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The Variational Modeling of Hierarchical Structured Deformations

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In memory of Jerry Ericksen, whose broad and deep scientific contributions and leadership never cease to evoke admiration and to provide inspiration.

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Abstract
Hierarchical (first-order) structured deformations are studied from the variational point of view. The main contributions of the present research are the first steps, at the theoretical level, to establish a variational framework to minimize mechanically relevant energies defined on hierarchical structured deformations. Two results are obtained here: (i) an approximation theorem and (ii) the assignment of an energy to a hierarchical structured deformation by means of an iterative procedure. This has the effect of validating the proposal made in Deseri and Owen (J. Elast. 135:149–182, 2019) to study deformations admitting slips and separations at multiple submacroscopic levels. An explicit example is provided to illustrate the behavior of the proposed iterative procedure and relevant directions for future research are highlighted.

Keywords Structured deformations · Hierarchies · Relaxation · Energy minimization · Integral representation

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1 Introduction

Refinements of classical continuum theories of elastic bodies have the potential to broaden their range of applicability by capturing phenomena of particular interest or to adapt the theories to specific physical contexts. Structured deformations [12] provide a mathematical framework to capture the effects at the macroscopic level of geometrical changes, such as slips and separations, at submacroscopic levels. The variational formulation of the theory of structured deformations [11] set the basis for the enrichment of the classes of energies and force systems of interest in variational and field-theoretic descriptions of deformations of elastic bodies, making it unnecessary to commit at the outset to any prototypical mechanical theories, such as elasticity, plasticity, or fracture.

Following [11], a (first-order) structured deformation is a pair $(g, G) \in SBV(\Omega;\mathbb{R}^d) \times L^1(\Omega;\mathbb{R}^{d\times N}) =: SD(\Omega)$, where $g\colon \Omega \to \mathbb{R}^d$ is the macroscopic deformation of the body

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and $G\colon \Omega \to \mathbb{R}^{d\times N}$ is a tensor field associated with submacroscopic deformation. In classical theories of mechanics, the field g and its gradient ∇g alone characterize the deformation of the body; in the framework of structured deformations, the additional geometrical field G captures the contribution at the macroscopic scale of smooth submacroscopic changes, and the difference $\nabla g - G$ captures the contribution at the macroscopic scale of non-smooth submacroscopic changes, such as slips and separations, which are referred to as *disarrangements* [13]. Accordingly, G is called the deformation without disarrangements, and, heuristically, the disarrangement tensor $M := \nabla g - G$ is an indication of how non-classical a structured deformation is: if g is a Sobolev field and $M = 0$, then the deformation without disarrangements $G = \nabla g$ is simply the classical deformation gradient; if $M \neq 0$, the effect of submacroscopic slips and separations, which are phenomena involving interfaces, is captured at the macroscopic level. This fact will be made precise in the Approximation Theorem 2.3.

The classical variational methods for continuum mechanics rely on energy minimization. In [11], the energy assigned to a structured deformation $(g, G) \in SD(\Omega)$ is the one arising from the most economical way to reach (g, G) by means of a sequence $\{u_n\} \subset SBV(\Omega;\mathbb{R}^d)$ approximating (g, G) in the following sense

$$u_n \to g \quad \text{in } L^1(\Omega;\mathbb{R}^d) \quad \text{and} \quad \nabla u_n \overset{*}{\rightharpoonup} G \quad \text{in } \mathcal{M}(\Omega;\mathbb{R}^{d\times N}), \tag{1.1}$$

where $\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ is the set of bounded matrix-valued Radon measures on Ω; the convergence in (1.1) will be denoted by $u_n \xrightarrow[SD]{*} (g, G)$.

We let the initial energy of a deformation $u \in SBV(\Omega;\mathbb{R}^d)$ be

$$E(u) := \int_\Omega W(x, \nabla u(x))\,\mathrm{d}x + \int_{\Omega\cap S_u} \psi(x, [u](x), \nu_u(x))\,\mathrm{d}\mathcal{H}^{N-1}(x), \tag{1.2}$$

which is determined by the bulk and surface energy densities $W\colon \Omega \times \mathbb{R}^{d\times N} \to [0, +\infty)$ and $\psi\colon \Omega \times \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty)$. In formula (1.2), $\mathrm{d}x$ and $\mathrm{d}\mathcal{H}^{N-1}(x)$ denote the N-dimensional Lebesgue and $(N-1)$-dimensional Hausdorff measures, respectively; $[u](x)$ and $\nu_u(x)$ denote the jump of u and the normal to the jump set for each $x \in S_u$, the jump set of u.

In mathematical terms, the process to assign an energy to a structured deformation $(g, G) \in SD(\Omega)$ reads

$$I(g, G) := \inf\left\{ \liminf_{n\to\infty} E(u_n) : u_n \in SBV(\Omega;\mathbb{R}^d),\ u_n \xrightarrow[SD]{*} (g, G)\right\}. \tag{1.3}$$

In the language of calculus of variations, the operation in (1.3) is known as *relaxation*, and an important goal is to prove that the functional I admits an integral representation, that is, that there exist functions $H\colon \Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N} \to [0, +\infty)$ and $h\colon \Omega \times \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty)$ such that

$$I(g, G) = \int_\Omega H(x, \nabla g(x), G(x))\,\mathrm{d}x + \int_{\Omega\cap S_g} h(x, [g](x), \nu_g(x))\,\mathrm{d}\mathcal{H}^{N-1}(x). \tag{1.4}$$

In the case where the initial bulk and surface energy densities W and ψ do not depend explicitly on the spatial variable x, this was the main result in [11], whose extension to include the explicit x dependence can be found in [17, Theorem 5.1].

In the recent contribution [14], Deseri and Owen extended the theory of [12] to hierarchical structured deformations in order to include the effects of disarrangements at more than one submacroscopic level. This extension is based on the fact that many natural and man-made materials exhibit different levels of disarrangements. Muscles, cartilage, bone, plants, and biomedical materials are just some of the materials whose mechanical behavior can be addressed within the generalized field theory proposed in [14]. Since a structured deformation (g, G) identifies two levels, the macroscopic one and the submacroscopic one, for $L \in \mathbb{N}$, we call an $(L+1)$-*level (first-order) structured deformation* an $(L+1)$-tuple $(g, G_1, \dots, G_L)$, where $g\colon \Omega \to \mathbb{R}^d$ is the macroscopic deformation and $G_\ell\colon \Omega \to \mathbb{R}^{d\times N}$, for $\ell = 1, \dots, L$, are the deformations without disarrangements at each of the L submacroscopic levels.

With the far-reaching goal of establishing a variational theory *à la* Choksi and Fonseca [11] for hierarchical structured deformations, the main results of the present research are: (i) an approximation theorem for $(L+1)$-level structured deformations (see Theorem 3.3 below) and (ii) a proposal for the assignment of an energy to the $(L+1)$-level structured deformation $(g, G_1, \dots, G_L)$ by means of a well-posed recursive relaxation process (see Theorem 3.4 and the example in Sect. 3.3 below).

After stating the notation that we use throughout this work, the plan of the paper is the following: in Sect. 2 we collect some results coming from the theory of structured deformations, together with some recent generalizations; we also state and prove Theorem 2.10 on the properties of the relaxed bulk and surface energy densities: this is the crucial result that allows the recursive relaxation process to continue through all levels. In Sect. 3, we present our main results on hierarchical structured deformations, namely the Approximation Theorem 3.3 and Theorem 3.4 on the well posedness of the recursive relaxation and, therefore, on the assurance that the energy assigned to an $(L+1)$-level structured deformation $(g, G_1, \dots, G_L)$ is well-defined. We also show an example in which explicit computations can be carried out. Finally, Sect. 4 offers an outlook for future directions of research.

1.1 Notation

We will use the following notations

- $\mathbb{N}$ denotes the set of natural numbers without the zero element;
- $\Omega \subset \mathbb{R}^N$ is a bounded connected open set;
- $\mathbb{S}^{N-1}$ denotes the unit sphere in $\mathbb{R}^N$;
- $Q := (-\frac{1}{2}, \frac{1}{2})^N$ denotes the open unit cube of $\mathbb{R}^N$ centred at the origin; for any $\nu \in \mathbb{S}^{N-1}$, Q_ν denotes any open unit cube in $\mathbb{R}^N$ with two faces orthogonal to ν;
- $\mathcal{A}(\Omega)$ is the family of all open subsets of Ω;
- $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ denote the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, respectively; the symbol $\mathrm{d}x$ will also be used to denote integration with respect to $\mathcal{L}^N$;
- $\mathcal{M}(\Omega; \mathbb{R}^{d\times N})$ is the set of finite matrix-valued Radon measures on Ω; $\mathcal{M}^+(\Omega)$ is the set of non-negative finite Radon measures on Ω; given $\mu \in \mathcal{M}(\Omega; \mathbb{R}^{d\times N})$, the measure $|\mu| \in \mathcal{M}^+(\Omega)$ denotes the total variation of μ;
- $SBV(\Omega; \mathbb{R}^d)$ is the set of vector-valued *special functions of bounded variation* defined on Ω. Given $u \in SBV(\Omega; \mathbb{R}^d)$, its distributional gradient Du admits the decomposition $Du = D^a u + D^s u = \nabla u \mathcal{L}^N + [u] \otimes \nu_u \mathcal{H}^{N-1} \llcorner S_u$, where S_u is the jump set of u, $[u]$ denotes the jump of u on S_u, and ν_u is the unit normal vector to S_u; finally, $\otimes$ denotes the dyadic product;

- $L^p(\Omega;\mathbb{R}^{d\times N})$ is the set of matrix-valued p-integrable functions;
- for $p \geqslant 1$, $SD^p(\Omega) := SBV(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^{d\times N})$ is the space of structured deformations (g, G) (notice that $SD^1(\Omega)$ is the space $SD(\Omega)$ introduced in [11]); the norm in $SD(\Omega)$ is defined by $\|(g, G)\|_{SD(\Omega)} := \|g\|_{BV(\Omega;\mathbb{R}^d)} + \|G\|_{L^1(\Omega;\mathbb{R}^{d\times N})}$;
- C represents a generic positive constant that may change from line to line.

