0+2\chi^2(\boldsymbol{\tau}\cdot\partial_{sd}^2\mathbf{u}_c^{(0)})(s,0)-\chi^2p_c^{(2)}(s,0)-2\chi^3(\boldsymbol{\tau}\cdot\partial_s\mathbf{u}^{(3)})(s,0)\\
&=\chi\mathcal{G}(x).
\end{aligned}\tag{4.17}$$

There we end the proof. □

5. Error estimates

In this section, we intend to complete the proof of our main result Theorem 1.2. In fact, the well-posedness of approximate solutions satisfying the BJSJ interface condition (1.9) have been verified in the previous sections. Now we are in the position to deal with the convergence rate of the error solution. Namely, we intend to prove the energy estimate of the error solution and derive the corresponding convergence rate about the parameter χ.

Noticing that the approximate solutions $\mathbf{u}_m^{app}$ in (4.5) lose the divergence free condition, which will in turn lead to the lost of the error solution. Motivated by [18], we introduce another function $\psi(x)$ satisfying the following system

$$\begin{cases}\Delta\psi=w(x)\quad\text{in }\Omega_m,\\ \psi\mid_{\Gamma_{cm}}=\partial_{x_2}\psi\mid_{\Gamma_m}=0,\end{cases}\tag{5.1}$$

where $w(x)$ is defined in (4.7). The existence and regularity of ψ can be obtained by the standard elliptic theory, see [17] for instance. More precisely, we have

$$\begin{aligned}\|\psi\|_{H^2(\Omega_m)} &\leq C\|w(x)\|_{L^2(\Omega_m)}\\ &\leq C\chi^{\frac{1}{2}}(\|\partial_s\tilde{\mathbf{u}}_m^{(3)}\|_{L^2(\Omega_m^\infty)}+\|\tilde{\mathbf{u}}_m^{(3)}\|_{L^2(\Omega_m^\infty)})\\ &\leq C\chi^{\frac{1}{2}},\end{aligned} \tag{5.2}$$

where we used $\|\tilde{\mathbf{u}}_m^{(3)}\|_{L^2(\Omega_m)} \leq C\chi^{\frac{1}{2}}\|\tilde{\mathbf{u}}_m^{(3)}\|_{L^2(\Omega_m^\infty)}$ and C is a generic constant, independent of χ. Similarly, we have

$$\|\psi\|_{H^3(\Omega_m)} \leq C\|w(x)\|_{H^1(\Omega_m)} \leq C\chi^{-\frac{1}{2}}. \tag{5.3}$$

Now we define $\boldsymbol{F} = (F_1, F_2) := \nabla\psi$, then we derive that $\nabla \cdot \boldsymbol{F} = w(x)$ satisfying $\partial_{x_2}F_1 \mid_{\Gamma_m} = F_2 \mid_{\Gamma_m} = 0$ with the regularity estimate

$$\begin{aligned}\|\boldsymbol{F}\|_{H^1(\Omega_m)} &\leq C\|\psi\|_{H^2(\Omega_m)} \leq C\chi^{\frac{1}{2}},\\ \|\boldsymbol{F}\|_{H^2(\Omega_m)} &\leq C\|\psi\|_{H^3(\Omega_m)} \leq C\chi^{-\frac{1}{2}}.\end{aligned} \tag{5.4}$$

Moreover, out of the following need we extend (one can refer to Theorem 7.25 in [17]) $\boldsymbol{F}$ from Ω_m to Ω such that $\boldsymbol{F}\big|_{\Gamma_c} = 0$ and

$$\begin{aligned}\|\boldsymbol{F}\|_{H^1(\Omega)} &\leq C\|\boldsymbol{F}\|_{H^1(\Omega_m)} \leq C\chi^{\frac{1}{2}},\\ \|\boldsymbol{F}\|_{H^2(\Omega)} &\leq C\|\boldsymbol{F}\|_{H^2(\Omega_m)} \leq C\chi^{-\frac{1}{2}}.\end{aligned} \tag{5.5}$$

Next we define the modified error functions for $j \in \{c, m\}$,

$$\mathbf{u}_c^{err} = \mathbf{u}_c - \mathbf{u}_c^{app}, \mathbf{u}_m^{err} = \mathbf{u}_m - \mathbf{u}_m^{app} - \chi^3\boldsymbol{F}, p_j^{err} = p_j - p_j^{app}. \tag{5.6}$$