2 Preliminaries on Structured Deformations

In this section, we present the two results of great importance in the theory of structured deformations, namely the Approximation Theorem (see Theorem 2.3 below) and the integral representation result of Theorem 2.8 generalizing that in [11, Theorems 2.16 and 2.17], see also [17, Theorem 5.1]. Since the energy for a structured deformation (g, G) is defined by means of a relaxation process, it is necessary to show that the set on which the relaxation is performed is non empty. At the same time, starting from an initial energy E of integral type, Theorem 2.8 guarantees that also the relaxed energy I is of integral type, and identifies the relaxed bulk and surface energy densities H and h, respectively. The most significant result of Sect. 2, Theorem 2.10, shows that the relaxed energy densities H and h, themselves, can serve as initial energy densities for the relaxation process, thus allowing the iterative process in Sect. 3 to provide a well-defined energy for $(L + 1)$-level structured deformations.

2.1 Approximation and Definition of the Energy

A fundamental result in the theory of structured deformations is the Approximation Theorem [12, Theorem 5.8], a counterpart of which was recovered in [11, Theorem 2.12] in the SBV framework and in [19] in a broader framework. In simple terms, [11, Theorem 2.12] states that given a structured deformation $(g, G) \in SBV(\Omega;\mathbb{R}^d) \times L^1(\Omega;\mathbb{R}^{d\times N})$, there exists a sequence $\{u_n\} \subset SBV(\Omega;\mathbb{R}^d)$ that approximates it, in the sense that the sequence of functions $n \mapsto u_n$ tends to the field g, and the sequence $n \mapsto \nabla u_n$ of the absolutely continuous parts of their gradients tends to the (matrix-valued) field G, in suitable senses. The proof of the approximation theorem rests on the following two results.

Theorem 2.1 ([1, Theorem 3]) *Let $f \in L^1(\Omega;\mathbb{R}^{d\times N})$. Then there exist $u \in SBV(\Omega;\mathbb{R}^d)$, a Borel function $\beta\colon \Omega \to \mathbb{R}^{d\times N}$, and a constant $C_N > 0$ depending only on N such that*

$$Du = f\,\mathcal{L}^N + \beta\mathcal{H}^{N-1}\llcorner S_u, \qquad \int_{S_u\cap\Omega} |\beta(x)|\,\mathrm{d}\mathcal{H}^{N-1}(x) \leqslant C_N\|f\|_{L^1(\Omega;\mathbb{R}^{d\times N})}. \tag{2.1}$$

Lemma 2.2 ([11, Lemma 2.9]) *Let $u \in BV(\Omega;\mathbb{R}^d)$. Then there exist piecewise constant functions $\bar{u}_n \in SBV(\Omega;\mathbb{R}^d)$ such that $\bar{u}_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and*

$$|Du|(\Omega) = \lim_{n\to+\infty} |D\bar{u}_n|(\Omega) = \lim_{n\to+\infty} \int_{S_{\bar{u}_n}} |[\bar{u}_n](x)|\,\mathrm{d}\mathcal{H}^{N-1}(x). \tag{2.2}$$

We are now ready to state, and prove for the reader's convenience, the approximation theorem for structured deformations. Even though its proof can be found in the literature, see, *e.g.*, [11, Theorem 2.12] or [17, Proposition 2.1], it is useful to show it here as a preparation for the proof of Theorem 3.3 below.

Theorem 2.3 (Approximation Theorem) *There exists $C > 0$ (depending only on the dimension N) such that for every $(g, G) \in SD(\Omega)$ there exists a sequence $\{u_n\} \subset SBV(\Omega; \mathbb{R}^d)$ converging to (g, G) according to*

$$u_n \to g \quad \text{in } L^1(\Omega; \mathbb{R}^d) \quad \text{and} \quad \nabla u_n \overset{*}{\rightharpoonup} G \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^{d\times N}) \tag{2.3}$$

and such that, for all $n \in \mathbb{N}$,

$$|Du_n|(\Omega) \leqslant C\|(g, G)\|_{SD(\Omega)}. \tag{2.4}$$

In particular, this implies that, up to a subsequence,

$$D^s u_n \overset{*}{\rightharpoonup} (\nabla g - G)\mathcal{L}^N + D^s g \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^{d\times N}). \tag{2.5}$$

Proof Let $(g, G) \in SD(\Omega)$ and, by applying Theorem 2.1 with $f := \nabla g - G$, let $u \in SBV(\Omega; \mathbb{R}^d)$ be such that $\nabla u = \nabla g - G$. Furthermore, let $\bar{u}_n \in SBV(\Omega; \mathbb{R}^d)$ be a sequence of piecewise constant functions approximating u, as per Lemma 2.2. Then the sequence of functions

$$u_n := g + \bar{u}_n - u \tag{2.6}$$

is easily seen to approximate (g, G) in the sense of (2.3). In fact, $u_n \to g$ in L^1 and $\nabla u_n(x) = G(x)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$. Invoking the triangle inequality, the inequality in (2.1), and (2.2), we obtain for $C = 3(1 + C_N)$

$$|Du_n|(\Omega) \leqslant C\|(g, G)\|_{SD(\Omega)} \quad \text{for all } n \in \mathbb{N} \text{ sufficiently large}, \tag{2.7}$$

so that (2.4) is proved for a suitable "tail" of the sequence u_n. The convergence of $u_n \to g$ in L^1 implies that Du_n converges to Dg in the sense of distributions. The uniform bound (2.7) ensures the existence of a (not relabeled) weakly-* converging subsequence such that $Du_n \overset{*}{\rightharpoonup} Dg$ in $\mathcal{M}(\Omega; \mathbb{R}^{d\times N})$, so that, since $\nabla u_n \overset{*}{\rightharpoonup} G$ in $\mathcal{M}(\Omega; \mathbb{R}^{d\times N})$, we have

$$D^s u_n \overset{*}{\rightharpoonup} (\nabla g - G)\mathcal{L}^N + D^s g \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^{d\times N}),$$

which is (2.5). The proof is concluded. □

Remark 2.4 An inspection of the proof of Theorem 2.3 (see also [11, Theorem 2.12] and [17, Proposition 2.1]) shows that the sequence $\{u_n\}$ in (2.6) satisfies

$$\nabla u_n = G. \tag{2.8}$$

Furthermore, if $p > 1$ and either the density W in (1.2) satisfies a coercivity condition as in (2.14) below with $q = p$, or a generic bound of the type $\sup_n\{\|\nabla u_n\|_{L^p(\Omega;\mathbb{R}^{d\times N})}\} \leqslant C$ is imposed on the sequences defining $I(g, G)$ in (1.3), then the second convergence in (2.3) can be strengthened to read $\nabla u_n \rightharpoonup G \in L^p(\Omega; \mathbb{R}^{d\times N})$. On the other hand, as observed in the Approximation Theorem, the existence of sequences approximating in this latter sense is guaranteed by (2.8). Hence, in the case $p > 1$, for $(g, G) \in SD^p(\Omega)$, we use the notation $u_n \xrightarrow[SD]{p} (g, G)$ to signify

$$u_n \to g \quad \text{in } L^1(\Omega; \mathbb{R}^d) \quad \text{and} \quad \nabla u_n \rightharpoonup G \quad \text{in } L^p(\Omega; \mathbb{R}^{d\times N}). \tag{2.9}$$

Let $p \geqslant 1$ and let the initial energy $E\colon SBV(\Omega;\mathbb{R}^d) \to [0,+\infty]$ be defined by (1.2), where $W\colon \Omega \times \mathbb{R}^{d\times N} \to [0,+\infty)$ and $\psi\colon \Omega \times \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0,+\infty)$ are the initial bulk and surface energy densities, respectively. For $(g,G) \in SD^p(\Omega)$, the energy $I(g,G)$ is defined by

$$I(g,G) \coloneqq \inf\Big\{ \liminf_{n\to\infty} E(u_n) : \{u_n\} \in \mathcal{R}_p(g,G)\Big\}, \tag{2.10}$$

where

$$\mathcal{R}_p(g,G) \coloneqq \Big\{\{u_n\} \subset SBV(\Omega;\mathbb{R}^d) : u_n \xrightarrow[SD]{p} (g,G)\Big\}. \tag{2.11}$$

2.2 Generalization of the Choksi–Fonseca Scheme

Here we define the class of initial bulk and surface energy densities W and ψ that can be used to define the initial energy E to be relaxed to structured deformations. Compared to the results in [11], we add the explicit dependence on the space variable and we prove that the relaxed energy densities still belong to the class of initial energies.

Definition 2.5 (Energy densities) For $q \geqslant 1$, we denote by $\mathcal{ED}(q)$ the collection of pairs (W,ψ) of bulk and surface energy densities, where $W\colon \Omega \times \mathbb{R}^{d\times N} \to [0,+\infty)$ and $\psi\colon \Omega \times \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0,+\infty)$ are continuous functions satisfying the following conditions

(1) there exists a continuous function $\omega_W\colon [0,+\infty) \to [0,+\infty)$ with $\omega_W(s) \to 0$ as $s \to 0^+$ such that, for every $x_0, x_1 \in \Omega$ and $A \in \mathbb{R}^{d\times N}$,

$$|W(x_1,A) - W(x_0,A)| \leqslant \omega_W(|x_1 - x_0|)(1 + |A|^q); \tag{2.12}$$

(2) (q-Lipschitz continuity) there exists $C_W > 0$ such that, for all $x \in \Omega$ and $A_1, A_2 \in \mathbb{R}^{d\times N}$,

$$|W(x,A_1) - W(x,A_2)| \leqslant C_W|A_1 - A_2|\big(1 + |A_1|^{q-1} + |A_2|^{q-1}\big); \tag{2.13}$$

(3) there exists $A_0 \in \mathbb{R}^{d\times N}$ such that $W(\cdot,A_0) \in L^\infty(\Omega)$;

(4) there exists $c_W > 0$ such that, for every $(x,A) \in \Omega \times \mathbb{R}^{d\times N}$,

$$c_W|A|^q - \frac{1}{c_W} \leqslant W(x,A); \tag{2.14}$$

(5) (symmetry) for every $x \in \Omega$, $\lambda \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{N-1}$,

$$\psi(x,\lambda,\nu) = \psi(x,-\lambda,-\nu);$$

(6) there exist $c_\psi, C_\psi > 0$ such that, for all $x \in \Omega$, $\lambda \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{N-1}$,

$$c_\psi|\lambda| \leqslant \psi(x,\lambda,\nu) \leqslant C_\psi|\lambda|; \tag{2.15}$$

(7) (positive 1-homogeneity) for all $x \in \Omega$, $\lambda \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{N-1}$, and $t > 0$

$$\psi(x,t\lambda,\nu) = t\psi(x,\lambda,\nu);$$

(8) (sub-additivity) for all $x \in \Omega$, $\lambda_1, \lambda_2 \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{N-1}$,

$$\psi(x,\lambda_1+\lambda_2,\nu) \leqslant \psi(x,\lambda_1,\nu) + \psi(x,\lambda_2,\nu);$$

(9) there exists a continuous function $\omega_\psi : [0, +\infty) \to [0, +\infty)$ with $\omega_\psi(s) \to 0$ as $s \to 0^+$ such that, for every $x_0, x_1 \in \Omega$, $\lambda \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{N-1}$,

$$|\psi(x_1, \lambda, \nu) - \psi(x_0, \lambda, \nu)| \leqslant \omega_\psi(|x_1 - x_0|)|\lambda|. \tag{2.16}$$

We say that $(W, \psi) \in \mathcal{ED}^{\rm nc}(q)$ if W satisfies only (1)–(3) while ψ satisfies (5)–(9). Clearly, $\mathcal{ED}(q)$ is a smaller class than $\mathcal{ED}^{\rm nc}(q)$. For energy densities $W : \mathbb{R}^{d\times N} \to [0, +\infty)$ and $\psi : \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty)$ (therefore *not* explicitly depending on the space variable), we define the class $\mathcal{ED}_{CF}(q)$ of pairs (W, ψ) satisfying (the obvious modifications of) (2), (4), (5), (6), (7), and (8); as above, we say that $(W, \psi) \in \mathcal{ED}^{\rm nc}_{CF}(q)$ if W satisfies only (2) while ψ satisfies only (5)–(8).

In this work, relevant values for q will be either $q = p > 1$ or $q = 1$.

Remark 2.6 We point out the following facts:

(i) as observed in [8, pages 298 and 305], the symmetry property (5) in Definition 2.5 ensures that $\psi(x, \lambda, \nu)$ can be equivalently seen as a function of x and $\lambda \otimes \nu$, for every $x \in \Omega$, $\lambda \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{N-1}$.

(ii) coercivity conditions (2.14) and (2.15) can rule out some physical relevant settings in particular, some arising from fracture mechanics (see [11, Remark 3.3]). Extra bounds on the norms of the admissible sequences can overcome these constraints to prove Theorem 2.8 below, as observed in the proof of [11, Theorem 2.16], see also Remark 2.9. Therefore, we can relax the request in property (6) of Definition 2.5 by requiring that

$$0 \leqslant \psi(x, \lambda, \nu) \leqslant C|\lambda|. \tag{2.17}$$

On the other hand (as evident also in the proof of [11, formula (4.22) and in the entire Sect. 5]), our coercivity assumptions on W (or milder ones of linear type) and ψ are crucial to deduce that densities H and h have the same type of properties as W and ψ, and thus it will be possible to apply Theorem 2.8, in turn, to H and h.

(iii) Notice that, if ψ does not depend explicitly on x, condition (2.17) need not be imposed as an assumption, since it follows immediately from the continuity and the positive 1-homogeneity of ψ: indeed, continuity and property (7) of Definition 2.5 imply that $\psi(0, \nu) = 0$ for every $\nu \in \mathbb{S}^{N-1}$ and so, for every $(\lambda, \nu) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{S}^{N-1}$, we have

$$0 \leqslant \psi(\lambda, \nu) = |\lambda|\psi\Big(\frac{\lambda}{|\lambda|}, \nu\Big) \leqslant C|\lambda|,$$

where $C = \max_{\mathbb{S}^{d-1}\times\mathbb{S}^{N-1}} \psi$, which is bounded by continuity.

For $A, B \in \mathbb{R}^{d\times N}$, let $a_A(x) := Ax$ be the linear function with (constant) gradient A and let

$$\mathcal{C}^{\rm bulk}(A, B) := \Big\{u \in SBV(Q; \mathbb{R}^d) : u|_{\partial Q} = a_A|_{\partial Q}, \int_Q \nabla u(x)\,{\rm d}x = B, |\nabla u| \in L^p(Q)\Big\}; \tag{2.18}$$

for $\lambda \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{N-1}$ let

$$\mathcal{C}^{\rm surf}(\lambda, \nu) := \big\{u \in SBV(Q_\nu; \mathbb{R}^d) : u|_{\partial Q_\nu} = s_{\lambda,0,\nu}|_{\partial Q_\nu}, \nabla u(x) = 0 \text{ for } \mathcal{L}^N\text{-a.e. } x \in Q_\nu\big\}, \tag{2.19}$$

where, for $\lambda_1, \lambda_2 \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{N-1}$, the function $s_{\lambda_1,\lambda_2,\nu}$ is defined by

$$s_{\lambda_1,\lambda_2,\nu}(x) := \begin{cases} \lambda_1 & \text{if } x \cdot \nu \geqslant 0, \\ \lambda_2 & \text{if } x \cdot \nu < 0. \end{cases} \tag{2.20}$$

Given $x \in \Omega$, $U \in \mathcal{A}(\Omega)$, and $u \in SBV(U; \mathbb{R}^d)$, we define the localized bulk and surface energies for the relaxation process

$$E_x^{\text{bulk}}(u; U) := \int_U W(x, \nabla u(y))\, \mathrm{d}y + \int_{S_u \cap U} \psi(x, [u](y), \nu_u(y))\, \mathrm{d}\mathcal{H}^{N-1}(y), \tag{2.21a}$$

$$E_x^{\text{surf}}(u; U) := \int_{S_u \cap U} \psi(x, [u](y), \nu_u(y))\, \mathrm{d}\mathcal{H}^{N-1}(y), \tag{2.21b}$$

and we define the relaxed bulk and surface energy densities $H\colon \Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N} \to [0, +\infty)$ and $h\colon \Omega \times \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty)$, respectively, by

$$H(x, A, B) := \inf \left\{ E_x^{\text{bulk}}(u; Q) : u \in \mathcal{C}^{\text{bulk}}(A, B) \right\}, \tag{2.22a}$$

$$h(x, \lambda, \nu) := \inf \left\{ E_x^{\text{surf}}(u; Q_\nu) : u \in \mathcal{C}^{\text{surf}}(\lambda, \nu) \right\}. \tag{2.22b}$$

The first result that we provide is the sequential characterization of the bulk energy density H defined in (2.22a).

Proposition 2.7 (Sequential characterization of H) *Under Assumptions* 2.5-(1), (2), (6), (8), *and* (9), *for every* $(x, A, B) \in \Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N}$, *we have* $H(x, A, B) = \widetilde{H}(x, A, B)$, *where*

$$\widetilde{H}(x, A, B) := \inf \left\{ \liminf_{n\to\infty} E_x^{\text{bulk}}(u_n; Q) : \{u_n\} \in \mathcal{C}_{\text{seq}}^{\text{bulk}}(A, B) \right\} \tag{2.23}$$

and

$$\mathcal{C}_{\text{seq}}^{\text{bulk}}(A, B) := \big\{ \{u_n\} \subset SBV(Q; \mathbb{R}^d) : u_n \to a_A \text{ in } L^1(Q; \mathbb{R}^d), \\ \nabla u_n \rightharpoonup B \text{ in } L^p(Q; \mathbb{R}^{d\times N}) \big\}. \tag{2.24}$$

Proof The proof follows the lines of that of [11, Proposition 3.1]. □

Theorem 2.8 (Integral representation theorem) *Let* $p \geqslant 1$ *and let* $(W, \psi) \in \mathcal{ED}^{\text{nc}}(q)$, *with either* $q = p$ *or* $q = 1$, *as in Definition* 2.5. *For* $(g, G) \in SD^p(\Omega)$, *let* $I(g, G)$ *be defined by* (2.10). *Then the functional* $I\colon SD^p(\Omega) \to [0, +\infty)$ *admits the integral representation*

$$I(g, G) = \int_\Omega H(x, \nabla g(x), G(x))\, \mathrm{d}x + \int_{\Omega \cap S_g} h(x, [g](x), \nu_g(x))\, \mathrm{d}\mathcal{H}^{N-1}(x), \tag{2.25}$$

where H *and* h *are defined in* (2.22a), (2.22b).

Proof This theorem can be found in [17, Theorem 5.1]. Here we just highlight some differences from the result in [17], which are due to the definition of the class $\mathcal{R}_p$ in (2.11). We notice that, for $(W, \psi) \in \mathcal{ED}_{CF}^{\text{nc}}(p)$, the result is [11, Theorem 2.16], which provides cell formulae for the relaxed bulk and surface energy densities H and h, respectively, which are as in (2.22a), (2.22b), but with no x-dependence. On the contrary, for $(W, \psi) \in \mathcal{ED}_{CF}^{\text{nc}}(1)$,

the result corresponds to [11, Theorem 2.17], which provides the cell formula for the relaxed bulk energy density H as in (2.22a), and for the relaxed surface energy density h a cell formula similar to that in (2.22b), but containing also a contribution of the recession function W^∞ of the initial bulk energy density. This is not the case in our context because the class $\mathcal{R}_p$ contains a uniform bound on the L^p norms of the gradients of the sequences converging to (g, G), and this is enough to prevent the appearance of W^∞ in the cell formula for h.