For simplicity, we also use the notation for $j \in \{c, m\}$, $\mathbf{u}_j^{err} = (u_j^{err}, v_j^{err})$. Then the error solution $(\mathbf{u}_j^{err}, p_j^{err}), j \in \{c, m\}$ respectively satisfy

$$\begin{cases} -\nabla \cdot \mathbb{T}(\mathbf{u}_c^{err}, p_c^{err}) = 0,\\ \nabla \cdot \mathbf{u}_c^{err} = 0,\\ \mathbf{u}_c^{err} \mid_{\Gamma_c} = 0,\end{cases} \tag{5.7}$$

and

$$\begin{cases} -\nabla \cdot \mathbb{T}(\mathbf{u}_m^{err}, p_m^{err}) + \dfrac{1}{\chi^3}\mathbf{u}_m^{err} = -\hat{\mathbf{g}}_m^{err},\\ \nabla \cdot \mathbf{u}_m^{err} = 0,\\ \dfrac{\partial u_m^{err}}{\partial x_2} \mid_{\Gamma_m} = v_m^{err} \mid_{\Gamma_m} = 0\end{cases} \tag{5.8}$$

with the interface condition on Γ_{cm}

$$\begin{cases} \mathbf{u}_c^{err}(s,0) = \mathbf{u}_m^{err}(s,0) + \chi^3 \boldsymbol{F}, \\ \mathbf{n}\cdot\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err})(s,0) = \mathbf{n}\cdot\mathbb{T}(\mathbf{u}_m^{err}, p_m^{err})(s,0) + \hat{\boldsymbol{\gamma}}^{err}(s,0), \end{cases} \tag{5.9}$$

where

$$\hat{\mathbf{g}}_m^{err} = \mathbf{g}_m^{err} - \chi^2 \Delta \boldsymbol{F} + \boldsymbol{F}, \tag{5.10}$$

and

$$\hat{\boldsymbol{\gamma}}^{err} = \boldsymbol{\gamma}^{err} + 2\chi^2 \mathbf{n}\cdot\mathbb{D}(\boldsymbol{F}). \tag{5.11}$$

Recall that $\boldsymbol{\gamma}^{\boldsymbol{err}}$ is defined in (4.9), then the order of $\hat{\boldsymbol{\gamma}}^{\boldsymbol{err}}$ about χ in (5.11) is $O(\chi^2)$, which is crucial to derive the error estimate in the following Theorem. Otherwise, the decay rate of $\mathbb{D}\mathbf{u}_c^{err}$ will be worse.

In order to prove Theorem 1.2, noting (5.5) and the definition of the error function (5.7), we only need to prove the following theorem.

Theorem 5.1. *We have*

$$\|\mathbb{D}\mathbf{u}_c^{err}\|_{L^2(\Omega_c)} \le C\chi^{\frac{3}{2}}, \quad \|\mathbf{u}_m^{err}\|_{L^2(\Omega_m)} \le C\chi^3, \quad \|\mathbb{D}\mathbf{u}_m^{err}\|_{L^2(\Omega_m)} \le C\chi^2,$$

here C is a generic constant independent of χ.

Proof. Multiplying (5.7) by $\mathbf{u}_c^{err}$ and (5.8) by $\mathbf{u}_m^{err}$ and integrating by parts leads to

$$-\int\limits_{\Gamma_{cm}} \left(\mathbf{n}\cdot\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err})\right)\cdot\mathbf{u}_c^{err} ds + 2\int\limits_{\Omega_c} |\mathbb{D}(\mathbf{u}_c^{err})|^2 dx = 0,$$

and

$$\int\limits_{\Gamma_{cm}} \left(\mathbf{n}\cdot\mathbb{T}(\mathbf{u}_m^{err}, p_m^{err})\right)\cdot\mathbf{u}_m^{err} ds + \frac{2}{\chi}\int\limits_{\Omega_m} |\mathbb{D}(\mathbf{u}_m^{err})|^2 dx + \frac{1}{\chi^3}\int\limits_{\Omega_m} |\mathbf{u}_m^{err}|^2 dx = \int\limits_{\Omega_m} \hat{\mathbf{g}}_m^{err}\cdot\mathbf{u}_m^{err} dx.$$

By adding the above two equalities together and using (5.9) we can get

$$\begin{aligned} &2\int\limits_{\Omega_c} |\mathbb{D}(\mathbf{u}_c^{err})|^2 dx + \frac{2}{\chi}\int\limits_{\Omega_m} |\mathbb{D}(\mathbf{u}_m^{err})|^2 dx + \frac{1}{\chi^3}\int\limits_{\Omega_m} |\mathbf{u}_m^{err}|^2 dx \\ &= \int\limits_{\Gamma_{cm}} \mathbf{n}\cdot[\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err}) - \mathbb{T}(\mathbf{u}_m^{err}, p_m^{err})]\cdot\mathbf{u}_m^{err} ds \\ &\quad + \chi^3 \int\limits_{\Gamma_{cm}} \mathbf{n}\cdot\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err})\cdot\boldsymbol{F} - \int\limits_{\Omega_m} \hat{\mathbf{g}}_m^{err}\cdot\mathbf{u}_m^{err} dx \\ &= \int\limits_{\Gamma_{cm}} \hat{\boldsymbol{\gamma}}^{err}\mathbf{u}_m^{err} ds + \chi^3 \int\limits_{\Gamma_{cm}} \mathbf{n}\cdot\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err})\cdot\boldsymbol{F} - \int\limits_{\Omega_m} \hat{\mathbf{g}}_m^{err}\cdot\mathbf{u}_m^{err} dx \\ &= \int\limits_{\Gamma_{cm}} \hat{\boldsymbol{\gamma}}^{err}\mathbf{u}_c^{err} ds + \chi^3 \int\limits_{\Gamma_{cm}} \hat{\boldsymbol{\gamma}}^{err}\cdot\boldsymbol{F} ds - \int\limits_{\Omega_m} \hat{\mathbf{g}}_m^{err}\cdot\mathbf{u}_m^{err} dx + \chi^3 \int\limits_{\Gamma_{cm}} \mathbf{n}\cdot\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err})\cdot\boldsymbol{F} dx \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{5.12}$$