As done in [17, Theorem 5.1], and previously in [7, Theorem 3.2], the explicit x-dependence is dealt with thanks to conditions (1) and (9) of Definition 2.5, borrowing techniques from [5]. □

Remark 2.9 Assuming coercivity of the bulk energy density W (see property (4) in Definition 2.5) would make the proofs a bit easier. Nonetheless, we prefer to state the hypotheses of Theorem 2.8 without this condition for the sake of generality. The integral representation theorem proved in [11] is also stated without the coercivity assumption, which is temporarily added in the proof to obtain L^p boundedness of the gradients of minimizing sequences, and then is removed. In our setting, the very definition of the class $\mathcal{R}_p$ carries the needed boundedness, so that our Theorem 2.8 can be safely stated without assuming condition (4) of Definition 2.5.

In the next theorem, we collect the properties of the relaxed energy densities H and h defined in (2.22a), (2.22b). We start from a pair $(W, \psi) \in \mathcal{ED}(q)$ and we prove that, via (2.22a), (2.22b), they generate a pair $(H^B, h) \in \mathcal{ED}(1)$. To show this, the coercivity of the initial bulk energy density W is not only needed, but will also hold true for the density $H^B(x, A) = H(x, A, B)$.

Theorem 2.10 (Properties of the relaxed energy densities) *Let $p > 1$ and let $(W, \psi) \in \mathcal{ED}(p)$. Let $H\colon \Omega \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N} \to [0, +\infty)$ and $h\colon \Omega \times \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty)$ be the functions defined in (2.22a), (2.22b). Let us define, for $B \in \mathbb{R}^{d\times N}$, the function $H^B\colon \Omega \times \mathbb{R}^{d\times N} \to [0, +\infty)$ by $(x, A) \mapsto H^B(x, A) := H(x, A, B)$. Then $(H^B, h) \in \mathcal{ED}(1)$. Moreover, the function H is p-Lipschitz continuous in the third component, namely for every $(x, A) \in \Omega \times \mathbb{R}^{d\times N}$ there exists a constant $C > 0$ such that for every $B_1, B_2 \in \mathbb{R}^{d\times N}$,*

$$|H(x, A, B_1) - H(x, A, B_2)| \leqslant C|B_1 - B_2|(1 + |B_1|^{p-1} + |B_2|^{p-1}). \tag{2.26}$$

Proof We start by proving that there exist constants $\bar{c}_H, \bar{C}_H > 0$ such that for every $x \in \Omega$ and $A, B \in \mathbb{R}^{d\times N}$ there holds

$$\bar{c}_H(|A| + |B|^p) - \frac{1}{\bar{c}_H} \leqslant H(x, A, B) \leqslant \bar{C}_H(1 + |A| + |B|^p). \tag{2.27}$$

The sequence $\{u_n\}$ provided by Theorem 2.3 to approximate the structured deformation (a_A, B) belongs to $\mathcal{C}^{\text{bulk}}_{\text{seq}}(A, B)$, so that, by Proposition 2.7, we obtain the upper bound in (2.27) by means of the following chain of inequalities

$$\begin{aligned} H(x, A, B) &\leqslant \liminf_{n\to\infty} \left(\int_Q W(x, \nabla u_n(y))\, \mathrm{d}y + \int_{S_{u_n}\cap Q} \psi(x, [u_n](y), \nu_{u_n}(y))\, \mathrm{d}\mathcal{H}^{N-1}(y) \right) \\ &\leqslant \liminf_{n\to\infty} \big(W(x, B) + C_\psi |D^s u_n|(Q) \big) \leqslant \bar{C}_H(1 + |A| + |B|^p), \end{aligned}$$

where we have used the fact that $\nabla u_n = B$ and (2.15) in the second inequality, and (2.13) jointly with property (3) in the third one; the constant $\bar{C}_H$ depends on N, on the norm of A_0, on $\|W(\cdot, A_0)\|_{L^\infty(\Omega)}$, and on C_W and C_ψ.

On the other hand, by (2.23) there exists a recovery sequence converging to (a_A, B) in the sense of (2.9). Thus, by the coercivity in (2.14) and (2.15) and the lower semicontinuity of the L^p norm and of the total variation of measures, we obtain that, for $\tilde{c}_H := \min\{c_W, c_\psi\}$,

$$H(x, A, B) \geqslant \tilde{c}_H(|B|^p + |A - B|) - \frac{1}{\tilde{c}_H} \geqslant \bar{c}_H(|A| + |B|^p) - \frac{1}{\bar{c}_H}, \tag{2.28}$$

for a suitable constant $\bar{c}_H \in (0, \tilde{c}_H)$, so that (2.27) is proved.

To show that H^B satisfies property (1) of Definition 2.5 with $q = 1$, we argue in the following way. Let $x_0, x_1 \in \Omega$ and $A \in \mathbb{R}^{d\times N}$ be given, and let $\varepsilon > 0$. By definition (2.22a), there exists $u^\varepsilon \in \mathcal{C}^{\text{bulk}}(A, B)$ such that

$$E^{\text{bulk}}_{x_1}(u^\varepsilon; Q) < H(x_1, A, B) + \varepsilon = H^B(x_1, A) + \varepsilon. \tag{2.29}$$

Therefore, since the same u^ε is also a competitor for $H(x_0, A, B)$, we have, defining the function $\tilde{\omega}_H(s) := \max\{\omega_W(s), \omega_\psi(s)\}$,

$$\begin{aligned}
&H^B(x_0, A) - H^B(x_1, A) = H(x_0, A, B) - H(x_1, A, B)\\
&\leqslant E^{\text{bulk}}_{x_0}(u^\varepsilon; Q) - E^{\text{bulk}}_{x_1}(u^\varepsilon; Q) + \varepsilon\\
&\leqslant \omega_W(|x_0 - x_1|)\int_Q (1 + |\nabla u^\varepsilon(y)|^p)\,\mathrm{d}y + \omega_\psi(|x_0 - x_1|)\int_{S_{u^\varepsilon}\cap Q} |[u^\varepsilon](y)|\,\mathrm{d}\mathcal{H}^{N-1}(y) + \varepsilon\\
&\leqslant \tilde{\omega}_H(|x_0 - x_1|)\Bigg(1 + \frac{1}{c_W^2} + \frac{1}{c_W}\int_Q W(x_1, \nabla u^\varepsilon(y))\,\mathrm{d}y\\
&\qquad\qquad + \frac{1}{c_\psi}\int_{S_{u^\varepsilon}\cap Q} \psi(x_1, [u^\varepsilon](y), \nu_{u^\varepsilon}(y))\,\mathrm{d}\mathcal{H}^{N-1}(y)\Bigg) + \varepsilon\\
&\leqslant \tilde{\omega}_H(|x_0 - x_1|)\max\left\{1 + \frac{1}{c_W^2}, \frac{1}{c_W}, \frac{1}{c_\psi}\right\}\left(1 + E^{\text{bulk}}_{x_1}(u^\varepsilon; Q)\right) + \varepsilon\\
&\leqslant \tilde{\omega}_H(|x_0 - x_1|)\max\left\{1 + \frac{1}{c_W^2}, \frac{1}{c_\psi}\right\}\left(1 + H(x_1, A, B) + \varepsilon\right) + \varepsilon\\
&\leqslant \tilde{\omega}_H(|x_0 - x_1|)\max\left\{1 + \frac{1}{c_W^2}, \frac{1}{c_\psi}\right\}\left(\bar{C}_H(1 + |A| + |B|^p) + \varepsilon\right) + \varepsilon,
\end{aligned}$$

where we have used, in sequence, (2.29) in the first line, (2.12) and (2.16) in the second line, the coercivity (2.14) and (2.15) in the third line, (2.29) again in the fifth line, and finally (2.27) in the sixth line. By exchanging the roles of x_0 and x_1 and by the arbitrariness of $\varepsilon > 0$, we have obtained that H^B satisfies (2.12) with $q = 1$ and modulus of continuity $\omega_{H^B}(s) := C(c_W, c_\psi, \bar{C}_H, |B|^p)\,\tilde{\omega}_H(s)$.

To show that H^B satisfies property (2) of Definition 2.5, we recall that by [11, Proposition 5.2] the relaxed bulk energy density is uniformly continuous in the A-variable. A closer inspection of the proof of that proposition, together with properties (1) and (9) of Definition 2.5 to control the x-dependence, shows that for every $(x, B) \in \Omega \times \mathbb{R}^{d\times N}$ the function $\mathbb{R}^{d\times N} \ni A \mapsto H(x, A, B)$ is indeed Lipschitz continuous, so that

$$|H^B(x, A_1) - H^B(x, A_2)| \leqslant C|A_1 - A_2|, \tag{2.30}$$

which is (2.13) with $q = 1$. Continuity with respect to A follows. (We note, in passing, that (2.13) is trivially satisfied also with a general $q \geqslant 1$.)

To show that H^B satisfies property (3) of Definition 2.5, let $A_0 \in \mathbb{R}^{d\times N}$ be the matrix that makes the property hold true for W and let $\{u_n\} \subset SBV(\Omega;\mathbb{R}^d)$ be the sequence provided by Theorem 2.3 to approximate the structured deformation (a_{A_0}, B). Then, since $\{u_n\} \in \mathcal{C}^{\text{bulk}}_{\text{seq}}(A_0, B)$, we apply Proposition 2.7 and for $x \in \Omega$ we have

$$\begin{aligned} H^B(x, A_0) &= \widetilde{H}(x, A_0, B) \\ &\leqslant \liminf_{n\to\infty} \left(\int_Q W(x, \nabla u_n(y))\,\mathrm{d}y + \int_{S_{u_n}\cap Q} \psi(x, [u_n](y), \nu_{u_n}(y))\,\mathrm{d}\mathcal{H}^{N-1}(y) \right) \\ &\leqslant \liminf_{n\to\infty} \big(W(x, B) + C_\psi |D^s u_n|(Q) \big) \\ &\leqslant W(x, A_0) + C_W |A_0 - B| \big(1 + |A_0|^{p-1} + |B|^{p-1} \big) + C_\psi C \left\| (a_{A_0}, B) \right\|_{SD(Q)}, \end{aligned}$$

where we have used (2.13) with $q = p$ and (2.4). This shows that $\left\| H^B(\cdot, A_0) \right\|_{L^\infty(\Omega)} \leqslant C + \|W(\cdot, A_0)\|_{L^\infty(\Omega)}$, so property (3) of Definition 2.5 holds for H^B with the same A_0 that was good for W.

Inequality (2.28) shows immediately that (2.14) holds for H^B with $q = 1$, and properties (1) and (2) of H^B imply that H^B is continuous.

For what concerns the surface energy density, we observe that, by its very definition (2.22b), h satisfies properties (5), (6) with the constants $c_h := c_\psi$ and $C_h := C_\psi$, (7), and (8) of Definition 2.5. In particular, in view of (5), it can be equivalently expressed as a function of x and $\lambda \otimes \nu$, see (i) in Remark 2.6. Therefore, as observed in [20, Theorem 2.3], the function h can be seen as the restriction to rank-one matrices of a function $\Phi\colon \Omega \times \mathbb{R}^{d\times N} \ni (x, M) \mapsto \Phi(x, M) \in [0, +\infty)$.

In particular, [20, equations (15), (16)] provide, for every $x \in \Omega$,

$$\begin{aligned} \Phi(x, M) = \inf \Bigg\{ &\sum_{i=1}^m \psi(x, \lambda_i, \nu_i) : m \in \mathbb{N}\backslash\{0\}, (\lambda_i, \nu_i) \in \mathbb{R}^d \times \mathbb{S}^{N-1}, \\ &i = 1, \dots, m, \sum_{i=1}^m \lambda_i \otimes \nu_i = M \Bigg\}, \end{aligned} \tag{2.31}$$

together with the subadditivity and the positive 1-homogeneity of the map $M \mapsto \Phi(x, M)$. Therefore, $\Phi(x, \cdot)$ is convex on $\mathbb{R}^{d\times N}$, for every $x \in \Omega$, and hence it is locally Lipschitzian and thus continuous, see [15, Sect. 6.3, Theorem 1].

To show that h satisfies property (9) of Definition 2.5, we argue in the following way. Let $x_0, x_1 \in \Omega$ and $(\lambda, \nu) \in \mathbb{R}^d \times \mathbb{S}^{N-1}$ be given, and let $\varepsilon > 0$. By definition (2.22b), there exists $u^\varepsilon \in \mathcal{C}^{\text{surf}}(\lambda, \nu)$ such that

$$E^{\text{surf}}_{x_1}(u^\varepsilon; Q_\nu) < h(x_1, \lambda, \nu) + \varepsilon. \tag{2.32}$$

Therefore, since the same u^ε is also a competitor for $h(x_0, \lambda, \nu)$, we have, defining $\tilde{\omega}_h(s) := c_\psi^{-1}\omega_\psi(s)$,

$$\begin{aligned} h(x_0, \lambda, \nu) - h(x_1, \lambda, \nu) &\leqslant E^{\text{surf}}_{x_0}(u^\varepsilon; Q_\nu) - E^{\text{surf}}_{x_1}(u^\varepsilon; Q_\nu) + \varepsilon \\ &\leqslant \omega_\psi(|x_0 - x_1|) \int_{S_{u^\varepsilon}\cap Q_\nu} |[u^\varepsilon](y)|\,\mathrm{d}\mathcal{H}^{N-1}(y) + \varepsilon \end{aligned}$$

$$
\begin{aligned}
&\leqslant \tilde{\omega}_h(|x_0 - x_1|) \int_{S_{u^\varepsilon} \cap Q_v} \psi(x_1, [u^\varepsilon](y), \nu_{u^\varepsilon}(y))\, \mathrm{d}\mathcal{H}^{N-1}(y) + \varepsilon \\
&= \tilde{\omega}_h(|x_0 - x_1|) E^{\mathrm{surf}}_{x_1}(u^\varepsilon; Q_v) + \varepsilon \\
&\leqslant \tilde{\omega}_h(|x_0 - x_1|)(h(x_1, \lambda, \nu) + \varepsilon) + \varepsilon \leqslant \tilde{\omega}_h(|x_0 - x_1|)(C_\psi |\lambda| + \varepsilon) + \varepsilon,
\end{aligned}
$$

where we have used, in sequence, (2.32) in the first line, (2.16) in the second line, the coercivity (2.15) in the third line, and (2.32) again and (2.15) in the fourth line. By exchanging the roles of x_0 and x_1 and by the arbitrariness of $\varepsilon > 0$, we have obtained that h satisfies (2.16) with modulus of continuity $\omega_h(s) := C_\psi \tilde{\omega}_h(s)$. Continuity of h in all of its variables follows.

We prove now (2.26). Let $(x, A) \in \Omega \times \mathbb{R}^{d\times N}$ be fixed, let $B_1, B_2 \in \mathbb{R}^{d\times N}$, and let $\{u_n\} \subset SBV(Q; \mathbb{R}^d)$ be a recovery sequence for $H(x, A, B_2)$. Let $\{v_n\} \subset SBV(Q; \mathbb{R}^d)$ be the sequence provided by Theorem 2.3 to approximate $(0, B_1 - B_2)$, so that the sequence $\{u_n + v_n\} \in \mathcal{C}^{\mathrm{bulk}}_{\mathrm{seq}}(A, B_1)$. Then we have, for C_p a positive constant depending only on p,

$$
\begin{aligned}
&H(x, A, B_1) - H(x, A, B_2) \\
&\leqslant \liminf_{n\to\infty} \big(E^{\mathrm{bulk}}_x(u_n + v_n; Q) - E^{\mathrm{bulk}}_x(u_n; Q)\big) \\
&\leqslant \liminf_{n\to\infty} \bigg(C_W |B_1 - B_2| \int_Q (1 + |\nabla u_n(y) + B_1 - B_2|^{p-1} + |\nabla u_n(y)|^{p-1})\, \mathrm{d}y \\
&\qquad + \int_{S_{v_n} \cap Q} \psi(x, [v_n](y), \nu_{v_n}(y))\, \mathrm{d}\mathcal{H}^{N-1}(y) \bigg) \\
&\leqslant \liminf_{n\to\infty} \bigg(C_W |B_1 - B_2| \Big(1 + C_p(|B_1|^{p-1} + |B_2|^{p-1}) + (1 + C_p) \int_Q |\nabla u_n(y)|^{p-1}\, \mathrm{d}y\Big) \\
&\qquad + C_\psi \, \|D^s v_n\| \, (Q) \bigg) \\
&\leqslant \liminf_{n\to\infty} \bigg(C_W (1 + C_p) |B_1 - B_2| \Big(1 + |B_1|^{p-1} + |B_2|^{p-1} + \Big(\int_Q |\nabla u_n(y)|^p\, \mathrm{d}y \Big)^{\frac{p-1}{p}} \Big) \\
&\qquad + C_\psi C_N |B_1 - B_2| \bigg),
\end{aligned}
$$

where we have used (2.13), the sub-additivity property (8), and the fact that $\nabla v_n = B_1 - B_2$ in the second line, inequalities on powers and (2.15) in the third line, Hölder's inequality and the inequality in (2.1) in the fourth line. To treat the term containing ∇u_n, we argue in the following way, letting $c_{W,p} := \big(\max\{c_W^{-1}, c_W^{-2}\}\big)^{\frac{p-1}{p}}$,

$$
\begin{aligned}
&\liminf_{n\to\infty} \bigg(\int_Q |\nabla u_n(y)|^p\, \mathrm{d}y \bigg)^{\frac{p-1}{p}} \\
&\leqslant \liminf_{n\to\infty} \bigg(\frac{1}{c_W^2} + \frac{1}{c_W} \int_Q W(x, \nabla u_n(y))\, \mathrm{d}y \bigg)^{\frac{p-1}{p}} \\
&\leqslant \liminf_{n\to\infty} c_{W,p} \big(1 + E^{\mathrm{bulk}}_x(u_n; Q)\big)^{\frac{p-1}{p}} = c_{W,p} (1 + H(x, A, B_2))^{\frac{p-1}{p}}
\end{aligned}
$$

$$\leqslant c_{W,p}\big(1+\bar{C}_H(1+|A|+|B_2|^p)\big)^{\frac{p-1}{p}} \leqslant K\big(1+|A|^{\frac{p-1}{p}}+|B_2|^{p-1}\big)$$

where we have used the coercivity property (2.14), the non-negativity of the surface energy, the fact that $\{u_n\}$ is a recovery sequence for $H(x, A, B_2)$, the upper bound in (2.27), and where we have called K a constant depending on p, $c_{W,p}$, and $\bar{C}_H$. Therefore we have, for a certain constant $C > 0$,

$$H(x, A, B_1) - H(x, A, B_2) \leqslant C|B_1 - B_2|(1+|B_1|^{p-1}+|B_2|^{p-1});$$

reversing the role of B_1 and B_2, we finally obtain (2.26), namely that H is p-Lipschitz continuous in the third variable. As a consequence, H is continuous in its arguments and the proof is concluded. □

Remark 2.11 We observe the following facts.

(i) We point out that the inequality from above in (2.27) could have been equivalently deduced by the fact that H satisfies (2.12), (2.13), and (2.26).