Now we are at the stage to prove from I_1 to I_4 in (5.12). First, utilizing the Trace inequality, Poincare inequality and the Young inequality and recalling the equality (4.9), we easily find

$$
\begin{aligned}
I_1 &= \int_{\Gamma_{cm}} \hat{\boldsymbol{\gamma}}^{err} \mathbf{u}_c^{err} ds \\
&= \chi^2 \int_{\Gamma_{cm}} [p_c^{(2)} \boldsymbol{n} + 2\mathbf{n} \cdot \mathbb{D}(\boldsymbol{F})] \cdot \mathbf{u}_c^{err} ds \\
&\le \frac{1}{2} \int_{\Omega_c} |\mathbb{D}(\mathbf{u}_c^{err})|^2 dx + \chi^4 (\|p_c^{(2)}\|^2_{L^2(\Gamma_{cm})} + \|\mathbb{D}\boldsymbol{F}\|^2_{L^2(\Gamma_{cm})}) \\
&\le \frac{1}{2} \int_{\Omega_c} |\mathbb{D}(\mathbf{u}_c^{err})|^2 dx + \chi^4 \|\boldsymbol{F}\|^2_{H^1(\Omega_c)} + C\chi^4, \\
&\le \frac{1}{2} \int_{\Omega_c} |\mathbb{D}(\mathbf{u}_c^{err})|^2 dx + C\chi^3,
\end{aligned} \tag{5.13}
$$

where we use the inequality (5.5).

Similarly, we derive that

$$
\begin{aligned}
I_2 &= \chi^3 \int_{\Gamma_{cm}} \hat{\boldsymbol{\gamma}}^{\boldsymbol{err}} \cdot \boldsymbol{F} ds \\
&\le \chi^3 \|\mathbf{u}_c^{err}\|_{L^2(\Gamma_{cm})} (\|p_c^{(2)}\|_{L^2(\Gamma_{cm})} + \|\nabla \boldsymbol{F}\|_{L^2(cm)}) \\
&\le \frac{1}{2} \|\mathbb{D}\mathbf{u}_c^{err}\|^2_{L^2} + C\chi^5.
\end{aligned} \tag{5.14}
$$

As for I_3 one has

$$
\begin{aligned}
I_3 &= \int_{\Omega_m} \hat{\mathbf{g}}_m^{err} \cdot \mathbf{u}_m^{err} dx \\
&\le \frac{1}{2\chi^3} \int_{\Omega_m} |\mathbf{u}_m^{err}|^2 dx + \chi^3 \int_{\Omega_m} |\hat{\boldsymbol{g}}_m^{err}|^2 dx \\
&\le \frac{1}{2\chi^3} \int_{\Omega_m} |\mathbf{u}_m^{err}|^2 dx + \chi^3 (\|\boldsymbol{g}_{\boldsymbol{m}}^{\boldsymbol{err}}\|^2_{L^2} + \chi^3 \|\boldsymbol{F}\|^2_{H^2} + \|\boldsymbol{F}\|^2_{L^2}) \\
&\le \frac{1}{2\chi^3} \int_{\Omega_m} |\mathbf{u}_m^{err}|^2 dx + C\chi^4.
\end{aligned} \tag{5.15}
$$

It follows from $(5.7)_1$, $F_1\big|_{\Gamma_c} = F_2\big|_{\Gamma_c} = 0$ and integration by parts that

$$
\begin{aligned}
I_4 &= \chi^3 \int_{\Gamma_{cm}} \mathbf{n}_{cm} \cdot \mathbb{T}(\mathbf{u}_c^{err}, p_c^{err}) \cdot \boldsymbol{F} ds \\
&= -\chi^3 \int_{\Gamma_c} \mathbf{n}_c \cdot \mathbb{T}(\mathbf{u}_c^{err}, p_c^{err}) \cdot \boldsymbol{F} ds + \chi^3 \int_{\Omega_c} \mathbb{T}(\mathbf{u}_c^{err}, p_c^{err}) \cdot \nabla \boldsymbol{F} dx
\end{aligned}
$$