(ii) We observe that (2.22b) is BV-elliptic (see [4, Definition 5.17]): indeed, since $x \in \Omega$ is fixed, it arises as a formula identical to the homogeneous version of [8, equation (11) and Proposition 2.2] and [2, equation (3.5), where the functional setting coincides with the current one]. In light of this coincidence, the density h guarantees the lower semicontinuity of surface integrals also with respect to the SBV_p convergence (see [3] and [4]). Roughly speaking one could say that h is the BV-elliptic envelope of ψ; besides, the notion has been introduced in the purely brittle fracture context, *i.e.*, in the SBV_p topology, (see [9, 10]). In particular, this observation goes in the direction that in the next relaxation iterations the h remains unchanged.

(iii) The above result still holds under weaker continuity conditions on the original bulk density W, thus enabling us to obtain a different, more analytically involved, energetic treatment of hierarchical (first-order) structured deformations. This will be the subject of a forthcoming paper.

3 Hierarchical First-Order Structured Deformations

In this section we extend the Approximation Theorem 2.3 and the integral representation Theorem 2.8 to hierarchical first-order structured deformations. In order to achieve the approximation result, Theorem 3.3 below, an appropriate notion of convergence of a (multi-indexed) sequence of SBV functions to an $(L+1)$-level (first-order) structured deformation $(g, G_1, \dots, G_L)$ is introduced in Definition 3.2. Then a recursive relaxation process is presented, in which Theorem 2.8 is recursively applied: in order to make this possible, Theorem 2.10 guarantees that after each relaxation process, the densities obtained by freezing the microscopic gradient are still in the class $\mathcal{ED}(1)$, so that Theorem 2.8 can be applied one more time.

Definition 3.1 For $L \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^N$ a bounded connected open set, we define

$$HSD_L(\Omega) := SBV(\Omega;\mathbb{R}^d) \times \underbrace{L^1(\Omega;\mathbb{R}^{d\times N}) \times \cdots \times L^1(\Omega;\mathbb{R}^{d\times N})}_{L\text{-times}}$$

the set of $(L+1)$-*level (first-order) structured deformations* on Ω. For $p > 1$, we define

$$HSD^p_L(\Omega) := SBV(\Omega;\mathbb{R}^d) \times \underbrace{L^p(\Omega;\mathbb{R}^{d\times N}) \times \cdots \times L^p(\Omega;\mathbb{R}^{d\times N})}_{L\text{-times}}.$$

An example of a three-level structured deformation is provided by a stack of bundled papers (see [14, Fig. 1]). Deformations at the macrolevel (level 1) correspond to changes in shape of the entire stack, deformations at the first submacroscopic level (level 2) correspond to changes in shape of each of the bundles, and deformations at the second submacroscopic level (level 3) correspond to changes in shape of each single sheet of paper. In this situation we have two levels of disarrangements, the first between bundled papers and the second between sheets of paper. Thus, one can consider a macroscopic deformation field g at each time, but also two tensor fields G_1 and G_2 that provide the effects at the macrolevel of geometrical changes without disarrangements at the submacroscopic levels 2 and 3, respectively, while the difference $\nabla g - G_1$ captures disarrangements in the form of slips or separations between adjacent bundles of papers, and the difference $G_1 - G_2$ captures disarrangements in the form of slips or separations between individual sheets of papers.

3.1 Approximation Theorem for Hierarchical (First-Order) Structured Deformations

The first result that we present is the definition of convergence of a sequence of SBV functions to an $(L+1)$-level structured deformation $(g, G_1, \dots, G_L)$ belonging to either $HSD_L(\Omega)$ or $HSD^p_L(\Omega)$.

Definition 3.2 Let $L \in \mathbb{N}$, let $p > 1$, let $(g, G_1, \dots, G_L) \in HSD^p_L(\Omega)$, and let $\mathbb{N}^L \ni (n_1, \dots, n_L) \mapsto u_{n_1,\dots,n_L} \in SBV(\Omega;\mathbb{R}^d)$ be a (multi-indexed) sequence. We say that $\{u_{n_1,\dots,n_L}\}$ converges in the sense of $HSD^p_L(\Omega)$ to $(g, G_1, \dots, G_L)$ if

(i) $\lim_{n_1\to\infty}\cdots\lim_{n_L\to\infty} u_{n_1,\dots,n_L} = g$, with each of the iterated limits in the sense of $L^1(\Omega;\mathbb{R}^d)$;

(ii) for all $\ell = 1, \dots, L-1$, $\lim_{n_{\ell+1}\to\infty}\cdots\lim_{n_L\to\infty} u_{n_1,\dots,n_L} =: g_{n_1,\dots,n_\ell} \in SBV(\Omega;\mathbb{R}^d)$ and

$$\lim_{n_1\to\infty}\cdots\lim_{n_\ell\to\infty} \nabla g_{n_1,\dots,n_\ell} = G_\ell,$$

with each of the iterated limits in the sense of weak convergence in $L^p(\Omega;\mathbb{R}^{d\times N})$;

(iii) $\lim_{n_1\to\infty}\cdots\lim_{n_L\to\infty} \nabla u_{n_1,\dots,n_L} = G_L$ with each of the iterated limits in the sense of weak convergence in $L^p(\Omega;\mathbb{R}^{d\times N})$;

we use the notation $u_{n_1,\dots,n_L} \xrightarrow[HSD_L]{p} (g, G_1, \dots, G_L)$ to indicate this convergence.

If $(g, G_1, \dots, G_L) \in HSD_L(\Omega)$ and if the weak L^p convergences above are replaced by weak-* convergences in $\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$, then we say that $\{u_{n_1,\dots,n_L}\}$ converges in the sense of $HSD_L(\Omega)$ to $(g, G_1, \dots, G_L)$ and we use the notation $u_{n_1,\dots,n_L} \xrightarrow[HSD_L]{*} (g, G_1, \dots, G_L)$ to indicate this convergence.

The sequential application of the idea behind the Approximation Theorem 2.3 provides the method for constructing a (multi-indexed) sequence $\{u_{n_1,\dots,n_L}\}$ that approximates an $(L+1)$-level structured deformation $(g, G_1, \dots, G_L)$.

Theorem 3.3 (Approximation Theorem for $(L+1)$-level structured deformations) *Let $(g, G_1, \dots, G_L)$ belong to either $HSD_L(\Omega)$ or $HSD^p_L(\Omega)$. Then there exists a sequence $(n_1, \dots, n_L) \mapsto u_{n_1,\dots,n_L} \in SBV(\Omega;\mathbb{R}^d)$ converging to $(g, G_1, \dots, G_L)$ in both of the senses of Definition 3.2.*

Proof We set $G_0 \coloneqq \nabla g$, for convenience. For every $\ell = 1, \dots, L$, let $u_\ell \in SBV(\Omega; \mathbb{R}^d)$ be the functions provided by Theorem 2.1 such that $\nabla u_\ell = G_{\ell-1} - G_\ell$ and let $n_\ell \mapsto \bar{u}_{n_\ell}$ be the piecewise constant sequence approximating u_ℓ in $L^1(\Omega; \mathbb{R}^d)$ provided by Lemma 2.2. We claim that the sequence of functions

$$u_{n_1,\dots,n_L} \coloneqq g + \sum_{\ell=1}^{L} (\bar{u}_{n_\ell} - u_\ell) \tag{3.1}$$

approximates $(g, G_1, \dots, G_L) \in HSD_L(\Omega)$ in the sense of the convergence $\xrightarrow[HSD_L]{*}$ or $(g, G_1, \dots, G_L) \in HSD^p_L(\Omega)$ in the sense of the convergence $\xrightarrow[HSD_L]{p}$, see Definition 3.2. Indeed,

$$\begin{aligned} &\lim_{n_1\to\infty} \cdots \lim_{n_L\to\infty} u_{n_1,\dots,n_L} = \lim_{n_1\to\infty} \cdots \lim_{n_L\to\infty} \left(g + \sum_{\ell=1}^{L} (\bar{u}_{n_\ell} - u_\ell) \right) \\ &= \lim_{n_1\to\infty} \cdots \lim_{n_{L-1}\to\infty} \left(g + \sum_{\ell=1}^{L-1} (\bar{u}_{n_\ell} - u_\ell) \right) = \cdots = g, \quad \text{in } L^1(\Omega; \mathbb{R}^d), \end{aligned} \tag{3.2}$$

proving (i). Using (3.2), we have that

$$\begin{aligned} g_{n_1,\dots,n_\ell} &\coloneqq \lim_{n_{\ell+1}\to\infty} \cdots \lim_{n_L\to\infty} u_{n_1,\dots,n_L} = \lim_{n_{\ell+1}\to\infty} \cdots \lim_{n_L\to\infty} \left(g + \sum_{j=1}^{L} (\bar{u}_{n_j} - u_j) \right) \\ &= g + \sum_{j=1}^{\ell} (\bar{u}_{n_j} - u_j) \in SBV(\Omega; \mathbb{R}^d), \end{aligned}$$

from which, observing that

$$\begin{aligned} \nabla g_{n_1,\dots,n_\ell} &= \nabla \left(g + \sum_{j=1}^{\ell} (\bar{u}_{n_j} - u_j) \right) \\ &= \left(\nabla g + \sum_{j=1}^{\ell} (G_j - G_{j-1}) \right) = G_0 + \sum_{j=1}^{\ell} (G_j - G_{j-1}) = G_\ell, \end{aligned} \tag{3.3}$$

we immediately obtain that

$$\lim_{n_1\to\infty} \cdots \lim_{n_\ell\to\infty} \nabla g_{n_1,\dots,n_\ell} = G_\ell,$$

both in the weak $L^p(\Omega; \mathbb{R}^{d\times N})$ sense and in the weak-* $\mathcal{M}(\Omega; \mathbb{R}^{d\times N})$ sense, which is condition (ii) (equality (3.3) tells us that the sequence $\{\nabla g_{n_1,\dots,n_\ell}\}$ is indeed a constant sequence). Finally, condition (iii) follows upon observing that

$$\nabla u_{n_1,\dots,n_L} = \nabla g + \sum_{\ell=1}^{L} (G_\ell - G_{\ell-1}) = G_L, \tag{3.4}$$

so that

$$\lim_{n_1\to\infty}\cdots\lim_{n_L\to\infty}\nabla u_{n_1,\ldots,n_L}=G_L,$$

both in the weak $L^p(\Omega;\mathbb{R}^{d\times N})$ sense and in the weak-* $\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ sense (equality (3.4) tells us that the sequence $\{\nabla u_{n_1,\ldots,n_L}\}$ is indeed a constant sequence). The proof is concluded. □