$$
\begin{aligned}
&= \chi^3 \int\limits_{\Omega_c} \mathbb{T}(\mathbf{u}_c^{err}, p_c^{err}) \cdot \nabla \boldsymbol{F} dx \\
&\leq \chi^3 \|\nabla \boldsymbol{F}\|_{L^2(\Omega_c)} (\|\mathbb{D}\mathbf{u}_c^{err}\|_{L^2} + \|p_c^{err}\|_{L^2}) \\
&\leq C\chi^3 \|\boldsymbol{F}\|_{H^1(\Omega_c)} \|\mathbb{D}\mathbf{u}_c^{err}\|_{L^2(\Omega_c)} \\
&\leq \frac{1}{2} \|\mathbb{D}\mathbf{u}_c^{err}\|^2_{L^2(\Omega_c)} + C\chi^7,
\end{aligned} \tag{5.16}
$$

where we assume that $\int_{\Omega_c} p_c^{err} dx = 0$ and utilize the generalized Poincaré inequality from [16] and (5.7), then we arrive at

$$
\|p_c^{err}\|_{L^2(\Omega_c)} \leq C \|\nabla p_c^{err}\|_{H^{-1}(\Omega_c)} \leq C \|\mathbb{D}\mathbf{u}_c^{err}\|_{L^2(\Omega_c)}.
$$

With the help of (5.13)-(5.16) one has

$$
\int\limits_{\Omega_c} |\mathbb{D}(\mathbf{u}_c^{err})|^2 dx + \frac{2}{\chi} \int\limits_{\Omega_m} |\mathbb{D}(\mathbf{u}_m^{err})|^2 dx + \frac{1}{2\chi^3} \int\limits_{\Omega_m} |\mathbf{u}_m^{err}|^2 dx \tag{5.17}
$$

$$
\leq \chi^3 \int\limits_{\Omega_m} |\hat{\mathbf{g}}_m^{err}|^2 dx + C\chi^4. \tag{5.18}
$$

By means of the explicit formula of $\hat{\mathbf{g}}_m^{err}$ in (5.10), it easily reduces that

$$
\chi^3 \int\limits_{\Omega_m} |\hat{\mathbf{g}}_m^{err}|^2 dx \leq C\chi^3, \tag{5.19}
$$

where we used the uniform estimate of $\|\boldsymbol{F}\|_{H^2(\Omega_m)}$ in (5.4).

At the end, (5.17) and (5.19) immediately imply

$$
\begin{aligned}
\|\mathbb{D}\mathbf{u}_c^{err}\|_{L^2(\Omega_c)} &\leq C\big(\chi^3 \|\hat{\mathbf{g}}_m^{err}\|_{L^2(\Omega_m)} + \chi^4\big)^{\frac{1}{2}} \leq C\chi^{\frac{3}{2}}, \\
\|\mathbf{u}_m^{err}\|_{L^2(\Omega_m)} &\leq C\big(\chi^6 \|\hat{\mathbf{g}}_m^{err}\|_{L^2(\Omega_m)} + \chi^7\big)^{\frac{1}{2}} \leq C\chi^3, \\
\|\mathbb{D}\mathbf{u}_m^{err}\|_{L^2(\Omega_m)} &\leq C\big(\chi^4 \|\hat{\mathbf{g}}_m^{err}\|_{L^2(\Omega_m)} + \chi^5\big)^{\frac{1}{2}} \leq C\chi^2.
\end{aligned}
$$

Therefore, we easily complete the proof of the theorem. □

At the end, we want to verify that the accurate solution of the coupled Stokes-Brinkman system satisfies the generalized Beaver-Joseph-Saffman-Jones interface boundary condition at the curved interface as the approximate solution in Lemma 4.1.

Theorem 5.2. *For the error solution* $(\mathbf{u}_c^{err}, p_c^{err})$*, we have*

$$
\left\| \boldsymbol{\tau}_{cm} \cdot (\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err}) \mathbf{n}_{cm}) + \frac{1}{\chi^2} \boldsymbol{\tau}_{cm} \cdot \mathbf{u}_c^{err} \right\|_{H^{-\frac{1}{2}}(\Gamma_{cm})} \leq C\chi^{\frac{1}{2}}. \tag{5.20}
$$

Moreover, the accurate solution $(\mathbf{u}_c, p_c)$ *of Stokes-Brinkman system satisfies*

$$
\left\| \boldsymbol{\tau}_{cm} \cdot (\mathbb{T}(\mathbf{u}_c, p_c) \mathbf{n}_{cm}) + \frac{1}{\chi^2} \boldsymbol{\tau}_{cm} \cdot \mathbf{u}_c \right\|_{H^{-\frac{1}{2}}(\Gamma_{cm})} \leq C\chi^{\frac{1}{2}}. \tag{5.21}
$$

Here C is a generic constant independent of χ.

Proof. Firstly we focus on the proof of (5.20). In step 1, we prove that

$$\left\|\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err})\mathbf{n}_{cm}\right\|_{H^{-\frac{1}{2}}(\Gamma_{cm})} \leq C\chi^{\frac{3}{2}}. \tag{5.22}$$

For any $\mathbf{q} \in H^{\frac{1}{2}}(\Gamma_{cm})$ which is x-periodic, we extend $\mathbf{q}$ to $\Omega_c \cup \Gamma_c \cup \Gamma_{cm}$ such that

$$\mathbf{q} \in H^1(\Omega_c), \quad \mathbf{q}\big|_{\Gamma_c} = 0.$$

Then with the help of $(5.7)_1$ we have

$$\begin{aligned} \int\limits_{\Gamma_{cm}} \left(\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err})\mathbf{n}_{cm}\right) \cdot \mathbf{q} ds &= \int\limits_{\Omega_c} \Big(\left(\nabla \cdot \mathbb{T}(\mathbf{u}_c^{err}, p_c^{err})\right) \cdot \mathbf{q} + \left(\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err}) : \nabla \mathbf{q}\right)\Big) dx \\ &= \left(\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err}) : \nabla \mathbf{q}\right)\Big) dx, \end{aligned} \tag{5.23}$$

which immediately implies

$$\begin{aligned} \left\|\left(\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err})\mathbf{n}_{cm}\right\|_{H^{-\frac{1}{2}}(\Gamma_{cm})} &\leq \left\|\mathbb{T}(\mathbf{u}_c^{err}, p_c^{err})\right\|_{L^2(\Omega_c)} \\ &\leq 2\left\|\mathbb{D}(\mathbf{u}_c^{err})\right\|_{L^2(\Omega_c)} + \left\|p_c^{err}\right\|_{L^2(\Omega_c)}. \end{aligned} \tag{5.24}$$

Assuming that $\int_{\Omega_c} p_c^{err} dx = 0$ and using the generalized Poincaré inequality of [16] lead to

$$\|p_c^{err}\|_{L^2(\Omega_c)} \leq C\|\nabla p_c^{err}\|_{H^{-1}(\Omega_c)} \leq C\|\mathbf{u}_c^{err}\|_{H^1(\Omega_c)} \leq C\chi^{\frac{3}{2}}.$$

In step 2, we prove

$$\left\|\mathbf{u}_c^{err}\right\|_{H^{-\frac{1}{2}}(\Gamma_{cm})} \leq C\chi^3. \tag{5.25}$$

Based on the similar arguments in (5.23)-(5.24), the first interface condition in (5.9), $\nabla \cdot \mathbf{u}_m^{err} = 0$ and Theorem 5.1, one has

$$\begin{aligned} \|\mathbf{u}_c^{err}\|_{L^2(\Gamma_{cm})} &= \|\mathbf{u}_m^{err}\|_{L^2(\Gamma_{cm})} + \chi^3\|\boldsymbol{F}\|_{L^2(\Gamma_{cm})} \\ &\leq C\|\mathbf{u}_m^{err}\|_{L^2(\Omega_m)}^{\frac{1}{2}}\|\mathbb{D}\mathbf{u}_m^{err}\|_{L^2(\Omega_m)}^{\frac{1}{2}} + \chi^3\|\boldsymbol{F}\|_{H^1(\Omega_m)} \\ &\leq C\chi^{\frac{5}{2}}. \end{aligned}$$

It follows from (5.22) and (5.25) that

$$\left\|\boldsymbol{\tau}_{cm} \cdot \mathbb{T}(\mathbf{u}_c^{err}, p_c^{err})\mathbf{n}_{cm}\right\|_{H^{-\frac{1}{2}}(\Gamma_{cm})} \leq C\chi^{\frac{3}{2}}, \quad \left\|\boldsymbol{\tau}_{cm} \cdot \mathbf{u}_c^{err}\right\|_{H^{-\frac{1}{2}}(\Gamma_{cm})} \leq C\chi^{\frac{5}{2}},$$

which completes the proof of (5.20).

Secondly, we will prove (5.21). From (4.10) and $\|H(x)\|_{L^2(\Gamma_{cm})} = O(1)$ one gets

$$\|\boldsymbol{\tau}_{cm} \cdot (\mathbb{T}(\mathbf{u}_c^{app}, p_c^{app})\mathbf{n}_{cm}) + \frac{1}{\chi^2}\boldsymbol{\tau}_{cm} \cdot \mathbf{u}_c^{app}\|_{L^2(\Gamma_{cm})} \leq C\chi^2,$$

which combined with (5.20) imply the result of (5.21).