3.2 Recursive Relaxation

The assignment of an energy to an $(L+1)$-level structured deformation $(g, G_1, \ldots, G_L)$ can be approached in a variety of ways. The simple approach taken here is through a recursive scheme whose initial step assigns an energetic response E_{L+1} to deformations u_{L+1} at level $L+1$ through a bulk energy density $(x, A)\mapsto W_0(x, A)$ and an interfacial energy density $(x,\lambda,\nu)\mapsto\psi_0(x,\lambda,\nu)$, with $(W_0,\psi_0)\in\mathcal{ED}(p)$, and then relaxes that energetic response to yield an energetic response E_L to structured deformations $(\tilde{g}_L,\tilde{G}_L)$ at level L. Here, the field $\tilde{g}_L$ represents a deformation at level L and $\tilde{G}_L$ represents the amount of deformation at level L that is due to smooth deformations at level $L+1$, *i.e.*, $\tilde{G}_L$ is the deformation at level L without $(L+1)$-level disarrangements. The energetic response E_L to structured deformations $(\tilde{g}_L,\tilde{G}_L)$ is determined according to Theorem 2.8 by means of the cell formulas (2.22a), (2.22b) for the relaxed bulk energy density W_1 and the relaxed interfacial density ψ_1 in terms of W_0 and ψ_0. Although the initial bulk energy density W_0 has domain $\Omega\times\mathbb{R}^{d\times N}$, the relaxed bulk density W_1 has domain $\Omega\times\mathbb{R}^{d\times N}\times\mathbb{R}^{d\times N}$, so that the initial recursive step results in the introduction of an additional matrix variable for the relaxed bulk density that appears within the integrand in the expression for $E_L(\tilde{g}_L,\tilde{G}_L)$ as the argument $\tilde{G}_L(x)$ of $W_1(x,\nabla\tilde{g}_L(x),\tilde{G}_L(x))$. The context of Theorem 2.8 assures that the arguments of ψ_1 are the same as those of ψ_0.

The simplicity of the present approach lies in the freezing of the additional argument $\tilde{G}_L(x)$ at a given but arbitrary matrix B_L for all subsequent steps which, accordingly, we describe as "partial relaxations". This allows us to replace $(x, A)\mapsto W_0(x, A)$ in the initial step by $(x, A)\mapsto W_1(x, A, B_L)$ and to replace ψ_0 by ψ_1 to start the next recursive step. Theorem 2.8 then can be applied to E_L, with densities $(x, A)\mapsto W_1(x, A, B_L)$ and ψ_1, because Theorem 2.10 guarantees that $(W_1(\cdot,\cdot,B_L),\psi_1)\in\mathcal{ED}(1)$, thereby accomplishing the first in the sequence of partial relaxations. We claim here that a single initial relaxation followed by L partial relaxations can be carried out, resulting in a pair $(W_L(\cdot,\cdot,B_1,\ldots,B_L),\psi_L)$ of multiply relaxed energy densities in terms of the L matrices $B_1,\ldots,B_L$ that were introduced, one-by-one, at successive steps during the recursive procedure. Consequently, the energy E_1 at level 1, the macroscopic level, can be assigned to an $(L+1)$-level structured deformation $(g, G_1, \ldots, G_L)$ via the formula

$$\begin{aligned}E_1(g,G_1,\ldots,G_L)\coloneqq&\int_\Omega W_L(x,\nabla g(x),G_1(x),\ldots,G_L(x))\,\mathrm{d}x\\&+\int_{\Omega\cap S_g}\psi_L(x,[g](x),\nu_g(x))\,\mathrm{d}\mathcal{H}^{N-1}(x).\end{aligned}$$

This assignment of energy corresponds to a step-wise optimization as the number of levels covered by the hierarchical structured deformation is increased by one at each recursive relaxation step, while, as is detailed in Sect. 4, other possible assignments of energy might

optimize without explicit passage from one level to another and without freezing the values of any of the fields $G_1, \dots, G_L$.

The discussion of recursive relaxation in the preceding paragraphs is preliminary to a recursive specification in which the index k for a recursive stage represents the level $L + 1 - k$ associated with submacroscopic refinement. For example, the recursive stage $k = 0$ corresponds to the initial level $L + 1$, the "finest" submacroscopic level, while the stage $k = L$ corresponds to level 1, the macroscopic level. In case the recursion should break down at some stage, we specify an integer $m \in \{1, \dots, L\}$ to denote that stage and, accordingly, require that $k \in \{0, \dots, m\}$. Given $p > 1$, $m \in \{0, \dots, L\}$, and $(W_0, \psi_0) \in \mathcal{ED}(p)$, we specify recursively pairs (W_k, ψ_k) for $k \in \{1, \dots, m\}$ with

$$W_k \colon \Omega \times \left(\mathbb{R}^{d\times N}\right)^{k+1} \to [0, +\infty) \quad \text{and} \quad \psi_k \colon \Omega \times \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty) \tag{3.5}$$

emerging from the following counterparts of the cell formulae (2.22a), (2.22b) in which, for notational convenience, we write $\mathbf{B}_k := (B_k, \dots, B_1) \in (\mathbb{R}^{d\times N})^k$ for $k \in \{1, \dots, m\}$. Hence the cell formulas at stage k are, for all $A_k, B_1, \dots, B_k \in \mathbb{R}^{d\times N}$,

$$\begin{aligned} W_k(x, A_k, \mathbf{B}_k) := \inf \Bigg\{ & \int_Q W_{k-1}(x, \nabla u(y), \mathbf{B}_{k-1})\, \mathrm{d}y \\ & + \int_{Q\cap S_u} \psi_{k-1}(x, [u](y), \nu_u(y))\, \mathrm{d}\mathcal{H}^{N-1}(y) : u \in \mathcal{C}^{\mathrm{bulk}}(A_k, B_k) \Bigg\}, \\ \psi_k(x, \lambda, \nu) := \inf \Bigg\{ & \int_{Q_\nu\cap S_u} \psi_{k-1}(x, [u](y), \nu_u(y))\, \mathrm{d}\mathcal{H}^{N-1}(y) : u \in \mathcal{C}^{\mathrm{surf}}(\lambda, \nu) \Bigg\}. \end{aligned} \tag{3.6}$$

We note that in the passage from the bulk energy W_{k-1} to W_k, the expression $W_{k-1}(x, A_{k-1}, B_{k-1}, \dots, B_1)$ is replaced by $W_k(x, A_k, B_k, B_{k-1}, \dots, B_1)$, so that the single matrix A_{k-1} in the arguments of W_{k-1} is replaced by the pair of matrices (A_k, B_k) in the arguments of W_k.

In particular, defining

$$\Omega \times \mathbb{R}^{d\times N} \ni (x, A) \mapsto W_k^{\mathbf{B}_k}(x, A) := W_k(x, A, \mathbf{B}_k), \tag{3.7}$$

by Theorem 2.10 $(W_k^{\mathbf{B}_k}, \psi_k) \in \mathcal{ED}(1)$ for every $k \in \{0, \dots, m\}$. Due to this fact, there is no stopping point to this process which allows us to consider energies associated to hierarchical structured deformations with as many levels as needed.

For each $(L + 1)$-level structured deformation $(g, G_1, \dots, G_L)$, for each level $\ell \in \{1, \dots, L\}$, and for each $\tilde{g}_\ell \in SBV(\Omega, \mathbb{R}^d)$, the $(L + 2 - \ell)$-tuple $(\tilde{g}_\ell, G_\ell, \dots, G_L)$ is an $(L + 2 - \ell)$-level structured deformation to which we assign the (partially) relaxed energy

$$\begin{aligned} E_\ell(\tilde{g}_\ell, G_\ell, \dots, G_L) := & \int_\Omega W_{L+1-\ell}(x, \nabla\tilde{g}_\ell(x), G_\ell(x), \dots, G_L(x))\, \mathrm{d}x \\ & + \int_{\Omega\cap S_g} \psi_{L+1-\ell}(x, [\tilde{g}_\ell](x), \nu_{\tilde{g}_\ell}(x))\, \mathrm{d}\mathcal{H}^{N-1}(x). \end{aligned} \tag{3.8}$$

We observe that the arguments given in the proof of Theorem 2.10 to obtain property (1) for the function $x \mapsto H(x, A, B)$, the Lipschitz continuity of $A \mapsto H(x, A, B)$ (see (2.30)), and the p-Lipschitz continuity of $B \mapsto H(x, A, B)$ (see (2.26)), applied to $W_{L+1-\ell}$, guarantee the measurability and the integrability of the integrand $x \mapsto W_{L+1-\ell}(x, \nabla\tilde{g}_\ell(x), G_\ell(x), \dots, G_L(x))$, so that the energy (3.8) is well defined.

The reasoning that we have just carried out provides a proof of the following statement.