Consequently we complete the proof of this theorem. □

6. Conclusion

We have derived the asymptotic behavior of the Stokes-Brinkman system with a curved interface at the physically important small porosity number regime. The expansion involves a boundary layer near the interface which renders the approximation a singular one. The asymptotic expansion is rigorously verified via energy methods. The explicitly constructed approximate solutions suggest that the leading order non-trivial behavior of the coupled Stokes-Brinkman system is independent of the curvature of the interface, at least in terms of a generalized Beavers-Joseph-Saffman-Jones interface boundary condition. Our work is the first of this kind with a curved interface. Furthermore, we observed that the exact scaling in terms of the dependence on the porosity in the BJSJ condition can be recovered if we adopt Allaire's version of the Brinkman system. Rigorous convergence of the leading order non-trivial solution to that of the Stokes-Darcy system with appropriate BJSJ interface condition will be furnished in a subsequent manuscript.

Acknowledgments

Fei is partly supported by NSF of China under Grant 11871075. Niu is partially supported by NSFC (No. 11871046, No. 11931010). In addition, Niu wish to thank the Institute for Mathematical Sciences (IMS) at Chinese University of Hong Kong, where this work was partially conducted, for their hospitality and support. Wang's work is supported in part by NSFC11871159 and Guangdong Provincial Key Laboratory via 2019B030301001. He also acknowledges the hospitality of the National Center for Theoretical Sciences in Taipei, China where part of the work was completed. Wang designed the problem. The three authors contributed equally to the work.

Appendix A. Derivation of the boundary layer system

A.1. Derive of system (2.12)–(2.16)

Here we derive the boundary layer system (2.12)–(2.16). First, we insert the ansatz (2.4) into (1.3) and collect different orders of χ, we arrive at

$$\begin{cases} -\dfrac{\partial^2}{\partial\eta^2}\tilde{\mathbf{u}}_m^{(0)} + \tilde{\mathbf{u}}_m^{(0)} = 0, \\ \partial_\eta \tilde{\mathbf{u}}_m^{(0)} \cdot \mathbf{n} = 0, \end{cases} \tag{A.1}$$

and

$$\begin{cases} -\dfrac{\partial^2}{\partial\eta^2}\tilde{\mathbf{u}}_m^{(1)} + \tilde{\mathbf{u}}_m^{(1)} = -\kappa\partial_\eta \tilde{\mathbf{u}}_m^{(0)}, \\ h^{-1}\boldsymbol{\tau} \cdot \partial_s \tilde{\mathbf{u}}_m^{(0)} + \partial_\eta \tilde{\mathbf{u}}_m^{(1)} \cdot \mathbf{n} = 0, \end{cases} \tag{A.2}$$

and

$$\begin{cases} -\frac{\partial^2}{\partial\eta^2}\tilde{\mathbf{u}}_m^{(2)} + \tilde{\mathbf{u}}_m^{(2)} + \mathbf{n}\partial_\eta \tilde{p}_m^{(0)} = \partial_{ss}^2 \tilde{\mathbf{u}}_m^{(0)} - \kappa\partial_\eta \tilde{\mathbf{u}}_m^{(1)} + \kappa\eta\partial_\eta \tilde{\mathbf{u}}_m^{(0)}, \\ \mathbf{n}\cdot\partial_\eta \tilde{\mathbf{u}}_m^{(2)} = -h^{-1}\boldsymbol{\tau}\cdot\partial_s \tilde{\mathbf{u}}_m^{(1)}, \end{cases} \tag{A.3}$$

and

$$\begin{cases} -\partial_{\eta\eta}^2 \tilde{\mathbf{u}}_m^{(3)} + \tilde{\mathbf{u}}_m^{(3)} + \mathbf{n}\partial_\eta \tilde{p}_m^{(1)} = -h^{-1}\kappa(\partial_\eta \tilde{\mathbf{u}}_m^{(2)} + \eta\partial_\eta \tilde{\mathbf{u}}_m^{(1)}) \\ \quad +h^{-2}(\partial_{ss}^2 \tilde{\mathbf{u}}_m^{(1)} + \eta\partial_{ss}^2 \tilde{\mathbf{u}}_m^{(0)}) + h^{-2}\eta\kappa'(s)\partial_s \tilde{\mathbf{u}}_m^{(0)} - h^{-1}\boldsymbol{\tau}\partial_s \tilde{p}_m^{(0)}, \\ \mathbf{n}\cdot\partial_\eta \tilde{\mathbf{u}}_m^{(3)} = -h^{-1}\boldsymbol{\tau}\cdot\partial_s \tilde{\mathbf{u}}_m^{(2)}. \end{cases} \tag{A.4}$$

Next we verify first half of (2.12), i.e.,

$$\tilde{\mathbf{u}}_m^{(0)} = 0. \tag{A.5}$$

In fact, according to the fact that $\mathbf{n}$ only depends on s, independent of η, we substitute the divergence free condition $(A.1)_2$ to $(A.1)_1$, then we easily derive that

$$\mathbf{n}\cdot\tilde{\mathbf{u}}_m^{(0)} = 0, \tag{A.6}$$

which implies that

$$\tilde{v}_m^{(0)} = 0 \quad \text{in} \quad \Omega_m^\infty.$$

Utilizing $(A.13)_1$ and $(A.1)_2$, we deduce that

$$\partial_\eta \tilde{\mathbf{u}}_m^{(0)}(s,0) = 0. \tag{A.7}$$

With the help of equation $\tilde{\mathbf{u}}_m^{(0)}(x,z)$ of $(A.1)_1$, which is

$$-\partial_{\eta\eta}^2 \tilde{\mathbf{u}}_m^{(0)} + \tilde{\mathbf{u}}_m^{(0)} = 0,$$

the boundary condition (A.7) and the decay property of $\tilde{\mathbf{u}}_m^{(0)}(s,\eta)$, i.e., $\tilde{\mathbf{u}}_m^{(0)}(s,\eta) \to 0$ as η goes to $-\infty$, we easily derive that

$$\tilde{\mathbf{u}}_m^{(0)} = 0 \quad \text{in} \quad \Omega_m^\infty. \tag{A.8}$$

Therefore it guarantees (A.5). A similar technique can be applied to (A.2) to derive the second half of (2.12), i.e.,

$$\tilde{\mathbf{u}}_m^{(1)} = 0. \tag{A.9}$$

We leave the details to the interested reader.

Now we claim that

$$\tilde{p}_m^{(0)} = 0$$

and system (2.13) of $\tilde{\mathbf{u}}_m^2$. First we will return to system (A.3) with $\tilde{\mathbf{u}}_m^{(0)} = \tilde{\mathbf{u}}_m^{(1)} = 0$, then (A.3) has the new form as

$$\begin{cases} -\dfrac{\partial^2}{\partial\eta^2}\tilde{\mathbf{u}}_m^{(2)} + \tilde{\mathbf{u}}_m^{(2)} + \mathbf{n}\partial_\eta \tilde{p}_m^{(0)} = 0, \\ \mathbf{n}\cdot\partial_\eta \tilde{\mathbf{u}}_m^{(2)} = 0. \end{cases} \tag{A.10}$$

Then by means of the divergence free condition $(A.10)_2$, differentiating $(A.10)_1$ with respect to η-variable, and multiplying by the normal vector $\mathbf{n}$, then using again the divergence free condition, we have

$$\partial^2_{\eta\eta}\tilde{p}_m^{(0)} = 0.$$

Moreover, with the help of the decay condition of $\tilde{p}_m^0$, we verify that

$$\tilde{p}_m^{(0)} = 0, \tag{A.11}$$

which finally complete (2.12). Then we easily verify the validity of (2.13).

A.2. Derive the interface condition on $\mathbf{\Gamma}_{cm}$

Applying the ansatz (2.4) into (1.4), comparing the order of χ and noting (2.6), (2.11) and (2.12), we directly obtain

$$\begin{cases} \mathbf{u}_c^{(0)}(s,0) = \tilde{\mathbf{u}}_m^{(0)}(s,0) = 0, \\ \mathbf{u}_c^{(1)}(s,0) = \tilde{\mathbf{u}}_m^{(1)}(s,0) = 0, \\ \mathbf{u}_c^{(2)}(s,0) = \tilde{\mathbf{u}}_m^{(2)}(s,0), \\ \mathbf{u}_c^{(i)}(s,0) = \mathbf{u}_m^{(i)}(s,0) + \tilde{\mathbf{u}}_m^{(i)}(s,0),\ i \geq 3, \end{cases} \tag{A.12}$$

and

$$\begin{cases} \partial_\eta \tilde{\mathbf{u}}_m^{(0)}(s,0) = 0, \\ \partial_\eta \tilde{\mathbf{u}}_m^{(1)}(s,0) = 0, \\ \partial_\eta \tilde{\mathbf{u}}_m^{(2)}(s,0) = \partial_d \mathbf{u}_c^{(0)}(s,0), \end{cases} \tag{A.13}$$

and for $i \geq 3$,

$$\begin{aligned} \partial_\eta \tilde{\mathbf{u}}_m^{(i)}(s,0) = &-(\mathbf{n}\cdot\partial_\eta \tilde{\mathbf{u}}_m^{(i-1)}(s,0))\mathbf{n} - \boldsymbol{\tau}(\mathbf{n}\cdot\partial_s \tilde{\mathbf{u}}_m^{(i-2)}(s,0)) \\ &- \boldsymbol{\tau}(\mathbf{n}\cdot\partial_s \mathbf{u}_m^{(i-1)}(s,0)) - \partial_d \mathbf{u}_m^{(i-1)}(s,0) \\ &- \mathbf{n}(\mathbf{n}\cdot\partial_d \mathbf{u}_m^{(i-1)}(s,0)) + (p_m^{(i-2)} + \tilde{p}_m^{(i-2)})(s,0)\mathbf{n} \\ &+ \boldsymbol{\tau}(\mathbf{n}\cdot\partial_s \mathbf{u}_c^{(i-2)}(s,0)) + \partial_d \mathbf{u}_c^{(i-2)}(s,0) \\ &+ \mathbf{n}(\mathbf{n}\cdot\partial_d \mathbf{u}_c^{(i-2)}(s,0)) - \mathbf{n}p_c^{(i-2)}(s,0). \end{aligned} \tag{A.14}$$