Theorem 3.4 (Well-posedness of the recursive relaxation) *Let $p > 1$ and let $(W_0, \psi_0) \in \mathcal{ED}(p)$; let $L \in \mathbb{N}$ and $(g, G_1, \dots, G_L) \in HSD^p_L(\Omega)$. Then for every level $\ell \in \{1, \dots, L\}$ and $\tilde{g}_\ell \in SBV(\Omega; \mathbb{R}^d)$ the (partially) relaxed energy $E_\ell(\tilde{g}_\ell, G_\ell, \dots, G_L)$ at level ℓ in (3.8) is well defined.*

3.3 An Explicit Example

We present here an example of the recursive process described in Sect. 3.2 that, although not strictly covered by the hypotheses of Theorem 3.4, is also well-posed in the sense of Theorem 3.4 and yields explicit formulas for the relaxed energy at every stage of the recursion. We set $d = N$, let $p > 1$, and consider initial energy densities $W_0 \colon \mathbb{R}^{N\times N} \to [0, +\infty)$ convex and satisfying property (2) of Definition 2.5 with $q = p$ and $\psi_0 \colon \mathbb{R}^N \times \mathbb{S}^{N-1} \to [0, +\infty)$ given by

$$\psi_0(\lambda, \nu) = |\lambda \cdot \nu| \,. \tag{3.9}$$

Because $p > 1$ and because ψ_0 satisfies (7) and (8) in Definition 2.5, the conclusions of Theorem 2.8, including the cell formulas (2.22a), (2.22b), apply in this example (see Remark 2.6). It is proved in [16, Proposition 3.6] that under these hypotheses we have the following form of the relaxed bulk energy density in (2.22a), which we will denote here by W_1,

$$W_1(A_1, B_1) = W_0(B_1) + \inf\left\{ \int_{Q\cap S_u} \psi_0([u](y), \nu_u(y))\, \mathrm{d}\mathcal{H}^{N-1}(y) : u \in \mathcal{C}^{\mathrm{bulk}}(A_1, B_1) \right\}. \tag{3.10}$$

The minimization problem in the right-hand side of formula (3.10) was solved in [6, 18], where it was proved that

$$\inf\left\{ \int_{Q\cap S_u} \psi_0([u](y), \nu_u(y))\, \mathrm{d}\mathcal{H}^{N-1}(y) : u \in \mathcal{C}^{\mathrm{bulk}}(A_1, B_1) \right\} = |\operatorname{tr}(A_1 - B_1)|$$

(see [20, Theorem 2.3] for a general formula). The results in [18, 20] also give the explicit expression for the relaxed surface energy density in (2.22b), which we will denote here by ψ_1,

$$\psi_1(\lambda, \nu) = \psi_0(\lambda, \nu) = |\lambda \cdot \nu|.$$

(It is worth to underline that ψ_0 is jointly convex, hence BV-elliptic (see [4, Definition 5.17 and Theorem 5.20] and Remark 2.11(ii)), hence $\psi_1 = h = \psi_0$.) Therefore we have

$$\begin{aligned} W_1(A_1, B_1) &= W_0(B_1) + |\operatorname{tr}(A_1 - B_1)| \\ \psi_1(\lambda, \nu) &= |\lambda \cdot \nu|. \end{aligned} \tag{3.11}$$

Upon noticing that the function $\mathbb{R}^{N\times N} \ni A \mapsto W_1^{B_1}(A) := W_1(A, B_1)$ is also convex and satisfies (2) in Definition 2.5 with $q = p$, we may then perform a second relaxation via the argument that led to (3.11) to obtain the relaxed densities

$$\begin{aligned} W_2(A_2, B_2, B_1) &= W_1^{B_1}(B_2) + |\operatorname{tr}(A_2 - B_2)| = W_1(B_2, B_1) + |\operatorname{tr}(A_2 - B_2)| \\ &= W_0(B_1) + |\operatorname{tr}(B_2 - B_1)| + |\operatorname{tr}(A_2 - B_2)| \\ \psi_2(\lambda, \nu) &= |\lambda \cdot \nu|. \end{aligned} \tag{3.12}$$

It is straightforward to show by induction that the recursive relaxation determined by W_0 and ψ_0 in this example admits relaxed densities at the k^{th} recursive stage given by

$$W_k(A_k, \mathbf{B}_k) = |\operatorname{tr}(A_k - B_k)| + \sum_{j=2}^{k} |\operatorname{tr}(B_j - B_{j-1})| + W_0(B_1) \tag{3.13a}$$

$$\psi_k(\lambda, \nu) = |\lambda \cdot \nu|. \tag{3.13b}$$

In this example, the L-fold relaxed bulk and interfacial energy densities W_L and ψ_L just obtained can be inserted in (3.8) to assign the energy E_1 to a hierarchical structured deformation $(g, G_1, \dots, G_L)$

$$E_1(g, G_1, \dots G_L) \coloneqq \int_\Omega W_L(\nabla g(x), G_1(x), \dots, G_L(x))\, \mathrm{d}x + \int_{\Omega \cap S_g} |[g](x) \cdot \nu_g(x)|\, \mathrm{d}\mathcal{H}^{N-1}(x). \tag{3.14}$$

If we put $G_0 \coloneqq \nabla g$ and use (3.13a) with $k = L$, the volume integrand $W_L(\nabla g, G_1, \dots, G_L)$ in (3.14) can be written as

$$\sum_{\ell=1}^{L} |\operatorname{tr}(G_\ell - G_{\ell-1})| + W_0(G_L). \tag{3.15}$$

The matrix-valued fields $G_{\ell-1} - G_\ell$ in (3.15) are the contributions to the macroscopic deformation gradient ∇g from the disarrangements arising in upscaling from submacroscopic level ℓ to submacroscopic level $\ell - 1$, so that the sum over ℓ represents a bulk energy density arising from hierarchical disarrangements. The remaining term $W_0(G_L)$ represents a bulk energy density arising from the contributions at the macrolevel of the deformation without disarrangements at submacroscopic level L.

4 Conclusions and Outlook

In this work we have taken the first steps towards a variational theory for hierarchical structured deformations, introduced by Deseri and Owen in [14]. In the spirit of the variational framework introduced by Choksi and Fonseca in [11], we proved two main results: (i) the Approximation Theorem 3.3, stating that each $(L + 1)$-level structured deformation $(g, G_1, \dots, G_L)$ can be approximated by a (multi-indexed) sequence $\{u_{n_1,\dots,n_L}\} \subset SBV(\Omega; \mathbb{R}^d)$ in the precise sense introduced in Definition 3.2; (ii) the recursive relaxation procedure described in Sect. 3.2 and culminating in Theorem 3.4, stating that the energy assigned to an $(L + 1)$-level structured deformation $(g, G_1, \dots, G_L)$ by means of iterated applications of the integral representation Theorem 2.8 is indeed well defined; this second result rests on Theorem 2.10, the main technical effort of this paper, in which we prove that the relaxed bulk and surface energy densities still belong to the class of energies that can be relaxed. It is important to notice that these findings provide a validation of the proposal initiated in [14], motivating the introduction of the notion of hierarchical structured deformations to model materials exhibiting multiple submacroscopic levels.

The Approximation Theorem 2.3 for first-order structured deformations (g, G) is crucial for the relaxation process (2.10), since it implies that the class $\mathcal{R}_p(g, G)$ defined in (2.11) is

not empty, so that (2.10) is a meaningful problem. The integral representation Theorem 2.8 then shows that starting from an initial energy E as in (1.2), the relaxed energy $I(g, G)$ assigned to (g, G) retains the integral form, and provides the cell formulae (2.22a), (2.22b) for the relaxed bulk and surface energy densities H and h, respectively.

At the moment, Theorems 3.3 and 3.4 work independently: the way we assign the energy to the $(L+1)$-level structured deformation $(g, G_1, \dots, G_L)$ is independent of the fact that $(g, G_1, \dots, G_L)$ can be approximated by means of the sequence $\{u_{n_1,\dots,n_L}\}$. Thanks to Theorem 2.10, our proposal for defining $E_1(g, G_1, \dots, G_L)$ via (3.8) (for $\ell = 1$) is meaningful, as formula (3.14) shows.

It would be amenable, and extremely interesting from the mathematical point of view, to define the energy of an $(L+1)$-level structured deformation $(g, G_1, \dots, G_L) \in HSD_L(\Omega)$ via relaxation of the initial energy E of (1.2) directly, namely by

$$I_1(g, G_1, \dots, G_L) := \inf \left\{ \liminf_{n\to\infty} E(u_{n_1,\dots,n_L}) : \{u_{n_1,\dots,n_L}\} \in \mathcal{R}_p(g, G_1, \dots, G_L) \right\}, \quad (4.1)$$

where a possible definition of the class $\mathcal{R}_p$ in this context could be

$$\mathcal{R}_p(g, G_1, \dots, G_L) := \Big\{ \{u_{n_1,\dots,n_L}\} \subset SBV(\Omega; \mathbb{R}^d) : u_{n_1,\dots,n_L} \xrightarrow[HSD_L]{*} (g, G_1, \dots, G_L), \\ \sup_{n_1,\dots,n_L \in \mathbb{N}^L} \left\| \nabla u_{n_1,\dots,n_L} \right\|_{L^p(\Omega;\mathbb{R}^{d\times N})} < +\infty \Big\}. \quad (4.2)$$

The Approximation Theorem 3.3 grants that the class $\mathcal{R}_p$ defined in (4.2) is non empty, so that, also in the context of hierarchical structured deformations, a definition by relaxation, such as (4.1) is meaningful. The properties of such a definition are object of research in progress. Whether the energy I_1 admits an integral representation and in which fashion the energy densities depend on the fields ∇g and $G_1, \dots, G_L$, and on the jump $[g]$ is still unknown at the moment, as is the relationship between the energies E_1 of (3.8) (for $\ell = 1$) and I_1. It seems not unreasonable to expect the latter to be smaller than the former: in the relaxation process defined in (4.1), the energy I_1 is given as the most energetically convenient way of reaching $(g, G_1, \dots, G_L)$, whereas in (3.8), the energy E_1 is the outcome of the iteration of the relaxation process (2.10), in which only one level at a time is upscaled, keeping the others frozen. This is evident in the way we define the bulk energy densities W_k at each stage by (3.7), by keeping the k fields $\mathbf{B}_k = (B_k, \dots, B_1)$ fixed, thus introducing some constraints. On the other hand, as pointed out in Remark 2.11(ii), the properties listed in Theorem 2.10 remain valid under weaker continuity assumptions on the original bulk density. The study of the energy I_1 is undertaken under these weaker conditions and is the subject of a forthcoming paper.

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Declarations

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