In particular, when $i = 3$, we have that

$$\begin{aligned} &\partial_\eta \tilde{\mathbf{u}}_m^{(3)}(s,0) \\ &= -\mathbf{n}[\mathbf{n}\cdot\partial_\eta \tilde{\mathbf{u}}_m^{(2)}(s,0) + \mathbf{n}\cdot\partial_d \mathbf{u}_m^{(2)}(s,0) - \mathbf{n}\cdot\partial_d \mathbf{u}_c^{(1)}(s,0)] \\ &\quad - \boldsymbol{\tau}\cdot[\mathbf{n}\cdot\partial_s \tilde{\mathbf{u}}_m^{(1)}(s,0) + \mathbf{n}\cdot\partial_s \mathbf{u}_m^{(2)}(s,0) - \mathbf{n}\cdot\partial_s \mathbf{u}_c^{(1)}(s,0)] \\ &\quad + (p_m^{(1)}(s,0) + \tilde{p}_m^{(1)}(s,0) - p_c^{(1)}(s,0))\mathbf{n} \\ &\quad + \partial_d \mathbf{u}_c^{(1)}(s,0) - \partial_d \mathbf{u}_m^{(2)}(s,0). \end{aligned} \tag{A.15}$$

Noting that $\tilde{\mathbf{u}}_m^{(2)}(s,\eta) = \partial_d \mathbf{u}_c^{(0)}(s,0)e^{\eta}$ by (3.9), we then have

$$\mathbf{n} \cdot \partial_\eta \tilde{\mathbf{u}}_m^{(2)}(s,0) = \mathbf{n} \cdot \partial_d \mathbf{u}_c^{(0)}(s,0). \tag{A.16}$$

Using the divergence-free condition of $\mathbf{u}_c^{(0)}$ and $\mathbf{u}_c^{(1)}$, we obtain that on Γ_{cm}

$$\begin{aligned} \boldsymbol{\tau} \cdot \partial_s \mathbf{u}_c^{(0)}(s,0) + \mathbf{n} \cdot \partial_d \mathbf{u}_c^{(0)}(s,0) = 0, \\ \boldsymbol{\tau} \cdot \partial_s \mathbf{u}_c^{(1)}(s,0) + \mathbf{n} \cdot \partial_d \mathbf{u}_c^{(1)}(s,0) = 0. \end{aligned} \tag{A.17}$$

Applying the fact that $\mathbf{u}_c^{(i)}(s,0) = 0(i = 0,1)$ to (A.17), we easily know that

$$\mathbf{n} \cdot \partial_d \mathbf{u}_c^i(s,0) = 0, i = 0,1. \tag{A.18}$$

Submitting (A.16) and (A.18) into (A.15) lead to

$$\partial_\eta \tilde{\mathbf{u}}_m^{(3)}(s,0) = \partial_d \mathbf{u}_c^{(1)}(s,0) + (p_m^{(1)}(s,0) + \tilde{p}_m^{(1)}(s,0) - p_c^{(1)}(s,0))\mathbf{n}, \tag{A.19}$$

here we have used $\tilde{\mathbf{u}}_m^{(1)} = \mathbf{u}_m^{(2)} = 0$.

It follows from $\mathbf{n} \cdot \partial_\eta \tilde{\mathbf{u}}_m^{(3)}(s,0) = -h^{-1}\boldsymbol{\tau} \cdot \partial_s \tilde{\mathbf{u}}_m^{(2)}(s,0) = -\boldsymbol{\tau} \cdot \partial_{sd}^2 \mathbf{u}_c^{(0)}(s,0)$, (A.18) $(i = 1)$ and (A.19) that

$$p_m^{(1)}(s,0) + \tilde{p}_m^{(1)}(s,0) - p_c^{(1)}(s,0) = -\boldsymbol{\tau} \cdot \partial_{sd}^2 \mathbf{u}_c^{(0)}(s,0) \tag{A.20}$$

which and (A.19) imply that

$$\partial_\eta \tilde{\mathbf{u}}_m^{(3)}(s,0) = \partial_d \mathbf{u}_c^{(1)}(s,0) - \mathbf{n}(\boldsymbol{\tau} \cdot \partial_{sd}^2 \mathbf{u}_c^{(0)})(s,0). \tag{A.21}$$

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